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# SPECTRAL DENSITY FOR MULTIPLICATION OPERATORS WITH APPLICATIONS TO FACTORIZATION OF $L^1$ FUNCTIONS

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Abstract. We give explicit formulae for the spectral density corresponding to $b(T)$ in terms of that associated with $T$, when $b$ is a finite Blaschke product and $T$ is an absolutely continuous contraction. As an application we obtain a decomposition of $L^1$ functions in terms of Hardy class functions. We use the decomposition of the spectral density corresponding to a particular multiplication operator in order to give a constructive proof of the fact that the classes $\mathbb{A}_{m,n}$ (occurring in dual algebra theory) are all distinct.

Keywords: *Spectral density, multiplication operator, dual algebra, invariant subspace.*

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## 1. INTRODUCTION

As usual, we define the Hardy space $H^2 = H^2(\mathbb{D})$ as the space of all functions $f : z \mapsto \sum\limits_{n=0}^{\infty} a_n z^n$ for which the norm $\|f\| = \Big(\sum\limits_{n=0}^{\infty} |a_n|^2\Big)^{1/2}$ is finite. It is well-known that $H^2(\mathbb{D})$ may be regarded isometrically as a closed subspace of $L^2(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle with normalized Lebesgue measure ([12], [10]) by identifying the Taylor coefficients of $f$ with the Fourier coefficients of an $L^2(\mathbb{T})$ function.

We use $P_{H_2}$ to denote the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2$, so that

$$P_{H^2} : \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\theta} \mapsto \sum_{n=0}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\theta}.$$

The complex conjugates of functions in $H^2$ form a closed subspace $\overline{H}^2$, and the associated orthogonal projection is given by

$$P_{\overline{H}^2} : \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\theta} \mapsto \sum_{n=-\infty}^{0} a_n \mathrm{e}^{\mathrm{i}n\theta}.$$

For a more general simply-connected domain $\Omega$ in $\mathbb{C}$, with at least two boundary points, and a conformal mapping $\alpha$ from $\mathbb{D}$ onto $\Omega$, we say that a function $f$ defined on $\Omega$ lies in $E^2(\Omega)$ if and only if the function $z \mapsto f(\alpha(z))(\alpha'(z))^{1/2}$ lies in $H^2$. We refer to [10] for the basic properties of these spaces. In particular, if $\Omega$ has rectifiable boundary $\Gamma$, and $f \in E^2(\Omega)$, then $f$ has nontangential limits almost everywhere on $\Gamma$, defining a function in $L^2(\Gamma)$. If $\Omega$ is a *Smirnov domain*, for example if $\Omega$ is starlike or has analytic boundary, then $E^2(\Omega)$ can be identified with the closure of the polynomials in $L^2(\Gamma)$.

Following [15], we say that a contraction $T$ on a Hilbert space $H$ lies in the class $C_{0,\cdot}$ if $\|T^n x\| \to 0$ as $n \to \infty$ for all $x \in H$, and in the class $C_{1,\cdot}$ if the only vector $x \in H$ for which $\|T^n x\| \to 0$ as $n \to \infty$ is the vector $x = 0$. Similarly, we say that $T \in C_{\cdot,0}$ if $T^* \in C_{0,\cdot}$, and $T \in C_{\cdot,1}$ if $T^* \in C_{1,\cdot}$. Finally for $\alpha, \beta \in \{0,1\}$ we define $C_{\alpha,\beta} = C_{\alpha,\cdot} \cap C_{\cdot,\beta}$.

Let $T$ be a nonzero absolutely continuous contraction on a separable infinite-dimensional Hilbert space $\mathcal{H}$, that is, one which either is completely non-unitary or has a unitary part with spectral measure absolutely continuous with respect to Lebesgue measure on the unit circle $\mathbb{T}$. For a function $f \in L^1(\mathbb{T})$, we say that $T$ *factorizes* $f$, if there exist $x, y \in \mathcal{H}$ such that $f = x \overset{T}{\cdot} y$. Explicitly, this means that the Fourier coefficients of $f$ satisfy

$$\widehat{f}(n) = (T^{*n} x, y) \quad \text{and} \quad \widehat{f}(-n) = (T^n x, y)$$

for all $n \geqslant 1$, while $\widehat{f}(0) = (x, y)$ (cf. [3], Proposition 8.3).

Recall that $T$ is said to lie in the class $\mathbb{A}$ whenever the Nagy-Foias functional calculus $\Phi_T$ associated with $T$ is an isometry. We denote by $[f]$ the equivalence class of a function $f \in L^1(\mathbb{T})$ in the quotient space $L^1(\mathbb{T})/H^1_0$ (with $H^1_0 = \{g \in H^1 : g(0) = 0\}$), which can be naturally identified with the predual of $H^\infty$. Whenever $m$ and $n$ are two indices with $1 \leqslant m, n \leqslant \aleph_0$, we denote by $\mathbb{A}_{m,n} = \mathbb{A}_{m,n}(\mathcal{H})$ the set of operators $T \in \mathcal{L}(\mathcal{H})$ lying in the class $\mathbb{A}$ for which, given any family $\{f_{ij} : 0 \leqslant i < m,\, 0 \leqslant j < m\}$ of functions in $L^1(\mathbb{T})$, one can find two sequences $(x_i)_{0 \leqslant i < m}$ and $(y_j)_{0 \leqslant j < n}$ of vectors of $\mathcal{H}$ such that $[f_{ij}] = \left[x_i \overset{T}{\cdot} y_j\right]$ (these classes were introduced in [3], and have been much studied since). It is known that the classes $\mathbb{A}_{m,n}$ are all distinct ([13]) but until now no explicit examples to show this had been given.

The paper is organized as follows. In Section 2 we recall some facts about the operator Poisson kernel, which provides a useful way of expressing the spectral density corresponding to an absolutely continuous contraction $T$. We also recall results from [8] concerning $S$, the unilateral shift of multiplicity one on the Hardy space $H^2$, and $U$, the corresponding bilateral shift on $L^2(\mathbb{T})$. It is known that $f = g \overset{S}{\cdot} h$ precisely when $f = g\overline{h}$, where $g, h \in H^2$, and $f = g \overset{U}{\cdot} h$ precisely when $f = g\overline{h}$, where $g, h \in L^2(\mathbb{T})$. Since $L^1(\mathbb{T}) = L^2(\mathbb{T})\overline{L^2(\mathbb{T})}$, any function $f \in L^1(\mathbb{T})$

can be factorized using $U$. For factorizations with $S$ the supplementary condition that $\log|f| \in L^1(\mathbb{T})$ is necessary and sufficient, by virtue of [4]. However, letting $u$ be an inner function which is not an automorphism of $\mathbb{D}$, we show that the operator $M_u$ of multiplication by $u$ on $H^2$ factorizes all $f \in L^1(\mathbb{T})$. This is in contrast to perturbation phenomena observed in the theory of dual algebras for absolutely continuous contractions $T$, where only the negative Fourier coefficients are to be recovered as soon as $T \in \mathbb{A}$ [1], [9].

In Section 3 we use the above results to obtain explicit formulae for the spectral density $x \overset{b(T)}{\cdot} y$, where $T$ is an absolutely continuous contraction and $b$ a finite Blaschke product, in terms of $x \overset{T}{\cdot} y$ and the finite cyclic group of continuous functions $u$ such that $b \circ u = b$ on $\mathbb{T}$. Applying this formula in the case $T = S$ and $b(z) = z^2$, we obtain the result that for any function $f \in L^1$ there exist $g, h \in H^2$ such that

$$f(\mathrm{e}^{\mathrm{i}t}) = \frac{(g\overline{h})(\mathrm{e}^{\mathrm{i}t/2}) + (g\overline{h})(-\mathrm{e}^{\mathrm{i}t/2})}{2}.$$

A general study of multiplication operators on spaces $E^2(\Omega)$ begins in Section 4, where it is shown that these are $C_{\cdot,0}$ contractions, and various unitary equivalences are established. These results are used in Section 5 in order to construct examples of operators lying in the classes $\mathbb{A}_{m,n} \setminus (\mathbb{A}_{m+1,1} \cup \mathbb{A}_{1,n+1})$. This is achieved by means of an explicit formula for $f \overset{M_\varphi}{\cdot} g$ when $f, g \in H^2$ and $\varphi$ is a conformal bijection from $\mathbb{D}$ to the left semi-disc $\mathbb{D}_L$. This resolves a conjecture which was formulated in [11] and provides the first concrete examples of operators showing that the classes $\mathbb{A}_{m,n}$ are all distinct.

## 2. PRELIMINARIES

2.1. THE OPERATOR POISSON KERNEL. Let $T$ be an absolutely continuous contraction on a Hilbert space $\mathcal{H}$. One can also define the function $x \overset{T}{\cdot} y$ via the operator Poisson kernel (see, for example, [14])

$$K_{r,t}(T) := (\mathrm{Id} - r\mathrm{e}^{-\mathrm{i}t}T)^{-1} + (\mathrm{Id} - r\mathrm{e}^{\mathrm{i}t}T^*)^{-1} - \mathrm{Id}$$

for $r \in (0,1)$ and $t \in [0, 2\pi)$ in the following way:

$$x \overset{T}{\cdot} y(\mathrm{e}^{\mathrm{i}t}) = \lim_{r\to 1^-} (K_{r,t}(T)x, y). \tag{2.1}$$

Note that $x \overset{T}{\cdot} y = \overline{y \overset{T}{\cdot} x}$.

LEMMA 2.1.

$$\begin{aligned} K_{r,t}(T) &= (\mathrm{Id} - r\mathrm{e}^{\mathrm{i}t}T^*)^{-1}(\mathrm{Id} - r^2T^*T)(\mathrm{Id} - r\mathrm{e}^{-\mathrm{i}t}T)^{-1} \\ &= (\mathrm{Id} - r\mathrm{e}^{-\mathrm{i}t}T)^{-1}(\mathrm{Id} - r^2TT^*)(\mathrm{Id} - r\mathrm{e}^{\mathrm{i}t}T^*)^{-1}. \end{aligned}$$

*Proof.* Note that

$$(\mathrm{Id} - r\mathrm{e}^{\mathrm{i}t}T^*)K_{r,t}(T)(\mathrm{Id} - r\mathrm{e}^{-\mathrm{i}t}T) = \mathrm{Id} - r^2T^*T$$

and

$$(\mathrm{Id} - r\mathrm{e}^{-\mathrm{i}t}T)K_{r,t}(T)(\mathrm{Id} - r\mathrm{e}^{\mathrm{i}t}T^*) = \mathrm{Id} - r^2TT^*.$$

The equalities of the above lemma are now clear. ∎

2.2. Factorizations in $L^1(\mathbb{T})$. We recall some results from [8] which will be needed later.

Lemma 2.2. *Let $f_0 \in L^\infty(\mathbb{T})$ be such that $1/f_0 \in L^\infty(\mathbb{T})$. Then $L^1(\mathbb{T}) = H^2\overline{H}^2 + \mathbb{C}f_0$.*

*Proof.* We leave this as an exercise for the reader. Alternatively, see [8]. ∎

Theorem 2.3. *Let $T = S \oplus A$, where $S \in \mathcal{L}(H^2)$ is the unilateral shift and $A \in \mathcal{L}(\mathcal{H})$ is an absolutely continuous contraction with $A \neq 0$. Suppose that $A$ factorizes some function $f_0 \in L^\infty(\mathbb{T})$ such that $1/f_0$ is also in $L^\infty(\mathbb{T})$. Then for all $f \in L^1(\mathbb{T})$ there exist $g \oplus x, h \oplus y \in H^2 \oplus \mathcal{H}$ such that $f = (g \oplus x) \overset{T}{\cdot} (h \oplus y)$.*

*Proof.* Let $f \in L^1(\mathbb{T})$. Using Lemma 2.2, there exist $g, h \in H^2$ and $\lambda \in \mathbb{C}$ such that $f = g\overline{h} + \lambda f_0$. Notice that $g\overline{h} = g \overset{S}{\cdot} h$. Our hypothesis implies that there exist $x, y \in \mathcal{H}$ such that $\lambda f_0 = x \overset{A}{\cdot} y$, and hence $f = (g \oplus x) \overset{T}{\cdot} (h \oplus y)$. ∎

Remark 2.4. The hypotheses on $A$ can be replaced by the identical hypotheses on its adjoint, since we always have

$$\Big(x \overset{A^*}{\cdot} y\Big)(\mathrm{e}^{\mathrm{i}\theta}) = \Big(x \overset{A}{\cdot} y\Big)(\mathrm{e}^{-\mathrm{i}\theta}).$$

Corollary 2.5. *If $T = S \oplus A$ where $A \neq 0$ and $\sigma_p(A) \cap \mathbb{D} \neq \emptyset$, then $T$ factorizes all functions in $L^1(\mathbb{T})$.*

*Proof.* Suppose that $\lambda \in \sigma_p(A) \cap \mathbb{D}$ and let $x$ be a unit vector in $\operatorname{Ker}(A - \lambda \operatorname{Id})$. Then $(A^n x, x) = \lambda^n$ and $(A^{*n} x, x) = \overline{\lambda}^n$ for $n \geqslant 1$, and hence $A$ factorizes the Poisson kernel

$$P_\lambda(z) = 1 + \sum_{n=1}^{\infty} \lambda^n z^n + \sum_{n=1}^{\infty} \lambda^n \overline{z}^n = \frac{1 - |\lambda|^2}{|1 - \lambda\overline{z}|^2}.$$

This is an invertible function in $L^\infty(\mathbb{T})$, and the result follows from Theorem 2.3. ∎

Remark 2.6. In particular, if $A \neq 0$, and $A$ is a finite-rank operator (and hence $A \in C_0$), then $S \oplus A$ factorizes any $L^1$ function directly. On the other hand (cf. [6]), the condition $T \oplus A \in \mathbb{A}_{m,n}$ implies that $T \in \mathbb{A}_{m,n}$, and so $A$ does not contribute to the possibility of recovering simultaneously the negative Fourier coefficients of a matrix of functions in $L^1(\mathbb{T})$.

Corollary 2.7. *If $T = u(S)$, where $u$ is an inner function which is not an automorphism of $\mathbb{D}$, then for all $f \in L^1(\mathbb{T})$ there exist $g, h \in H^2$ such that $f = g \overset{u(S)}{\cdot} h$.*

*Proof.* From [6], the operator $u(S)$ is unitarily equivalent to $S \oplus \cdots \oplus S$ with $n$ summands if $u$ is rational of degree $n$, and to a countably infinite direct sum of copies of $S$ otherwise. Therefore $u(S)$ is unitarily equivalent to $S \oplus A$ where $\sigma_p(A^*) \neq \emptyset$. Hence the result follows by Theorem 2.3 and Remark 2.4. ∎

## 3. THE LINK BETWEEN $x \overset{b(T)}{\cdot} y$ AND $x \overset{T}{\cdot} y$

THEOREM 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be any absolutely continuous contraction, and let $b$ be a finite Blaschke product. Then, for every $x, y \in \mathcal{H}$, we have:*

$$\Big(x \overset{b(T)}{\cdot} y\Big)(\mathrm{e}^{\mathrm{i}t}) = \sum_{j=1}^{d} \frac{(x \overset{T}{\cdot} y)(\xi_j)}{|b'(\xi_j)|} \quad a.e.,$$

*where $\xi_1, \dots, \xi_d$ are the solutions of $b(z) = \mathrm{e}^{\mathrm{i}t}$.*

*Proof.* We may partition $\mathbb{T}$ into intervals $J_1, \dots, J_d$, each of which is mapped onto $\mathbb{T}$ by $b$ (see, for example, [5]). Then, by (2.1),

$$\begin{aligned}\Big(x \overset{b(T)}{\cdot} y\Big)(\mathrm{e}^{\mathrm{i}t}) &= \lim_{r\to 1^-} (K_{r,t}(b(T))x, y)\\ &= \lim_{r\to 1^-} ((\mathrm{Id} - r\mathrm{e}^{-\mathrm{i}t} b(T))^{-1}x, y) + (x, (\mathrm{Id} - r\mathrm{e}^{-\mathrm{i}t} b(T))^{-1}y) - (x, y)\\ &= \lim_{r\to 1^-} \Big\langle (1 - r\mathrm{e}^{-\mathrm{i}t} b)^{-1}, x \overset{T}{\cdot} y\Big\rangle + \Big\langle (1 - r\mathrm{e}^{\mathrm{i}t}\overline{b})^{-1}, x \overset{T}{\cdot} y\Big\rangle - \Big\langle 1, x \overset{T}{\cdot} y\Big\rangle,\end{aligned}$$

since $(f(T)x, y) = \Big\langle f, x \overset{T}{\cdot} y\Big\rangle$ for all $f \in L^\infty(\mathbb{T})$, by the functional calculus ([2]).

Hence

$$\begin{aligned}\Big(x \overset{b(T)}{\cdot} y\Big)(\mathrm{e}^{\mathrm{i}t}) &= \lim_{r\to 1^-} \int_0^{2\pi} \Big(x \overset{T}{\cdot} y\Big)(\mathrm{e}^{\mathrm{i}\theta}) \frac{1 - r^2}{|1 - r\mathrm{e}^{-\mathrm{i}t} b(\mathrm{e}^{\mathrm{i}\theta})|^2} \frac{\mathrm{d}\theta}{2\pi}\\ &= \lim_{r\to 1^-} \sum_{j=1}^{d} \int_{J_j} \Big(x \overset{T}{\cdot} y\Big)(\mathrm{e}^{\mathrm{i}\theta}) \frac{1 - r^2}{|1 - r\mathrm{e}^{-\mathrm{i}t} b(\mathrm{e}^{\mathrm{i}\theta})|^2} \frac{\mathrm{d}\theta}{2\pi}\\ &= \lim_{r\to 1^-} \sum_{j=1}^{d} \int_0^{2\pi} \frac{\Big(x \overset{T}{\cdot} y\Big)(b^{-1}(\mathrm{e}^{\mathrm{i}\alpha}))\mathrm{e}^{\mathrm{i}\alpha}}{b'(b^{-1}(\mathrm{e}^{\mathrm{i}\alpha}))b^{-1}(\mathrm{e}^{\mathrm{i}\alpha})} \frac{1 - r^2}{|1 - r\mathrm{e}^{-\mathrm{i}t}\mathrm{e}^{\mathrm{i}\alpha}|^2} \frac{\mathrm{d}\alpha}{2\pi},\end{aligned}$$

where $\mathrm{e}^{\mathrm{i}\alpha} = b(\mathrm{e}^{\mathrm{i}\theta})$. Since $\frac{\mathrm{d}\theta}{\mathrm{d}\alpha} \geqslant 0$, we have

$$\begin{aligned}\Big(x \overset{b(T)}{\cdot} y\Big)(\mathrm{e}^{\mathrm{i}t}) &= \lim_{r\to 1^-} \sum_{j=1}^{d} \int_0^{2\pi} \frac{\Big(x \overset{T}{\cdot} y\Big)(b^{-1}(\mathrm{e}^{\mathrm{i}\alpha}))}{|b'(b^{-1}(\mathrm{e}^{\mathrm{i}\alpha}))|} \frac{1 - r^2}{|1 - r\mathrm{e}^{-\mathrm{i}t}\mathrm{e}^{\mathrm{i}\alpha}|^2} \frac{\mathrm{d}\alpha}{2\pi}\\ &= \sum_{j=1}^{d} \frac{\Big(x \overset{T}{\cdot} y\Big)(\xi_j)}{|b'(\xi_j)|}. \quad ∎\end{aligned}$$

REMARK 3.2. If one takes $\theta \in [0, 2\pi)$ such that $\mathrm{e}^{\mathrm{i}t} = b(\mathrm{e}^{\mathrm{i}\theta})$ it follows from [5] that the solutions of $b(z) = b(\mathrm{e}^{\mathrm{i}\theta})$ are given by $\xi_j = u^{(j)}(\mathrm{e}^{\mathrm{i}\theta})$, $1 \leqslant j \leqslant d$ where $u$ is a continuous function on $\mathbb{T}$ and where $u^{(j)} = \underbrace{u \circ \cdots \circ u}_{j}$. Therefore we get

$$\Big(x \overset{b(T)}{\cdot} y\Big)(\mathrm{e}^{\mathrm{i}t}) = \Big(x \overset{b(T)}{\cdot} y\Big)(b(\mathrm{e}^{\mathrm{i}\theta})) = \sum_{j=1}^{d} \frac{\Big(x \overset{T}{\cdot} y\Big)(u^{(j)}(\mathrm{e}^{\mathrm{i}\theta}))}{|b'(u^{(j)}(\mathrm{e}^{\mathrm{i}\theta}))|},$$

i.e.

$$\Big(x \overset{b(T)}{\cdot} y\Big) \circ b = \sum_{j=1}^{d} \Big(\frac{x \overset{T}{\cdot} y}{|b'|}\Big) \circ u^{(j)}.$$

The following corollary is immediate.

Corollary 3.3. *Let $b$ be a finite Blaschke product of degree $d$ and for each point $\mathrm{e}^{\mathrm{i}t} \in \mathbb{T}$ denote by $\xi_1, \ldots, \xi_d$ the solutions of $b(z) = \mathrm{e}^{\mathrm{i}t}$. Let $M_b : H^2 \to H^2$ be the operator defined by $M_b(f) = bf$. Then for each $f, g \in H^2$ one has:*

$$\Big(f \overset{M_b}{\cdot} g\Big)(\mathrm{e}^{\mathrm{i}t}) = \sum_{j=1}^{d} \frac{(f\,\overline{g})(\xi_j)}{|b'(\xi_j)|}.$$

The next corollary follows from Corollary 2.7 and Corollary 3.3.

Corollary 3.4. *Let $b$ be a finite Blaschke product of degree $d \geqslant 2$. Then for any function $f \in L^1$ there exist $g, h \in H^2$ such that*

$$f(\mathrm{e}^{\mathrm{i}t}) = \sum_{j=1}^{d} \frac{(g\overline{h})(\xi_j)}{|b'(\xi_j)|},$$

*where $\xi_1, \ldots, \xi_d$ are the solutions of $b(z) = \mathrm{e}^{\mathrm{i}t}$.*

*In particular, taking $b(z) = z^2$, for any function $f \in L^1$ there exist $g, h \in H^2$ such that*

$$f(\mathrm{e}^{\mathrm{i}t}) = \frac{(g\overline{h})(\mathrm{e}^{\mathrm{i}t/2}) + (g\overline{h})(-\mathrm{e}^{\mathrm{i}t/2})}{2}.$$

## 4. GENERAL PROPERTIES OF MULTIPLICATION OPERATORS

We now extend the ideas of the previous section in order to discuss contractive operators of multiplication by general functions in $H^\infty$, also known as *analytic Toeplitz operators*. We begin with some basic properties of these operators.

Lemma 4.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant holomorphic function and define $M_\varphi : H^2 \to H^2$ by*

$$(M_\varphi f)(z) = \varphi(z) f(z), \quad z \in \mathbb{D}.$$

*Then the operator $M_\varphi$ is a $C_{\cdot,0}$ contraction.*

*Proof.* Since $\|M_\varphi^k\| \leqslant 1$, it is clearly sufficient to verify that

$$\lim_{n\to\infty} \|(M_\varphi^*)^n e_k\| = 0 \quad \text{for } k = 0, 1, 2, \ldots,$$

where $e_k$ is the function $z \mapsto z^k$. Now

$$((M_\varphi^*)^n e_k, e_l) = (e_k, \varphi^n e_l) = 0 \quad \text{for } l > k,$$

i.e.

$$\|(M_\varphi^*)^n e_k\|^2 = \sum_{l=0}^{k} |(e_{k-l}, \varphi^n)|^2 = \sum_{r=0}^{k} |(\varphi^n, e_r)|^2.$$

Therefore it is sufficient to show that $\lim_{n\to\infty} |(\varphi^n, e_r)| = 0$ for each $r$. Write $\varphi(z) = a_0 + \varphi_1(z)$, where $a_0 = \varphi(0)$ and so $|a_0| < 1$. Then we have

$$|(\varphi^n, e_r)| \leqslant \sum_{k=0}^{r} \binom{n}{k} |a_0|^{n-k} |(\varphi_1^k, e_r)|.$$

The last term is bounded independently of $n$ by $M$. Therefore we have

$$|(\varphi^n, e_r)| \leqslant \sum_{k=0}^{r} \frac{n^k}{k!} |a_0|^{n-k} M,$$

which tends to zero as $n$ tends to $\infty$. ∎

Similar results hold on more general domains, as in the next result.

Corollary 4.2. *Let $\Omega$ be a bounded simply connected domain in $\mathbb{C}$ and $\Psi : \Omega \to \mathbb{D}$ analytic. Define $M_\Psi : E^2(\Omega) \to E^2(\Omega)$ by $M_\Psi f = \Psi f$. Then $M_\Psi$ is a $C_{\cdot,0}$ contraction.*

*Proof.* This is immediate from Lemma 4.1, since $M_\Psi$ is unitarily equivalent to the analytic Toeplitz operator $M_{\Psi\circ\alpha}$ on $H^2$ where $\alpha : \mathbb{D} \to \Omega$ is a conformal bijection. ∎

In particular, if $T$ is the operator of multiplication by the independent variable on $E^2(\mathbb{D}_L)$ where $\mathbb{D}_L$ is the left hand half-unit-disc, then $T$ is unitarily equivalent to the operator $M_\alpha$ of multiplication by $\alpha$ on $H^2$ where $\alpha : \mathbb{D} \to \mathbb{D}_L$ is a conformal bijection.

Lemma 4.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic. If $|\varphi(\mathrm{e}^{\mathrm{i}t})| < 1$ a.e. then $M_\varphi : H^2 \to H^2$ defined by $M_\varphi(f) = \varphi f$ is a $C_{00}$ contraction; otherwise it is $C_{10}$.*

*Proof.* Suppose $|\varphi(\mathrm{e}^{\mathrm{i}t})| < 1$ a.e. and take $f \in H^2$. Then for each $\varepsilon > 0$ there is a constant $\delta > 0$ such that

$$\Bigl( \int_C |f(\mathrm{e}^{\mathrm{i}t})|^2 \,\mathrm{d}t \Bigr)^{1/2} < \frac{\varepsilon}{\sqrt{2}}$$

if $C$ is a subinterval of $(0, 2\pi)$ whose length $\ell(C)$ is less than $\delta$. Now choose $\beta$ with $0 < \beta < 1$ such that $\ell(A_\beta) < \delta$ where $A_\beta = \{t : |\varphi(\mathrm{e}^{\mathrm{i}t})| > \beta\}$. Then,

$$\|\varphi^n f\|_2^2 \leqslant \int_{A_\beta} |\varphi^n f|^2(\mathrm{e}^{\mathrm{i}t}) \,\mathrm{d}t + \int_{(0,2\pi)\setminus A_\beta} |\varphi^n f|^2(\mathrm{e}^{\mathrm{i}t}) \,\mathrm{d}t \leqslant \frac{\varepsilon^2}{2} + k^{2n} \|f\|_2^2 \leqslant \varepsilon^2$$

for $n$ sufficiently large. Conversely, if $|\varphi(\mathrm{e}^{\mathrm{i}t})| = 1$ on a set $E$ of positive Lebesgue measure and if $f \in H^2$, then $\|\varphi^n f\|^2 \geqslant \int_E |f|^2(\mathrm{e}^{\mathrm{i}t}) \,\mathrm{d}t$ for all $n$ and this is nonzero for all $f \in H^2 \setminus \{0\}$. ∎

## 5. OPERATORS IN $\mathbb{A}_{m,n} \setminus (\mathbb{A}_{m+1,1} \cup \mathbb{A}_{1,n+1})$

### 5.1. The operator $T_L \oplus T_R^*$.

5.1.1. Definition and remarks. Let $L = \{z \in \mathbb{C} : (|z| = 1, \operatorname{Re}(z) \leqslant 0)$ or $(\operatorname{Re}(z) = 0, |\operatorname{Im}(z)| \leqslant 1)\}$. Thus $L$ is the boundary of the open left half unit disc; let $\mathbb{D}_L$ denote this left half disc (so $\mathbb{D}_L$ is the simply connected component of $\mathbb{C} \setminus L$). Put arc-length measure $\ell$ on $L$, and define $L^2(L, \mathrm{d}\ell)$ to be the space of (equivalence classes of) square integrable complex functions on $L$. As noted earlier, we may regard $E^2(L, \mathrm{d}\ell)$ as the closure of the polynomials in $L^2(L, \mathrm{d}\ell)$. Similarly, let us define

$$R = \{z \in \mathbb{C} : |z| = 1, \operatorname{Re}(z) \geqslant 0\} \cup \{z \in \mathbb{C} : \operatorname{Re}(z) = 0, |\operatorname{Im}(z)| \leqslant 1)\}.$$

So $R$ is the boundary of the open right half unit disc $\mathbb{D}_R$ and we define $L^2(R, \mathrm{d}\ell)$ and $E^2(R, \mathrm{d}\ell)$ in the analogous way.

Let $N_L$ be the (normal) operator of multiplication by $z$ on $L^2(L, \mathrm{d}\ell)$, and $T_L$ its (subnormal) restriction to $E^2(L, \mathrm{d}\ell)$. Let $N_R$ and $T_R$ be defined similarly relative to $R$.

Let $w = \varphi(z)$ be a map that takes the disc to the left semi-disc $\mathbb{D}_L$; for example, $z = \varphi^{-1}(w) = (w^2 - 2w - 1)/(w^2 + 2w - 1)$. Note that $\varphi'(z) = (w^2 + 2w - 1)^2/(4(w^2 + 1))$. Then, as we saw in Corollary 4.4, we have an unitary equivalence between $T_L$ and the operator $M_\varphi$ on $H^2$.

The following assertion is an immediate consequence of the fact that the membership of the classes $\mathbb{A}_{m,n}$ is invariant under unitary equivalences and the fact that an operator $T$ belongs to a class $\mathbb{A}_{m,n}$ if and only if its adjoint $T^*$ belongs to the class $\mathbb{A}_{n,m}$ with $1 \leqslant m, n \leqslant \aleph_0$.

Lemma 5.1. *Set $\widetilde{T} = T_R \oplus {T_L}^*$. Then the operator $\widetilde{T}$ belongs to the class $\mathbb{A}_{1,2}$ if and only if it belongs to the class $\mathbb{A}_{2,1}$.*

5.1.2. Factorization using $M_\varphi$. The following formula will be of use in interpreting factorizations with $M_\varphi$. In what follows $\varphi : \mathbb{D} \to \mathbb{D}_L$ is a conformal bijection, and we shall choose it so that left-hand arc $[\mathrm{e}^{\mathrm{i}\pi/2}, \mathrm{e}^{3\mathrm{i}\pi/2}]$ is mapped to itself (to achieve this, consider instead $\varphi \circ \mu$ for a suitable Möbius map $\mu$).

Lemma 5.2. *For $\pi/2 < t < 3\pi/2$ and $f, g \in H^2(\mathbb{T})$, we have*

$$f \overset{M_\varphi}{\cdot} g(\mathrm{e}^{\mathrm{i}t}) = \frac{f\overline{g}(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))\mathrm{e}^{\mathrm{i}t}}{\varphi'(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})} + \int_{-\pi/2}^{\pi/2} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})} \frac{1 - |\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2}{|1 - \mathrm{e}^{-\mathrm{i}t}\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2} \,\mathrm{d}\theta,$$

*and the second term has an analytic extension to the left-hand half plane $\{\operatorname{Re} z < 0\}$.*

*Proof.* Using Lemma 2.1, we have, for $\pi/2 < t < 3\pi/2$,

$$
\begin{aligned}
f \overset{M_\varphi}{\cdot} g(\mathrm{e}^{\mathrm{i}t})
&= \lim_{r\to 1^-} \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})} \frac{1-r^2|\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2}{|1-r\mathrm{e}^{-\mathrm{i}t}\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta \\
&= \lim_{r\to 1^-} \int_{\pi/2}^{3\pi/2} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})} \frac{1-r^2}{|1-r\mathrm{e}^{-\mathrm{i}t}\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta + \int_{-\pi/2}^{\pi/2} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})} \frac{1-|\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2}{|1-\mathrm{e}^{-\mathrm{i}t}\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta.
\end{aligned}
$$

Define a new function $\psi : \mathbb{T} \to \mathbb{T}$ by

$$
\psi(\mathrm{e}^{\mathrm{i}\theta}) = \begin{cases} \varphi(\mathrm{e}^{\mathrm{i}\theta}) & \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2}, \\ \mathrm{e}^{\mathrm{i}\theta} & \text{otherwise.} \end{cases}
$$

Then, since $\lim\limits_{r\to 1^-} \frac{1-r^2}{|1-r\mathrm{e}^{-\mathrm{i}t}\psi(\mathrm{e}^{\mathrm{i}\theta})|^2} = 0$ for $\theta \in (0, 2\pi) \setminus \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, we get

$$
\begin{aligned}
f \overset{M_\varphi}{\cdot} g(\mathrm{e}^{\mathrm{i}t})
&= \lim_{r\to 1^-} \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})} \frac{1-r^2}{|1-r\mathrm{e}^{-\mathrm{i}t}\psi(\mathrm{e}^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta + \int_{-\pi/2}^{\pi/2} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})} \frac{1-|\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2}{|1-\mathrm{e}^{-\mathrm{i}t}\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta \\
&= \frac{f\overline{g}(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))\mathrm{e}^{\mathrm{i}t}}{\varphi'(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})} + \int_{-\pi/2}^{\pi/2} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})} \frac{1-|\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2}{|1-\mathrm{e}^{-\mathrm{i}t}\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2}\,\mathrm{d}\theta,
\end{aligned}
$$

using a change of variables and the standard properties of the Poisson kernel, as in the proof of Theorem 3.1. Note that the first term involves the value of $f\overline{g}$ at just one point of the left-hand semi-circle, and the second term involves values only on the right-hand semi-circle.

Since, for $\mathrm{e}^{\mathrm{i}t}$ on the unit circle,

$$
|1-\mathrm{e}^{-\mathrm{i}t}w|^{-2} = |\mathrm{e}^{\mathrm{i}t}-w|^{-2} = (\mathrm{e}^{\mathrm{i}t}-w)^{-1}(\mathrm{e}^{-\mathrm{i}t}-\overline{w})^{-1} = \frac{\mathrm{e}^{\mathrm{i}t}}{(\mathrm{e}^{\mathrm{i}t}-w)(1-\overline{w}\mathrm{e}^{\mathrm{i}t})},
$$

it is clear that the second term has an analytic extension to the left-hand half plane $\{\operatorname{Re} z < 0\}$. ∎

REMARK 5.3. Similar formulae for $f \overset{M_\varphi}{\cdot} g(\mathrm{e}^{\mathrm{i}t})$ hold in the case where $\varphi$ lies in the disc algebra, $\|\varphi\|_\infty = 1$, and for all $\mathrm{e}^{\mathrm{i}t}$ in some subarc of $\mathbb{T}$ we have $\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}) \cap \mathbb{T}$ finite and nonempty.

We are now ready for the main result of this section.

THEOREM 5.4. *$T_L \oplus T_R^* \in \mathbb{A} \setminus (\mathbb{A}_{1,2} \cup \mathbb{A}_{2,1})$.*

*Proof.* Clearly $\sigma(T_L \oplus T_R^*) = \overline{\mathbb{D}}$, which is a sufficient condition for membership in $\mathbb{A}$. By Lemma 5.1, it is sufficient to prove that $M_\varphi \oplus T_R^* \notin \mathbb{A}_{2,1}$. Let $\delta = 1/2$ and $C_\delta = \{z \in \mathbb{C} : \operatorname{Re} z < -\delta\}$. Now consider the function $\gamma$ defined on $\mathbb{T}$ by

$$\gamma(\mathrm{e}^{\mathrm{i}t}) = \frac{\mathrm{e}^{\mathrm{i}t}}{\varphi'(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})}.$$

Take $\Omega_1$ and $\Omega_2$ to be closed subarcs of $C_\delta \cap \mathbb{T}$ such that $\ell(\Omega_1^{\mathrm{c}} \cap \Omega_2^{\mathrm{c}} \cap C_\delta) > 0$, $\ell(\Omega_1 \cap \Omega_2) = 0$, and $\ell(\Omega_j) > 0$ for $j = 1, 2$. Suppose that $M_\varphi \oplus T_R^* \in \mathbb{A}_{2,1}$. It follows that there exist functions $f_1, f_2, g$ in $H^2$ and $x_1, x_2, y$ in $E^2(R, \mathrm{d}\ell)$ such that:

$$\begin{cases} \left[f_1 \overset{M_\varphi}{\cdot} g\right] + \left[x_1 \overset{T_R{}^*}{\cdot} y\right] = [\chi_{\Omega_1}] \\ \left[f_2 \overset{M_\varphi}{\cdot} g\right] + \left[x_2 \overset{T_R{}^*}{\cdot} y\right] = [\chi_{\Omega_2}]. \end{cases}$$

Since $\sigma(T_R^*) \cap \mathbb{T} = R \cap \mathbb{T}$, it follows from the proof of Lemma 5.1 in [6] that $x_1 \overset{T_R{}^*}{\cdot} y$ and $x_2 \overset{T_R{}^*}{\cdot} y$ have analytic extensions to $C_\delta$. Using Lemma 5.2 there exist $h_1$ and $h_2$ in $H^2$ and $k_1, k_2$ analytic on $C_\delta$, such that, on $C_\delta \cap \mathbb{T}$,

$$\begin{cases} \chi_{\Omega_1}(\mathrm{e}^{\mathrm{i}t}) + \mathrm{e}^{\mathrm{i}t}h_1(\mathrm{e}^{\mathrm{i}t}) = f_1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))\overline{g(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))}\gamma(\mathrm{e}^{\mathrm{i}t}) + k_1(\mathrm{e}^{\mathrm{i}t}) \\ \chi_{\Omega_2}(\mathrm{e}^{\mathrm{i}t}) + \mathrm{e}^{\mathrm{i}t}h_2(\mathrm{e}^{\mathrm{i}t}) = f_2(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))\overline{g(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))}\gamma(\mathrm{e}^{\mathrm{i}t}) + k_2(\mathrm{e}^{\mathrm{i}t}). \end{cases} \tag{5.1}$$

On $\Omega_1^{\mathrm{c}} \cap \Omega_2^{\mathrm{c}} \cap C_\delta$ we have

$$[\mathrm{e}^{\mathrm{i}t}h_1(\mathrm{e}^{\mathrm{i}t}) - k_1(\mathrm{e}^{\mathrm{i}t})]f_2(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) = [\mathrm{e}^{\mathrm{i}t}h_2(\mathrm{e}^{\mathrm{i}t}) - k_2(\mathrm{e}^{\mathrm{i}t})]f_1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})),$$

and hence the same inequality holds on $C_\delta \cap \mathbb{T}$ since both sides of the equation are $E^1$ (Hardy class) functions on $C_\delta \cap \mathbb{D}$. Multiplying the first equation of (5.1 ) by $f_2(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))$ and the second by $f_1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))$ and subtracting, we see that

$$\chi_{\Omega_1}(\mathrm{e}^{\mathrm{i}t})f_2(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) = \chi_{\Omega_2}(\mathrm{e}^{\mathrm{i}t})f_1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) \quad \text{on } C_\delta \cap \mathbb{T},$$

which implies that $f_1$ and $f_2$ are identically zero, since they vanish on subsets of positive measure. This is absurd as $\chi_{\Omega_1}$ and $\chi_{\Omega_2}$ are not the restrictions of analytic functions. ∎

5.2. The operator $T_L^{(m)} \oplus T_R^{*(n)}$. For every pair of positive integers $m$ and $n$, denote by $T_L^{(m)} \oplus T_R^{*(n)}$ the operator given by the direct sum of $m$ copies of $T_L$ and $n$ copies of $T_R^*$.

We extend the methods of the previous section to obtain the following result, establishing Conjecture 3.5 of [11], which provides a constructive proof of the fact that the classes $\mathbb{A}_{m,n}$ are distinct. Recall that a non constructive proof was given in [13].

THEOREM 5.5. $T_L^{(m)} \oplus T_R^{*(n)} \in \mathbb{A}_{m,n} \setminus (\mathbb{A}_{m+1,1} \cup \mathbb{A}_{1,n+1})$.

*Proof.* The fact that $T_L^{(m)} \oplus T_R^{*(n)} \in \mathbb{A}_{m,n}$ is a consequence of [7] but was proved previously in Proposition 3.2 of [11]. It is sufficient to prove that $T_L^{(m)} \oplus T_R^{*(n)} \notin \mathbb{A}_{m+1,1}$; the assertion concerning $\mathbb{A}_{1,n+1}$ follows similarly, after taking the adjoint and exchanging $L$ and $R$.

With the methods of the previous section, we prove that the unitarily equivalent operator $M_\varphi^{(m)} \oplus T_R^{*(n)} \notin \mathbb{A}_{m+1,1}$. Let $\delta, C_\delta$ and $\gamma$ be as in the proof of Theorem 5.4. Let $(\Omega_l)_{1\leqslant l\leqslant m+1}$ be closed disjoint subarcs of $C_\delta \cap \mathbb{T}$, each of positive Lebesgue measure, such that $\ell\Big(\bigcap\limits_{1\leqslant l\leqslant m+1} \Omega_l \cap C_\delta\Big) > 0$. Suppose that $M_\varphi^{(m)} \oplus T_R^{*(n)} \in \mathbb{A}_{m+1,1}$. Then, as in the proof of Theorem 5.4, there exist arrays $(f_l^j)_{1\leqslant l\leqslant m+1, 1\leqslant j\leqslant m}$, $(g^j)_{1\leqslant j\leqslant m}$ and $(h_l)_{1\leqslant l\leqslant n+1}$ in $H^2$, and $(k_l)_{1\leqslant l\leqslant n+1}$ analytic on $C_\delta$ such that the following system of equations is satisfied on $C_\delta \cap \mathbb{T}$:

$$\chi_{\Omega_l}(\mathrm{e}^{\mathrm{i}t}) + \mathrm{e}^{\mathrm{i}t} h_l(\mathrm{e}^{\mathrm{i}t}) = \sum_{j=1}^{m} f_l^j(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))\overline{g^j(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t}))}\gamma(\mathrm{e}^{\mathrm{i}t}) + k_l(\mathrm{e}^{\mathrm{i}t}), \tag{5.2}$$

for $1 \leqslant l \leqslant m+1$.

So, for each $t$, the following determinant is equal to 0:

$$0 = \begin{vmatrix} f_1^1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & \cdots & f_1^m(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & (-\chi_{\Omega_1}(\mathrm{e}^{\mathrm{i}t}) + q_1(\mathrm{e}^{\mathrm{i}t})) \\ f_2^1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & \cdots & f_2^m(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & (-\chi_{\Omega_2}(\mathrm{e}^{\mathrm{i}t}) + q_2(\mathrm{e}^{\mathrm{i}t})) \\ \vdots & \vdots & \vdots & \vdots \\ f_{m+1}^1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & \cdots & f_{m+1}^m(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & (-\chi_{\Omega_{m+1}}(\mathrm{e}^{\mathrm{i}t}) + q_{m+1}(\mathrm{e}^{\mathrm{i}t})) \end{vmatrix},$$

where, for $1 \leqslant l \leqslant m+1$, we have $q_l(\mathrm{e}^{\mathrm{i}t}) = -\mathrm{e}^{\mathrm{i}t} h_l(\mathrm{e}^{\mathrm{i}t}) + k_l(\mathrm{e}^{\mathrm{i}t}))$, which is holomorphic on $C_\delta \cap \mathbb{D}$. Thus,

$$0 = \begin{vmatrix} f_1^1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & \cdots & f_1^m(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & (-\chi_{\Omega_1}(\mathrm{e}^{\mathrm{i}t})) \\ f_2^1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & \cdots & f_2^m(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & (-\chi_{\Omega_2}(\mathrm{e}^{\mathrm{i}t})) \\ \vdots & \vdots & \vdots & \vdots \\ f_{m+1}^1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & \cdots & f_{m+1}^m(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & (-\chi_{\Omega_{m+1}}(\mathrm{e}^{\mathrm{i}t})) \end{vmatrix} + H(\mathrm{e}^{\mathrm{i}t}),$$

with $H$ analytic on $C_\delta \cap \mathbb{D}$ and equal to 0 on $\Omega^{\mathrm{c}} \cap C_\delta$, where $\Omega = \bigcup\limits_{1\leqslant l\leqslant m+1} \Omega_l$. Since $\ell(\Omega^{\mathrm{c}} \cap C_\delta) > 0$, we get that the function $H$ is identically equal to 0. Therefore, for each $t$ such that $\mathrm{e}^{\mathrm{i}t} \in \Omega_{m+1}$, we get:

$$\begin{vmatrix} f_1^1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & \cdots & f_1^m(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) \\ f_2^1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & \cdots & f_2^m(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) \\ \vdots & \vdots & \vdots \\ f_m^1(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) & \cdots & f_m^m(\varphi^{-1}(\mathrm{e}^{\mathrm{i}t})) \end{vmatrix} = 0.$$

Since $\ell(\Omega_{m+1}) > 0$ and since the above determinant, say $D$, is an analytic function in the Nevanlinna class, we obtain that $D$ is identically equal to 0. It follows from (5.2) that there exists an integer $l_0 \in \{1, \ldots, m\}$ such that $\chi_{\Omega_{l_0}}$ is a linear combination of $\{\chi_{\Omega_l} : l \neq l_0, 1 \leqslant l \leqslant m\}$ up to an analytic function in the Nevanlinna class on $C_\delta \cap \mathbb{D}$. The choice of the subarcs $\Omega_l$, for $1 \leqslant l \leqslant m$, makes the last assertion absurd and this completes the proof. ∎

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