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ENDOMORPHISMS OF TYPE I VON NEUMANN ALGEBRAS WITH DISCRETE CENTER

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Abstract. A version of Cuntz-Krieger algebras associated with infinite, possibly infinite valued matrices with any number of zero entries correspond to C^*-algebras of directed graphs with any number of edges, sources, sinks, and isolated vertices. We show that the correspondence established previously between representations and $*$-endomorphisms involving the original Cuntz-Krieger algebras extends to this setting, so to a correspondence between representations of Cuntz-Krieger algebras for infinite matrices and $*$-endomorphisms of a direct sum of type I factors.

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ENDOMORPHISMS OF TYPE I VON NEUMANN ALGEBRAS WITH DISCRETE CENTER

In [3] a correspondence between unital $*$-endomorphisms of finite direct sums of type I_∞ von Neumann factors and representations of Cuntz-Krieger algebras $\mathcal{O}_A$ ([5]) is established, extending the well known correspondence ([1]) between unital $*$-endomorphisms of the algebra $\mathcal{B}(\mathcal{H})$, of all bounded operators on a Hilbert space $\mathcal{H}$, and nondegenerate representations of the Cuntz algebras $\mathcal{O}_n$. Here A is a finite square, nonnegative integer valued matrix, with no zero rows or columns. In [4], Cuntz-Krieger algebras $\mathcal{O}_B$ are introduced where B is a possibly infinite and infinite valued square matrix with no restriction on the entries. This family of algebras coincides with the family of graph algebras $G^*(E)$, E an arbitrary directed graph, also introduced in [4]. References to other approaches to Cuntz-Krieger algebras for various families of matrices, and also to graph C^*-algebras may be found in [4] and [7]. The correspondence is determined by $\mathcal{O}_B \cong G^*(E)$ where B is the vertex matrix of the graph E. In this note we describe an aspect of these algebras which underscores their suitability for further investigations. Namely, the above mentioned correspondence between endomorphisms and representations of

Cuntz and Cuntz-Krieger algebras ([3]), extends to this enlarged class of Cuntz-Krieger or graph algebras. Thus there is a correspondence between a class of $*$-endomorphisms, not necessarily unital or injective, of countable sums of type I factors and representations of the graph C^*-algebras $G^*(E)$ for arbitrary directed graphs E. The presence of such a correspondence may be taken as a guiding principle to further investigate other generalizations of Cuntz-Krieger algebras, especially to continuous situations (Section, [9], [10]).

We begin in Section 1 with a description of the graph C^*-algebra $C^*(E)$ for arbitrary directed graphs E of [7] as a relative Cuntz-Pimsner algebra, and then indicate how the graph C^*-algebra $G^*(E)$ of [4] may be viewed as an "unaugmented" relative Cuntz-Pimsner algebra. This characterization is used later in Section 3 to show that the endomorphism associated with a given representation of $G^*(E)$ does not depend on the chosen basis of the Hilbert bimodule. In Section 2 we quickly describe how an arbitrary, possibly infinite, matrix — or index — with nonnegative integer or infinite values can be associated with an endomorphism of a countable direct sum of type I factors. Although the endomorphisms we consider are not necessarily unital, or injective, there are several conditions we impose on the endomorphisms. The first condition requires that the endomorphism be unital on certain finite parts of the domain. The second condition is a reflection of the nondegeneracy of the representations we consider, while the third condition maximizes the domain of the endomorphism by collecting the factors in the kernel of the endomorphism into one containing factor.

Section 3 describes the correspondence between these endomorphisms and the nondegenerate representations of graph C^*-algebras $G^*(E)$ for arbitrary directed graphs E, extending the results of [3]. In Section 4 we use the established structure of endomorphisms to see how a condition on the graph of the endomorphism determines structure of the domain, ensuring that no finite factors appear.

NOTATION. If Y is a locally compact topological space, then $C_{\mathrm{c}}(Y)$, $C_0(Y)$, and $C_{\mathrm{b}}(Y)$ respectively denote the space of continuous complex valued functions on Y that have compact support, vanish at infinity, and are bounded. They are normed linear spaces with the sup norm $\|\cdot\|_\infty$ and an involution $*$ given by complex conjugation. If S is a subset of a set Y then χ_s denotes the characteristic function of S, and $\delta_x = \chi_{\{x\}}$. For S a linear subspace of a Hilbert space $\mathcal{H}$, $[S]$ denotes its closure in $\mathcal{H}$ and $[S]^\perp$ the closed subspace of vectors in $\mathcal{H}$ orthogonal to S. The C^*-algebra of all bounded operators on $\mathcal{H}$ is denoted $\mathcal{B}(\mathcal{H})$, and if $\mathcal{A}$ is a $*$-closed subset of $\mathcal{B}(\mathcal{H})$ then $\mathcal{A}'$ is the von Neumann algebra of bounded operators commuting with $\mathcal{A}$.

1. GRAPH ALGEBRAS AS CUNTZ-PIMSNER ALGEBRAS

Assume E is a directed graph (E^1, E^0, r, s) where the edge set E^1 and the vertex set E^0 are countable, and r, s are the range and source maps of E^1 to E^0. Let I be the set of isolated vertices, $I = E^0 \setminus (R \cup G)$ where $R = r(E^1)$ and $G = s(E^1)$. The maps r and s define maps $r_\#$ and $s_\#$ of $A = C_0(E^0)$ to $C_{\mathrm{b}}(E^1)$ where, for example, $r_\#(f) = f \circ r$. We may also view $r_\#$ as a map of $C_0(R) \to C_{\mathrm{b}}(E^1)$ and $s_\# : C_0(G) \to C_{\mathrm{b}}(E^1)$. Since $C_{\mathrm{c}}(E^1)$ is an ideal in the algebra $C_{\mathrm{b}}(E^1)$ we may define a right $C_0(R)$-module structure, or a right $C_0(E^0)$-module structure on the space $X_{\mathrm{c}} = C_{\mathrm{c}}(E^1)$ by $f \cdot h = f \cdot r_\#(h)$ for $f \in X_{\mathrm{c}}$, and $h \in C_0(R)$ or $C_0(E^0)$. Similarly X_{c} can be viewed as a left $C_0(G)$, or a left $C_0(E^0)$-module with the left action defined using the map $s_\#$.

One can check that the map $\psi_r : X_{\mathrm{c}} \to C_0(R)$ defined by $\psi_r(\delta_e) = \delta_{r(e)}$ for $e \in E^1$ is a conditional expectation with respect to the right module structure on X_{c}; so $\psi_r(f^*) = \psi_r(f)^*$ and $\psi_r(f \cdot h) = \psi_r(f) \cdot h$ for $f \in X_{\mathrm{c}}$ and $h \in C_0(R)$. We may also view ψ_r as having values in $C_0(E^0) = A$ in which case it is a conditional expectation with respect to the right A-module structure on X_{c}. With this structure there is a routine method of forming a Hilbert bimodule, or a right A-rigged, left A-module ([6]). Define an A-valued inner product on X_{c} via ψ_r by $\langle x, y\rangle = \psi_r(x^* \cdot y)$ which yields a norm on X_{c}, $\|x\|^2 = \|\langle x, x\rangle\|_\infty$. If X is the completion of X_{c} with respect to this norm, then X is a Hilbert A-module. This Hilbert bimodule is used in [7], [8]. In fact, if X_v is the Hilbert A-module $L^2(r^{-1}(v))$ for $v \in R$, where $\langle f, f\rangle = \sum\{|f(e)|^2\delta_v : r(e) = v\}$ for $f \in X_v$, then X is the direct sum $\bigoplus_{v \in R} X_v$ of the Hilbert A-modules. We have $X = \Big\{ f : E^1 \to \mathbb{C} : \sum_{v \in R} \Big(\Big\{ \sum |f_e|^2 : r(e) = v \Big\} \Big) \delta_v \in A \Big\}$. The left action of A is an action by adjointable maps on X, so by elements of $\mathcal{L}(X)$. We note that the left action $\varphi : A \to \mathcal{L}(X)$ of A on X is not faithful. In fact $A = C_0(G) \oplus C_0(F)$ as C^*-algebras, where $F = E^0 \setminus G$ are the sinks — including the isolated points I — of the graph E. The elements of $C_0(F)$ act as 0 on X, while the left action of $C_0(G)$ on X is faithful. Note that for $x = \delta_e \in X$ and $a = \delta_v \in A$ we have:

$$\begin{array}{rcllll} \varphi(a)x &=& \varphi(\delta_v) \cdot \delta_e = \chi_{s^{-1}(v)} \cdot \delta_e = \delta_e & \text{if } s(e) = v, & \text{zero otherwise;} \\ x \cdot a &=& \delta_e \cdot \delta_v = \delta_e \cdot \chi_{r^{-1}(v)} = \delta_e & \text{if } r(e) = v, & \text{zero otherwise;} \\ \langle \delta_e, \delta_l \rangle &=& \psi(\delta_e) = \delta_{r(e)} & \text{if } e = l, & \text{zero otherwise.} \end{array}$$

The first identity implies that the left A-module X is essential, so $\mathrm{Span}\{\varphi(a)f : a \in A, f \in X\}$ is dense in X. Also, for $l \in E^1$, the compact operator $\delta_l \otimes \delta_l^*$ on X, maps δ_e to $\delta_l\langle \delta_l, \delta_e\rangle = \delta_l \cdot r_\#\psi(\delta_l\delta_e) = \delta_e$ if $e = l$, and to zero if $e \neq l$. By Proposition 4.4 of [8] the ideal $\varphi^{-1}(\mathcal{K}(X)) = J$ of A is $C_0(\{v \in E^0 : |s^{-1}(v)| < \infty\})$, so since φ is injective on $C_0(G)$, it is also injective on the ideal $J_0 = C_0(\{v \in E^0 : 0 < |s^{-1}(v)| < \infty\})$ contained in J. By Theorem 1.5 of [11] there is a unique C^*-algebra $\widetilde{\mathcal{O}}(J_0, X)$, the relative Cuntz-Pimsner algebra determined by the ideal J_0 satisfying the following universal property:

For $T : X \to \mathcal{B}(\mathcal{H})$ a linear map and $\sigma : A \to \mathcal{B}(\mathcal{H})$ a nondegenerate $*$-homomorphism satisfying:

(1) $T(f \cdot a) = T(f)\sigma(a)$;

(2) $T(\varphi(a)\cdot f)=\sigma(a)T(f)$;
(3) $T^*(f)T(g)=\sigma(\langle f,g\rangle)$ for $f,g\in X$, $a\in A$; and
(4) $\sigma^1(\varphi(a))=\sigma(a)$, $a\in J_0$, where $\sigma^1:\mathcal{K}(X)\to\mathcal{B}(\mathcal{H})$ is defined by $\sigma^1(f\otimes g^*)=T(f)T(g)^*$;

there is a unique nondegenerate representation $\pi:\widetilde{\mathcal{O}}(J_0,X)\to\mathcal{B}(\mathcal{H})$ with $\pi(qT_f)=T(f)$ for $f\in X$ and $\pi(q(\varphi_\infty(a)))=\sigma(a)$ for $a\in A$. Here $q:\mathcal{T}(X)\to\widetilde{\mathcal{O}}(J_0,X)$ is a quotient map determined by J_0 of the Toeplitz C^*-algebra $\mathcal{T}(X)$ associated with the bimodule X, namely the C^*-subalgebra of the adjointable operators on the Fock space $\mathcal{F}(X)$ over X generated by the creation operators T_f, $f\in X$, and the diagonal action of A on $\mathcal{F}(X)$. The representation π is denoted by $T\times\sigma$, and any representation π of $\widetilde{\mathcal{O}}(J_0,X)$ arises in this manner. The elements qT_f are denoted by S_f, $f\in X$. We point out that the realization of the universal Toeplitz C^*-algebra on Fock space is an isomorphism ([8]).

Note that the fourth condition is equivalent to $\sum\limits_{l\in s^{-1}(v)}T(\delta_l)T(\delta_l)^*=\sigma(\delta_v)$ for $v\in E^0$ emitting a finite, nonzero number of edges, since $\varphi(\delta_v)$ is the compact operator $\sum\limits_{l\in s^{-1}(v)}\delta_l\otimes\delta_l^*$ for such v ([8]). Using that A is generated by $\{\delta_v: v\in E^0\}$ and $X=\overline{\mathrm{Span}}\{\delta_e: e\in E^1\}$, along with the bimodule structure of X, we may restate the universal property of $\widetilde{\mathcal{O}}(J_0,X)$ to be:

For a family $\{T(e): e\in E^1\}$ of elements in $\mathcal{B}(\mathcal{H})$ and an orthogonal family of projections $\{\rho_v: v\in E^0\}$ in $\mathcal{B}(\mathcal{H})$ with $\sum\rho_v=I_{\mathcal{H}}$ satisfying:

(1) $T(e)\rho_v=T(e)$ if $r(e)=v$, and zero otherwise;
(2) $\rho_v\cdot T(e)=T(e)$ if $s(e)=v$, and zero otherwise;
(3) $T(e)^*T(l)=\rho_{r(e)}$ if $e=l$, and zero otherwise;
(4) $\sum\limits_{l\in s^{-1}(v)}T(l)T(l)^*=\rho_v$ if $0<|s^{-1}(v)|<\infty$;

there is a nondegenerate representation π of $\widetilde{\mathcal{O}}(J_0,X)$ in $\mathcal{B}(\mathcal{H})$ such that $\pi(S_{\delta_e})=T(e)$ and $\pi(q\varphi_\infty(\delta_v))=\rho_v$.

Since the third condition states that the $T(e)$ are partial isometries with orthogonal final ranges, the first condition follows from the third and the fact that the projections ρ_v are orthogonal. Also Condition (2) can be restated as $T(e)T(e)^*\leqslant\rho_{s(e)}$. To see this we need to be assured that such a family of partial isometries with an orthogonal family of projections defines a covariant representation (T,σ) of the Hilbert bimodule X. We may define σ by extending the map ρ linearly to $C_{\mathrm{c}}(E^0)$ and noting that this map is continuous. The same is true for T: if $f=\sum f_e\delta_e\in X_0$ then $T(f)=\sum f_eT(e)=\sum\limits_{v\in R}\Big(\sum\limits_{e\in r^{-1}(v)}f_e\Big)T(e)\rho_v$ which has norm $\sup\limits_{v\in R}\Big\|\sum\limits_{e\in r^{-1}}f_eT(e)\Big\|$ since the partial isometries $T(e)$, $T(l)$ have orthogonal initial spaces if $r(e)\neq r(l)$, and otherwise have identical initial spaces. Now the final spaces of the partial isometries $T(e)$ are orthogonal, so the operator $\sum\limits_{r(e)=v}f(e)T(e)$ has operator norm bounded by $\Big[\sum|f(e)|^2\Big]^{1/2}$, and so $T:X_{\mathrm{c}}\to\mathcal{B}(\mathcal{H})$ is continuous.

Thus we see that $\widetilde{\mathcal{O}}(J_0, X)$ is the universal C^*-algebra $C^*(E)$ of [7] generated by a family $\{S_e : e \in E^1\}$ of partial isometries with orthogonal final ranges, and a family of orthogonal projections $\{p_v : v \in E^0\}$ such that $S_e S_e^* \leqslant p_{s(e)}$, $S_e^* S_e = p_{r(e)}$, and $\Big\{ \sum S_l S_l^* : l \in s^{-1}(v) \Big\} = p_v$ if $0 < |s^{-1}(v)| < \infty$.

Theorem 1.1. *If E is a directed graph and X the Hilbert A-bimodule described above, then the graph C^*-algebra $C^*(E)$ is isomorphic to the relative Cuntz-Pimsner algebra $\widetilde{\mathcal{O}}(J_0, X)$ determined by the ideal $J_0 = C_0(\{v \in E^0 : 0 < |s^{-1}(v)| < \infty\})$ of $C_0(E^0)$.*

In [7] a corresponding statement for the Cuntz-Pimsner algebra $\mathcal{O}_X$ is given under the assumption that the graph E has no sinks.

In his original approach to the algebra $\mathcal{O}_X$ associated with a Hilbert bimodule X over a C^*-algebra A, Pimsner ([9]) considered the image of X as generating the algebra $\mathcal{O}_X$, while the algebra generated by both X and A he referred to as the augmented algebra $\widetilde{\mathcal{O}}_X$. We may take the same approach here, considering the relative algebra $\widetilde{\mathcal{O}}(J_0, X)$ as the augmented algebra determined by the ideal J_0, and denote the relative algebra generated by X alone as $\mathcal{O}(J_0, X)$.

Recall ([4], Remark 2.6) that if E is a directed graph then E_{ess}, the essential part of E, is $E \setminus I$ where I are the isolated points or vertices of E, and that the graph C^*-algebra $G^*(E) \cong G^*(E_{\mathrm{ess}}) \oplus C_0(I)$. For a graph E with no isolated vertices the graph C^*-algebra $G^*(E)$ of [4] is the universal C^*-algebra generated by partial isometries $\{S_e : e \in E^1\}$ with orthogonal range projections such that

$$S_e^* S_e = \sum \{S_l S_l^* : l \in s^{-1}(r(e))\} \quad \text{if } 0 < |s^{-1}(r(e))| < \infty,$$

$$S_e^* S_e S_l^* S_l = \begin{cases} S_l^* S_l & \text{if } r(e) = r(l), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad S_e^* S_e S_l S_l^* = \begin{cases} S_l S_l^* & \text{if } r(e) = s(l), \\ 0 & \text{otherwise.} \end{cases}$$

We are interested here in viewing $G^*(E_{\mathrm{ess}})$ as such a relative Cuntz-Pimsner algebra $\mathcal{O}(J_0, X)$ generated by X alone. We do not digress to define these Cuntz-Pimsner algebras and fit them into a general context, but briefly describe a universal property:

Given $T : X \to \mathcal{B}(\mathcal{H})$ a linear map so that the C^*-algebra generated by $T(X)$ acts nondegenerately on $\mathcal{H}$ and $\sigma : \langle X, X\rangle \to \mathcal{B}(\mathcal{H})$ a $*$-homomorphism satisfying:

(1) $T(f \cdot a) = T(f)\sigma(a)$, $f \in X$, $a \in \langle X, X\rangle$;
(2) $T(\varphi(a) \cdot f) = \sigma(a)T(f)$, $f \in X$, $a \in \langle X, X\rangle$;
(3) $T^*(f)T(g) = \sigma(\langle f, g\rangle)$, $f, g \in X$;
(4) $\sigma^1(\varphi(a)) = \sigma(a)$ for $a \in J_0 \cap \langle X, X\rangle$;

then there is a unique nondegenerate representation $\pi : \mathcal{O}(J_0, X) \to \mathcal{B}(\mathcal{H})$ with $\pi(qT_f) = T(f)$ for $f \in X$ and $\pi(q\varphi_\infty(a)) = \sigma(a)$ for $a \in \langle X, X\rangle$. Here $\langle X, X\rangle$ is the closed linear span of $\{\langle f, g\rangle : f, g \in X\}$, a two-sided $*$-ideal of the C^*-algebra A, and q, φ_∞ are as above.

Let X be the Hilbert bimodule over $A = C_0(E^0)$ corresponding to a directed graph E with no isolated vertices. Then $\langle X, X\rangle$ is the ideal of functions of A that vanish at the sources of E^0, so can be identified with $C_0(R)$. As before, Condition (3) implies that the $T(\delta_e)$, $e \in E^1$ are partial isometries with orthogonal final ranges, and Condition (1) follows from (3). We also see from Condition (2) that $T^*(\delta_e)T(\delta_e)T(\delta_l)T(\delta_l^*) = \sigma(\delta_{r(e)})T(\delta_l)T(\delta_l)^* = T(\varphi(\delta_{r(e)}) \cdot \delta_l)T(\delta_l)^*$. Since $\varphi(\delta_{r(e)}) \cdot \delta_l = \delta_l$ if and only if $s(l) = r(e)$ and zero otherwise, the later expression is $T(\delta_l)T(\delta_l)^*$ if $s(l) = r(e)$, and zero otherwise. Thus the final projection $T(\delta_l)T(\delta_l)^* \leqslant T^*(\delta_e)T(\delta_e)$ if and only if $s(l) = r(e)$. It now follows that $\mathcal{O}(J_0, X)$ is the universal C^*-algebra $G^*(E)$.

THEOREM 1.2. *If E is a directed graph with no isolated vertices with vertex matrix B and X the $C_0(E^0)$-Hilbert bimodule associated with E then the graph C^*-algebra $G^*(E)$, or the Cuntz-Krieger algebra $\mathcal{O}_B$, is isomorphic to the "unaugmented" Cuntz-Pimsner algebra $\mathcal{O}(J_0, X)$ generated by X.*

Since $G^*(X)$ is the C^*-subalgebra of $C^*(E)$ generated by the partial isometries $\{S_e : e \in E^1\}$ alone ([4]) we have that $\mathcal{O}(J_0, X)$ is the C^*-subalgebra of $\widetilde{\mathcal{O}}(J_0, X)$ generated by $\{S_f : f \in X\}$. By analogy with the situation for Cuntz algebras we may view X as a Hilbert module generating the C^*-algebra $G^*(E)$ via a linear contraction S.

REMARK 1.3. In the following the representations $\pi : G^*(E) \to \mathcal{B}(\mathcal{H})$ that we deal with are chosen so that $\pi(S_e) \neq 0$ for $e \in E^1$. This is equivalent to requiring that the projections $p_{[e]} = \pi(S_e^* S_e)$ are nonzero, or equivalently that the $*$-homomorphism $\sigma : C_0(R) \to \mathcal{B}(\mathcal{H})$ occurring in the pair (T, σ) corresponding to π is injective.

2. ENDOMORPHISMS OF SUMS OF TYPE I FACTORS

Consider $*$-endomorphisms of a countable direct sum $\mathcal{R} = \bigoplus \mathcal{R}_i$ of type I factors $\mathcal{R}_i$, where $\mathcal{R}_i = p_i \mathcal{R} p_i$ is type I_{n_i} and p_i are orthogonal central and minimal countably decomposable projections of $\mathcal{R}$. Although we need not assume that the endomorphisms are either unital or injective, there are some restrictions we impose.

We form a matrix φ_* from the endomorphism φ by slightly modifying the approach in [2]. Consider the map $\gamma = p_i \varphi | \mathcal{R}_k$ from $\mathcal{R}_k$ to $\mathcal{R}_i$, which must either be injective or the zero map. In the later case set $\varphi_*(i, k) = 0$. Otherwise, let Γ_i be a representation of $\mathcal{R}_i$. It is unitarily equivalent to an isomorphism of $\mathcal{R}_i$ with $\mathcal{B}(\mathcal{H}_i) \otimes I_{\mathcal{L}_i}$ for Hilbert spaces $\mathcal{H}_i$, $\mathcal{L}_i$. The representation $\Gamma_i \gamma$ of $\mathcal{R}_k$ is unitarily equivalent to a representation $(\pi_{ik} \oplus 0) \otimes I_{\mathcal{L}_i}$ where π_{ik} is a nondegenerate representation of $\mathcal{R}_k$ on a subspace $\mathcal{H}_{ik}$ of $\mathcal{H}_i$. The commutant $\pi_{ik}(\mathcal{R}_k)'$ is a type I_m factor where $m = m_{ik}$ is the multiplicity of the representation π_{ik}. Since $\pi_{ik}(\mathcal{R}_k)'$ is isomorphic to $p_i \varphi(p_k)[\varphi(\mathcal{R}_k)' \cap \mathcal{R}_i]$ the multiplicity is well defined, independently of the representation Γ_i chosen. Set the (i, k) entry of the matrix φ_*, $\varphi_*(i, k) = m_{ik}$.

The von Neumann algebra $\varphi(\mathcal{R}) \cap \mathcal{R}_i$ is isomorphic to $\Big[\bigoplus\limits_k \pi_{ik}(\mathcal{R}_k) \oplus 0\Big] \otimes I_{\mathcal{L}_i}$ where 0 denotes the zero representation on the subspace $\Big[\bigoplus\limits_k \mathcal{H}_{i_k}\Big]^{\perp}$ of $\mathcal{H}_i$ corresponding to (the image under Γ_i of) the projection $I_{\mathcal{R}_i} - p_i \varphi(I)$. The projection $p_i \varphi(I)$ is the unique largest central projection in $\varphi(\mathcal{R})' \cap \mathcal{R}_i$ with $(I_{\mathcal{R}_i} - p_i \varphi(I))(\varphi(\mathcal{R}) \cap \mathcal{R}_i) = 0$.

If p is a minimal central projection of $\mathcal{R}$ then the projection $p\varphi(I)$ of $\mathcal{R}_p$ is in general a proper subprojection of p. We will consider those endomorphisms φ of $\mathcal{R}$ which are described by certain conditions.

Condition 2.1. *If p is a minimal central projection of $\mathcal{R}$ with $\varphi(p)$ and $p\varphi(I)$ both nonzero, then $p\varphi(I) \subsetneqq p$ implies that $p\varphi(I)[\varphi(\mathcal{R})' \cap \mathcal{R}_p]$ has infinite projections.*

Loosely speaking this first condition requires that certain finite parts of the endomorphism φ are unital, so $p\varphi(I)$, if not zero, must be all of p if $\varphi(p)$ is not zero and $p\varphi(I)(\varphi(\mathcal{R})' \cap \mathcal{R}_p)$ is a finite von Neumann algebra. This, as we shall later see, is a reflection of the relation occuring in the definition of $G^*(E)$ that requires initial projections of certain generating partial isometries to be equal to finite sums of certain other final projections. Indeed, note that if φ is a unital endomorphism, or if $\varphi(I)$ is required to be a central projection of $\mathcal{R}$, then φ vacuously satisfies this first condition since $p\varphi(I)$ must either be 0 or p for each p.

The second condition involves the relation of $p\varphi(I)$ to p if p is a minimal central projection of $\mathcal{R}$ in the kernel of φ with $p\varphi(I)$ nonzero. If the projection $h = p - p\varphi(I)$ is nonzero then, since φ maps the factor $p\mathcal{R}$ to zero, φ maps the subspaces $h\mathcal{R}$ and $\mathcal{R}h$ to zero. Also, since $1 - h = 1 - p + p\varphi(I)$, we have that $\varphi(x) = (1-h)\varphi(x)(1-h)$ for $x \in \mathcal{R}$. Thus information on φ is retained if we restrict the domain of φ by replacing the factor $p\mathcal{R}$ with the factor $p\varphi(I)\mathcal{R}p\varphi(I)$. Basically with this condition we arrange that $p = p\varphi(I)$ for these particular central projections in the kernel of φ.

Condition 2.2. *If p is a minimal central projection of $\mathcal{R}$ with $\varphi(p) = 0$ and $p\varphi(I)$ nonzero then $p = p\varphi(I)$.*

Again note that if φ is unital, or if $\varphi(I)$ is central, then φ also satisfies the second condition. The third condition again involves a simplifying condition on the domain $\mathcal{R}$ of φ. If p and q are minimal central projections of $\mathcal{R}$ in the kernel of φ with $p\varphi(I)$ and $q\varphi(I)$ nonzero then we may enlarge the domain of φ by replacing the sum of factors $p\mathcal{R} \oplus q\mathcal{R}$ with a single type I factor in the kernel of φ with unit $p+q$. We will see that this simplification is a reflection of Proposition 3.1 below where maximal domains defining an endomorphism are chosen. This last condition is different from the first two as it affects the actual matrix φ_*, namely φ_* may only have one zero column if φ satisfies Condition (3).

Condition 2.3. *If p, q are central projections in $\mathcal{R}$ with $\varphi(p) = \varphi(q) = 0$ and $p\varphi(I)$, $q\varphi(I)$ are nonzero, then $p = q$.*

The entries of the matrix φ_* satisfy a compatibility condition with the sequence (n_k) of nonzero elements of $\mathbb{N} \cup \{\infty\}$, where $\mathcal{R}_k$ is a factor of type I_{n_k}; namely $\sum \varphi_*(i,k)n_k \leqslant n_i$. Since the statement that $p_i\varphi(I)[\varphi(\mathcal{R})' \cap \mathcal{R}_i]$ has infinite projections is equivalent to $\sum_k \varphi_*(i,k) = \infty$ we see that Conditions 2.1 and 2.2 imply that $\sum \varphi_*(i,k)n_k = n_i$ whenever the i-row of φ_* is nonzero, so whenever the left hand side is nonzero. If $\mathcal{R}$ is a sum of type I_∞ factors, so $n_k = \infty$ for all k, then this stronger compatibility condition is automatically satisfied for any matrix φ_* associated with an endomorphism φ of $\mathcal{R}$.

If the projection $p_i\varphi(I)$ of $\mathcal{R}_i$ is nonzero, and not equal to p_i, then the matrix φ_* contains no information on how large the projection $p_i - p_i\varphi(I)$, so it may be finite or infinite in $\varphi(\mathcal{R})' \cap \mathcal{R}_i$. The size of these projections are preserved under inner automorphisms of $\mathcal{R}$, so if φ and ψ are two endomorphisms of $\mathcal{R}$ with $\varphi_* = \psi_*$ then it does not follow that there is an inner automorphism α of $\mathcal{R}$ with $\alpha \cdot \varphi = \psi$. If φ is a unital endomorphism, or if $\varphi(I)$ is a central projection of $\mathcal{R}$ then, as in [2], the matrix φ_* does determine the endomorphism up to an inner automorphism.

3. REPRESENTATIONS AND ENDOMORPHISMS

We describe a correspondence between endomorphisms of sums of type I factors and representations of graph algebras $G^*(E)$ extending the earlier such correspondences ([1], [3]).

If E is a directed graph then $G^*(E)$ is of the form $G^*(E_{\text{ess}}) \oplus C_0(I)$ where $E_{\text{ess}} = E \setminus I$, I the set of isolated vertices, and $G^*(E_{\text{ess}})$ is generated by partial isometries S_e, $e \in E^1$ satisfying the relations listed after Theorem 1.1 in ([4]). For π a nondegenerate representation of $G^*(E)$ on a Hilbert space $\mathcal{H}$ and $\mathcal{H}_v$ the orthogonal subspaces of $\mathcal{H}$ corresponding to the projections $\pi(\delta_v)$, $v \in I$, set $\mathcal{H}_e$ to be the subspace $\big(\oplus \mathcal{H}_v\big)^{\perp}$ of $\mathcal{H}$ corresponding to the projection $1_{\mathcal{H}} - \sum\limits_{v \in I} \pi(\delta_v)$. Then π defines a nondegenerate representation of $G^*(E_{\text{ess}})$ on $\mathcal{H}_e$.

Given π a nondegenerate representation of $G^*(E)$ on $\mathcal{H}$ define a $*$-linear map φ_π of $\mathcal{B}(\mathcal{H}_e) \oplus \bigoplus\limits_{v \in I} \mathcal{B}(\mathcal{H}_v)$ by $\varphi_\pi(x) = \sum\limits_{e \in E^1} \pi(S_e) x \pi(S_e)^*$. We note that $\varphi_\pi | \bigoplus\limits_{v \in I} \mathcal{B}(\mathcal{H}_v) = 0$, so we restrict attention from now on to $\varphi_\pi | \mathcal{B}(\mathcal{H}_e)$ and assume that E has no isolated points, so $\mathcal{H} = \mathcal{H}_e$. Since the partial isometries $\pi(S_e)$ have orthogonal final ranges the sum defining φ_π converges in the strong operator topology, and furthermore $\|\varphi_\pi(x)h\| \leqslant \|x\|\,\|h\|$ for $h \in \mathcal{H}$, so φ_π is a linear contraction on $\mathcal{B}(\mathcal{H})$.

Note that $\varphi_\pi(x) = \sum \pi(S_e S_e^*) \pi(S_e) x \pi(S_e)^* \pi(S_e S_e^*)$ is contained in

$$\begin{aligned}\bigoplus_{e \in E^1} \pi(S_e S_e^*) \mathcal{B}(\mathcal{H}) \pi(S_e S_e^*) = & \bigoplus_{v \in r(E^1)} \bigoplus_{e \in E^1} \{\pi(S_e S_e^*) \mathcal{B}(\mathcal{H}) \pi(S_e S_e^*) : s(e) = v\} \\ & \oplus \bigoplus_{e \in E^1} \{\pi(S_e S_e^*) \mathcal{B}(\mathcal{H}) \pi(S_e S_e^*) : s(e) \text{ a source}\}.\end{aligned}$$

Since $S_e S_e^* \leqslant S_l^* S_l$ if $r(l) = s(e)$ this is a subalgebra of $\bigoplus\limits_{l \in E^1/\sim} \pi(S_l^* S_l) \mathcal{B}(\mathcal{H}) \pi(S_l^* S_l) \oplus q\mathcal{B}(\mathcal{H})q$, where $q = I_{\mathcal{H}} - \sum\limits_{l \in E^1/\sim} \pi(S_l^* S_l)$ and where two edges $l, k \in E^1$ are equivalent, $l \sim k$, if $r(l) = r(k)$. Thus φ_π has range in the subalgebra $\{\pi(S_e^* S_e) : e \in E^1\}'$ of $\mathcal{B}(\mathcal{H})$.

As in [3] there is a maximal domain $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$, (or $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H}_e) \oplus \bigoplus\limits_{v \in I} \mathcal{B}(\mathcal{H}_v)$ if we keep track of the isolated points I of E) for φ_π so that $\varphi_\pi | \mathcal{R}$ is a $*$-homomorphism. The proof of the following proposition is the same as Proposition 2.1 of [3].

Proposition 3.1. *The maximal domain $\mathcal{R}$ on which φ_π is a $*$-homomorphism is the von Neumann algebra $\{\pi(S_e^*S_e) : e \in E^1\}'$.*

Thus φ_π is a $*$-endomorphism of $\mathcal{R}$. Since two projections in $\{\pi(S_e^*S_e) : e \in E^1\}$ are either orthogonal or equal, $\mathcal{R}'$ is abelian and $\mathcal{R}$ is a type I von Neumann algebra with center countably generated by the orthogonal projections $\{\pi(S_e^*S_e) : e \in E^1/\sim\} \cup \{q\}$. The subalgebra $q\mathcal{B}(\mathcal{H})q$ is evidently in the kernel of φ_π, in fact φ_π is injective if and only if $q = 0$. Thus the endomorphism φ_π has at most one type I factor summand of the domain $\mathcal{R}$ in its kernel, and

$$q\varphi(I) = q\Big(\sum_{e\in E^1} \pi(S_e)\pi(S_e)^*\Big) = q\Big(\sum_{e\in E^1} \{\pi(S_e)\pi(S_e)^* : s(e) \text{ a source of } E\}\Big).$$

If we assume that π is a nondegenerate representation of $G^*(E)$, we have that $I_\mathcal{H} = \sum_{l\in E^1/\sim} \pi(S_l^*S_l) + \sum_{e\in E^1} \{\pi(S_e)\pi(S_e)^* : s(e) \text{ a source of } E\}$, so

$$q = \sum_{e\in E^1} \{\pi(S_e)\pi(S_e)^* : s(e) \text{ a source}\}.$$

Thus φ satisfies Conditions 2.2 and 2.3 of Section 2. We shall see later that φ satisfies Condition 2.1 as well.

For π the given nondegenerate representation of $G^*(E)$ there is a unique isometric covariant representation (T, σ) of $(X, \langle X, X\rangle)$ on $\mathcal{H}$ where X is the Hilbert bimodule over $A = C_0(E^0)$ associated with E and $T(\delta_e) = \pi(S_e)$ for $e \in E^1$. Note that we are assuming here that E has no isolated points, which we may do by previous comments.

To show that the endomorphism φ_π depends only on the image $T(X)$ of the Hilbert bimodule we first show that the domain $\mathcal{R}$ is determined by T.

Proposition 3.2. *The domain $\mathcal{R}$ of φ_π is $\{T(f)^*T(f) : f \in X\}'$.*

Proof. Clearly $\mathcal{R} \supseteq \{T(f)^*T(f) : f \in X\}'$. If $f = \sum f_e\delta_e \in C_c(E^1)$ then $T(f)^*T(f) \in \operatorname{Span}\{\pi(S_e^*S_e) : e \in E^1\}$ since $\pi(S_e)^*\pi(S_l) = 0$ unless $e = l$. Since T is a continuous map of X to $\mathcal{B}(\mathcal{H})$, of norm 1, the result follows. ∎

As in [1] and [3], set $\mathcal{L}_\varphi = \{M \in \mathcal{B}(\mathcal{H}) : \varphi_\pi(x)M = Mx, x \in \mathcal{R}\}$ a σ-weakly closed linear subspace of $\mathcal{B}(\mathcal{H})$ which is a right $\mathcal{A}$-module, with $\mathcal{A}$ the abelian algebra $\mathcal{R}'$.

Lemma 3.3. *With $q = 1_\mathcal{H} - \sum_{e\in E^1/\sim} \pi(S_e^*S_e)$, $p = 1_\mathcal{H} - \sum_{e\in E^1} \pi(S_eS_e^*)$, and $M \in \mathcal{L}_\varphi$ we have $Mq = pM = 0$.*

Proof. We have $\varphi_\pi(q) = 0$ since $q\mathcal{B}(\mathcal{H})q \subseteq \ker\varphi_\pi$, so for $M \in \mathcal{L}_\varphi$, $0 = \varphi_\pi(q)M = Mq$. Since range $\varphi_\pi \subseteq \bigoplus_{e\in E^1} \pi(S_eS_e^*)\mathcal{B}(\mathcal{H})\pi(S_eS_e^*)$ we have $\varphi_\pi(x)p = p\varphi_\pi(x) = 0$, so for $M \in \mathcal{L}_\varphi$, $0 = p\varphi_\pi(I_\mathcal{H})M = pM$. ∎

Since $M^*S \in \mathcal{R}' = \mathcal{A}$ for $M, S \in \mathcal{L}_\varphi$, $\langle M, S\rangle = M^*S$ defines a Hilbert $\mathcal{A}$-module structure on $\mathcal{L}_\varphi$.

PROPOSITION 3.4. *$\mathcal{L}_\varphi = \overline{\mathrm{Span}_{\mathcal{A}} T(X)}$ where the closure is in the σ-weak topology and $\mathrm{Span}_{\mathcal{A}} T(X)$ is the right $\mathcal{A}$-module generated by $T(X)$.*

Proof. We have

$$\varphi(x)\pi(S_e)=\sum \pi(S_l)x\pi(S_l)^*\pi(S_e)=\pi(S_e)x\pi(S_e)^*\pi(S_e)=\pi(S_eS_e^*S_e)x=\pi(S_e)x$$

for $x \in \mathcal{R}$, so $T(\delta_e) = \pi(S_e) \in \mathcal{L}_\varphi$, for $e \in E^1$ and $\mathcal{L}_\varphi \supseteq \overline{\mathrm{Span}_{\mathcal{A}} T(X)}$. The other inclusion follows by noting that for $M \in \mathcal{L}_\varphi$,

$$M = (1-p)M = \sum_{e\in E^1} \pi(S_eS_e^*)M = \sum \pi(S_e)\langle \pi(S_e), M\rangle,$$

where the first sum converges in the strong topology on the unit ball of $\mathcal{B}(\mathcal{H})$ since the projections $\pi(S_eS_e^*)$ are orthogonal. ∎

We had remarked in the previous lemma that $\varphi_\pi(a)p = 0$ for $a \in \mathcal{R}$ and $p = 1 - \sum \pi(S_e)\pi(S_e)^* = 1 - \sum T(\delta_e)T(\delta_e)^*$. Since T is a linear contraction

$$[T(X)\mathcal{H}] = \Big[\sum_{e\in E^1} T(\delta_e)\mathcal{H}\Big] = \sum T(\delta_e)T(\delta_e)^*\mathcal{H} = (1-p)\mathcal{H},$$

so the kernel of $\varphi_\pi(a)$ contains $[T(X)\mathcal{H}]^\perp$ for each $a \in \mathcal{R}$.

PROPOSITION 3.5. *Let E be a directed graph with no isolated points and X the Hilbert bimodule associated with E generating the C^*-algebra $G^*(E)$. Let π be a nondegenerate representation of $G^*(E)$ on $\mathcal{H}$ with (T,σ) the associated isometric covariant representation of $(X, \langle X, X\rangle)$, and φ_π the $*$-endomorphism of $\mathcal{R} = \{T(f)^*T(f) : f \in X\}'$. If $bM = Ma$ $(M \in T(X))$ for some $a \in \mathcal{R}$, $b \in \mathcal{B}(\mathcal{H})$ with $b|_{[T(X)\mathcal{H}]^\perp} = 0$ then $b = \varphi_\pi(a)$.*

Proof. Since $T(X) \subseteq \mathcal{L}_\varphi$ we have $\varphi_\pi(a)M = Ma$ for $M \in T(X)$, so $(b - \varphi_\pi(a))T(f) = 0$ for $f \in X$ and $(b-\varphi_\pi(a))|_{T(X)\mathcal{H}} = 0$. Since $(b-\varphi_\pi(a))|_{[T(X)\mathcal{H}]^\perp} = 0$ it follows that $b = \varphi_\pi(a)$. ∎

These results show that although the endomorphism was initially defined using specific generators of X, it only depends on X and the representation π. If U is a unitary of the Hilbert $\mathcal{A}$-bimodule X, and (T,σ) an isometric covariant representation of $(X, \langle X, X\rangle)$, then so is $(T\circ U, \sigma)$. The endomorphism $\varphi_{\pi\circ U}$ is given by $\varphi_{\pi\circ U}(x) = \sum T(U\delta_e)xT(U\delta_e)^*$ with domain $\{T(Uf)^*T(Uf) : f \in X\}'$ which is the same as $\mathcal{R}$, the domain of φ_π, since

$$T(Uf)^*T(Uf) = \sigma(\langle Uf, Uf\rangle) = \sigma(\langle f, f\rangle) = T(f)^*T(f)$$

for $f \in X$. Also, for $a \in \mathcal{R}$, $\ker \varphi_{\pi\circ U}(a)$ contains $[T(UX)\mathcal{H}]^\perp = [T(X)\mathcal{H}]^\perp$. Proposition 3.4 implies $\mathcal{L}_{\varphi_{\pi\circ U}} = \mathcal{L}_{\varphi_\pi}$ so $\varphi_{\pi\circ U}(a)M = Ma$ for $M \in \mathcal{L}_\varphi$ and $a \in \mathcal{R}$. Proposition 3.5 applies to show $\varphi_{\pi\circ U} = \varphi_\pi$.

The fixed point algebra of the endomorphism φ_π is determined by the representation π of the bimodule X.

THEOREM 3.6. *Let E be a directed graph with no isolated points, π a nondegenerate representation of $G^*(E)$ and T the corresponding representation of the bimodule X associated with the directed graph E. If φ_π is the endomorphism of $\mathcal{R} = \{T^*(f)T(f) : f \in X\}'$ associated with π then the fixed point subalgebra of $\varphi_\pi = \{T(X) \cup T^*(X)\}' \cap (1-p)\mathcal{R}(1-p)$ where $1 - p = \sum_{e \in E^1} \pi(S_e S_e^*)$.*

Proof. If $a \in (1-p)\mathcal{R}(1-p)$ commutes with $T(X)$ then

$$\varphi_\pi(a) = \sum_{e \in E^1} T(\delta_e) a T(\delta_e)^* = a \sum T(\delta_e)T(\delta_e)^* = a(1-p) = a,$$

so the set is contained in the fixed point subalgebra. For the other inclusion first recall that we have already noted, in Lemma 3.3, that range $\varphi_\pi \subseteq (1-p)\mathcal{R}(1-p)$, and that φ_π is a $*$-homomorphism. It is therefore enough to show that the fixed point algebra is contained in $T(X)'$. This follows from Proposition 3.4 since the fixed point algebra of φ_π is contained in $\mathcal{L}'_\varphi$. ∎

Although $(1-p)\mathcal{R}(1-p)$ may not be zero, the fixed point algebra of the endomorphism φ_π may certainly be zero, as the example with the directed graph containing exactly one edge e with $r(e) \neq s(e)$ shows.

Given a directed graph E, with no isolated points, we may coalesce all sources of E into one single source to obtain a directed graph $\widetilde{E}$. Since the defining relations for $G^*(E)$ only involve the edge set E^1, and do not involve sources, and since $E^1 = \widetilde{E}^1$, it is clear that $G^*(E) = G^*(\widetilde{E})$.

REMARK 3.7. The graph $\widetilde{E}$ is a partial in-amalgamation of the graph E in the terminology of [4]. Also, if B and $\widetilde{B}$ are the vertex matrices for E and $\widetilde{E}$ respectively, then the complete in-split matrices B_w and $\widetilde{B}_w$ are equal, so once again $calO_B \cong \mathcal{O}_{\widetilde{B}}$. Note that the complete in-amalgamations of B_w and $\widetilde{B}_w$ are both equal to $\widetilde{B}$.

Theorem 4.7 of [3] holds in the current, more general, situation. That situation dealt with finite graphs, and unital injective endomorphisms; so in particular no sources. To see that the former situation is a special case of Theorem 3.8 below recall that the Cuntz-Krieger algebras $\mathcal{O}_B$ are defined using the complete in-split matrix B_w of B ([4]).

THEOREM 3.8. *If E is a directed graph with no isolated points and π is a nondegenerate representation of $G^*(E)$ on a separable Hilbert space with φ_π its associated endomorphism, then the endomorphism satisfies Conditions 2.1, 2.2, and 2.3, and the matrix $(\varphi_\pi)_* = \widetilde{B}$, the vertex matrix of the graph $\widetilde{E}$ where all sources of E are coalesced into one source. In particular, if E has no sources, or only one source, then $(\varphi_\pi)_* = B$, the vertex matrix of E.*

Proof. By the comments preceding Remark 3.7 we may assume that $E = \widetilde{E}$, so that E has at most one source. Setting two edges e, l to be equivalent, $e \sim l$, if and only if $r(e) = r(l)$ we have that the projections $p_{[e]} = \pi(S_e^* S_e)$, $[e] \in E^1/\sim$, are minimal central projections of the domain $\mathcal{R}$ of φ_π, and $\mathcal{R} = \bigoplus_{e \in E^1/\sim} p_{[e]}\mathcal{R} \oplus q\mathcal{R}$

with $q = 1 - \sum p_{[e]}$. Note that $q \neq 0$ if and only if E has a source. In this case set $a \in E^0$ to be the source of E.

Now $(\varphi_\pi)_*$ is a square matrix over the index set $(E^1/\sim) \cup \{q\}$ and B is a square matrix over the index set E^0, with $B(v,w) = |\{k \in E^1 : s(k) = v,\ r(k) = w\}|$ for $v, w \in E^0$. The map of $(E^1/\sim) \cup \{q\} \to E^0$ defined by $[e] \to r(e)$ and, if $q \neq 0$, $q \to a$ is a well defined set bijection.

Since the q-th column of $(\varphi_\pi)_*$ describes the map $\varphi_\pi|_{q\mathcal{R}}$ we see that the q-th column of $(\varphi_\pi)_*$ is the same as the a column of B, namely both are zero. Now choose $w \in E^0$ and $l \in E^1$ with $r(l) = w$. To identify the entries in the $[l]$ column of $(\varphi_\pi)_*$, fix $x \in p_{[l]}\mathcal{R}$. Then

$$\varphi_\pi(x) = \sum_{k \in E^1} \pi(S_k)x\pi(S_k)^* = \sum \pi(S_k)x\pi(S_l^*S_lS_k^*) = \sum_{k\sim l} \pi(S_k)x\pi(S_k)^*$$

since $S_l^*S_lS_k^*S_k = 0$ if $r(k) \neq r(l)$, and $S_k^*S_k$ otherwise. If $v \in E^0$ and $v \neq a$ then there is an $e \in E^1$ with $r(e) = v$. The element

$$\begin{aligned} p_{[e]}\varphi_\pi(x) &= \sum_{k\sim l} \pi(S_e^*S_eS_k)x\pi(S_k)^* = \sum_{k\sim l} \pi(S_e^*S_eS_kS_k^*S_k)x\pi(S_k)^* \\ &= \sum_k \{\pi(S_k)x\pi(S_k)^* : k \sim l, s(k) = r(e) = v\} \end{aligned}$$

by the relations for $G^*(E)$, so $(\varphi_\pi)_*([e],[l]) = |\{k \in E^1 : r(k) = w, s(k) = v\}| = B(v,w)$. We also compute

$$\begin{aligned} q\varphi_\pi(x) &= \sum_{k\sim l} \Big(I - \sum_g \pi(S_g^*S_g)\Big)\pi(S_k)x\pi(S_k)^* \\ &= \sum \{\pi(S_k)x\pi(S_k)^* : k \sim l, s(k) = a\} \end{aligned}$$

since E has only one source a and π is nondegenerate. Thus $(\varphi_\pi)_*(q,[l]) = B(a,w)$, and we have shown $(\varphi_\pi)_* = \widetilde{B}$.

That φ_π satisfies Conditions 2.2 and 2.3 was noted earlier. To show Condition 2.1 is satisfied, it is enough to show, since $\varphi_\pi(q) = 0$, that if for a fixed $e \in E^1$, $p_{[e]}\varphi_\pi(I) \neq 0$ and $p_{[e]}\varphi_\pi(I)(\varphi(\mathcal{R})' \cap \mathcal{R}p_{[e]})$ is a finite von Neumann algebra then $p_{[e]} = p_{[e]}\varphi_\pi(I)$. However the indicated algebra is finite if and only if $\sum_{k \in (E^1/\sim)\cup\{q\}} (\varphi_\pi)_*([e],k) < \infty$, so, since the q-column of $(\varphi_\pi)_*$ is zero, exactly when $\sum_{l \in E^1/\sim} (\varphi_\pi)_*([e],[l]) < \infty$. The later sum is

$$\sum_{l \in E^1/\sim} |\{k \in E^1 : r(k) = r(l), s(k) = r(e)\}| = |\{k \in E^1 : s(k) = r(e)\}| = |s^{-1}(r(e))|.$$

We compute

$$p_{[e]}\varphi_\pi(I) = \pi(S_e^*S_e) \sum_{l \in E^1} \pi(S_lS_l^*) = \sum_{l \in s^{-1}(r(e))} \pi(S_lS_l^*),$$

which is nonzero precisely when $0 < |s^{-1}(r(e))|$. To show Condition 2.1 we are thus reduced to showing that if $0 < |s^{-1}(r(e))| < \infty$ then $\pi(S_e^*S_e) = \sum_{l \in s^{-1}(r(e))} \pi(S_lS_l^*)$. This is however π applied to a defining relation for $G^*(E)$. ∎

THEOREM 3.9. *Let* $\mathcal{R} = \bigoplus_{l\in J} \mathcal{R}_l$ *be a countable direct sum of countably decomposable type* I *factors* $\mathcal{R}_l$, $l \in J$. *Assume that each* $\mathcal{R}_l$ *is represented irreducibly on a Hilbert space* $\mathcal{H}_l$ *and that* $\mathcal{R}$ *is thus represented on* $\mathcal{H} = \bigoplus \mathcal{H}_l$. *If* φ *is a* $*$*-endomorphism of* $\mathcal{R}$ *satisfying Conditions* 2.1, 2.2 *and* 2.3 *then there is a nondegenerate representation* $\pi : \mathcal{O}_B \to \mathcal{B}(\mathcal{H})$ *of the generalized Cuntz-Krieger algebra* $\mathcal{O}_B$ *with* $B = \varphi_*$ *so that* φ_π, *the* $*$*-endomorphism associated with* π, *is equal to* φ.

Proof. Let $\mathcal{R}_l = \mathcal{R}_{p_l}$ where p_l is the unit of $\mathcal{R}_l$, a minimal central projection of $\mathcal{R}$. Condition 2.3 implies that there is at most one projection q with $q\varphi(I) \neq 0$ and $\varphi(q) = 0$, so in other words where the q-column of φ_* is zero and the q-row of φ_* is nonzero. If I is the subset of J with $l \in I$ if and only if both the l-column and l-row of φ_* are zero, then set $\mathcal{H}_e$ to be the subspace $\left(\bigoplus_{v\in I} \mathcal{H}_v\right)^\perp$ of $\mathcal{H}$. We will define a representation π of $G^*(E)$ on $\mathcal{H}$, where E is a directed graph with vertex set $E^0 = J$, so that the isolated vertices of E consist of the set I, and q is the one possible vertex of E^0 emitting edges but not receiving edges. By the definition of φ_π it now suffices to construct a representation π of $G^*(E_{\text{ess}})$ on $\mathcal{H}_e$ where $E_{\text{ess}} = E \setminus I$, E_{ess} is a directed graph with no isolated vertices. Thus, without loss of generality, we may suppose that $I = \phi$.

As in the proof of Theorem 3.9 of [3] there are, for $\varphi_*(l,k) \neq 0$, partial isometries $\{T_{lik} : 1 \leqslant i \leqslant \varphi_*(l,k), l,k \in J\}$ with orthogonal final spaces, where T_{lik} has initial space $\mathcal{H}_k$, final space a subspace of $\mathcal{H}_l$ and

$$\varphi(x) = \sum_{l,k} p_l\varphi(p_k x) = \sum_{l,k\in J} \sum_{i=1}^{\varphi_*(l,k)} T_{lik}(p_k x)T^*_{lik} = \sum_{l,k\in J} \sum_{i=1}^{\varphi_*(l,k)} T_{lik} x T^*_{lik}.$$

Define a directed graph E with $E^0 = J$, and for each $T_{lik} \neq 0$ there is an edge $e \in E^1$ with $r(e) = k$, $s(e) = l$. Clearly the vertex matrix B for E is the matrix φ_*. Set $T_e = T_{lik}$. Note that the condition $\sum_k \varphi_*(l,k) < \infty$ is equivalent to $|s^{-1}(l)| < \infty$, $\varphi(p_k) \neq 0$ is equivalent to $r^{-1}(k) \neq \phi$, $p_l\varphi(I) \neq 0$ means $s^{-1}(l) \neq \phi$, and $p_l\varphi(I) = p_l$ is equivalent to $S^*_e S_e = \sum\{S_g S^*_g : g \in s^{-1}(l)\}$ if $r(e) = l$. Thus the T_e, $e \in E^1$, are partial isometries satisfying the defining relations for $G^*(E)$ if φ satisfies Condition 2.1. This defines a representation π of $G^*(E)$ on $\mathcal{H}$ with $\varphi_\pi = \varphi$. The representation π is nondegenerate since φ satisfies Condition 2.2. ∎

4. ENDOMORPHISMS OF SUMS OF I_∞ FACTORS

Let φ be an endomorphism of $\mathcal{R}$ where $\mathcal{R}$ is a countable sum of type I factors, and φ satisfies Conditions 2.1, 2.2, and 2.3 of Section 2. If E is the directed graph with vertex matrix $B = \varphi_*$ we give sufficient conditions on E that ensure that $\mathcal{R}$ must be a sum of type I_∞ factors. By Theorem 3.9 we know that $\varphi = \varphi_\pi$ where π is a nondegenerate representation of $G^*(E)$.

For $n \in \mathbb{N}$ let E^n be the paths in E of length n, so $E^n = \{(\alpha_1, \ldots, \alpha_n) : s(\alpha_{i+1}) = r(\alpha_i)$ for $1 \leqslant i \leqslant n-1$ and $\alpha_i \in E^1$ for $1 \leqslant i \leqslant n\}$, and $E^* = \bigcup\limits_{n \geqslant 1} E^n$. The conditions on E are most concisely stated in terms of the infinite path space $E^\infty = \Big\{\alpha \in \prod\limits_{\mathbb{N}} E^1 : s(\alpha_{i+1}) = r(\alpha_i)$ for $i \in \mathbb{N}\Big\}$ with the subspace topology from $\prod\limits_{\mathbb{N}} E^1$, although Theorem 4.2 below could be stated and proved using only E^*, without reference to E^∞.

Proposition 4.1. *If E has no sinks and $(\alpha_1, \ldots, \alpha_n) \in E^n$ then there is an $\alpha \in E^\infty$ with $(\alpha)_i = \alpha_i$ for $i = 1, \ldots, n$.*

Proof. First note that since E has no sinks, if $f \in E^n$ then there is $g \in E^{n+1}$ which extends f, in the sense that the domain of f is contained in that of g, and g agrees with f on the domain of f. We write $f \leqslant g$. This defines a partial order on $\mathcal{P} = \{h \in E^* : (\alpha_1, \ldots, \alpha_n) \leqslant h\}$, and the Hausdorff maximality principle yields a maximal chain $\mathcal{C}$ in $\mathcal{P}$. Since E has no sinks, the principle of induction shows $\bigcup\{\text{domain } f : f \in \mathcal{C}\} = \mathbb{N}$. Define $\alpha \in E^\infty$ by $\alpha(n) = f(n)$ for $n \in$ domain f, $f \in \mathcal{C}$. ∎

The principle condition on E, other than E having no sinks, is: given $\delta \in E^n$ there is an $m > n$, and distinct $\alpha, \beta \in E^m$ with $\delta \leqslant \alpha$, $\delta \leqslant \beta$. By the last Proposition we may choose $\alpha, \beta \in E^\infty$, and the condition may be restated in terms of E^∞ by saying that E^∞ has no isolated points. The main tool used in the proof of the following theorem is the compatibility condition of Section 2, namely $\sum \varphi_*(i,k) n_k \leqslant n_i$ for φ any endomorphism of $\mathcal{R} = \bigoplus \mathcal{R}_k$, with $\mathcal{R}_k$ a type I_{n_k} factor.

Theorem 4.2. *If the directed graph E has no sinks and E^∞ has no isolated points then the domain $\mathcal{R}$ of the endomorphism $\varphi = \varphi_\pi$ associated with a nondegenerate representation π of $G^*(E)$ is a direct sum of type I_∞ factors.*

Proof. As in Theorem 3.8, $\mathcal{R} = \bigoplus\limits_{e \in E^1/\sim} p_{[e]}\mathcal{R} \oplus q\mathcal{R}$ where $p_{[e]} = \pi(S_e^* S_e)$ and $q = 1 - \sum p_{[e]}$. We first show that $p_{[e]}\mathcal{R}$, a type $\mathrm{I}_{n_{[e]}}$ factor, is a type I_∞ factor for each $e \in E^1$. Fix an arbitrary $e \in E^1$ and note that since E has no sinks, there is an $l \in E^1$ with $r(e) = s(l)$. Let $\alpha \in E^\infty$ with $\alpha_1 = e$ and $\alpha_2 = l$. Note that if $r(e) = s(e)$ then it is entirely possible that $l = e$. We have

$$n_{[\alpha_{m+1}]} \leqslant \varphi_*([\alpha_m], [\alpha_{m+1}]) n_{[\alpha_{m+1}]} \leqslant n_{[\alpha_m]}$$

for all m so $n_{[\alpha_m]}$ is a decreasing sequence of natural numbers (or infinity) bounded above by $n_{[e]}$. Since $\alpha \in E^\infty$ is not isolated, for each $m \in N$ there is an $r_m > m$ and an edge $k_m \neq \alpha_{r_m+1}$ with $s(k_m) = r(\alpha_{r_m})$. We have

$$n_{[\alpha_{r_m}+1]} \lneqq \varphi_*([\alpha_{r_m}], [\alpha_{r_m+1}])n_{[\alpha_{r_m+1}]} + \varphi_*([\alpha_{r_m}], [k_m])n_{[k_m]} \leqslant n_{[\alpha_{r_m}]}$$

if $n_{[\alpha_{r_m}+1]}$ is finite. Thus, if $n_{[e]}$ is finite we have that it must bound the strictly decreasing sequence $n_{[\alpha_{r_m}]}$ of positive natural numbers, which is not possible.

We have shown that $p_{[e]}\mathcal{R}$ is a type I_∞ factor for all $e \in E^1$, so it only remains to show that $q\mathcal{R}$ is also for $q \neq 0$. If $q \neq 0$, then E has a source a and if $e \in E^1$ with $s(e) = a$ then

$$n_{[e]} \leqslant \varphi_*(a, [e])n_{[e]} \leqslant \sum \varphi_*(a, j)n_j \leqslant n_a,$$

so n_a must also be infinite. ∎

We show that if E has no isolated points and a sink p, and has at most one source then we may always find an endomorphism φ of a sum of type I factors with φ_* the vertex matrix of E and $\mathcal{R}$ containing finite type I factor summands. If π is a nondegenerate representation of $G^*(E)$ on $\mathcal{H}$ and $\varphi = \varphi_\pi$ its associated endomorphism, then φ_* is the vertex matrix of E and φ is an endomorphism of $\mathcal{R} = \bigoplus_{e\in E^1/\sim} p[e]\mathcal{R} \oplus q\mathcal{R}$. If $l \in E^1$ is an edge with $r(l) = p$ then by the defining relations for $G^*(E)$ the projection $p_{[l]} = \pi(S_l^* S_l)$ is orthogonal to each projection $\pi(S_e S_e^*)$, $e \in E^1$, so $p_{[l]}\varphi(\mathcal{R}) = 0$ and $p_{[l]}\mathcal{R} \cap \varphi(\mathcal{R}) = 0$. If ψ is the restriction of φ to $\mathcal{M} = \bigoplus_{\substack{e\in E^1/\sim \\ e\neq l}} p_{[e]}\mathcal{R} \oplus \mathbb{C}p_{[l]} \oplus q\mathcal{R}$ then $\psi_* = \varphi_*$, the vertex matrix of E and ψ is an endomorphism of a sum of type I factors that includes at least one finite factor. Note also that ψ satisfies Conditions 2.1, 2.2, and 2.3 since the first two conditions do not apply to $p_{[l]}$ as $p_{[l]}\psi(I) = p_{[l]}\varphi(I) = 0$.

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