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ANALYTIC FINITE BAND WIDTH REPRODUCING KERNELS AND OPERATOR WEIGHTED SHIFTS

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Abstract. This paper realizes the reproducing kernel Hilbert spaces with orthonormal bases of the form $\{(a_{n,0} + a_{n,1}z + \cdots + a_{n,J}z^J)z^n : n \geqslant 0\}$ in a matrix valued kernel setting. The question of when multiplication by φ, denoted M_φ, is a bounded operator is investigated and it is shown that M_φ can be viewed as the compression of a matrix valued Toeplitz type operator.

Keywords: *Reproducing kernels, multiplication operators, orthogonal polynomials, operator weighted shifts.*

MSC (2000): Primary 47B38; Secondary 46E20.

1. INTRODUCTION

In Shields [7] multiplication operators on reproducing kernel Hilbert spaces with kernels of the form $K(z,w) = \sum\limits_{n=0}^{\infty} a_n z^n \overline{w}^n$ were extensively studied. In these spaces the monomials $\{\sqrt{a_n} z^n\}$ form an orthonormal basis, and the operator M_z of multiplication by z is a forward unilateral shift. In Adams and McGuire ([1]), a beginning was made on the study of the reproducing kernel Hilbert spaces with kernels of the form $K(z,w) = \sum\limits_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$ where $f_n(z) = (a_{n,0} + a_{n,1}z + \cdots + a_{n,J}z^J)z^n$ and J is fixed. These spaces are known as bandwidth J spaces since the Taylor series expansion of $K(z,w) = \sum\limits_{i,j=0}^{\infty} a_{i,j} z^i \overline{w}^j$ satisfies $a_{i,j} = 0$ outside the band $|i-j| > J$. Also the polynomials $\{f_n(z)\}$ form an orthonormal basis. It was shown in Adams, McGuire [2] that the behavior of the multiplication operators on these spaces can be markedly different from the Shields case ($J = 0$). Aside from being a natural generalization of the spaces treated by Shields and providing a natural setting for studying classes of orthogonal polynomials, we will show in this paper a connection of these spaces and their multiplication operators to spaces of vector valued functions and matrix valued weighted shifts. As relatively little is known about operator valued weighted shifts, the connection to multiplication by z on bandwidth J spaces provides an insight to an important class of examples of operator valued weighted shifts.

Section 2 recalls some basic properties of reproducing kernel Hilbert spaces.

2. REPRODUCING KERNEL HILBERT SPACES

If $\mathcal{D}$ is a set and K is a function from $\mathcal{D} \times \mathcal{D}$ into the bounded linear operators $\mathcal{B}(\mathcal{C})$ on a Hilbert space $\mathcal{C}$, then K is *positive definite* (denoted $K \gg 0$) on $\mathcal{D}$ provided $\sum\limits_{i,j=0}^{n} \langle K(z_i, z_j)\vec{x}_j, \vec{x}_i\rangle_{\mathcal{C}} \geqslant 0$ for any finite subsets $\{z_1, \ldots, z_n\} \subset \mathcal{D}$ and $\{\vec{x}_1, \ldots, \vec{x}_n\} \subset \mathcal{C}$ with strict inequality unless all the $\vec{x}_i$'s are zero. It is well known that if $K \gg 0$ on $\mathcal{D}$, then the set of generating functions $\mathcal{G} = \Big\{ \sum\limits_{j=1}^{n} K(z, z_j)\vec{x}_j : z_1, \ldots, z_n \in \mathcal{D}, \vec{x}_1, \ldots, \vec{x}_n \in \mathcal{C} \Big\}$ is dense in a Hilbert space $H(K)$ of functions on $\mathcal{D}$ with inner product $\langle K(z, z_1)\vec{x}_1, K(z, z_2)\vec{x}_2\rangle_{H(K)} = \langle K(z_2, z_1)\vec{x}_1, \vec{x}_2\rangle_{\mathcal{C}}$ and norm $\Big\| \sum\limits_{j=1}^{n} K(z, z_j)\vec{x}_j \Big\|^2 = \sum\limits_{i,j=0}^{n} \langle K(z_i, z_j)\vec{x}_j, \vec{x}_i\rangle_{\mathcal{C}}$.

A fundamental property of $H(K)$ is that for each w in $\mathcal{D}$ the evaluation map $E_w : H(K) \to \mathcal{C}$ defined by $E_w f = f(w)$ is a bounded linear map. In this case the function $K : \mathcal{D} \times \mathcal{D} \to \mathcal{B}(\mathcal{C})$ is given by $K(z, w) = E_z E_w^*$. Moreover, if $f \in H(K)$, $\vec{x} \in \mathcal{C}$, and $w \in \mathcal{D}$, then $\langle f(z), K(z, w)\vec{x}\rangle_{H(K)} = \langle f(w), \vec{x}\rangle_{\mathcal{C}}$. Conversely, if H is a Hilbert space of $\mathcal{C}$-valued functions defined on $\mathcal{D}$ such that for each $w \in \mathcal{D}$ evaluation at w is a bounded linear map into $\mathcal{C}$, then there is a unique K defined on $\mathcal{D} \times \mathcal{D}$ such that $H = H(K)$. For general properties of reproducing kernel Hilbert spaces, the reader is referred to N. Aronszajn [4], H. Dym [5], and Adams, McGuire, Paulsen [1], and Adams, Froelich, McGuire, and Paulsen [3].

If Ω is an open subset of $\mathbb{C}^n$ and the function $K : \Omega \times \Omega \to \mathcal{B}(\mathcal{C})$ defined by $K(z, w) = E_z E_w^*$ is analytic in z and coanalytic in w, then $H(K)$ is an *analytic* reproducing kernel Hilbert space.

With $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of non-negative integers and n a fixed positive integer, the set $\mathbb{N}^n$ is partially ordered by setting $I = (i_1, \ldots, i_n) \geqslant (j_1, \ldots, j_n) = J$ if and only if $i_k \geqslant j_k$ for $k = 1, \ldots, n$. If $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ then we set $z^I = z_1^{i_1} \cdots z_n^{i_n}$.

Let H be an analytic reproducing kernel space of $\mathcal{C}$-valued functions on Ω and assume that Ω contains 0. Then E_z has a power series expansion, $E_z = \sum\limits_{I \geqslant 0} z^I B_I$, where $B_I \in \mathcal{B}(H, \mathcal{C})$ and consequently, $K(z, w) = \sum\limits_{I,J \geqslant 0} z^I \overline{w}^J B_I B_J^*$ for z, w in some neighborhood of 0. The matrix $A = (B_I B_J^*)_{I,J \geqslant 0}$ is formally positive in the sense that if $\{\vec{x}_J\}_{J \in \mathbb{N}^n}$ is any collection of vectors in $\mathcal{C}$ with only finitely many non-zero terms, then $\sum_{I,J}\langle B_I B_J^* \vec{x}_J, \vec{x}_I\rangle_{\mathcal{C}} \geqslant 0$. Conversely, if $A = (A_{I,J})_{I,J\geqslant 0}$, $A_{I,J}$ in $\mathcal{B}(\mathcal{C})$, is formally positive and $K(z,w) = \sum z^I \overline{w}^J A_{I,J}$ converges on some polydisk, then $K(z,w)$ is positive definite on that polydisk.

Let $\{e_i\}_{i=0}^{\infty}$ denote the standard orthonormal basis for l^2 and set $l^2(n) = l^2 \otimes \cdots \otimes l^2$ (n copies) which has orthonormal basis $e_I = e_{i_1} \otimes \cdots \otimes e_{i_n}$, $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$. Every vector $\vec{x}$ in $l^2(n) \otimes \mathcal{C}$ has a unique representation as $\vec{x} = \sum_I e_I \otimes \vec{x}_I$ with $\vec{x}_I \in \mathcal{C}$ and $\sum \|\vec{x}_I\|^2 < \infty$. If $A \in \mathcal{B}(l^2(n) \otimes \mathcal{C})$ then A has a representation as $A = (A_{I,J})$ where each $A_{I,J} \in \mathcal{B}(\mathcal{C})$ and $A\vec{x} = \sum_I e_I \otimes (\sum_J A_{I,J}\vec{x}_J)$. Moreover if A is a positive operator, then $(A_{I,J})$ is formally positive, and $K(z,w) = \sum z^I \overline{w}^J A_{I,J}$ converges for z and w in the unit polydisk $\mathbb{D}^n$.

Thus to every positive operator $A = (A_{I,J})$ in $\mathcal{B}(l^2(n) \otimes \mathcal{C})$ we have an associated analytic reproducing kernel Hilbert space of $\mathcal{C}$-valued functions on $\mathbb{D}^n$ which we will denote by $H(A)$.

Conversely, if H is an analytic reproducing kernel Hilbert space of $\mathcal{C}$-valued functions on a connected domain Ω in $\mathbb{C}^n$, then by a translation and rescaling we may assume Ω contains the closed unit polydisk and that $K(z,w) = \sum z^I \overline{w}^J A_{I,J}$ where $A = (A_{I,J})$ defines a bounded positive operator on $\mathcal{B}(l^2(n) \otimes \mathcal{C})$. Letting A range over the positive operators on $\mathcal{B}(l^2(n) \otimes \mathcal{C})$, up to some equivalence, one obtains all the reproducing kernel Hilbert spaces of $\mathcal{C}$-valued functions on any connected domain Ω in $\mathbb{C}^n$. If $A = BB^*$, then the space $H(K)$ is given by $\Big\{ \sum_I y_I z^I : y = Bx, x \in l^2(n) \otimes \mathcal{C} \Big\}$.

Let $\widehat{K}(z,w) = \sum_{j=0}^{\infty} L_j L_j^* (\overline{w}z)^j$ where L_j is a lower triangular $n \times n$ matrix with trivial kernel and $w, z \in \mathcal{D} \subset \mathbb{C}$. By Theorem 2.1 of Adams, Froelich, McGuire, and Paulsen [3] $H(\widehat{K})$ can also be characterized by $R(L)$, the range space of L, where $L = \begin{pmatrix} L_0 & 0 & \cdots \\ 0 & L_1 & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}$ acts on $l^2 \otimes \mathbb{C}^n = \Big\{ \sum_{j=0}^{\infty} e_j \otimes \vec{x}_j : \vec{x}_j \in \mathbb{C}^n, \sum_{j=0}^{\infty} \|\vec{x}_j\|^2 < \infty \Big\}$. Here $\sum_{j=0}^{\infty} L_j \vec{x}_j z^j$ is identified with $\sum_{j=0}^{\infty} e_j \otimes L_j \vec{x}_j = L\Big(\sum_{j=0}^{\infty} e_j \otimes \vec{x}_j\Big)$ and $\Big\| \sum_{j=0}^{\infty} L_j \vec{x}_j z^j \Big\|_{H(\hat{K})}^2 = \Big\| L\Big(\sum_{j=0}^{\infty} e_j \otimes \vec{x}_j\Big)\Big\|_{R(L)}^2 = \sum_{j=0}^{\infty} \|\vec{x}_j\|^2$. An orthonormal basis for $H(\widehat{K})$ is given by $\{L_j(\vec{e}_j)z^j : j = 0, 1, 2, \ldots\}$ where $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is the standard basis for $\mathbb{C}^n$. Note also that if $\widehat{M}_z$ denotes multiplication by z, then formally $\widehat{M}_z L_j \vec{x} z^j = L_j \vec{x} z^{j+1} = L_{j+1}(L_{j+1}^{-1} L_j)\vec{x} z^{j+1}$. Hence if each L_j is invertible, then $\widehat{M}_z$ is an operator valued weighted shift with weights $\{L_{j+1}^{-1} L_j : j = 0, 1, \ldots\}$ and has the matrix form

$$\widehat{M}_z = \begin{pmatrix} 0 & 0 & \cdots \\ L_1^{-1} L_0 & 0 & 0 \\ 0 & L_2^{-1} L_1 & 0 \\ \vdots & \ddots & \ddots \end{pmatrix}.$$

3. THE MAIN RESULTS

THEOREM 3.1. *If $\widehat{K}(z,w)=\sum\limits_{n=0}^{\infty}R_nR_n^*(\overline{w}z)^n$ is positive definite on $\mathcal{D}\times\mathcal{D}$ for a sequence $\{R_n\}$ of $J\times J$ matrices, $h(z)=(h_1(z),\dots,h_J(z))$ is any holomorphic function from $\mathbb{C}$ into $\mathbb{C}^J$, $H(\widehat{K}_h)$ is the subspace of $H(\widehat{K})$ spanned by $\Big\{\sum\limits_{j=1}^{n}\alpha_j\widehat{K}(z,w_j)h(w_j)^* : w_j\in\mathcal{D},\alpha_j\in\mathbb{C},n\in\mathbb{N}\Big\}$, and K is the scalar valued kernel defined on $\mathcal{D}\times\mathcal{D}$ by $K(z,w)=h(z)\widehat{K}(z,w)h(w)^*$, then the map $V:H(\widehat{K}_h)\to H(K)$ defined by $(Vf)(z)=h(z)f(z)$ is an isomorphism.*

Proof. First note that $\Big\{\sum\limits_{j=1}^{n}\alpha_j\widehat{K}(z,w_j)h(w_j)^* : w_j\in\mathcal{D},\alpha_j\in\mathbb{C}\Big\}$ and $\Big\{\sum\limits_{j=1}^{n}\alpha_jh(z)\widehat{K}(z,w_j)h(w_j)^* : w_j\in\mathcal{D},\alpha_j\in\mathbb{C}\Big\}$ are dense in $H(\widehat{K}_h)$ and $H(K)$ respectively. Since $V\Big(\sum\limits_{j=1}^{n}\alpha_j\widehat{K}(z,w_j)h(w_j)^*\Big)=\sum\limits_{j=1}^{n}\alpha_jh(z_j)\widehat{K}(z,w_j)h(w_j)^*$ and

$$\Big\|\sum_{j=1}^{n}\alpha_j\widehat{K}(z,w_j)h(w_j)^*\Big\|^2_{H(\hat{K})}=\sum_{i,j}\alpha_i\overline{\alpha}_j\langle\widehat{K}(w_j,w_i)h(w_i)^*,h(w_j)^*\rangle_{\mathbb{C}^J}$$
$$=\sum_{i,j}\alpha_i\overline{\alpha}_jh(w_j)\widehat{K}(w_j,w_i)h(w_i)^*=\Big\|\sum_{j=1}^{n}\alpha_jh(z)\widehat{K}(z,w_j)h(w_j)^*\Big\|^2_{H(K)},$$

V is an isometry on the dense generating sets. Hence V extends to an isomorphism. ∎

DEFINITION 3.2. If $K(z,w)=\sum\limits_{n=0}^{\infty}f_n(z)\overline{f_n(w)}$ is a J bandwidth kernel with $f_n(z)=(a_{n,0}+a_{n,1}z+\cdots+a_{n,J}z^J)z^n$, then K is *strictly positive* if $a_{n,0}\neq0$ for all n.

Note that if $K(z,w)$ is written as $K(z,w)=(1,z,z^2,\dots)LL^*(1,w,w^2,\dots)^*$

$$=\begin{pmatrix}1 & z & z^2 & \dots\end{pmatrix}\begin{pmatrix}a_{0,0} & 0 & \cdots\\ a_{0,1} & a_{1,0} & \cdots\\ \vdots & a_{1,1} & \ddots\\ a_{0,J} & \vdots & \ddots\\ 0 & a_{1,J} & \ddots\\ \vdots & \ddots & \ddots\end{pmatrix}\begin{pmatrix}\overline{a}_{0,0} & \cdots & \overline{a}_{0,J} & 0 & \\ 0 & \overline{a}_{1,0} & \cdots & \overline{a}_{1,J} & \ddots\\ \vdots & \ddots & \ddots & \ddots & \ddots\end{pmatrix}\begin{pmatrix}1\\ \overline{w}\\ \overline{w}^2\\ \vdots\end{pmatrix},$$

then K strictly positive means the diagonal entries of L are all nonzero. If $a_{n,0}=0$ for some n, then the Cholesky decomposition of LL^* would result in a lower triangular matrix Q such that $QQ^*=LL^*$ and such that both the nth row and nth column of Q are identically zero. The effect on $H(K)$ is a splitting of the space into polynomials of degree less than n and z^{n+1} times a J bandwidth space. For this reason our attention will be focused on strictly positive kernels of finite bandwidth.

DEFINITION 3.3. The strictly positive reproducing kernel K is *properly* J *bandwidth* if K is J bandwidth and there does not exist a strictly positive kernel K_r of bandwidth $r < J$ such that $K_r \ll K$.

If $f_n(z) = (a_{n,0} + a_{n,1}z + \cdots + a_{n,J}z^J)z^n$, then $K(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$ can always be put into the form $K(z,w) = h(z)\left(\sum_{n=0}^{\infty} L_n L_n^*(\overline{w}z)^n\right)h(w)^*$ where $h(z) = (1, z, \ldots, z^J)$ and

$$L_n = \begin{pmatrix} a_{0,0} & 0 & \cdots & 0 \\ a_{0,1} & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ a_{0,J} & 0 & \cdots & 0 \end{pmatrix}.$$

Clearly this particular L_n is not invertible. Our next result leads to alternate choices for L_n that maximizes its rank based on its proper bandwidth.

THEOREM 3.4. *If* $K(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$ *is a strictly positive* J *bandwidth kernel with* $f_n(z) = (a_{n,0} + a_{n,1}z + \cdots + a_{n,J}z^J)z^n$, *then the following are equivalent:*

(i) *there exists a strictly positive properly* r *bandwidth kernel* K_r *where* $0 \leqslant r < J$ *such that* $K \gg K_r$;

(ii) $K(z,w) = K_0(z,w) + K_1(z,w) + \cdots + K_{J-r}(z,w)$ *where* $K_0(z,w)$ *is a* J *bandwidth kernel and* $K_m(z,w)$ *is* $(\overline{w}z)^m$ *times a strictly positive properly bandwidth* r *kernel for* $1 \leqslant m \leqslant J-r$, $0 \leqslant r < J$;

(iii) $K(z,w) = \sum_{n=0}^{\infty} h(z)L_nL_n^*h(w)^*(\overline{w}z)^n$, *where* $h(z) = (1, z, \ldots, z^J)$ *and each* L_n *is a* $J+1 \times J+1$ *matrix of the form*

$$\begin{pmatrix} \alpha_{0,0} & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ \alpha_{1,0} & \alpha_{1,1} & 0 & \ddots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \cdots & \vdots \\ \alpha_{r+1,0} & \alpha_{r+1,1} & \ddots & \ddots & 0 & 0 & \cdots & 0 \\ \vdots & 0 & \alpha_{r+2,2} & \ddots & \alpha_{J-r,J-r} & \vdots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{J,0} & 0 & \cdots & 0 & \alpha_{J,J-r} & 0 & \cdots & 0 \end{pmatrix}$$

with $\alpha_{m,m} \neq 0$ *for* $0 \leqslant m \leqslant J-r$.

Proof. Since the case $J = 0$ is trivial, assume $J \geqslant 1$. Recall that if $K(z,w) = \sum_{i,j=0}^{\infty} a_{i,j}z^i\overline{w}^j$, then the corresponding coefficient matrix $A = [a_{i,j}]$ satisfies $a_{i,j} = 0$ if $|i-j| > J$. Conversely if $A = [a_{i,j}] > 0$ satisifes $a_{i,j} = 0$ if $|i-j| > J$, then the Cholesky decomposition $A = LL^*$ produces a lower triangular matrix

L of bandwidth J. Now suppose $K_r(z,w) = \sum\limits_{n=0}^{\infty} g_n(z)\overline{g_n(w)}$ is a strictly positive properly r bandwidth kernel with $0 \leqslant r < J$ such that $K \gg K_r$. Note the associated matrices A and A_r satisfy $A - A_r > 0$ and $A - A_r$ is of the same bandwidth as A. Hence the Cholesky algorithm produces L_1 of bandwidth J such that $A - A_r = L_1 L_1^*$ which implies $K - K_r$ is of bandwidth J. Similarly $K_0(z,w) = K(z,w) - K_r(z,w) + g_0(z)\overline{g_0(w)}$ is of bandwidth J. For $1 \leqslant m \leqslant J-r-1$, let

$$K_m(z,w) = (\overline{w}z)^m \Big[\frac{g_m(z)\overline{g_m(w)}}{2^{m-1}(\overline{w}z)^m} + \sum_{n=m+1}^{\infty} \frac{g_n(z)\overline{g_n(w)}}{2^m(\overline{w}z)^m}\Big]$$

and let $K_{J-r}(z,w) = (\overline{w}z)^{J-r} \sum\limits_{n=J-r}^{\infty} \frac{g_n(z)\overline{g_n(w)}}{2^{J-r-1}(\overline{w}z)^{J-r}}$. Since $K_r(z,w)$ is a strictly positive r bandwidth kernel, $K_m(z,w)$ is $(\overline{w}z)^m$ times a strictly positive r bandwidth kernel for each $m = 1,\dots,J-r$. It is routine to observe that $K(z,w) = K_0(z,w) + K_1(z,w) + \cdots + K_{J-r}(z,w)$. That condition (ii) implies (i) is trivial.

Assume L_n is as is in the statement of part (iii) of the theorem. The equivalence of (ii) and (iii) follows from the observation that

$$K(z,w) = \sum_{n=0}^{\infty} h(z)L_nL_n^*h(w)^*(\overline{w}z)^n = \sum_{m=0}^{J-r}\Big(\sum_{n=0}^{\infty}(h(z)L_n\vec{e}_m)(\vec{e}_m^{\,*}L_n^*h(w)^*)(\overline{w}z)^n\Big)$$

where $\vec{e}_m$ is the mth canonical basis vector of $\mathbb{C}^J$. Also note that

$$K_m(z,w) = \sum_{n=0}^{\infty}(h(z)L_n\vec{e}_m)(\vec{e}_m^{\,*}L_n^*h(w)^*)(\overline{w}z)^n$$

is $(\overline{w}z)^m$ times a strictly positive bandwidth r kernel for $m = 1,\dots,J-r$ while $K_0(z,w)$ is a strictly positive bandwidth J kernel. ∎

Of particular interest is the case where $K \gg K_0$ for some strictly positive bandwidth 0 kernel. In this case $H(K)$ contains all the powers of z and hence all the polynomials. Additionally each L_n is a lower triangular invertible matrix. The next theorem summarizes this case and provides a more concrete necessary and sufficient condition for $H(K)$ to contain the polynomials. It is important to note that the choice of L_n is not unique. It will be shown later that there is a range of choices for L_n and that a judicious choice is not only possible but necessary for the boundedness of $\widehat{M}_z$.

THEOREM 3.5. *If $f_n(z) = (a_{n,0} + a_{n,1}z + \cdots + a_{n,J}z^J)z^n$ with $a_{n,0} \neq 0$ and $K(z,w) = \sum\limits_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$, then the following are equivalent:*

(i) *$K(z,w)$ may be written in the form*

$$K(z,w) = h(z)\Big(\sum_{n=0}^{\infty} L_nL_n^*(\overline{w}z)^n\Big)h(w)^*,$$

where $h(z) = (1, z, \dots, z^J)$ and L_n is an invertible $J+1 \times J+1$ lower triangular matrix;

(ii) *the space $H(K)$ contains the polynomials;*

(iii) *for* $j \geqslant 0$, *the sequence* $\left\{1, \frac{D_{j,1}}{a_{j+1,0}}, \frac{D_{j,2}}{a_{j+1,0}a_{j+2,0}}, \frac{D_{j,3}}{a_{j+1,0}a_{j+2,0}a_{j+3,0}}, \cdots\right\}$ *is square summable where* $D_{j,k}$ *is the* $k \times k$ *determinant of the submatrix of*

$$L = \begin{pmatrix} a_{0,0} & 0 & 0 & \cdots \\ a_{0,1} & a_{1,0} & 0 & \cdots \\ \vdots & a_{1,1} & \ddots & \ddots \\ a_{0,J} & \ddots & \ddots & \ddots \\ 0 & a_{1,J} & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix}$$

which has $a_{j,1}$ *in the upper left corner.*

Proof. The equivalence of (i) and (ii) is a consequence of Theorem 3.4. To establish (iii) recall z^n belongs to $H(K)$ if and only if the range of L contains the canonical basis vector $\vec{e}_n$. This reduces the problem to finding a square summable sequence $\{\lambda_{n+k}\}_{k=0}^{\infty}$ such that

$$\begin{pmatrix} a_{n,0} & 0 & 0 & \cdots \\ a_{n,1} & a_{n+1,0} & 0 & \cdots \\ \vdots & a_{n+1,1} & \ddots & \ddots \\ a_{n,J} & \ddots & \ddots & \ddots \\ 0 & a_{n+1,J} & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \lambda_n \\ \lambda_{n+1} \\ \lambda_{n+2} \\ \lambda_{n+3} \\ \lambda_{n+4} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

Since L is upper triangular it is straightforward to use Cramer's Rule to determine that

$$\lambda_n = \frac{1}{a_{n,0}}, \quad \lambda_{n+1} = \frac{\begin{vmatrix} a_{n,0} & 1 \\ a_{n,1} & 0 \end{vmatrix}}{\begin{vmatrix} a_{n,0} & 0 \\ a_{n,1} & a_{n+1,0} \end{vmatrix}}, \quad \lambda_{n+2} = \frac{\begin{vmatrix} a_{n,0} & 0 & 1 \\ a_{n,1} & a_{n+1,0} & 0 \\ a_{n,2} & a_{n+1,1} & 0 \end{vmatrix}}{\begin{vmatrix} a_{n,0} & 0 & 0 \\ a_{n,1} & a_{n+1,0} & 0 \\ a_{n,2} & a_{n+1,1} & a_{n+2,0} \end{vmatrix}},$$

and so on. Expanding the top determinants along the last column shows $a_{n,0}\lambda_{n+k} = (-1)^k \frac{D_{n,k}}{a_{n+1,0}\cdots a_{n+k,0}}$ and hence the sequence $\{\lambda_{n+k}\}$ is square summable if and only if the sequence $\left\{\frac{D_{n,k}}{a_{n+1,0}\cdots a_{n+k,0}}\right\}$ is square summable. ∎

Our next result relates the multiplication operators on a bandwidth J space $H(K)$ containing the polynomials to compressions of Toeplitz type matrix operators on $H(\widehat{K})$. In particular the operator of multiplication by z on $H(K)$ is realized as the compression of a matrix valued weighted shift. Before stating and proving the result a preliminary lemma is necessary.

LEMMA 3.6. *If $\{g_n\}_{n=0}^{\infty}$ converges in norm to g in $H(\widehat{K})$ where $\widehat{K}$ is an analytic $J+1 \times J+1$ matrix valued reproducing kernel in a domain $\mathcal{D} \times \mathcal{D}$, then:*

(i) *$g_n^{(k)}(z) \to g^{(k)}(z)$ uniformly on compact subsets of $\mathcal{D}$ for all k;*

(ii) *the Taylor series coefficients of g_n converge to the Taylor series coefficients of g.*

Proof. For each $\vec{x} \in \mathbb{C}^J$ and $w \in \mathcal{D}$,

$$\begin{aligned}|\langle g_n(w) - g(w), \vec{x}\rangle_{\mathbb{C}^J}| &= |\langle g_n - g, \widehat{K}(z,w)\vec{x}\rangle_{H(\hat{K})}| \leqslant \|g_n - g\|_{H(\hat{K})} \|\widehat{K}(z,w)\vec{x}\|_{H(\hat{K})} \\ &\leqslant \|g_n - g\|_{H(\hat{K})} \|\widehat{K}(w,w)^{1/2}\|_{\mathbb{C}^J} \|\vec{x}\|_{\mathbb{C}^J}.\end{aligned}$$

Since $\|K(w,w)^{1/2}\|_{\mathbb{C}^J}$ is uniformly bounded on compact subsets of $\mathcal{D}$ and $\|g_n - g\|_{H(\hat{K})} \to 0$, $g_n(w) \to g(w)$ uniformly on compact subsets of $\mathcal{D}$. Hence all the derivatives converge uniformly and consequently the Taylor series coefficients. ∎

THEOREM 3.7. *If $K(z,w)$ is a strictly positive J bandwidth kernel such that $H(K)$ contains the polynomials, then there exists a sequence of invertible $J+1 \times J+1$ matrices $\{R_n\}_{n=0}^{\infty}$ such that:*

(i) *the map $V : H(\widehat{K}_h) \to H(K)$ given by $V(f(z)) = h(z)f(z)$ is an isomorphism where $\widehat{K}(z,w) = \sum\limits_{n=0}^{\infty} R_n R_n^*(\overline{w}z)^n$, $h(z) = (1, z, \ldots, z^J)$, and $H(\widehat{K}_h)$ is the subspace of $H(\widehat{K})$ spanned by $\mathcal{G} = \Big\{ \sum\limits_{i=1}^{n} \alpha_i \widehat{K}(z,w_i)h(w_i)^* : \alpha_i \in \mathbb{C}, w_i \in \mathcal{D}, n \in \mathbb{N}\Big\}$;*

(ii) *if $\varphi(z) = \sum\limits_{n=0}^{\infty} \varphi_n z^n$ is an analytic function, then φ is a multiplier of $H(K)$ if and only if the compression of $\widehat{M}_\varphi$ to $H(\widehat{K}_h)$ is bounded where $\widehat{M}_\varphi$ is formally defined on $H(\widehat{K})$ by the matrix*

$$\widehat{M}_\varphi = \begin{pmatrix} \varphi_0 I & 0 & 0 & \cdots \\ \varphi_1 R_1^{-1} R_0 & \varphi_0 I & 0 & \ddots \\ \varphi_2 R_2^{-1} R_0 & \varphi_1 R_2^{-1} R_1 & \varphi_0 I & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

In this case $H(\widehat{K}_h)$ is invariant under $\widehat{M}_\varphi^*$ and $M_\varphi^* = V\widehat{M}_\varphi^* V^*$.

Proof. Part (i) is a simple consequence of Theorem 3.1 and Theorem 3.5.

Before proceeding note that if $f(z) = \sum\limits_{n=0}^{\infty} R_n \vec{y}_n z^n$ is in $H(\widehat{K})$, then $\widehat{M}_\varphi$ and $\widehat{M}_\varphi^*$ are formally defined by $\widehat{M}_\varphi(f(z)) = \sum\limits_{n=0}^{\infty} R_n \Big(\sum\limits_{j=0}^{n} \varphi_j R_n^{-1} R_{n-j} \vec{y}_{n-j} \Big) z^n$ and

$\widehat{M}_\varphi^*(f(z)) = \sum\limits_{n=0}^{\infty} R_n\Big(\sum\limits_{j=0}^{\infty} \overline{\varphi}_j R_n^* R_{n+j}^{*^{-1}} \vec{y}_{n+j}\Big) z^n$. Also for $w \in \mathcal{D}$,

$$\begin{aligned}
&\widehat{M}_\varphi^*(\widehat{K}(z,w)h(w)^*)\\
&\quad = \widehat{M}_\varphi^*\Big(\sum_{n=0}^{\infty} R_n R_n^* h(w)^*(\overline{w}z)^n\Big) = \sum_{n=0}^{\infty} R_n\Big(\sum_{j=0}^{\infty} \overline{\varphi}_j R_n^* R_{n+j}^{*^{-1}} R_{n+j}^* h(w)^* \overline{w}^{n+j}\Big) z^n\\
&\quad = \sum_{n=0}^{\infty} R_n R_n^* \Big(\sum_{j=0}^{\infty} \overline{\varphi}_j \overline{w}^j\Big) h(w)^*(\overline{w}z)^n = \overline{\varphi(w)}\widehat{K}(z,w)h(w)^*.
\end{aligned}$$

To establish (ii) first assume that M_φ is bounded on $H(K)$. Since $M_\varphi^* K(z,w) = \overline{\varphi(w)}K(z,w)$, $V^* M_\varphi^* V \widehat{K}(z,w)h(w)^* = \overline{\varphi(w)}\widehat{K}(z,w)h(w)^*$. Thus $\widehat{M}_\varphi^*$ and $V^* M_\varphi^* V$ agree on $\mathcal{G}$.

Now suppose $\{g_n\} \subset \mathcal{G}$ and g_n converges to g in norm. Write $g_n(z) = \sum\limits_{j=0}^{\infty} R_j \vec{y}_{n,j} z^j$ and $g(z) = \sum\limits_{j=0}^{\infty} R_j \vec{y}_j z^j$. By Lemma 3.6, $\vec{y}_{n,j} \to \vec{y}_j$ for each j. Also since $V^* M_\varphi^* V g_n \to V^* M_\varphi^* V g$ and

$$V^* M_\varphi^* V g_n = \widehat{M}_\varphi^* g_n = \sum_{j=0}^{\infty} R_j \Big(\sum_{k=0}^{\infty} \overline{\varphi}_k R_j^* R_{j+k}^{*^{-1}} \vec{y}_{n,j+k}\Big) z^j$$

for each j, the sequence $\Big\{R_j \sum\limits_{k=0}^{\infty} \overline{\varphi}_k R_j^* R_{j+k}^{*^{-1}} \vec{y}_{n,j+k}\Big\}_{n=0}^{\infty}$ converges to the jth coefficient of $V^* M_\varphi^* V g$. For each k, $\overline{\varphi}_k R_j^* R_{j+k}^{*^{-1}} \vec{y}_{n,j+k} \to \overline{\varphi}_k R_j^* R_{j+k}^{*^{-1}} \vec{y}_{j+k}$. As the series $\sum\limits_{k=0}^{\infty} \overline{\varphi}_k R_j^* R_{j+k}^{*^{-1}} \vec{y}_{n,j+k}$ converges for each j,

$$\lim_{n\to\infty} R_j\Big(\sum_{k=0}^{\infty} \overline{\varphi}_k R_j^* R_{j+k}^{*^{-1}} \vec{y}_{n,j+k}\Big) = R_j \sum_{k=0}^{\infty} \overline{\varphi}_k R_j^* R_{j+k}^{*^{-1}} \vec{y}_{j+k}.$$

Thus $V^* M_\varphi^* V g = \sum\limits_{j=0}^{\infty} R_j \Big(\sum\limits_{k=0}^{\infty} \overline{\varphi}_k R_j^* R_{j+k}^{*^{-1}} \vec{y}_{j+k}\Big) z^j = \widehat{M}_\varphi^* g$.

For the converse of (ii), let A denote the compression of $\widehat{M}_\varphi$ to $H(\widehat{K}_h)$ and assume A is bounded. Note VAV^* is bounded on $H(K)$ and

$$\begin{aligned}
\langle VAV^* f(z), K(z,w)\rangle &= \langle f(z), VA^*V^*K(z,w)\rangle = \langle f(z), VA^*\widehat{K}(z,w)h(w)^*\rangle\\
&= \langle f(z), V\overline{\varphi(w)}\widehat{K}(z,w)h(w)^*\rangle = \varphi(w)\langle f(z), K(z,w)\rangle = \varphi(w)f(w)
\end{aligned}$$

for all $w \in \mathcal{D}$. Thus $VAV^* = M_\varphi$ is bounded on $H(K)$. ∎

DEFINITION 3.8. A sequence $\{R_n\}_{n=0}^{\infty}$ that satisfies Theorem 3.7 above is called a *lifting* for $H(K)$.

It is important to realize two things about Theorem 3.7 above. First the lifting sequences $\{R_n\}_{n=0}^{\infty}$ are not unique and second it is possible that the compression of $\widehat{M}_\varphi$ is bounded even though $\widehat{M}_\varphi$ is not bounded. The example below illustrates these points in that two liftings are produced for the same $H(K)$. With one of the liftings $\widehat{M}_z$ is bounded and with one it is not. The lack of uniqueness of the liftings will be addressed further in the next section. Recall the form of $\widehat{M}_z$ when the polynomials are present is

$$\widehat{M}_z = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ L_1^{-1}L_0 & 0 & \ddots & \ddots \\ 0 & L_2^{-1}L_1 & 0 & \ddots \\ \vdots & \ddots & L_3^{-1}L_2 & \ddots \end{pmatrix}.$$

EXAMPLE 3.9. Let

$$K(z,w) = \sum_{n=0}^{\infty}(2+z)(2+\overline{w})(\overline{w}z)^n = \sum_{n=0}^{\infty}\begin{pmatrix} 1 & z \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \overline{w} \end{pmatrix}(\overline{w}z)^n$$

and note the space $H(K)$ contains exactly the same functions as $(2+z)H^2 = H^2$. Thus the multipliers of $H(K)$ are the bounded analytic functions. In particular M_z is bounded on $H(K)$ and is unitarily equivalent to the unilateral shift. One lifting of $H(K)$ is to write

$$\begin{aligned} &K(z,w) \\ &\quad = \begin{pmatrix} 1 & z \end{pmatrix}\left[\begin{pmatrix} 2 & 0 \\ 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & \sqrt{2} \end{pmatrix} + \sum_{k=1}^{\infty}\begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 1 \end{pmatrix}(\overline{w}z)^k\right]\begin{pmatrix} 1 \\ \overline{w} \end{pmatrix}. \end{aligned}$$

In this case $L_{k+1}^{-1}L_k = I$ for $k \geqslant 1$ and hence $\widehat{M}_z$ is bounded. Another lifting of $H(K)$ is to write

$$\begin{aligned} &K(z,w) \\ &\quad = \begin{pmatrix} 1 & z \end{pmatrix}\left[\begin{pmatrix} 2 & 0 \\ 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & \sqrt{2} \end{pmatrix} + \sum_{k=1}^{\infty}\begin{pmatrix} \alpha_k & 0 \\ \beta_k & \gamma_k \end{pmatrix}\begin{pmatrix} \alpha_k & \beta_k \\ 0 & \gamma_k \end{pmatrix}(\overline{w}z)^k\right]\begin{pmatrix} 1 \\ \overline{w} \end{pmatrix}, \end{aligned}$$

where $\alpha_{2k-1}^2 = 2$, $\alpha_{2k}^2 = 3 - \frac{1}{k^2}$, $\beta_k = \frac{2}{\alpha_k}$, $\gamma_{2k-1}^2 = \frac{1}{k^2}$, and $\gamma_{2k}^2 = 3 - \frac{4k^2}{3k^2-1}$ if $k \geqslant 1$. In this case $L_{k+1}^{-1}L_k = \begin{pmatrix} \frac{\alpha_k}{\alpha_{k+1}} & 0 \\ \frac{\beta_k}{\gamma_{k+1}} - \frac{\alpha_k\beta_{k+1}}{\gamma_{k+1}\alpha_{k+1}} & \frac{\gamma_k}{\gamma_{k+1}} \end{pmatrix}$ for $k \geqslant 1$. Since $\frac{\gamma_{2k}^2}{\gamma_{2k+1}^2} \geqslant (k+1)^2$, $\widehat{M}_z$ is unbounded.

4. EXAMPLES AND THE LIFTING MAP

While Theorems 3.4, 3.5, and 3.7 guarantee the existence of a lifting whenever $H(K)$ contains the polynomials, they are not constructive. By Theorem 3.4, L_n may be taken to be of the form

$$L_n = \begin{pmatrix} \alpha_{n,0} & 0 & 0 & \cdots & 0 \\ \alpha_{n,1} & \beta_{n,1} & 0 & \ddots & \vdots \\ \alpha_{n,2} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \alpha_{n,J} & 0 & \cdots & 0 & \beta_{n,J} \end{pmatrix}.$$

We assume the diagonal entries have been normalized so they are non-negative. By Theorem 3.1, $V(\widehat{K}(z,w)h(w)^*) = K(z,w)$ which leads to the following recursive conditions on collecting like coefficients of z and $\overline{w}$:

(1) $\alpha_{0,0}^2 = a_{0,0}^2$;

(2) $\sum\limits_{i=0}^{k} |\alpha_{k-i,i}|^2 + \sum\limits_{i=1}^{k} \beta_{k-i,i}^2 = \sum\limits_{i=0}^{k} |a_{k-i,i}|^2$ if $k = 1, \ldots, J-1$;

(3) $\sum\limits_{i=0}^{J} |\alpha_{k-i,i}|^2 + \sum\limits_{i=1}^{J} \beta_{k-i,i}^2 = \sum\limits_{i=0}^{J} |a_{k-i,i}|^2$ if $k \geqslant J$;

(4) $\sum\limits_{i=0}^{k} \alpha_{k-i,i+l}\overline{\alpha}_{k-i,i} = \sum\limits_{i=0}^{k} a_{k-i,i+l}\overline{a}_{k-i,i}$ if $l > 0$ and $k + l < J$;

(5) $\sum\limits_{i=0}^{J-l} \alpha_{k-i,i+l}\overline{\alpha}_{k-i,i} = \sum\limits_{i=0}^{J-l} a_{k-i,i+l}\overline{a}_{k-i,i}$ if $l > 0$ and $k + l \geqslant J$.

Although the general solution, where J is arbitrary, of this non-linear recursion is yet out of reach, the next result provides a constructive solution in the case $J = 1$.

Theorem 4.1. *If* $K(z,w) = \sum\limits_{n=0}^{\infty} (a_n + b_n z)(a_n + \overline{b}_n \overline{w})(\overline{w}z)^n$, *then the following are equivalent:*

(i) *$H(K)$ contains the polynomials;*

(ii) $s_n^2 = 1 + \left|\frac{b_n}{a_{n+1}}\right|^2 + \left|\frac{b_n b_{n+1}}{a_{n+1}a_{n+2}}\right|^2 + \cdots$ *is finite for all n;*

(iii) *the recursion*

$$\left\{\begin{array}{ll} \alpha_0^2 = a_0^2 & \\ \alpha_1^2 + \gamma_0^2 = a_1^2 & \\ \alpha_n^2 + |\beta_{n-1}|^2 + \gamma_{n-1}^2 = a_n^2 + |b_{n-1}|^2 & \text{if } n \geqslant 2 \\ \alpha_n \beta_n = a_n b_n & \text{for } n \geqslant 0 \end{array}\right\}$$

can be satisfied for $\alpha_n \neq 0$, β_n, and $\gamma_n \neq 0$ enabling

$$K(z,w) = \sum_{n=0}^{\infty} \begin{pmatrix} 1 & z \end{pmatrix} \begin{pmatrix} \alpha_n & 0 \\ \beta_n & \gamma_n \end{pmatrix} \begin{pmatrix} \alpha_n & \overline{\beta}_n \\ 0 & \gamma_n \end{pmatrix} \begin{pmatrix} 1 \\ \overline{w} \end{pmatrix} (\overline{w}z)^n$$

with $\alpha_n \neq 0$ and $\gamma_n \neq 0$.

Proof. Since the equivalence of condition (ii) and condition (i) follows immediately from part (iii) of Theorem 3.5 it suffices to show (ii) and (iii) are equivalent. If $s_n < \infty$ for all n, then we show that the α's can be chosen sequentially to satisfy $\alpha_0^2 = a_0^2$, $a_1^2\left(1 - \frac{1}{s_1^2}\right) < \alpha_1^2 < a_1^2$, and $a_n^2\left(1 - \frac{1}{s_n^2}\right) < \alpha_n^2 < a_n^2 + |b_{n-1}|^2 - \left|\frac{a_{n-1}b_{n-1}}{\alpha_{n-1}}\right|^2$, for $n \geqslant 2$. If we have chosen $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ to satisfy their respective inequalities, then in order to choose an α_n we need to know that

$$a_n^2\left(1 - \frac{1}{s_n^2}\right) < a_n^2 + |b_{n-1}|^2 - \left|\frac{a_{n-1}b_{n-1}}{\alpha_{n-1}}\right|^2.$$

This inequality is obvious if $b_{n-1} = 0$ and otherwise is equivalent to $\left|\frac{a_{n-1}b_{n-1}}{\alpha_{n-1}}\right|^2 < \frac{a_n^2}{s_n^2} + |b_{n-1}|^2$ or

$$\alpha_{n-1}^2 > \frac{|a_{n-1}b_{n-1}|^2}{\frac{a_n^2}{s_n^2} + |b_{n-1}|^2} = \frac{|a_{n-1}b_{n-1}|^2}{\frac{|b_{n-1}|^2}{s_{n-1}^2 - 1} + |b_{n-1}|^2} = a_{n-1}^2\left(1 - \frac{1}{s_{n-1}^2}\right)$$

which was satisfied with the selection of α_{n-1}. The β's and γ's are determined immediately from the α's.

Conversely, suppose the system (iii) is solvable for α's, β's, and γ's with $\alpha_n \neq 0$ and $\gamma_n \neq 0$. We will show this implies $s_n < \infty$ for all n. Since $|\beta_{n-1}|^2 < a_n^2 + |b_{n-1}|^2$ and $\alpha_{n-1}\beta_{n-1} = a_{n-1}b_{n-1}$, we have $\frac{|a_{n-1}b_{n-1}|^2}{\alpha_{n-1}^2} < a_n^2 + |b_{n-1}|^2$. Thus, for $n \geqslant 2$

$$\alpha_{n-1}^2 > \frac{|a_{n-1}b_{n-1}|^2}{a_n^2 + |b_{n-1}|^2} = a_{n-1}^2\left(1 - \frac{1}{1 + \frac{|b_{n-1}|^2}{a_n^2}}\right)$$

or for $n \geqslant 1$, $\alpha_n^2 > a_n^2\left(1 - \frac{1}{1 + \frac{|b_n|^2}{a_{n+1}^2}}\right)$. We also see that for $n \geqslant 1$,

$$\alpha_n^2 < a_n^2 + |b_{n-1}|^2 - |\beta_{n-1}|^2 = a_n^2 + |b_{n-1}|^2 - \frac{a_{n-1}^2|b_{n-1}|^2}{\alpha_{n-1}^2}.$$

By combining the last two inequalities we obtain that for $n \geqslant 1$,

$$a_n^2\left(1 - \frac{1}{1 + \frac{|b_n|^2}{a_{n+1}^2}}\right) < a_n^2 + |b_{n-1}|^2 - \frac{a_{n-1}^2|b_{n-1}|^2}{\alpha_{n-1}^2}.$$

This becomes

$$\frac{a_{n-1}^2|b_{n-1}|^2}{\alpha_{n-1}^2} < |b_{n-1}|^2 + \frac{a_n^2}{1 + \frac{|b_n|^2}{a_{n+1}^2}}$$

or

$$\alpha_{n-1}^2 > \frac{a_{n-1}^2|b_{n-1}|^2}{|b_{n-1}|^2 + \frac{a_n^2}{1 + \frac{|b_n|^2}{a_{n+1}^2}}} = \frac{a_{n-1}^2}{1 + \frac{a_n^2}{|b_{n-1}|^2\left(1 + \frac{|b_n|^2}{a_{n+1}^2}\right)}},$$

assuming $b_{n-1} \neq 0$. Thus

$$\alpha_{n-1}^2 > \frac{a_{n-1}^2}{1+\frac{1}{\frac{|b_{n-1}|^2}{a_n^2}+\frac{|b_{n-1}|^2|b_n|^2}{a_n^2 a_{n+1}^2}}} = a_{n-1}^2\left(1-\frac{1}{1+\frac{|b_{n-1}|^2}{a_n^2}+\frac{|b_{n-1}|^2|b_n|^2}{a_n^2 a_{n+1}^2}}\right),$$

which is valid even if $b_{n-1}=0$. We have arrived at

$$a_n^2\left(1-\frac{1}{1+\frac{|b_n|^2}{a_{n+1}^2}+\frac{|b_n|^2|b_{n+1}|^2}{a_{n+1}^2 a_{n+2}^2}}\right) < \alpha_n^2 < a_n^2 + |b_{n-1}|^2 - \frac{a_{n-1}^2|b_{n-1}|^2}{\alpha_{n-1}^2}.$$

Proceeding inductively, we get

$$a_n^2\left(1-\frac{1}{s_n^2}\right) \leqslant \alpha_n^2 < a_n^2 + |b_{n-1}|^2 - \frac{a_{n-1}^2|b_{n-1}|^2}{\alpha_{n-1}^2}.$$

However, notice that $\alpha_1^2 < a_1^2$, which implies $|\beta_1|^2 = \frac{a_1^2|b_1|^2}{\alpha_1^2} > |b_1|^2$, which in turn implies $\alpha_2^2 < a_2^2$, etc. Since $\alpha_n^2 < a_n^2$, we cannot have $s_n^2 = \infty$. ∎

The bandwidth 1 example $K(z,w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$ where $f_n(z) = \left(1 + \frac{n+1}{n+2}z\right)z^n$ was extensively studied in Adams, McGuire [1]. In particular it was shown that M_z is bounded and $\varphi(z) = \sum_{n=0}^{\infty} \varphi_n z^n$ was determined to be a multiplier of $H(K)$ if and only if $\varphi \in H^\infty$ and $\sum_{m=1}^{\infty}\left[\frac{(P_m\varphi)'(-1)}{m+1}\right]^2 < \infty$ where $(P_m\varphi)(z) = \sum_{n=0}^{m} \varphi_n z^n$. It is worth observing that a bounded lifting exists for this example and one such is obtained by taking $R_n = \begin{pmatrix} \alpha_n & 0 \\ \beta_n & \gamma_n \end{pmatrix}$ where $\alpha_0 = 1$, $\beta_0 = 1/2$, $\gamma_0 = 1/\sqrt{3}$, and thereafter $\alpha_n^2 = \beta_n^2 = \frac{n+1}{n+2}$, $\gamma_n^2 = \frac{1}{(n+3)(n+2)^2}$. In this case $\widehat{K}(z,w) = \sum_{n=0}^{\infty} R_n R_n^*(\overline{w}z)^n$ and M_z on $H(K)$ is a compression of $\widehat{M}_z$ on $H(\widehat{K})$ where

$$\widehat{M}_z = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \cdots & \cdots & \cdots \\ \begin{pmatrix} \sqrt{\frac{3}{2}} & 0 \\ -3 & 2\sqrt{3} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \ddots & \ddots \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{2}{3}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{5} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \ddots \\ \vdots & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{\sqrt{5}}{4}\begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{6} \end{pmatrix} & \ddots \end{pmatrix}.$$

5. OPEN QUESTIONS

(1) If M_φ is bounded on $H(K)$ must there always exist a lifting for which $\widehat{M}_\varphi$ is bounded?

(2) If K is J bandwidth and M_z is bounded, is M_z unitarily equivalent to a "small" perturbation of a weighted shift?

(3) What role if any do the cluster points of the sequence of zeros of $f_n(z)$ play in the characterization of the functions in $H(K)$?

(4) To what extent can Theorems 3.5 and 3.7 be modified for spaces that do not contain the polynomials?

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