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THE TOEPLITZ ALGEBRA ON THE BERGMAN SPACE COINCIDES WITH ITS COMMUTATOR IDEAL

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Abstract. Let L_a^2 be the Bergman space of the unit disk and $\mathfrak{T}(L_a^2)$ be the Banach algebra generated by Toeplitz operators T_f, with $f \in L^\infty$. We prove that the closed bilateral ideal of $\mathfrak{T}(L_a^2)$ generated by operators of the form $T_fT_g - T_gT_f$ coincides with $\mathfrak{T}(L_a^2)$.

Keywords: *Commutator ideal, Toeplitz operators, Bergman space.*

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1. INTRODUCTION

If $0 < p \leqslant \infty$ let $L^p = L^p(\mathbb{D}, \mathrm{d}A)$, where $\mathbb{D}$ is the open unit disk and $\mathrm{d}A(z) = (1/\pi)\mathrm{d}x\mathrm{d}y$, with $z = x + \mathrm{i}y$, is the normalized area measure on $\mathbb{D}$. The Bergman space L_a^p is formed by the analytic functions in L^p. If $1 < p < \infty$ then

$$(Pf)(z) = \int_{\mathbb{D}} \frac{f(\omega)}{(1-\overline{\omega}z)^2}\,\mathrm{d}A(\omega)$$

is a bounded projection from L^p onto L_a^p. This is the usual Bergman projection. For $a \in L^\infty$ let $M_a : L^p \to L^p$ be the operator of multiplication by a and $P_a = PM_a$. Then $\|P_a\| \leqslant C_p\|a\|_\infty$, where C_p is the norm of P acting on L^p. The Toeplitz operator $T_a : L_a^p \to L_a^p$ is the restriction of P_a to the space L_a^p. If B is a Banach space, we will write $\mathfrak{L}(B)$ for the algebra of all bounded operators on B and $\mathfrak{T}(L_a^p)$ for the closed subalgebra of $\mathfrak{L}(L_a^p)$ generated by $\{T_a : a \in L^\infty\}$.

If A is a Banach algebra, its commutator ideal $\mathfrak{C}A$ is the closed bilateral ideal generated by elements of the form $[x, y] \stackrel{\mathrm{def}}{=} xy - yx$, with $x, y \in A$. It is clear that $\mathfrak{C}A$ is the smallest closed ideal of A such that $A/\mathfrak{C}A$ is a commutative Banach algebra. There is an extensive literature on commutator ideals and abelianizations of Toeplitz algebras acting on the Hardy space H^2. The book of Nikoslki

([5]) contains plentiful information and further references. In contrast with this situation, we only have a handful of results for Toeplitz algebras of operators on L_a^2. Probably the most relevant papers on the subject are [2], [4] and [1].

If H is a Hilbert space of dimension greater than one then $\mathfrak{C}\mathfrak{L}(H) = \mathfrak{L}(H)$. Although this situation is very unusual for Toeplitz algebras, the purpose of this paper is to prove the following

THEOREM 1.1. *The Toeplitz algebra on L_a^2 coincides with its commutator ideal.*

In [3] it is shown that if $\phi(z) = \exp(\mathrm{i}\log\log|z|^{-2})$ then the semicommutator $T_{\overline{\phi}\phi} - T_{\overline{\phi}}T_\phi$ is a nontrivial scalar multiple of the identity. Analogously, it could happen that there are two simple functions $a, b \in L^\infty$ such that $T_aT_b - T_bT_a$ is easily seen to be invertible. This would immediately prove Theorem 1.1. Since I was unable to find such functions or even prove their existence, the proof here is considerably more complicated.

2. SEGMENTATION

For $z \in \mathbb{D}$ let $\varphi_z(\omega) = (z - \omega)/(1 - \overline{z}\omega)$, the special automorphism of $\mathbb{D}$ that interchanges 0 and z. The pseudo-hyperbolic metric is defined by $\rho(z, \omega) = |\varphi_z(\omega)|$ for $z, \omega \in \mathbb{D}$. It is well known that ρ is invariant under the action of $\mathrm{Aut}(\mathbb{D})$. We will also use that

$$\rho(z,\omega) \geqslant \frac{\rho(z,u) - \rho(u,\omega)}{1 - \rho(z,u)\rho(u,\omega)} \quad \text{for all } z, \omega, u \in \mathbb{D}.$$

If $0 < r < 1$ write $K(z, r) \stackrel{\text{def}}{=} \{\omega \in \mathbb{D} : \rho(\omega, z) \leqslant r\}$ for the closed ball of center z and radius r with respect to ρ. A sequence $\mathcal{S} = \{z_n\}$ in $\mathbb{D}$ will be called *separated* if $\inf\limits_{i \neq j} \rho(z_i, z_j) > 0$. Although I have not found the next result in its present form in the literature, it is a well known feature of separated sequences. We sketch here a proof.

LEMMA 2.1. *Let $\mathcal{S}$ be a separated sequence and $0 < \sigma < 1$. Then there is a finite decomposition $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_N$ such that for every $1 \leqslant i \leqslant N$: $\rho(z, \omega) > \sigma$ for all $z \neq \omega$ in $\mathcal{S}_i$.*

Proof. Since $\mathcal{S}$ is separated, there is some positive integer N depending only on σ and the degree of separation of $\mathcal{S}$, such that $K(z, \sigma) \cap \mathcal{S}$ has no more than N points for every $z \in \mathbb{D}$. Let $\mathcal{S}_1 \subset \mathcal{S}$ be a maximal sequence such that $\rho(z, \omega) > \sigma$ for every $z, \omega \in \mathcal{S}_1$ with $z \neq \omega$. The maximality implies that $\mathcal{S} \subset \bigcup\limits_{z \in \mathcal{S}_1} K(z, \sigma)$. If $\mathcal{S} = \mathcal{S}_1$ we are done. Otherwise suppose that $n \geqslant 2$, $\mathcal{S}_1, \ldots, \mathcal{S}_{n-1}$ are chosen and $\mathcal{S} \setminus (\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{n-1}) \neq \emptyset$. Let $\mathcal{S}_n \subset \mathcal{S} \setminus (\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{n-1})$ be a maximal sequence such that $\rho(z, \omega) > \sigma$ for every $z, \omega \in \mathcal{S}_n$ with $z \neq \omega$. By the maximality at the previous steps, if $z \in \mathcal{S}_n$ there is some $z_i \in \mathcal{S}_i$ such that $z \in K(z_i, \sigma)$ for every $1 \leqslant i \leqslant n - 1$. Therefore $\{z, z_1, \ldots, z_{n-1}\} \subset K(z, \sigma) \cap \mathcal{S}$, and consequently $n \leqslant N$. ∎

Lemma 2.2. *For $1 \leqslant k \leqslant m$ let $\{a_j^k\}_{j\geqslant 1}$ be sequences in the unit ball of L^∞ such that $\operatorname{supp} a_j^k \subset K(\alpha_j, r)$, where $K(\alpha_j,r)\cap K(\alpha_i,r)=\emptyset$ if $i\neq j$. Suppose that $1<p<\infty$ and $\{R_j\}_{j\geqslant 1}$ is a bounded sequence in $\mathfrak{L}(L^p)$. If $f\in L^p$ is such that $\sum_{j\geqslant 1} M_{a_j^m} R_j f \in L^p$ then*

$$\Big\|\sum_{j\geqslant 1} P_{a_j^1}\cdots P_{a_j^m} R_j f\Big\|_p \leqslant C_p^m \Big\|\sum_{j\geqslant 1} M_{a_j^m} R_j f\Big\|_p,$$

where C_p is the norm of the projection P acting on L^p.

Proof. Write $Q_j = P_{a_j^2}\cdots P_{a_j^{m-1}} P$ for all $j\geqslant 1$ and $S=\sum_{j\geqslant 1} M_{a_j^1} Q_j M_{a_j^m} R_j$. Then $\|Q_j\|\leqslant C_p^{m-1}$ and for $f\in L^p$ we have

$$\begin{aligned}\|Sf\|_p^p &= \Big\|\sum_{j\geqslant 1} M_{a_j^1} Q_j M_{a_j^m} R_j f\Big\|_p^p = \sum_{j\geqslant 1}\|M_{a_j^1} Q_j M_{a_j^m} R_j f\|_p^p \\ &\leqslant C_p^{(m-1)p}\sum_{j\geqslant 1}\|M_{a_j^m} R_j f\|_p^p = C_p^{(m-1)p}\Big\|\sum_{j\geqslant 1}(M_{a_j^m} R_j) f\Big\|_p^p.\end{aligned} \tag{2.1}$$

If the last quantity is finite then $Sf\in L^p$ and the sums $S_n f=\sum_{j=1}^n M_{a_j^1} Q_j M_{a_j^m} R_j f$ converge to Sf in L^p-norm when $n\to\infty$. Therefore

$$\Big\|\sum_{j\geqslant 1} P_{a_j^1}\cdots P_{a_j^m} R_j f\Big\|_p^p = \lim_n \Big\|\sum_{j=1}^n P_{a_j^1}\cdots P_{a_j^m} R_j f\Big\|_p^p = \lim_n \|PS_n f\|_p^p \leqslant C_p\|Sf\|_p^p.$$

The lemma follows combining this inequality with (2.1). ∎

Corollary 2.3. *Taking $R_j = I$ for every j in Lemma 2.2 we obtain*

$$\Big\|\sum_{j\geqslant 1} P_{a_j^1}\cdots P_{a_j^m}\Big\|_{\mathfrak{L}(L^p)} \leqslant C_p^m.$$

Proof. By the lemma,

$$\Big\|\sum_{j\geqslant 1} P_{a_j^1}\cdots P_{a_j^m} f\Big\|_p \leqslant C_p^m\Big\|\sum_{j\geqslant 1} M_{a_j^m} f\Big\|_p \leqslant C_p^m\, \|M_{(\sum_{j\geqslant 1} a_j^m)} f\|_p \leqslant C_p^m \|f\|_p$$

for every $f\in L^p$. ∎

The next result is a particular case of Lemma 4.2.2 in [6].

Lemma 2.4. *If $t>-1$, c is real and*

$$F_{c,t}(z) = \int_{\mathbb{D}} \frac{(1-|\omega|^2)^t}{|1-z\overline{\omega}|^{2+t+c}}\, \mathrm{d}A(\omega) \quad z\in\mathbb{D},$$

then $F_{c,t}$ is bounded when $c<0$ and $|F_{c,t}(z)|\leqslant C(1-|z|^2)^{-c}$ when $c>0$.

LEMMA 2.5. *Let $0 < r < 1$ and $\{\alpha_j\}_{j\geqslant 1} \subset \mathbb{D}$ such that $K(\alpha_j, r)\cap K(\alpha_i, r) = \emptyset$ if $i \neq j$. If $r < R < 1$ and $0 < \beta < 1$ then*

$$\int_{\mathbb{D}} \sum_j [\chi_{K(\alpha_j,r)}(z)\chi_{D\setminus K(\alpha_j,R)}(\omega)] \frac{(1-|\omega|^2)^{-\beta}}{|1-z\overline{\omega}|^2}\, \mathrm{d}A(\omega) \leqslant c_\beta(R)(1-|z|^2)^{-\beta}, \tag{2.2}$$

where $c_\beta(R) \to 0$ when $R \to 1$.

Proof. If $z \in K(\alpha_j, r)$ and $\omega \in \mathbb{D}\setminus K(\alpha_j, R)$ then

$$\rho(\omega, z) \geqslant \frac{\rho(\omega,\alpha_j) - \rho(\alpha_j, z)}{1-\rho(\alpha_j, z)\rho(\omega,\alpha_j)} > \frac{R-r}{1-Rr} = \delta,$$

where $\delta = \delta(R) \to 1$ when $R \to 1$. Therefore $\mathbb{D}\setminus K(\alpha_j, R) \subset \mathbb{D}\setminus K(z, \delta)$ and

$$\sum_j \chi_{K(\alpha_j,r)}(z)\chi_{D\setminus K(\alpha_j,R)}(\omega) \leqslant \sum_j \chi_{K(\alpha_j,r)}(z)\, \chi_{D\setminus K(z,\delta)}(\omega).$$

Hence, the integral in (2.2) is bounded by

$$\begin{aligned}
&\sum_j \chi_{K(\alpha_j,r)}(z) \int_{\mathbb{D}} \chi_{D\setminus K(z,\delta)}(\omega) \frac{(1-|\omega|^2)^{-\beta}}{|1-z\overline{\omega}|^2}\, \mathrm{d}A(\omega)\\
&\quad = \sum_j \chi_{K(\alpha_j,r)}(z) \int_{|v|>\delta} \frac{(1-|\varphi_z(v)|^2)^{-\beta}}{|1-z\overline{v}|^2}\, \mathrm{d}A(v)\\
&\quad \leqslant \int_{|v|>\delta} \frac{(1-|v|^2)^{-\beta}}{|1-z\overline{v}|^{2-2\beta}}(1-|z|^2)^{-\beta}\, \mathrm{d}A(v),
\end{aligned} \tag{2.3}$$

where the equality comes from the change of variables $v = \varphi_z(\omega)$ and the inequality because $K(\alpha_j, r)$ are pairwise disjoint. Pick some $p = p(\beta) > 1$ satisfying simultaneously the conditions $p\beta < 1$ and $p(2-\beta) < 2$. If $p^{-1} + q^{-1} = 1$, Holder's inequality gives

$$\int_{|v|>\delta} \frac{(1-|v|^2)^{-\beta}}{|1-z\overline{v}|^{2-2\beta}}\, \mathrm{d}A(v) \leqslant \Big(\int_{\mathbb{D}} \frac{(1-|v|^2)^{-p\beta}}{|1-z\overline{v}|^{2p(1-\beta)}}\, \mathrm{d}A(v)\Big)^{1/p} (1-\delta^2)^{1/q}.$$

Since $2p(1-\beta) = 2 - p\beta + [p(2-\beta) - 2] < 2 - p\beta$, then Lemma 2.4 says that the last expression is bounded by $C_\beta(1-\delta^2)^{1/q}$, where C_β depends only on β. Going back to (2.3) we see that the integral in (2.2) is bounded by

$$C_\beta(1-\delta(R)^2)^{1/q(\beta)}(1-|z|^2)^{-\beta},$$

proving the lemma. ∎

LEMMA 2.6. *Let $0 < r < 1$ and $\alpha_j \in \mathbb{D}$, $j \geqslant 1$, such that $K(\alpha_j, r)$ are pairwise disjoint. Suppose that $R \in (r, 1)$ and $a_j, A_j \in L^\infty$ are functions of norm $\leqslant 1$ such that*

$$\operatorname{supp} a_j \subset K(\alpha_j, r) \quad \textit{and} \quad \operatorname{supp} A_j \subset \mathbb{D}\setminus K(\alpha_j, R).$$

Then $\sum_{j\geqslant 1} M_{a_j} P M_{A_j}$ *is bounded on* L^p *for every* $1 < p < \infty$, *with norm bounded by some constant* $k_p(R) \to 0$ *when* $R \to 1$.

Proof. Write

$$\Phi(z,\omega) = \sum_{j\geqslant 1} \chi_{K(\alpha_j,r)}(z)\chi_{D\setminus K(\alpha_j,R)}(\omega)\,\frac{1}{|1-\overline{\omega}z|^2}.$$

Let $f \in L^p$. Since $\|a_j\|_\infty, \|A_j\|_\infty \leqslant 1$ for all j, then

$$\Big|\Big(\sum_{j\geqslant 1} M_{a_j} P M_{A_j} f\Big)(z)\Big| = \Big|\sum_{j\geqslant 1} a_j(z)\int\limits_{\mathbb{D}} A_j(\omega)f(\omega)\frac{\mathrm{d}A(\omega)}{(1-\overline{\omega}z)^2}\Big|$$
$$\leqslant \int\limits_{\mathbb{D}} \Phi(z,\omega)|f(\omega)|\,\mathrm{d}A(\omega).$$

Taking $h(z) = (1-|z|^2)^{-1/pq}$, where $p^{-1}+q^{-1} = 1$, Lemma 2.5 asserts that

$$\int\limits_{\mathbb{D}} \Phi(z,\omega)h(\omega)^q\,\mathrm{d}A(\omega) \leqslant c_{p^{-1}}(R)h(z)^q$$

and Lemma 2.4 implies that there is some $C > 0$ such that

$$\int\limits_{\mathbb{D}} \Phi(z,\omega)h(z)^p\,\mathrm{d}A(z) \leqslant Ch(\omega)^p.$$

By Schur's theorem ([6], p. 42) the integral operator with kernel $\Phi(z,\omega)$ is bounded on L^p and its norm is bounded by $(c_{p^{-1}}(R))^{1/q}C^{1/p} \to 0$ as $R \to 1$. ∎

Let $a_j^i, b_j \in L^\infty$, $j \geqslant 1$ and $1 \leqslant i \leqslant m$, be functions of norm at most 1 supported on $K(\alpha_j, r)$, where the pseudo-hyperbolic disks are pairwise disjoint. By Lemma 2.1 for any $\sigma \in (r,1)$ there is some $n = n(\sigma) \geqslant 1$ and a partition of the positive integers $\mathbb{N} = N_1 \cup \cdots \cup N_n$ such that

$$\rho(\alpha_i,\alpha_j) > \sigma \quad \text{for } i \neq j,\ i,j \in N_k,\ 1 \leqslant k \leqslant n.$$

LEMMA 2.7. *If* $1 < p < \infty$ *then*

$$\sum_{1\leqslant k\leqslant n}\Big[\Big(\sum_{j\in N_k} P_{a_j^1}\cdots P_{a_j^m}\Big)P_{\big(\sum\limits_{i\in N_k} b_i\big)}\Big] \to \sum_{j\geqslant 1} P_{a_j^1}\cdots P_{a_j^m}P_{b_j} \tag{2.4}$$

in operator norm when $\sigma \to 1$.

Proof. Write $B_j = \sum\limits_{\substack{i\in N_k\\ i\neq j}} b_i$ when $j \in N_k$ for some $1 \leqslant k \leqslant n$. Since $P_{\big(\sum\limits_{i\in N_k} b_i\big)} = P_{b_j} + P_{B_j}$, the first term in (2.4) can be decomposed as

$$\sum_{k=1}^{n}\Big[\sum_{j\in N_k} P_{a_j^1}\cdots P_{a_j^m}P_{b_j} + \sum_{j\in N_k} P_{a_j^1}\cdots P_{a_j^m}P_{B_j}\Big] = S_1 + S_2,$$

where

$$S_1 = \sum_{k=1}^{n} \sum_{j\in N_k} P_{a_j^1} \cdots P_{a_j^m} P_{b_j} = \sum_{j\geqslant 1} P_{a_j^1} \cdots P_{a_j^m} P_{b_j}$$

and

$$S_2 = \sum_{k=1}^{n} \sum_{j\in N_k} P_{a_j^1} \cdots P_{a_j^m} P_{B_j} = \sum_{j\geqslant 1} P_{a_j^1} \cdots P_{a_j^m} P_{B_j}.$$

Let $f \in L^p$. By Lemmas 2.2 and 2.6

$$\|S_2 f\|_p \leqslant C_p^m \Big\| \sum_{j\geq 1} M_{a_j^m} P_{B_j} f \Big\|_p. \tag{2.5}$$

If $\omega \in \operatorname{supp} B_j$ for $j \in N_k$ with $1 \leqslant k \leqslant n$, then there is $i \neq j$ in N_k such that $\omega \in K(\alpha_i, r)$. Then

$$\rho(\omega, \alpha_j) \geqslant \frac{\rho(\alpha_j, \alpha_i) - \rho(\omega, \alpha_i)}{1 - \rho(\alpha_j, \alpha_i)\rho(\omega, \alpha_i)} > \frac{\sigma - r}{1 - \sigma r} = R(\sigma),$$

meaning that $\operatorname{supp} B_j \subset \mathbb{D} \setminus K(\alpha_j, R(\sigma))$. Since $R(\sigma) \to 1$ when $\sigma \to 1$, (2.5) and Lemma 2.6 prove (2.4). ∎

COROLLARY 2.8. *Under the conditions of Lemma* 2.7,

$$\sum_{1\leqslant k\leqslant n} \Big[\Big(\sum_{j\in N_k} T_{a_j^1} \cdots T_{a_j^m} \Big) T_{\left(\sum_{i\in N_k} b_i\right)}\Big] \to \sum_{j\geqslant 1} T_{a_j^1} \cdots T_{a_j^m} T_{b_j} \tag{2.6}$$

and

$$\sum_{1\leqslant k\leqslant n} \Big[T_{\left(\sum_{i\in N_k} b_i\right)} \Big(\sum_{j\in N_k} T_{a_j^1} \cdots T_{a_j^m} \Big)\Big] \to \sum_{j\geqslant 1} T_{b_j} T_{a_j^1} \cdots T_{a_j^m} \tag{2.7}$$

in operator norm when $\sigma \to 1$.

Proof. We obtain (2.6) by restricting the operators of (2.4) to L_a^p. To prove (2.7) use (2.6) with

$$\sum_{1\leqslant k\leqslant n} \Big[\Big(\sum_{j\in N_k} T_{\overline{a}_j^m} \cdots T_{\overline{a}_j^1} \Big) T_{\left(\sum_{i\in N_k} \overline{b}_i\right)}\Big]$$

acting on L_a^q and then take adjoints. ∎

PROPOSITION 2.9. *Let* $1 < p < \infty$ *and* $c_j^1, \ldots, c_j^l, a_j, b_j, d_j^1, \ldots, d_j^m \in L^\infty$ *be functions of norm* $\leqslant 1$ *supported on* $K(\alpha_j, r)$ *for* $j \geqslant 1$, *where* $K(\alpha_j, r) \cap K(\alpha_i, r) = \emptyset$ *if* $i \neq j$. *Then*

$$\sum_{j\geqslant 1} T_{c_j^1} \cdots T_{c_j^l} (T_{a_j} T_{b_j} - T_{b_j} T_{a_j}) T_{d_j^1} \cdots T_{d_j^m} \in \mathfrak{CT}(L_a^p).$$

Proof. For $r < \sigma < 1$ decompose $\mathbb{N} = N_1 \cup \cdots \cup N_n$ as in the paragraph that precedes Lemma 2.7. By Corollary 2.8,

$$\sum_{1\leqslant k\leqslant n} \big[T_{\left(\sum_{j\in N_k} a_j\right)} T_{\left(\sum_{i\in N_k} b_i\right)} - T_{\left(\sum_{i\in N_k} b_i\right)} T_{\left(\sum_{j\in N_k} a_j\right)} \big] \to \sum_{j\geqslant 1} (T_{a_j} T_{b_j} - T_{b_j} T_{a_j})$$

in operator norm when $\sigma \to 1$. Since the first operators belong to the commutator ideal, so does their limit. Thus,

$$\sum_{j\in F}(T_{a_j}T_{b_j} - T_{b_j}T_{a_j}) \in \mathfrak{CT}(L^p_a)$$

for any subset $F \subset \mathbb{N}$. In particular, this hold for $F = N_k$, $1 \leqslant k \leqslant n$. Then

$$\sum_{1\leqslant k\leqslant n}\Big[\Big(\sum_{j\in N_k}(T_{a_j}T_{b_j} - T_{b_j}T_{a_j})\Big)T_{\left(\sum\limits_{i\in N_k} d^1_i\right)}\Big] \in \mathfrak{CT}(L^p_a),$$

and since (2.6) says that the above operators converge to

$$\sum_{j\geqslant 1}(T_{a_j}T_{b_j} - T_{b_j}T_{a_j})T_{d^1_j}$$

when $\sigma \to 1$, this operator is also in $\mathfrak{CT}(L^p_a)$. Clearly, the same holds if the sum is over any set $F \subset \mathbb{N}$. We can repeat this process $m-1$ more times using (2.6) and then l times using (2.7) to obtain the desired result. ∎

3. AN INVERTIBLE OPERATOR IN $\mathfrak{CT}(L^2_a)$

Let $a \in L^\infty$ be a real-valued function such that $a(\omega) \geqslant \delta > 0$ for every $\omega \in \mathbb{D}$. Then T_a is self-adjoint and

$$\langle T_a f, f\rangle = \int\limits_{\mathbb{D}} a|f|^2\,\mathrm{d}A \geqslant \delta\int\limits_{\mathbb{D}}|f|^2\,\mathrm{d}A = \delta\|f\|_2^2$$

for every $f \in L^2_a$. Therefore T_a is invertible. Theorem 1.1 will be proved by constructing a function a as above such that $T_a \in \mathfrak{CT}(L^2_a)$.

We need to summarize several basic features of Toeplitz operators. If $a, b \in L^\infty$ then $T_aT_b = T_{ab}$ when $\overline{a} \in H^\infty$ or $b \in H^\infty$. If $z \in \mathbb{D}$ then $U_z f = (f\circ\varphi_z)\varphi'_z$ defines a unitary self-adjoint operator on L^2_a. Therefore, if $a \in L^\infty$ and $f, g \in L^2_a$,

$$\langle U_zT_aU_zf, g\rangle = \langle T_aU_zf, U_zg\rangle = \langle a(f\circ\varphi_z)\varphi'_z, (g\circ\varphi_z)\varphi'_z\rangle = \langle (a\circ\varphi_z)f, g\rangle,$$

where the last equality comes from changing variables inside the integral. Thus

$$U_zT_{a_1}\cdots T_{a_n}U_z = U_zT_{a_1}U_z\cdots U_zT_{a_n}U_z = T_{a_1\circ\varphi_z}\cdots T_{a_n\circ\varphi_z} \tag{3.1}$$

for $a_j \in L^\infty$, $1 \leqslant j \leqslant n$. By diagonal operator we always mean diagonal with respect to the orthonormal basis $\{\sqrt{n+1}z^n\}_{n\geqslant 0}$.

A straightforward calculation with polar coordinates shows that if $a \in L^\infty$ is a radial function (i.e. $a(z) = a(|z|)$), then T_a is diagonal with n-entry

$$\lambda_n(a) = \int\limits_0^1 a(t^{1/2})(n+1)t^n\,\mathrm{d}t. \tag{3.2}$$

If χ_r denotes the characteristic function of the ball $\{|\omega| \leqslant r\}$, where $0 < r < 1$, then (3.2) yields $T_{\chi_r}\omega^n = r^{2(n+1)}\omega^n$.

LEMMA 3.1. *Let $a \in L^\infty$ be a radial function and $0 < r < 1$. Then*

$$T_{\chi_r} T_a = T_{\chi_r(\omega) a(\omega/r)}.$$

Proof. The operator $T_{\chi_r(\omega)a(\omega/r)}$ is diagonal, and its n-entry is

$$\int_0^1 \chi_{[0,r]}(t^{1/2}) a\Big(\frac{t^{1/2}}{r}\Big)(n+1)t^n \,\mathrm{d}t = \int_0^{r^2} a\Big(\frac{t^{1/2}}{r}\Big)(n+1)t^n \,\mathrm{d}t$$
$$= r^{2n+2} \int_0^1 a(u^{1/2})(n+1)u^n \,\mathrm{d}u,$$

where the last equality comes from the change of variables $u = t/r^2$. By (3.2) $T_{\chi_r} T_a$ is also diagonal and has the same entries. ∎

A simple calculation shows that if $n \geqslant 1$ then $\langle T_{\overline{\omega}} \omega^n, \omega^k \rangle = \langle \omega^n, \omega^{k+1} \rangle = \langle (n/n+1)\omega^{n-1}, \omega^k \rangle$. A recursive argument then gives that for every nonnegative integer k,

$$T_{\overline{\omega}^k} \omega^n = \Big(\frac{n+1-k}{n+1}\Big)\omega^{n-k} \quad \text{if } n \geqslant k$$

and $T_{\overline{\omega}^k}\omega^n = 0$ if $n < k$. Thus

$$T_{\overline{\omega}^k} T_{\chi_r} \omega^n = r^{2(n+1)} \Big(\frac{n+1-k}{n+1}\Big)\omega^{n-k} \quad \text{if } n \geqslant k,$$

and since $T_{\chi_r} T_{\omega^k} \omega^n = r^{2(n+k+1)}\omega^{n+k}$ then

$$(T_{\overline{\omega}^k} T_{\chi_r})(T_{\chi_r} T_{\omega^k})\omega^n = r^{4(n+k+1)} \Big(\frac{n+1}{n+k+1}\Big)\omega^n = r^{4k} T_{\chi_{r^2}} T_{\overline{\omega}^k} T_{\omega^k} \omega^n, \tag{3.3}$$

where the second equality comes from the limit case $r = 1$ in the first equality and from $T_{\chi_{r^2}} \omega^n = r^{4(n+1)}\omega^n$. Since T_{χ_r} and $T_{\omega^k} T_{\overline{\omega}^k}$ are diagonal, they commute, and since $T_{\chi_r}^2 = T_{\chi_{r^2}}$ then

$$T_{\chi_r} T_{\omega^k} T_{\overline{\omega}^k} T_{\chi_r} = T_{\chi_r}^2 T_{\omega^k} T_{\overline{\omega}^k} = T_{\chi_{r^2}} T_{\omega^k} T_{\overline{\omega}^k}. \tag{3.4}$$

By (3.3), (3.4) and Lemma 3.1,

$$S_k \overset{\text{def}}{=} [T_{\omega^k \chi_r}, T_{\overline{\omega}^k \chi_r}] = T_{\chi_{r^2}}(T_{\omega^k} T_{\overline{\omega}^k} - r^{4k} T_{\overline{\omega}^k} T_{\omega^k}) = T_{\chi_{r^2}} T_{\omega^k} T_{\overline{\omega}^k} - T_{\chi_{r^2}|\omega|^{2k}}. \tag{3.5}$$

Let $P_0 \in \mathfrak{L}(L_a^2)$ be the operator $P_0 f = f(0)$. Straightforward evaluations on the basis $\{z^n\}_{n \geqslant 0}$ give the following identities

$$T_\omega T_{\overline{\omega}} = T_{1+\log|\omega|^2}, \quad T_{\omega^2} T_{\overline{\omega}^2} = T_{1+2\log|\omega|^2} + P_0 \quad \text{and} \quad T_{\chi_{r^2}} P_0 = r^4 P_0. \tag{3.6}$$

Then

$$\begin{aligned} 2S_1 - S_2 &\overset{\text{by (3.5)}}{=} T_{\chi_{r^2}}(2T_\omega T_{\overline{\omega}} - T_{\omega^2} T_{\overline{\omega}^2}) + T_{\chi_{r^2}(|\omega|^4 - 2|\omega|^2)} \\ &\overset{\text{by (3.6)}}{=} T_{\chi_{r^2}(1+|\omega|^4-2|\omega|^2)} - r^4 P_0 = T_{\chi_{r^2}(1-|\omega|^2)^2} - r^4 P_0. \end{aligned} \tag{3.7}$$

Since $2S_1 - S_2$, T_{χ_r} and P_0 are diagonal operators, they commute. Consequently

$$P_0 T_{\chi_r} T_\omega = T_{\chi_r} P_0 T_\omega = 0,$$

which together with Lemma 3.1 and (3.7) gives

$$T_{\chi_r \overline{\omega}}(2S_1 - S_2)T_{\chi_r \omega} = T_{\overline{\omega}} T_{\chi_r}(2S_1 - S_2)T_{\chi_r} T_\omega = T_{\chi_{r^4}(1-|\omega|^2/r^4)^2|\omega|^2}. \tag{3.8}$$

If $\alpha \in \mathbb{D}$ then (3.1), (3.5) and (3.8) yield

$$\begin{aligned} &T_{(\chi_r\circ\varphi_\alpha)\overline{\varphi}_\alpha}(2[T_{(\chi_r\circ\varphi_\alpha)\varphi_\alpha}, T_{(\chi_r\circ\varphi_\alpha)\overline{\varphi}_\alpha}] - [T_{(\chi_r\circ\varphi_\alpha)\varphi_\alpha^2}, T_{(\chi_r\circ\varphi_\alpha)\overline{\varphi}_\alpha^2}])T_{(\chi_r\circ\varphi_\alpha)\varphi_\alpha} \\ &\quad = U_\alpha T_{\chi_r\overline{\omega}}(2S_1 - S_2)T_{\chi_r\omega}U_\alpha = T_{(\chi_{r^4}\circ\varphi_\alpha)(1-|\varphi_\alpha|^2/r^4)^2|\varphi_\alpha|^2}. \end{aligned} \tag{3.9}$$

Suppose that $0 < r < 1$ and $\{\alpha_j\} \subset \mathbb{D}$ is a sequence such that $K(\alpha_i, r) \cap K(\alpha_j, r) = \emptyset$ for $i \neq j$. Since $(\chi_{r^4} \circ \varphi_\alpha)(\omega) = \chi_{K(\alpha, r^4)}(\omega)$, the characteristic function of $K(\alpha, r^4)$, then

$$A(\omega) \stackrel{\text{def}}{=} \sum_{j\geqslant 1} \chi_{r^4}(\varphi_{\alpha_j}(\omega))\Big(1 - \frac{|\varphi_{\alpha_j}(\omega)|^2}{r^4}\Big)^2 |\varphi_{\alpha_j}(\omega)|^2$$

is in L^∞ with $\|A\|_\infty \leqslant 1$. In conjunction with (3.9), Proposition 2.9 tells us that

$$T_A = \sum_{j\geqslant 1} T_{(\chi_{r^4}\circ\varphi_{\alpha_j})(1-|\varphi_{\alpha_j}|^2/r^4)^2|\varphi_{\alpha_j}|^2} \in \mathfrak{CT}(L_a^2). \tag{3.10}$$

When $\omega \in \mathbb{D}$ satisfies $r^4/4 < \rho(\omega, \alpha_j) \leqslant (3/4)r^4$ for some α_j we have

$$\Big(1 - \frac{|\varphi_{\alpha_j}(\omega)|^2}{r^4}\Big)^2 |\varphi_{\alpha_j}(\omega)|^2 \geqslant \Big(1 - \frac{3^2 r^8}{4^2 r^4}\Big)^2 \frac{r^8}{4^2} \geqslant \frac{r^8}{2^8},$$

meaning that

$$A(\omega) \geqslant \Big(\frac{r}{2}\Big)^8 \quad \text{when } \omega \in K\Big(\alpha_j, \Big(\frac{3}{4}\Big)r^4\Big) \setminus K\Big(\alpha_j, \frac{r^4}{4}\Big) \text{ for some } \alpha_j. \tag{3.11}$$

LEMMA 3.2. *Given $0 < \sigma < 1$ there is a separated sequence $\{\alpha_j\}$ in $\mathbb{D}$ such that every $z \in \mathbb{D}$ is in $K(\alpha_j, 3\sigma/4) \setminus K(\alpha_j, \sigma/4)$ for some α_j.*

Proof. Take a sequence $\{\alpha_j\} \subset \mathbb{D}$ such that $\rho(\alpha_i, \alpha_j) > \sigma/100$ if $i \neq j$ and

$$\rho(\{\alpha_j\}_{j\geqslant 1}, \omega) \leqslant \frac{\sigma}{8} \quad \text{for every } \omega \in \mathbb{D}. \tag{3.12}$$

For an arbitrary $z \in \mathbb{D}$ write $\beta_j = \varphi_z(\alpha_j)$. The conformal invariance of ρ implies that $\{\beta_j\}_{j\geqslant 1}$ satisfies (3.12). We claim that there is some β_j such that $\sigma/4 < |\beta_j| \leqslant (3/4)\sigma$. Otherwise

$$\rho\Big(\frac{\sigma}{2}, \{\beta_j\}_{j\geqslant 1}\Big) \geqslant \rho\Big(\frac{\sigma}{2}, \mathbb{D}\setminus\Big\{\frac{\sigma}{4} < |\omega| \leqslant \frac{3}{4}\sigma\Big\}\Big) = \rho\Big(\frac{\sigma}{2}, \Big\{\frac{\sigma}{4}, \frac{3\sigma}{4}\Big\}\Big) \geqslant \frac{\frac{\sigma}{4}}{1 - \frac{\sigma}{4}\cdot\frac{\sigma}{2}} > \frac{\sigma}{4}.$$

This contradicts (3.12) with respect to $\{\beta_j\}_{j\geqslant 1}$ for $\omega = \sigma/2$. If $\sigma/4 < |\beta_{j_0}| \leqslant (3/4)\sigma$ then

$$\rho(\alpha_{j_0}, z) = \rho(\varphi_z(\alpha_{j_0}), \varphi_z(z)) = \rho(\beta_{j_0}, 0) = |\beta_{j_0}| \in \Big(\frac{\sigma}{4}, \frac{3\sigma}{4}\Big],$$

and since $z \in \mathbb{D}$ is arbitrary, the lemma follows. ∎

Returning to our construction, fix $0 < r < 1$ and suppose that $\mathcal{S} = \{\alpha_j\}_{j \geqslant 1}$ is a sequence satisfying Lemma 3.2 for $\sigma = r^4$. Since $\mathcal{S}$ is separated, by Lemma 2.1 we can decompose $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_N$, where for each $1 \leqslant k \leqslant N$, $K(\alpha_i, r) \cap K(\alpha_j, r) = \emptyset$ if $\alpha_i, \alpha_j \in \mathcal{S}_k$ with $i \neq j$. For $1 \leqslant k \leqslant N$ write

$$A_k(\omega) = \sum_{\alpha_j \in \mathcal{S}_k} \chi_{r^4}(\varphi_{\alpha_j}(\omega)) \Big(1 - \frac{|\varphi_{\alpha_j}(\omega)|^2}{r^4}\Big)^2 |\varphi_{\alpha_j}(\omega)|^2.$$

Then $\|A_k\|_\infty \leqslant 1$ and (3.10) says that $T_{A_k} \in \mathfrak{CT}(L^2_a)$. Consequently

$$\sum_{k=1}^{N} T_{A_k} = T_{\left(\sum\limits_{k=1}^{N} A_k\right)} \in \mathfrak{CT}(L^2_a).$$

In addition, (3.11) says that for every $1 \leqslant k \leqslant N$,

$$A_k(\omega) \geqslant \Big(\frac{r}{2}\Big)^8 \quad \text{when } \omega \in K\Big(\alpha_j, \Big(\frac{3}{4}\Big) r^4\Big) \setminus K\Big(\alpha_j, \frac{r^4}{4}\Big) \text{ for some } \alpha_j \in \mathcal{S}_k,$$

and since Lemma 3.2 asserts that

$$\mathbb{D} = \bigcup_{1 \leqslant k \leqslant N} \bigcup_{\alpha_j \in \mathcal{S}_k} K\Big(\alpha_j, \Big(\frac{3}{4}\Big) r^4\Big) \setminus K\Big(\alpha_j, \frac{r^4}{4}\Big)$$

then $\sum\limits_{k=1}^{N} A_k(\omega) \geqslant (r/2)^8$ for every $\omega \in \mathbb{D}$. This completes the construction and proves Theorem 1.1.

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