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REPRODUCING KERNELS AND INVARIANT SUBSPACES OF THE BERGMAN SHIFT

GEORGE CHAILOS

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Abstract. In this article we consider index 1 invariant subspaces M of the operator of multiplication by $\zeta(z) = z$, M_ζ, on the Bergman space $L^2_{\mathrm{a}}(\mathbb{D})$ of the unit disc. It turns out that there is a positive sesquianalytic kernel l_λ defined on $\mathbb{D}\times\mathbb{D}$ which determines M uniquely. We set $\sigma(M^*_\zeta|M^\perp)$ to be the spectrum of M^*_ζ restricted to $M^\perp$, and we consider a conjecture due to Hedenmalm which states that if $M \neq L^2_{\mathrm{a}}(\mathbb{D})$, then $\operatorname{rank} l_\lambda$ equals the cardinality of $\sigma(M^*_\zeta|M^\perp)$. In this direction we show that *cardinality* $\sigma(M^*_\zeta|M^\perp)\cap\mathbb{D} \leqslant \operatorname{rank} l_\lambda \leqslant$ *cardinality* $\sigma(M^*_\zeta|M^\perp)$ and furthermore, we resolve the conjecture in the case of zero based invariant subspaces. Moreover, we describe the structure of l_λ for finite zero based invariant subspaces.

Keywords: *Invariant subspaces, Bergman spaces, Bergman shift, reproducing kernels, Bergman type kernels.*

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1. INTRODUCTION

Let $\Omega \subset \mathbb{C}$ be a region and let k be a positive sesquianalytic kernel on Ω; that is for each $\lambda \in \Omega$ the function k_λ is an analytic function on Ω such that $\sum\limits_{i,j=1}^{n} a_i\overline{a}_j k_{\lambda_i}(\lambda_j) \geqslant 0$ for all $n \in \mathbb{N}$, $a_i \in \mathbb{C}$, $\lambda_j \in \Omega$, $i, j \in \{1, \ldots, n\}$.

It is well known that every positive sesquianalytic kernel k on Ω is the reproducing kernel for a unique Hilbert space $\mathcal{H}(k)$ of analytic functions on Ω (see [3]). In particular, if $\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}(k)}$ denotes the Hilbert space inner product, $f(\lambda) = \langle f, k_\lambda\rangle_{\mathcal{H}(k)}$ for every $f \in \mathcal{H}(k)$, $\lambda \in \Omega$.

Denote by $\mathcal{M}(k)$ the set of multipliers of $\mathcal{H}(k)$. If $\zeta \in \mathcal{M}(k)$, denote by M_ζ the multiplication operator associated with the identity function $\zeta(z) = z$, $z \in \mathbb{D}$. Let also $\operatorname{Lat}(M_\zeta, \mathcal{H}(k))$ be the lattice of the invariant subspaces of $(M_\zeta, \mathcal{H}(k))$.

Given a subspace M of $\mathcal{H}(k)$, denote by P_M the (orthogonal) projection onto M. Furthermore, M is called a *multiplier invariant subspace* of $\mathcal{H}(k)$ if $gf \in M$ for all $f \in M$ and all $g \in \mathcal{M}(k)$. If $0 \in \Omega$ and if M_ζ is bounded below, set $M \ominus \zeta M \equiv M \cap (\zeta M)^\perp$ and define the *index* of M to be the dimension of $M \ominus \zeta M$. That is, $\operatorname{ind} M = \dim M \ominus \zeta M$.

From now on, by an invariant subspace, unless it is stated otherwise, we will always mean an invariant subspace of $(M_\zeta, \mathcal{H}(k))$. For a subset S of $\mathcal{H}(k)$ write $[S]$ for the smallest invariant subspace which contains all of S. For a single nonzero function $f \in \mathcal{H}(k)$ simply write $[f]$ for $[\{f\}]$. Such invariant subspaces are called *cyclic* and a function $f \in \mathcal{H}(k)$ such that $[f] = \mathcal{H}(k)$ is called a *cyclic vector* in $\mathcal{H}(k)$.

Suppose that $A = \{\alpha_k\}_{k\in\mathbb{N}} \in \Omega$ is a $\mathcal{H}(k)$*-zero sequence*; that is the sequence of zeros, repeated according to multiplicity, of some nonidentically vanishing function in $\mathcal{H}(k)$. Write

$$\mathcal{H}_A(k) = \{f \in \mathcal{H}(k) : f(\alpha) = 0 \text{ for each } \alpha \in A, \text{ accounting for multiplicities}\}$$

for the set of functions in $\mathcal{H}(k)$ that vanish in the sequence A to at least the prescribed multiplicity. Note that if A is a $\mathcal{H}(k)$-zero sequence, $\mathcal{H}_A(k)$ is a nontrivial invariant subspace of $\mathcal{H}(k)$. Such spaces are called *zero-based* invariant subspaces. If $0 \in \Omega$ and if $M_{\zeta-\alpha}$ is bounded below for all $\alpha \in \Omega$, then it is well known that $\mathcal{H}_A(k)$ has index one (see [14], Corollary 3.4).

For $\alpha \in \Omega$ the space $\mathcal{H}_\alpha(k) = \{f \in \mathcal{H}(k) : f(\alpha) = 0\}$ is a multiplier invariant subspace of $\mathcal{H}(k)$. Furthermore, if P_α denotes the projection onto $\mathcal{H}_\alpha(k)$ and $k_\alpha(\alpha) \neq 0$, then $P_\alpha f = f - \frac{f(\alpha)}{k_\alpha(\alpha)} k_\alpha$.

Here we would like to mention that in some occasions we also use the symbol u for positive sesquianalytic kernels. If $u_\lambda(z)$ is a positive sesquianalytic kernel on Ω then there are analytic functions $u_n, n \geqslant 1$ on Ω such that $u_\lambda(z) = \sum_{n\geqslant 1} \overline{u_n(\lambda)} u_n(z)$, where the sum converges uniformly on compact subsets of $\Omega \times \Omega$. For example one can take $\{u_n\}_{n\geqslant 1}$ to be an orthonormal basis for $\mathcal{H}(u)$.

Definition 1.1. If $u_\lambda(z)$ is a positive sesquianalytic kernel on Ω, $u_\lambda(z) \neq 0$, the *rank* of $u_\lambda(z)$ is defined to be the least number of (nonidentically vanishing) functions u_n in Ω such that $u_\lambda(z) = \sum_{n\geqslant 1} \overline{u_n(\lambda)} u_n(z)$, where the sum converges uniformly on compact subsets of $\Omega \times \Omega$. If $u_\lambda(z)$ is identically zero, set its rank to be zero.

In this article we are mainly interested in the case of the classical Bergman space on the unit disc $\mathbb{D}$; that is the space $L^2_{\mathrm{a}}(\mathbb{D})$ of all analytic functions on $\mathbb{D}$ that are square integrable with respect to the Lebesgue area measure on $\mathbb{D}$. We suppose that $M \in \operatorname{Lat}(M_\zeta, L^2_{\mathrm{a}}(\mathbb{D}))$, $\operatorname{ind} M = 1$, and G is a unit vector in $M \ominus \zeta M$. Hence, see [1], $M = [G]$. From this it follows easily that M/G is the closure of the analytic polynomials in $L^2_{\mathrm{a}}(|G|^2 \mathrm{d}\mathcal{A})$, where $\mathcal{A}$ is the normalized Lebesgue measure on $\mathbb{D}$. Moreover, it is not hard to see that the point evaluations are bounded on M/G and hence M/G has a reproducing kernel which we denote by k^G_λ. If $k_\lambda(z)$ denotes the reproducing kernel for the Bergman space, then it is elementary to show that $k^G_\lambda(z) = \frac{P_M k_\lambda(z)}{\overline{G(\lambda)} G(z)}$. The following theorem, which was proved by

Hedenmalm, Jakobsson and Shimorin ([11], Theorem 6.3) and the remark following it, are not only essential for the development of this study, but also constitute the main motivation for this article.

Theorem 1.2. *Suppose that $M \in \operatorname{Lat}(M_\zeta, L^2_{\mathrm{a}}(\mathbb{D}))$, $\operatorname{ind} M = 1$. If G is a unit vector in $M \ominus \zeta M$, then there is a positive definite sesquianalytic kernel l^M_λ defined on $\mathbb{D} \times \mathbb{D}$, such that*

$$\frac{P_M k_\lambda(z)}{\overline{G(\lambda)} G(z)} = (1 - \overline{\lambda} z l^M_\lambda(z)) k_\lambda(z),$$

where $k_\lambda(z) = (1 - \overline{\lambda} z)^{-2}$, $\lambda, z \in \mathbb{D}$, is the Bergman kernel.

The following remark shows that l^M_λ determines the invariant subspace M uniquely.

Remark 1.3. If M_1, M_2 are index 1 invariant subspaces of $L^2_{\mathrm{a}}(\mathbb{D})$ with $l^{M_1}_\lambda = l^{M_2}_\lambda$, then $M_1 = M_2$.

Indeed, if G_{M_i} are unit vectors in $M_i \ominus \zeta M_i$, $i = 1, 2$, then $\frac{M_1}{G_{M_1}} = \frac{M_2}{G_{M_2}}$, with equality of norms, since the kernel defines the space uniquely. Moreover, $\frac{M_i}{G_{M_i}}$ is the closure of the analytic polynomials in $L^2_{\mathrm{a}}(|G_{M_i}|^2 \mathrm{d}\mathcal{A})$, $i = 1, 2$. The result now follows from Theorem 1 of [15].

From now on l^M_λ will denote the kernel function which appears in the expression for the reproducing kernel of M/G. We will simply call l^M_λ the *associated kernel* for M. Whenever there is no ambiguity, the superscript M in l^M_λ will be excluded from the notation.

Now we shall try to give the reader some more intuition and motivation for our article.

Since the kernel l^M_λ defines the subspace M uniquely, it seems natural to ask about the structural properties of l^M_λ and relate them to common properties of the functions in M. This type of study of l^M_λ is recent and we would like to note that prior to our article very few results were known about the form of l^M_λ.

The exact expression of l^M_λ is known whenever M is a single zero based invariant subspace. For example one can derive this easily from Lemma 6.6 of [2]. In particular, if $\alpha \in \mathbb{D}$, $\alpha \neq 0$, $m \in \mathbb{N}$ and if

$$M = \{f \in L^2_{\mathrm{a}}(\mathbb{D}) : f^{(j)}(\alpha) = 0,\ 0 \leqslant j \leqslant m-1\},$$

then

$$l^M_\lambda(z) = \frac{c}{(z - A)(\overline{\lambda} - \overline{A})},$$

where $c = \frac{m(m+1)(1-|\alpha|^2)^2}{|\alpha|^2}$ and $A = \frac{1+m(1-|\alpha|^2)}{\overline{\alpha}}$. In this case the above form of l^M_λ implies that $\operatorname{rank} l^M_\lambda = 1$. We also note that if M is zero based and its zero sequence contains more than one (distinct) point, then a similar approach for the calculation of the form of l^M_λ does not appear to be practical. In such case the manipulations are becoming extremely complicated since they involve factorizations of higher order polynomials. For more information about the form of l^M_λ and formal calculations in the case where M is a finite zero based invariant subspace, we refer to Section 4 (Theorem 4.2).

H. Hedenmalm, see [10], stated the following conjecture regarding the rank l^M_λ.

HEDENMALM'S CONJECTURE. *Suppose that* $M \neq L^2_{\mathrm{a}}(\mathbb{D})$, $\operatorname{ind} M = 1$ *and* $M \in \operatorname{Lat}(M_\zeta, L^2_{\mathrm{a}}(\mathbb{D}))$. *Then*

$$\operatorname{rank} l^M_\lambda = \operatorname{card} \sigma(M^*_\zeta | M^\perp),$$

where $\operatorname{card} \sigma(M^*_\zeta | M^\perp)$ *is the cardinality, in the sense that is defined to be finite or infinite, of the spectrum of* M^*_ζ *restricted to* $M^\perp$.

Given a function $f \in L^2_{\mathrm{a}}(\mathbb{D})$, its lower zero set (or liminf zero set) written $\underline{Z}(f)$ consists of all actual zeros of f inside $\mathbb{D}$, and all points λ on the unit circle $\mathbb{T}$ for which $\varliminf\limits_{\substack{z\to\lambda \\ z\in\mathbb{D}}} |f(z)| = 0$. We extend this notion to collections of functions S in $L^2_{\mathrm{a}}(\mathbb{D})$ by declaring $\underline{Z}(S) \equiv \bigcap\{\underline{Z}(f) : f \in S\}$ and we set $\overline{\underline{Z}(S)} \equiv \{\lambda \in \mathbb{C} : \overline{\lambda} \in \underline{Z}(S)\}$. It is shown in [9] that $\overline{\underline{Z}(M)} = \sigma(M^*_\zeta | M^\perp)$. This result will be used many times throughout this article.

A major part of this article is devoted to the investigation of Hedenmalm's Conjecture.

Our first result (see Theorem 3.4) resolves *Hedenmalm's Conjecture* in the case of zero based invariant subspaces of $L^2_{\mathrm{a}}(\mathbb{D})$; that is

THEOREM 1.5. *If* M *is a zero based invariant subspace of the Bergman shift, then*

$$\operatorname{rank} l^M_\lambda = \operatorname{card} \sigma(M^*_\zeta | M^\perp). \tag{1.1}$$

Furthermore in Theorem 3.8 and in Theorem 3.11 we prove

THEOREM 1.6. *If* $M \in \operatorname{Lat}(M_\zeta, \mathcal{H}(k))$, $\operatorname{ind} M = 1$, *then*

$$\operatorname{card}\,\big(\sigma(M^*_\zeta | M^\perp) \cap \mathbb{D}\big) \leqslant \operatorname{rank} l_\lambda \leqslant \operatorname{card} \sigma(M^*_\zeta | M^\perp).$$

In the last section, and as an application of Theorem 3.4 we give a description of the structure of l_λ for finite zero based invariant subspaces (see Theorem 4.2). More specifically Theorem 4.2 implies

THEOREM 1.7. *If* M *is a finite zero based invariant subspace of the Bergman shift with* n *distinct zeros, then*

$$\overline{\lambda} z l_\lambda(z) = \frac{p(\overline{\lambda}, z)}{\prod\limits_{j=1}^{r} (z - A_j)(\overline{\lambda} - \overline{A}_j)},$$

where p *is a symmetric polynomial,* $\deg p = n$, $p(0, z) = 0$ *for every* $z \in \mathbb{D}$ *and*

(i) *if* $0 \in \Lambda$, $A_i \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $i = 1, \dots, r$, $r = n - 1$;

(ii) *if* $0 \notin \Lambda$, $A_i \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $i = 1, \dots, r$, $r = n$.

2. BERGMAN TYPE KERNELS AND POSITIVE OPERATORS

In this section we briefly present some of the main results from the theory of *Bergman type-kernels* as developed by McCullough and Richter in [13]. Even though we mainly need the results when the Bergman type-kernel is the classical Bergman kernel, we state the theorems in full generality since we obtain some interesting results which hold in this context. For example, in Theorem 2.7 and Corollary 2.8 we show that if k is a Bergman type kernel, $M \in \mathrm{Lat}(M_\zeta, \mathcal{H}(k))$ and $\operatorname{ind} M = 1$, then $\operatorname{rank} l_\lambda = \operatorname{rank} Q$, where Q is some positive operator on $\mathcal{H}(k)$ which depends on M. Hence the study of the $\operatorname{rank} l_\lambda$ carries over to the study of the range of the operator Q. Then we apply these results to the case of the Bergman kernel and we prove results (see Corollary 2.12 and Lemma 2.13) which will be important for the development of the main results of this article.

We start with the definition of the Bergman type kernels.

DEFINITION 2.1. A function k in $\mathbb{D} \times \mathbb{D}$ is a Bergman type kernel if there is an outer function $\rho \in H^\infty$, $\frac{1}{2} \leqslant |\rho|^2 \leqslant 1$, and functions $u_n \in H^\infty$ such that $u_n(0) = 0$ for all $n \in \mathbb{N}$, and, with $\varphi = \frac{\zeta}{\rho}$,

$$|\varphi(z)|^2\Big(1 - \sum_{n\geqslant 1} |u_n(z)|^2\Big) = 1 \quad \text{a.e. } |z| = 1,$$

$$|\varphi(z)|^2\Big(1 - \sum_{n\geqslant 1} |u_n(z)|^2\Big) < 1 \quad \text{for all } |z| < 1,$$

and

$$\frac{1}{k_\lambda(z)} = 1 - \overline{\varphi(\lambda)}\varphi(z)\Big(1 - \sum_{n\geqslant 1} \overline{u_n(\lambda)} u_n(z)\Big).$$

Furthermore, in [13], it is shown that if k is a Bergman type kernel, then $\mathcal{M}(k) = H^\infty$ with equality of norms, and that $H^2 \subseteq \mathcal{H}(k) \subseteq L^2_{\mathrm{a}}$. If $\varphi(z) = \sqrt{2}z$, $u_\lambda(z) = \frac{1}{2}\overline{\lambda}z$, then $k_\lambda(z) = (1 - \overline{\lambda}z)^{-2}$ is the classical Bergman kernel, and if $\varphi(z) = z$, $u = 0$, then $k_\lambda(z) = (1 - \overline{\lambda}z)^{-1}$ is the classical Szegö kernel.

The following two theorems are fundamental for the development of the main results of this article and shall be used extensively in the sequel. The first is an immediate consequence of Corollary 0.8(a) and Lemma 1.4 from [13]. The second is the Wandering Subspace Theorem 0.17 from [13].

THEOREM 2.2. *Let k be a Bergman type kernel and let M be a multiplier invariant subspace of index* 1. *In addition define $\mathcal{L} \equiv \mathcal{H}_0 \ominus \zeta M$ and the operators $T \equiv \sum\limits_{n\geqslant 1} P_M M_{u_n} P_{M^\perp} M^*_{u_n} P_M$ and $S \equiv P_M M_{1/\varphi} P_{\mathcal{L}} M^*_{1/\varphi} P_M$, where u_n, $n \geqslant 1$, and φ are as in Definition* 2.1. *If G denotes a unit vector in $M \ominus \zeta M$, and $Q = T + S$, then there is a positive definite kernel $l_\lambda(z)$ such that*

$$\frac{P_M k_\lambda(z)}{\overline{G(\lambda)}G(z)} = (1 - \overline{\lambda} z l_\lambda(z)) k_\lambda(z), \tag{2.1}$$

where

$$\overline{\lambda} z l_\lambda(z) = \frac{\overline{\varphi(\lambda)}\varphi(z)}{\overline{G(\lambda)}G(z)} \langle Qk_\lambda, k_z\rangle. \tag{2.2}$$

THEOREM 2.3. (Wandering Subspace Theorem) *If k is a Bergman type kernel and M is a multiplier invariant subspace of $\mathcal{H}(k)$, then the span of the set $\{\zeta^n f : n \geqslant 0,\ f \in M \ominus \zeta M\}$ is dense in M.*

REMARK 2.4. In the case of the Bergman kernel $k_\lambda(z) = (1-\overline{\lambda} z)^{-2}$, formula (2.1) of Theorem 2.2 was proved in Theorem 6.3 of [11], and Theorem 2.3 was proved in Theorem 3.5 of [1].

Let $\mathcal{H}$ be a Hilbert space, and denote by $\mathcal{B}(\mathcal{H}), \mathcal{B}_+(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$, and its positive elements respectively. If $Q \in \mathcal{B}(\mathcal{H})$, then $\operatorname{rank} Q$ denotes the Hilbert space dimension of the closure of the range of Q and $\sigma(Q)$ denotes the *spectrum* of Q. The abbreviations SOT, WOT, refer to the strong and weak operator topologies respectively.

If $U \subseteq \mathcal{H}$ then we let $\bigvee\{U\}$ to denote the closed linear span of the set U and by $\operatorname{cl} U$ we denote the closure of U under the norm topology of $\mathcal{H}$. In addition, we use the symbols I, J to denote subsets of $\mathbb{N}$. Also for $f, g \in \mathcal{H}$ we let $f \otimes g$ to denote the rank one operator on $\mathcal{H}$ that is defined by

$$(f \otimes g)(h) = \langle h, g\rangle_{\mathcal{H}} f.$$

In the rest of this section we prove results which shall be used in Section 3. The following two lemmas will be used in the proof of Theorem 2.7. The proof of the next lemma is an elementary application of Theorem I in [3].

LEMMA 2.5. *Suppose that $u_\lambda(z) = \sum\limits_{i\in I} \overline{u_i(\lambda)} u_i(z)$ is the reproducing kernel for the Hilbert space $\mathcal{H}(u)$. Then $u_i \in \mathcal{H}(u)$ for every $i \in I$.*

LEMMA 2.6. *If $Q \in \mathcal{B}_+(\mathcal{H})$, then $Q = \sum\limits_{i\in I} \sqrt{Q} e_i \otimes \sqrt{Q} e_i$ in the SOT sense, where $\{e_i\}_{i\in I}$ is an orthonormal basis for $\operatorname{cl}\operatorname{range} Q$.*

Proof. First define $f_i = \sqrt{Q} e_i,\ i \in I$, and assume that $I = \mathbb{N}$. If I is finite then the proof is similar. Since $\operatorname{cl}\operatorname{range} Q = (\ker \sqrt{Q})^\perp$, an elementary argument shows that $\{f_i\}_{i\in I}$ is a linearly independent subset of $\mathcal{H}$.

Now for each $n \in \mathbb{N}$ we define $P_n = \sum\limits_{i=1}^{n} e_i \otimes e_i$ and we show that $\Big\{\sum\limits_{i=1}^{n} f_i \otimes f_i\Big\}_{n\in\mathbb{N}} \to Q$ in the WOT.

Indeed, if $f, g \in \mathcal{H}$, then

$$\begin{aligned} \Big\langle \Big(\sum_{i=1}^{n} f_i \otimes f_i\Big) f, g\Big\rangle &= \sum_{i=1}^{n} \langle f, f_i\rangle\langle f_i, g\rangle = \sum_{i=1}^{n} \big\langle \sqrt{Q} f, e_i\big\rangle \langle e_i, \sqrt{Q} g\rangle \\ &= \Big\langle \Big(\sum_{i=1}^{n} e_i \otimes e_i\Big) \sqrt{Q} f, \sqrt{Q} g\Big\rangle = \langle P_n \sqrt{Q} f, \sqrt{Q} g\rangle. \end{aligned} \tag{2.3}$$

We use elementary functional analysis results to get $\{P_n\} \to P_{\vee\{P_n\mathcal{H}:n\in\mathbb{N}\}}$ in the SOT. Furthermore note that since $\{e_i\}_{i\in I}$ is an orthonormal basis of $\operatorname{cl}\operatorname{range} Q$, we have $\bigvee\{P_n\mathcal{H} : n \in \mathbb{N}\} = \operatorname{cl}\operatorname{range} Q$, and so from (2.3),

$$\Big\{\Big\langle\Big\{\sum_{i=1}^{n} f_i \otimes f_i\Big\}f, g\Big\rangle\Big\}_{n\in\mathbb{N}} \to \langle P_{\operatorname{cl}\operatorname{range} Q}\sqrt{Q}f, \sqrt{Q}g\rangle.$$

Since $\operatorname{cl}\operatorname{range} Q = \operatorname{cl}\operatorname{range}\sqrt{Q}$ and since $\Big\{\sum\limits_{i=1}^{n} f_i \otimes f_i\Big\}_{n\in\mathbb{N}}$ is an increasing sequence of positive operators, $\Big\{\sum\limits_{i=1}^{n} f_i \otimes f_i\Big\}_{n\in\mathbb{N}} \to Q$ in the SOT. This concludes the proof of the lemma. ∎

THEOREM 2.7. *If* $Q \in \mathcal{B}_+(\mathcal{H}(k))$ *and* $u_\lambda(z) = \langle Qk_\lambda, k_z\rangle_{\mathcal{H}(k)}$, *then*

$$\operatorname{rank} u_\lambda(z) = \operatorname{rank} Q.$$

Proof. If $\{e_i\}_{i\in I}$ is an orthonormal basis for $\operatorname{cl}\operatorname{range} Q$, then the above lemma and the defining property of the reproducing kernels imply that

$$u_\lambda(z) = \langle Qk_\lambda, k_z\rangle = \sum_{i\in I} \overline{f_i(\lambda)} f_i(z), \tag{2.4}$$

where $f_i = \sqrt{Q}e_i$, $i \in I$, are linearly independent vectors in $\mathcal{H}(u)$ and where the sum converges uniformly on compact subsets of $\Omega \times \Omega$. The above equation implies that $\operatorname{rank} u \leqslant \operatorname{rank} Q$. If $\mathcal{H}(u)$ is the Hilbert space of functions with reproducing kernel u, then by (2.4) and Lemma 2.5, $f_i \in \mathcal{H}(u)$, $i \in I$. Since $\{f_i\}_{i\in I}$ is a linearly independent set, $\operatorname{card} I \leqslant \dim \mathcal{H}(u)$. Moreover, the definition of the rank and an elementary argument given by Aronszajn in pp. 346–347 of [3], imply easily that $\dim \mathcal{H}(u) \leqslant \operatorname{rank} u$. The proof now is complete since $\operatorname{card} I = \operatorname{rank} Q$. ∎

Roughly speaking, the next result shows that the study of the rank of $l_\lambda(z)$ carries over to the study of the rank of the operator Q.

COROLLARY 2.8. *If* $k, M, l_\lambda(z), G, Q$ *are as in Theorem* 2.2, *then*

$$\operatorname{rank} l_\lambda(z) = \operatorname{rank} Q.$$

Proof. From (2.2) we have $\overline{\lambda} z l_\lambda(z) = \frac{\overline{\varphi(\lambda)}\varphi(z)}{\overline{G(\lambda)}G(z)}\langle Qk_\lambda, k_z\rangle$, where $l_\lambda(z)$ is a positive definite kernel. This and the definition of the rank (Definition 1.1) imply that $\operatorname{rank} l_\lambda(z) = \operatorname{rank}\langle Qk_\lambda, k_z\rangle$. The result now follows from the above theorem. ∎

LEMMA 2.9. *Suppose* $T, S \in \mathcal{B}_+(\mathcal{H})$. *If* $Q = T + S$, *then the following hold:*

(i) $\operatorname{cl}\operatorname{range} T \subseteq \operatorname{cl}\operatorname{range} Q$*;*

(ii) *if* $\operatorname{range} S \subseteq \operatorname{range} T$, *then* $\operatorname{cl}\operatorname{range} Q = \operatorname{cl}\operatorname{range} T$ *and in particular* $\operatorname{rank} Q = \operatorname{rank} T$*;*

(iii) *if* $f_i \in \mathcal{H}$ *for every* $i \in I$ *and* $Q = \sum\limits_{i\in I} f_i \otimes f_i$, *where the convergence is in the* SOT, *then* $\operatorname{cl}\operatorname{range} Q = \bigvee\limits_{i\in I}\{f_i\}$.

Proof. (i) Since $T, S \in \mathcal{B}_+(\mathcal{H})$, $\langle Qx, x\rangle = \left\|T^{1/2}x\right\|^2 + \left\|S^{1/2}x\right\|^2$ for all $x \in \mathcal{H}$. For $y \in \ker Q$, $\|T^{1/2}y\| = 0$, hence $y \in \ker T$. From this we get that $\operatorname{cl}\operatorname{range} T \subseteq \operatorname{cl}\operatorname{range} Q$.

(ii) It follows from (i).

(iii) Note that $\operatorname{cl}\operatorname{range} Q \subseteq \bigvee_{i\in I}\{f_i\}$.

Fix $j \in I$ and set $T_j = f_j \otimes f_j$, $S_j = \sum_{\substack{i\in I\\ i\neq j}} f_i \otimes f_i$. Now write $Q = T_j + S_j$. Since $T_j, S_j \in \mathcal{B}_+(\mathcal{H})$, from part (i), $\operatorname{cl}\operatorname{range} T_j \subseteq \operatorname{cl}\operatorname{range} Q$. In particular $f_j \in \operatorname{cl}\operatorname{range} Q$. Since j is arbitrary in I we get $\bigvee_{i\in I}\{f_i\} \subseteq \operatorname{cl}\operatorname{range} Q$. ∎

Using the above lemma (part (iii)) and Lemma 2.6 it is elementary to show the following

LEMMA 2.10. *Suppose that $L \in \mathcal{B}(\mathcal{H})$ and M is a closed subspace of $\mathcal{H}$. If P_M denotes the projection onto M and $R = LP_ML^*$, then* $\operatorname{cl}\operatorname{range} R = \operatorname{cl} LM$.

We close this section with results which shall be used in the proofs of Theorem 3.4 and Theorem 3.11.

REMARK 2.11. Note that in the case of the Bergman kernel, the expressions of the operators T and S, as defined in Theorem 2.2, are becoming:

$$T = P_M M_u P_{M^\perp} M_u^* P_M, \quad S = P_M M_{1/\varphi} P_{\mathcal{L}} M_{1/\varphi}^* P_M,$$

where $u(z) = z/\sqrt{2}$, $\varphi(z) = \sqrt{2}z$, and $\mathcal{L} = \mathcal{H}_0 \ominus \zeta M$.

Now, in light of the above remark, the following result is an immediate application of Lemma 2.10.

COROLLARY 2.12. *If k is the Bergman kernel, then for the operators T and S which are defined in Theorem* 2.2, *we have:* $\operatorname{range} T = P_M M_\zeta M^\perp$ *and* $\operatorname{range} S = P_M M_{1/\zeta}\mathcal{L}$.

LEMMA 2.13. *Suppose that for $n \in \mathbb{N}$, $M_n, M \in \operatorname{Lat}(M_\zeta, L^2_{\mathrm{a}}(\mathbb{D}))$ are nontrivial with index* 1 *and that $P_{M_n} \to P_M$ in the WOT. Then for the associated kernels $l_\lambda^{M_n}, l_\lambda^M$, the following holds:*

$$\operatorname{rank} l_\lambda^M \leqslant \varliminf_{n\to\infty} \operatorname{rank} l_\lambda^{M_n}.$$

Proof. Suppose that k is a Bergman type kernel on $\mathbb{D}\times\mathbb{D}$, N is an index 1 invariant subspace of $\mathcal{H}(k)$ and G_N is a unit vector in $N \ominus \zeta N$. Then from Theorem 2.2 (just combine (2.1) and (2.2)) we conclude that

$$\frac{P_N k_\lambda(z)}{\overline{G_N(\lambda)}G_N(z)} = \Big(1 - \frac{\overline{\varphi(\lambda)}\varphi(z)}{\overline{G_N(\lambda)}G_N(z)}\langle Q_N k_\lambda, k_z\rangle\Big)k_\lambda(z) \tag{2.5}$$

for some $Q_N \in \mathcal{B}_+(\mathcal{H}(k))$ and for some meromorphic function φ defined on $\mathbb{D}$.

Since $P_{M_n} \to P_M$ in the WOT, $P_{M_n}k_\lambda \to P_M k_\lambda$ uniformly on compact subsets of $\mathbb{D}\times\mathbb{D}$. Therefore, if for all $n \geqslant 1$, G_{M_n}, G_M are properly normalized, for example by choosing a point $x_o \in \mathbb{D}$ such that they are all real valued functions, then $G_{M_n} \to G_M$ uniformly on compact subsets of $\mathbb{D}$. Thus, from (2.5) it follows that

$$\langle Q_{M_n} k_\lambda, k_z\rangle \to \langle Q_M k_\lambda, k_z\rangle \quad \text{in } \mathbb{D}\times\mathbb{D}.$$

Additionally, in light of Remark 2.11 we conclude that $\{Q_{M_n}\}_{n=1}^{\infty}$ is a uniformly bounded sequence. Furthermore, and since finite linear combinations of $\{k_z : z \in \mathbb{D}\}$ are dense in $L^2_{\rm a}(\mathbb{D})$, $Q_{M_n} \to Q_M$ in the WOT. Moreover, it is well known that the rank function (for a bounded linear operator on a Hilbert space $\mathcal{H}$) is weakly lower semicontinuous, in the sense that if $\{T_i\}_{i\in I}$ is a net in $\mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$ such that $T_i \to T$ in the WOT, then $\operatorname{rank} T \leqslant \varliminf_{n\to\infty} \operatorname{rank} T_i$ (see Appendix, [8]).

The proof now is complete since for every $N \in \operatorname{Lat}(M_\zeta, L^2_{\rm a}(\mathbb{D}))$, $\operatorname{ind} N = 1$, we have $\operatorname{rank} l_\lambda^N = \operatorname{rank} Q_N$ (see Corollary 2.8). ∎

Remark 2.14. (i) It is worthwhile to observe that in the case of Bergman type kernels an analogous result holds, provided that the associated sequence $\{Q_{M_n}\}_{n\in\mathbb{N}}$ is uniformly bounded.

(ii) A result due to Shimorin (see Theorem 5, [16]), states that if M is an index 1 invariant subspace of $L^2_{\rm a}(\mathbb{D})$, then there is always a sequence of finite zero based invariant subspaces $\{M_n\}_{n\in\mathbb{N}}$ such that $P_{M_n} \to P_M$ in the WOT. The above lemma implies that $\operatorname{rank} l_\lambda^M \leqslant \varliminf_{n\to\infty} \operatorname{rank} l_\lambda^{M_n}$.

3. HEDENMALM'S CONJECTURE

In this section we resolve Hedenmalm's Conjecture in the case of zero based invariant subspaces of the Bergman shift, and in addition we show that for every $M \in \operatorname{Lat}(M_\zeta, L^2_{\rm a}(\mathbb{D}))$, $\operatorname{ind} M = 1$, the following holds:

$$\operatorname{card}\big(\sigma(M_\zeta^*|M^\perp)\cap\mathbb{D}\big) \leqslant \operatorname{rank} l_\lambda^M \leqslant \operatorname{card}\sigma(M_\zeta^*|M^\perp).$$

In order to show our main results we need the following two theorems which were proved originally by S. Walsh; see [17], Theorems 1, 2. The first of them was proved for a larger class of spaces and for hyponormal operators.

Theorem 3.1. *If M_ζ denotes the multiplication by z on the $L^2_{\rm a}(\mathbb{D})$ and if f is an analytic function in a neighborhood of $\overline{\mathbb{D}}$, then the following holds.*

Either *f is cyclic for M_ζ^* (that is $[f]_{M_\zeta^*} = L^2_{\rm a}(\mathbb{D})$)* or *$f$ belongs to a finite dimensional M_ζ^* invariant subspace of $L^2_{\rm a}(\mathbb{D})$.*

Theorem 3.2. *Suppose that $f \in L^2_{\rm a}(\mathbb{D})$. Then f is in a finite dimensional invariant subspace of M_ζ^* if and only if it is rational with zero residues at its poles.*

Lemma 3.3. *Let $\overline{\partial}^l k_\lambda$ denote the kernel of the evaluation of the l^{th} derivative at λ; that is $f^{(l)}(\lambda) = \langle f, \overline{\partial}^l k_\lambda\rangle$, $\lambda \in \mathbb{D}$, $l \geqslant 0$.*

Let $\lambda \in \mathbb{D}$, $\lambda \neq 0$, and let ρ be a fixed positive integer. Set $W = \bigvee_{j=1}^{\rho}\{\overline{\partial}^{j-1}k_\lambda\}$, where $k_\lambda(z) = (1-\overline{\lambda}z)^{-2}$, $z \in \mathbb{D}$. Then

$$W = \bigvee_{j=1}^{\rho}\{(1-\overline{\lambda}z)^{-(j+1)}\} = \{k_\lambda(z)\} \vee \bigvee_{j=2}^{\rho}\{z(1-\overline{\lambda}z)^{-(j+1)}\}.$$

Proof. For $k \geqslant 1$,

$$z(1-\overline{\lambda}z)^{-(k+1)} = \frac{1}{\overline{\lambda}}(1-(1-\overline{\lambda}z))(1-\overline{\lambda}z)^{-(k+1)} = \frac{1}{\overline{\lambda}}[(1-\overline{\lambda}z)^{-(k+1)} - (1-\overline{\lambda}z)^{-k}].$$

The above leads to the second equality.

Since $\overline{\partial}^{j-1}k_\lambda(z) = c_{j-1}z^{j-1}(1-\overline{\lambda}z)^{-(j+1)}$, $c_{j-1} = j!$, $j \geqslant 1$, a repeated application of $z = \frac{1}{\overline{\lambda}}(1-(1-\overline{\lambda}z))$ to $z^j(1-\overline{\lambda}z)^{-(j+2)}$, as in the above calculation, proves the first equality. ∎

We now state and prove our first main result, which resolves Hedenmalm's Conjecture in the case of zero based invariant subspaces.

THEOREM 3.4. *Let $\Lambda = \{\lambda_i\}_{i\in I}$ be a nonempty sequence of points in $\mathbb{D}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$, $i,j \in I$. Suppose that for $i \in I$, ρ_i is a positive integer. Set $M = \{f \in L^2_{\mathrm{a}}(\mathbb{D}) : f^{(m)}(\lambda_i) = 0,\ i \in I,\ 0 \leqslant m \leqslant \rho_i - 1\}$ and assume that M is nontrivial.*

If $l_\lambda(z)$ is the associated kernel for M, then

$$\operatorname{rank} l_\lambda(z) = \operatorname{card}\Lambda.$$

Proof. From the hypothesis, $M = \bigcap_{i\in I}\bigcap_{m=0}^{\rho_i-1}\{f \in L^2_{\mathrm{a}}(\mathbb{D}) : f^{(m)}(\lambda_i) = 0\}$. Thus

$$M^\perp = \bigvee_{i\in I}\bigvee_{j=1}^{\rho_i}\{\overline{\partial}^{j-1}k_{\lambda_i}\}. \tag{3.1}$$

CLAIM 3.5. range $S \subseteq$ range T *and* $\operatorname{rank} T \leqslant \operatorname{card}\Lambda$.

For each $\lambda_i \neq 0$, $i \in I$, set

$$R_i = \bigvee_{j=2}^{\rho_i}\{z(1-\overline{\lambda}_i z)^{-(j+1)}\}. \tag{3.2}$$

Note that if for some $i \in I, \rho_i = 1$, then $R_i = (0)$.

We divide the proof of this claim into two cases.

Case 1. $0 \notin \Lambda$.

By (3.1) and Lemma 3.3, for $i \in I$,

$$M^\perp = \bigvee_{i\in I}\Big(\{k_{\lambda_i}\}\bigcup\frac{1}{\zeta}R_i\Big) = \bigvee_{i\in I}\big(\{k_{\lambda_i}\}\bigcup R_i\big). \tag{3.3}$$

Since range $T = P_M M_\zeta M^\perp$, and since $R_i \subseteq M^\perp$, $i \in I$,

$$\operatorname{range} T = P_M M_\zeta \bigvee_{i\in I}\Big(\{k_{\lambda_i}\}\bigcup\frac{1}{\zeta}R_i\Big) = \bigvee_{i\in I}\{P_M M_\zeta k_{\lambda_i}\}. \tag{3.4}$$

This implies that $\operatorname{rank} T \leqslant \operatorname{card}\Lambda$.

Now consider $\mathcal{L} = \mathcal{H}_0 \ominus (\zeta M)$ and observe that $\zeta M = M \cap \operatorname{span}\{k_0\}^\perp$. Hence

$$\mathcal{L} = (M^\perp \vee \{k_0\}) \ominus \operatorname{span}\{k_0\}. \tag{3.5}$$

Moreover, for every $i \in I$ we see that $k_0 = 1$, $\langle k_{\lambda_i} - k_0, k_0 \rangle = 0$, and that for $n \geqslant 0$, $\langle z(1 - \overline{\lambda}_i z)^{-n}, k_0 \rangle = 0$. Now use (3.5) and (3.3) to conclude that

$$\mathcal{L} = \bigvee_{i \in I} (\{k_{\lambda_i} - 1\} \cup R_i)\,. \tag{3.6}$$

Since range $S = P_M M_{1/\zeta} \mathcal{L}$, and since for every $i \in I$, $\frac{R_i}{\zeta} \subseteq M^{\perp}$,

$$\operatorname{range} S = P_M \bigvee_{i \in I} \Big(\Big\{ \frac{k_{\lambda_i} - 1}{\zeta} \Big\} \cup \frac{R_i}{\zeta} \Big) = P_M \bigvee_{i \in I} \Big\{ \frac{k_{\lambda_i} - 1}{\zeta} \Big\}. \tag{3.7}$$

Note that for $i \in I$, $\frac{k_{\lambda_i}(z) - 1}{z} = \overline{\lambda}_i \frac{2 - \overline{\lambda}_i z}{(1 - \overline{\lambda}_i z)^2}$ and $k_{\lambda_i} \in M^{\perp}$.

Thus, $P_M \frac{k_{\lambda_i} - 1}{\zeta} = -\overline{\lambda}_i^2 P_M M_{\zeta} k_{\lambda_i}$, and hence from (3.7), range $S = \bigvee_{i \in I} \{P_M M_{\zeta} k_{\lambda_i}\}$.

From the above and (3.4), range S = range T and thus the proof of Case 1 of the proof of Claim 3.5 is complete.

Case 2. $0 \in \Lambda$.

We assume, without loss of generality, that $\lambda_0 = 0$ has multiplicity $\rho > 0$ in M and that $\lambda_i \neq 0$ for $i \neq 0$, $i \in I$. Using a similar argument as developed in the proof of Case 1 we show that

$$\operatorname{range} T = \{P_M \overline{\partial}^{\rho} k_0\} \vee \bigvee_{\substack{i \in I \\ i \neq 0}} \{P_M \zeta k_{\lambda_i}\}, \tag{3.8}$$

and

$$\operatorname{range} S = \bigvee_{\substack{i \in I \\ i \neq 0}} \{P_M M_{\zeta} k_{\lambda_i}\}. \tag{3.9}$$

Thus, range $S \subseteq$ range T and rank $T \leqslant$ card Λ. The proof of Claim 3.5 is now complete.

By Lemma 2.9 and the above claim, range Q = range T and rank $Q \leqslant$ card Λ. Now it follows (see (3.4) and (3.8)) that

$$\operatorname{range} Q = \begin{cases} \bigvee_{i \in I} \{P_M M_{\zeta} k_{\lambda_i}\} & 0 \notin \Lambda, \\ \{P_M \overline{\partial}^{\rho} k_0\} \vee \bigvee_{\substack{i \in I \\ i \neq 0}} \{P_M M_{\zeta} k_{\lambda_i}\} & 0 \in \Lambda \text{ of multiplicity } \rho. \end{cases} \tag{3.10}$$

To conclude the proof of the theorem, since rank Q = rank l_{λ} (see Corollary 2.8), it remains to show that the sets $\{P_M \overline{\partial}^{\rho} k_0, P_M M_{\zeta} k_{\lambda_i}\}_{i \in I,\, i \neq 0}$ and $\{P_M M_{\zeta} k_{\lambda_i}\}_{i \in I}$ are linearly independent subsets of $L^2_{\mathrm{a}}(\mathbb{D})$.

To this end we only consider the set $\{P_M \overline{\partial}^{\rho} k_0, P_M M_{\zeta} k_{\lambda_i}\}_{i \in I, i \neq 0}$. The other case follows from this.

Assume that J is a finite subset of $I \setminus \{0\}$ such that

$$b P_M \overline{\partial}^{\rho} k_0 + \sum_{j \in J} a_j P_M M_{\zeta} k_{\lambda_j} = 0, \quad a_j, b \in \mathbb{C},\ j \in J. \tag{3.11}$$

It remains to show that $a_j = b = 0,\ j \in J$. We define

$$f = b\overline{\partial}^\rho k_0 + \sum_{j\in J} a_j M_\zeta k_{\lambda_j}, \quad a_j, b \in \mathbb{C},\ j \in J. \tag{3.12}$$

In light of (3.11), $P_M f = 0$ and hence

$$f \in M^\perp. \tag{3.13}$$

Furthermore, (3.12) implies that f is analytic in a neighborhood of $\overline{\mathbb{D}}$. Consequently, Walsh's theorems (see Theorem 3.1 and Theorem 3.2) apply and we have two options.

Option 1. f is contained in a finite dimensional M_ζ^* invariant subspace of $L_{\mathrm{a}}^2(\mathbb{D})$.

Another expression for f is

$$f(z) = b(-1)^\rho(\rho+1)!z^\rho + \sum_{j\in J} \frac{a_j}{\overline{\lambda}_j^2}\Big[\Big(z - \frac{1}{\overline{\lambda}_j}\Big)^{-1} + \frac{1}{\overline{\lambda}_j}\Big(z - \frac{1}{\overline{\lambda}_j}\Big)^{-2}\Big], \quad z \in \mathbb{C}. \tag{3.14}$$

Thus, $\operatorname{Res}\Big(f, \frac{1}{\overline{\lambda}_j}\Big) = \frac{a_j}{\overline{\lambda}_j^2}$, where $\operatorname{Res}\Big(f, \frac{1}{\overline{\lambda}_j}\Big)$ denotes the residue of f at $\frac{1}{\overline{\lambda}_j}$, $j \in J$. Now Theorem 3.2 forces $\operatorname{Res}\Big(f, \frac{1}{\overline{\lambda}_j}\Big)$ to be zero and hence $a_j = 0$. Additionally, since $\zeta^\rho \notin M^\perp$ and since $f \in M^\perp$, we obtain $b = 0$.

Option 2. f is cyclic for M_ζ^*.

Thus, $[f]_{M_\zeta^*} = L_{\mathrm{a}}^2(\mathbb{D})$. Since $M^\perp$ is an M_ζ^* invariant subspace of $L_{\mathrm{a}}^2(\mathbb{D})$, $L_{\mathrm{a}}^2(\mathbb{D}) = [f]_{M_\zeta^*} \subseteq M^\perp$ and hence $M \equiv 0$. This leads to contradiction because M is a nonzero invariant subspace. The proof of the theorem is now complete. ∎

Corollary 3.6. *Suppose that M, N are invariant subspaces of $L_{\mathrm{a}}^2(\mathbb{D})$, M is zero based, $M \subseteq N$ and* $\operatorname{ind} N = 1$. *If l_λ^M, l_λ^N denote the associated kernels, then*

$$\operatorname{rank} l_\lambda^N \leqslant \operatorname{rank} l_\lambda^M.$$

Proof. Under the hypothesis of the corollary, N is zero based and the zero sequence of N is contained in the zero sequence of M, see Corollary 10.3, [11]. An application of the above theorem concludes the proof. ∎

Remark 3.7. In the proof of Theorem 3.4, in order to show that the set $\{P_M\overline{\partial}^\rho k_0, P_M M_\zeta k_{\lambda_i}\}_{i\in I}$ is a linearly independent subset of $L_{\mathrm{a}}^2(\mathbb{D})$ we only used that $M \in \operatorname{Lat}(M_\zeta, L_{\mathrm{a}}^2(\mathbb{D}))$ and $\overline{\partial}^\rho k_0 \notin M^\perp$. Hence the set $\{P_M\overline{\partial}^\rho k_0, P_M M_\zeta k_{\lambda_i}\}_{i\in I}$ is a linearly independent subset of $L_{\mathrm{a}}^2(\mathbb{D})$ whenever $M \in \operatorname{Lat}(M_\zeta, L_{\mathrm{a}}^2(\mathbb{D}))$, $\overline{\partial}^\rho k_0 \notin M^\perp$ and $\lambda_i \in \mathbb{D}$, $i \in I$.

Theorem 3.8. *If $M \in \operatorname{Lat}(M_\zeta, L_{\mathrm{a}}^2(\mathbb{D}))$,* $\operatorname{ind} M = 1$, *then*

$$\operatorname{card}(\sigma(M_\zeta^*|M^\perp) \cap \mathbb{D}) \leqslant \operatorname{rank} l_\lambda(z).$$

Proof. Without loss of generality we suppose that

$$\underline{Z}(M) = \{\lambda_i\}_{i\in I} \cup \{\zeta_j\}_{j\in J}, \tag{3.15}$$

where for $i \in I$, λ_i are distinct points in $\mathbb{D}$, and for $j \in J$, ζ_j are distinct points in $\mathbb{T}$. This implies that $M \subseteq L$, where

$$L^\perp = \bigvee_{i\in I} \bigvee_{l=1}^{\rho_i} \{\overline{\partial}^{l-1} k_{\lambda_i}\}$$

for some positive integers ρ_i, $i \in I$. Since $\underline{Z}(M) = \overline{\sigma(M_\zeta^*|M^\perp)}$ (cf. [10]), $\overline{\sigma(M_\zeta^*|M^\perp)} \cap \mathbb{D} = \{\lambda_i\}_{i\in I}$. For the rest of the proof we use the same notation as in the proof of Theorem 3.4.

In light of Remark 2.11 and Corollary 2.12, $Q = T + S$, where $\operatorname{range} T = P_M M_\zeta M^\perp$, $\operatorname{range} S = P_M M_\zeta \mathcal{L}$ and $\mathcal{L} = \mathcal{H}_0 \ominus \zeta M$.

Since $\operatorname{cl}\operatorname{range} T \supseteq P_M M_\zeta L^\perp$, Lemma 2.9(i) implies that

$$\operatorname{cl}\operatorname{range} Q \supseteq P_M M_\zeta L^\perp. \tag{3.16}$$

As in the proof of the Claim 3.5 of Theorem 3.4, and since $\bigvee_{i\in I} \bigvee_{l=1}^{\rho_i} \{\overline{\partial}^{l-1} k_{\lambda_i}\} \subseteq M^\perp$, we have

$$P_M M_\zeta L^\perp = \begin{cases} \bigvee_{i\in I} \{P_M M_\zeta k_{\lambda_i}\} & 0 \notin Z(M), \\ \bigvee_{\substack{i\in I\\ i\neq 0}} \{P_M \overline{\partial}^\rho k_0, P_M M_\zeta k_{\lambda_i}\} & \lambda_0 = 0 \in Z(M) \text{ of multiplicity } \rho. \end{cases}$$

From Remark 3.7, $\{P_M \overline{\partial}^\rho k_0,\ P_M M_\zeta k_{\lambda_i}\}_{i\in I}$ is a linearly independent subset of $L_{\mathrm{a}}^2(\mathbb{D})$. Consequently,

$$\operatorname{rank} P_M M_\zeta L^\perp \geqslant \operatorname{card} I = \operatorname{card}(\sigma(M_\zeta^*|M^\perp) \cap \mathbb{D}).$$

In light of (3.16) the proof of the theorem is complete, since $\operatorname{rank} l_\lambda^M = \operatorname{rank} Q$ (see Corollary 2.8). ∎

The following facts shall be used in the proof of the next main result of this article (Theorem 3.11), where we show that for any $M \in \operatorname{Lat}(M_\zeta, L_{\mathrm{a}}^2(\mathbb{D}))$ with index 1, $\operatorname{rank} l_\lambda(z) \leqslant \operatorname{card} \sigma(M_\zeta^*|M^\perp)$.

The first of them is due to Shimorin (see Lemma 2, [16]).

LEMMA 3.9. *Let $\mathcal{H}$ be a Hilbert space and M, M_n, $n \in \mathbb{N}$, are closed subspaces of $\mathcal{H}$. With P_M, P_{M_n} we denote the orthogonal projections onto M and M_n respectively. If $P_{M_n} \to P_M$ in the* WOT *and $x_n \in M_n$, $x \in \mathcal{H}$ such that $x_n \to x$ weakly, then $x \in M$.*

Suppose that $M \in \operatorname{Lat}(M_\zeta, L_{\mathrm{a}}^2(\mathbb{D}))$ and that $\operatorname{ind} M = 1$. Let also n_M denote the smallest nonnegative integer such that there is $f \in M$ with $f^{(n_M)}(0) \neq 0$. Then a well known and simple argument shows that $G \in M \ominus \zeta M$, $G^{(n_M)} \neq 0$, if and only if G is a solution to the extremal problem

$$\sup\{\operatorname{Re} f^{(n_M)}(0) : f \in M,\ \|f\| \leqslant 1\}.$$

Additionally, one can prove (for a proof see Proposition 3.5, [12]) that this solution is unique. For this reason the above function G is also called the *extremal function* for M.

Now set

$$B_w(z) = \frac{|w|}{w}\frac{w-z}{1-\overline{w}z}, \quad \text{and} \quad S_\zeta(z) = \exp\Big(-\frac{\zeta+z}{\zeta-z}\Big), \quad w \neq 0,\, w, z \in \mathbb{D}\,, \zeta \in \mathbb{T}.$$

Then for every $b \in (0, +\infty)$ and $\zeta \in \mathbb{T}$ it is elementary to show that

$$B^n_{(1-(b/n))\zeta} \to S^b_\zeta \quad \text{as } n \to \infty \tag{3.17}$$

uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{\zeta\}$. Furthermore it is well known that if $k \in \mathbb{N}$, then for every $j \in \{1, \ldots, k\}$, $\beta_j \in (0, +\infty)$ and $\zeta_j \in \mathbb{T}$, the following holds:

$$\Big[\prod_{j=1}^k S^{\beta_j}_{\zeta_j}\Big] = \bigcap_{j=1}^k [S^{\beta_j}_{\zeta_j}]. \tag{3.18}$$

To prove this, one could show, using the contractive divisor property (see Theorem 3.34, [12]), that $\bigcap\limits_{j=1}^k [S^{\beta_j}_{\zeta_j}] \subseteq \Big[\prod\limits_{j=1}^k S^{\beta_j}_{\zeta_j}\Big]$. The other inclusion is trivial. The proof of the next result is essentially due to Atzmon and can be found in Theorem 1.6 of [5]. The theorem we are referring to holds in greater generality and for a larger class of Hilbert spaces of analytic functions.

Lemma 3.10. *Suppose that $M \in \operatorname{Lat}(M_\zeta, L^2_{\mathrm{a}}(\mathbb{D}))$ and $k \in \mathbb{N}$ such that*

$$\sigma(M^*_\zeta|M^\perp) = \{\zeta_j\}_{j=1}^k, \quad \zeta_j \in \mathbb{T},\, 1 \leqslant j \leqslant k.$$

Then for every $j \in \{1, \ldots, k\}$ there is a $\beta_j \in (0, +\infty)$ such that

$$M = \Big[\prod_{j=1}^k S^{\beta_j}_{\zeta_j}\Big], \quad S^{\beta_j}_{\zeta_j}(z) = \exp\Big(-\beta_j\frac{\zeta_j+z}{\zeta_j-z}\Big) \quad \textit{for } z \in \mathbb{D},\, j \in \{1, \ldots, k\}.$$

It is also worthwhile to mention that even though the proof given in Theorem 1.6 from [5] applies for the case where $\sigma(M^*_\zeta|M^\perp)$ is a singleton on $\mathbb{T}$, a minor modification leads to the proof of the previous result. One can also derive the above lemma from Theorem 2 of [4].

Theorem 3.11. *If $M \in \operatorname{Lat}(M_\zeta, L^2_{\mathrm{a}}(\mathbb{D}))$, $\operatorname{ind} M = 1$, then*

$$\operatorname{card}\big(\sigma(M^*_\zeta|M^\perp) \cap \mathbb{D}\big) \leqslant \operatorname{rank} l^M_\lambda \leqslant \operatorname{card} \sigma(M^*_\zeta|M^\perp).$$

Proof. Observe that the first inequality in the result is exactly Theorem 3.8 and that the second one holds trivially if $\operatorname{card} \underline{Z}(M) = \infty$.

So we suppose that there are $s, k \in \mathbb{N}$, such that

$$\overline{\underline{Z}(M)} = \{\overline{\alpha_i}\}_{i=1}^s \cup \{\zeta_j\}_{j=1}^k, \tag{3.19}$$

where for $i = 1, \ldots, s$, α_i are distinct points in $\mathbb{D}$, and for $j = 1, \ldots, k$, ζ_j are distinct points in $\mathbb{T}$.

In light of (3.19) it is not hard to prove that $M = L \cap N$, where

$$L^{\perp} = \bigvee_{i=1}^{s} \bigvee_{l=1}^{\rho_i} \{\overline{\partial}^{l-1} k_{\alpha_i}\}$$

for some positive integers $\rho_i,\ i \in \{1, \dots, s\}$, and $N = \frac{M}{G_L}$, where G_L is the extremal function for L (the proof of the above becomes elementary once we observe that $G_L \in H^{\infty}$). We also note that

$$\sigma(M_{\zeta}^{*}\,\big|N^{\perp}) = \overline{Z(N)} = \{\zeta_j\}_{j=1}^{k}.$$

Now use Lemma 3.10 to obtain $\beta_j \in (0, +\infty)$, $j = 1, \dots, k$, such that

$$N = \Big[\prod_{j=1}^{k} S_{\zeta_j}^{\beta_j}\Big].$$

For $n \in \mathbb{N}$ write

$$S = \prod_{j=1}^{k} S_{\zeta_j}^{\beta_j} \quad \text{and} \quad B_n = \prod_{j=1}^{k} B_{(1-\beta_j/n)\zeta_j}^{n}$$

and hence,

$$M = L \cap [S]. \tag{3.20}$$

Furthermore, equation (3.17) obviously implies that

$$B_n \to S, \tag{3.21}$$

uniformly on compact subsets of $\overline{\mathbb{D}} \setminus \{\zeta_j\}_{j=1}^{k}$.

For $n \in \mathbb{N}$ set

$$M_n = L \cap [B_n] \tag{3.22}$$

and denote by G_n, G the extremal functions for M_n and M respectively.

In the following we show that $G_n \to G$ weakly. Since for $n \in \mathbb{N}$, $\|G_n\| = 1$, there is $F \in L_{\mathrm{a}}^2(\mathbb{D})$ with $\|F\| \leqslant 1$ such that an appropriate subsequence of $\{G_n\}_{n\in\mathbb{N}}$ converges to F weakly. The following argument applies to any convergent subsequence of $\{G_n\}_{n\in\mathbb{N}}$, and moreover, a standard argument implies the result for the full sequence. Thus, in order to simplify the notation, we shall assume that $G_n \to F$. Consequently, it is enough to show that F is extremal for M. We set $B = \prod_{i=1}^{s} B_{\alpha_i}^{\rho_i}$ and without loss of generality we consider only the case where $\alpha_i \neq 0$, $i = 1, \dots, s$, since the other case can be proved in a similar way. For the proof of this we need the following two claims.

Claim 3.12. $\operatorname{Re} F(0) \geqslant \operatorname{Re} G(0)$.

Since for $n \in \mathbb{N}$, G_n is extremal for M_n, $\frac{\operatorname{Re}(pBB_n(0))}{\|pBB_n\|} \leqslant \operatorname{Re} G_n(0)$. Now use (3.21) to obtain $\frac{\operatorname{Re}(pBS(0))}{\|pBS\|} \leqslant \operatorname{Re} F(0)$, and take the supremum over the set of all analytic polynomials which are defined on $\mathbb{D}$. Furthermore, one easily concludes from (3.20) and the expressions of L and B, that the set $\{pBS\}$, where p ranges over the set of all analytic polynomials, is dense in M. Hence, since G is the extremal function for M, we obtain that $\operatorname{Re} F(0) \geqslant \operatorname{Re} G(0)$.

CLAIM 3.13. $F \in [G] = M$.

It is clear that $F \in L$. In light of (3.18) it is enough to show that for every $j \in \{1, \dots, k\}$, $F \in [S_{\zeta_j}^{\beta_j}]$. Now fix $j \in \{1, \dots, k\}$ and for $n \in \mathbb{N}$ denote by g_n, g the extremal functions for $[B_{(1-\beta_j/n)\zeta_j}]$ and $[S_{\zeta_j}^{\beta_j}]$ respectively.

It is shown in pp. 256–257, [7], that $g_n \to g$ uniformly on compact subsets of $\mathbb{D}$ and particularly $\lim_n g_n(z) = g(z)$ pointwise in $\mathbb{D}$. Consequently, by Theorem 1A from [16], we get $P_{[g_n]} \to P_{[g]}$ in the WOT.

Observe that $G_n \in [g_n]$ for every $n \in \mathbb{N}$, and since $G_n \to F$ weakly, use Lemma 3.9 to conclude that $F \in [g] = [S_{\zeta_j}^{\beta_j}]$. Since j is arbitrary in $\{1, \dots, k\}$, the proof of the claim is complete.

The above claim implies that $\frac{\operatorname{Re} F(0)}{\|F\|} \leqslant \operatorname{Re} G(0)$, and hence by Claim 3.12 we get $\|F\| \geqslant 1$. Since $\|F\| \leqslant 1$, $\|F\| = 1$. This together with Claim 3.12 imply that F is the extremal function for M. Thus, $G_n \to G$ weakly and particularly $\lim_{n\to\infty} G_n(z) = G(z)$ pointwise in $\mathbb{D}$. Consequently, Theorem 1A of [16] implies that $P_{[G_n]} \to P_{[G]}$ in the WOT and equivalently $P_{M_n} \to P_M$ in the WOT. Hence Lemma 2.13 applies and therefore

$$\operatorname{rank} l_\lambda^M \leqslant \varliminf_{n\to+\infty} \operatorname{rank} l_\lambda^{M_n}.$$

Furthermore, and for sufficiently large values of n, we can assume that α_i, $\Big(1 - \frac{\beta_j}{n}\Big)\zeta_j$ for $i = 1, \dots, s$, $j = 1, \dots, k$, are distinct points in $\mathbb{D}$; hence, from Theorem 3.4 we get that for every $n \in \mathbb{N}$, $\operatorname{rank} l_\lambda^{M_n} = s + k$. Thus,

$$\operatorname{rank} l_\lambda^M \leqslant \operatorname{card} \underline{Z}(M) = \operatorname{card} \sigma(M_\zeta^* | M^\perp). \qquad \blacksquare$$

4. APPLICATIONS

In the present section we are concerned with the structure of the associated kernel l_λ^M when M is a finite zero based invariant subspace of the Bergman shift. Such spaces are of special interest in the theory of Bergman spaces. For example, a result due to Shimorin (see Theorem 5, [16]) states that any invariant subspace can be "approximated" (in some sense) by a sequence of finite zero based invariant subspaces (see also Remark 2.14 part (ii)).

Before we state the main result of this section we give the following definition.

DEFINITION 4.1. A nonzero sesquianalytic polynomial $p(\overline{\lambda}, z)$ defined on $\mathbb{D} \times \mathbb{D}$; that is a polynomial which is analytic in z and conjugate analytic in λ, for $\lambda, z \in \mathbb{D}$, is called *symmetric*, if $p(\overline{\lambda}, z) = \overline{p(\overline{z}, \lambda)}$ for every $\lambda, z \in \mathbb{D}$. In such a case $\deg p$ denotes the degree of p with respect to either of the variables, $\overline{\lambda}, z$.

Let $I = \{1, 2, \dots, n\}$ and set $\Lambda = \{\lambda_i\}_{i\in I}$ to be a nonempty sequence of points in $\mathbb{D}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$, $i, j \in I$. Suppose that for $i \in I$, ρ_i is a positive integer and set $M = \{f \in L_{\mathrm{a}}^2(\mathbb{D}) : f^{(m)}(\lambda_i) = 0,\ i \in I,\ 0 \leqslant m \leqslant \rho_i - 1\}$.

Theorem 4.2. *The reproducing kernel of M/G is of the form*

$$k_\lambda^G(z) = \Bigg(1 - \frac{p(\overline{\lambda}, z)}{\prod_{i=1}^{r}(z - A_i)(\overline{\lambda} - \overline{A}_i)}\Bigg) k_\lambda(z),$$

where p is a symmetric polynomial, $\deg p = n$, $p(0,z) = 0$ *for every* $z \in \mathbb{D}$ *and*

(i) *if* $0 \in \Lambda$, $A_i \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $i = 1, \dots, r$, $r = n-1$;

(ii) *if* $0 \notin \Lambda$, $A_i \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $i = 1, \dots, r$, $r = n$.

Proof. First we show that $\deg p(\overline{\lambda}, z) \leqslant n$, and then using Theorem 3.4 we prove that the degree of p is exactly n.

Claim 4.3. *The reproducing kernel of M/G has the form as stated in the theorem with* $\deg p(\overline{\lambda}, z) \leqslant n$.

We treat the case where $0 \in \Lambda$. The result for the case where $0 \notin \Lambda$ has an almost identical proof with fewer technicalities.

We assume that $\lambda_n = 0$ with multiplicity ρ_n in M and that $\lambda_i \neq 0$ for $i = 1, \dots, n-1$. Hence, as was done in Lemma 3.3,

$$M^\perp = \bigvee_{i=1}^{n} \bigvee_{j=1}^{\rho_i} \{\overline{\partial}^{j-1} k_{\lambda_i}\} = \{1, z, \dots, z^{\rho_n - 1}\} \vee \bigvee_{i=1}^{n-1} \bigvee_{j=1}^{\rho_i} \left\{\frac{1}{(1 - \overline{\lambda}_i z)^{j+1}}\right\}. \tag{4.1}$$

Let

$$A = \{1, z, \dots, z^{\rho_n - 1}\} \cup \bigcup_{i=1}^{n-1} \bigcup_{j=1}^{\rho_i} \left\{\frac{1}{(1 - \overline{\lambda}_i z)^{j+1}}\right\} \quad \text{and} \quad \alpha = \operatorname{card} A.$$

Let also $\{f_k\}_{k=1}^{\alpha}$ be an enumeration of A and note that $\{f_k\}_{k=1}^{\alpha}$ is an ordered basis of $M^\perp$. If $\{e_k\}_{k=1}^{\alpha}$ is the dual basis of $\{f_k\}_{k=1}^{\alpha}$, then $P_{M^\perp} = \sum_{k=1}^{\alpha} e_k \otimes f_k$. Moreover, for every $k \in \{1, \dots, \alpha\}$, $e_k = \sum_{j=1}^{\alpha} a_{kj} f_j$ for some $a_{kj} \in \mathbb{C}$, $k, j \in \{1, \dots, \alpha\}$. Thus $P_{M^\perp} = \sum_{k,j=1}^{\alpha} a_{kj}(f_j \otimes f_k)$. Note that $P_M k_\lambda(z) = \overline{P_M k_z(\lambda)}$ and write $P_M k_\lambda(z) = k_\lambda(z) - P_{M^\perp} k_\lambda(z)$ for every $\lambda, z \in \mathbb{D}$. Thus, $P_M k_\lambda(z) = k_\lambda(z) - \sum_{k,j=1}^{\alpha} a_{kj} \overline{f_k(\lambda)} f_j(z)$, and by doing some elementary calculations we have

$$P_M k_\lambda(z) = \frac{p_1(\overline{\lambda}, z)}{(1 - \overline{\lambda} z)^2 \prod_{i=1}^{n-1}(1 - \overline{\lambda}_i z)^{\rho_i + 1}(1 - \lambda_i \overline{\lambda})^{\rho_i + 1}}, \tag{4.2}$$

where p_1 is a symmetric polynomial with $\deg p_1 \leqslant n + \sum_{i=1}^{n} \rho_i$.

Since $P_M k_\lambda \in M$,

$$p_1(\overline{\lambda}, z) = \overline{\lambda}^{\rho_n} z^{\rho_n} \prod_{i=1}^{n-1} (z - \lambda_i)^{\rho_i} (\overline{\lambda} - \overline{\lambda}_i)^{\rho_i} p_2(\overline{\lambda}, z) \tag{4.3}$$

for some symmetric polynomial p_2 with $\deg p_2 \leqslant n$.

In the rest of the proof, c_1, c_2, c_3 are constants in $\mathbb{C}$. Since 0 has multiplicity ρ_n, it is not hard to see that $G(z) = c_1 P_M \overline{\partial}^{\rho_n} k_0(z)$. Moreover, $P_M \overline{\partial}^{\rho_n} k_\lambda(z) = \overline{\partial}^{\rho_n} k_\lambda(z) - P_{M^\perp} \overline{\partial}^{\rho_n} k_\lambda(z)$. Consequently, by (4.1), we obtain after simple algebraic manipulations

$$P_M \overline{\partial}^{\rho_n} k_0(z) = \frac{q(z)}{\prod_{i=1}^{n-1} (1 - \overline{\lambda}_i z)^{\rho_i + 1}}, \tag{4.4}$$

where q is a polynomial in $\mathbb{D}$ with

$$\deg q = n - 1 + \sum_{i=1}^{n} \rho_i. \tag{4.5}$$

It is well known that G has exactly n zeros in $\mathbb{D}$ with the right multiplicity, and that $|G| \geqslant 1$ on $\mathbb{T}$; hence G has no extra zeros in $\overline{\mathbb{D}}$. (For a proof of these results we refer to Theorem 1 and Lemma 5, [7].) Therefore (4.5) implies that

$$q(z) = c_2 z^{\rho_n} \prod_{i=1}^{n-1} (z - \lambda_i)^{\rho_i} \prod_{j=1}^{n-1} (z - A_j), \quad \text{where } A_j \in \mathbb{C} \setminus \overline{\mathbb{D}},\ j = 1, \ldots, n-1.$$

We use (4.2), (4.3), (4.4) and the above to get

$$\begin{aligned} k_\lambda^G(z) &= c_3 \frac{P_M k_\lambda(z)}{P_M \overline{\partial}^{\rho_n} k_0(z) P_M \overline{\partial}^{\rho_n} k_\lambda(0)} \\ &= a \frac{p_2(\overline{\lambda}, z)}{\prod_{j=1}^{n-1} (z - A_j)(\overline{\lambda} - \overline{A}_j)} k_\lambda(z) \quad \text{for some constant } a, \end{aligned}$$

where p_2 is the symmetric polynomial appeared in (4.3) with $\deg p_2 \leqslant n$.

Now recall that the reproducing kernel of M/G which is identically 1 for $\lambda = 0$ and all $z \in \mathbb{D}$, is of the form $k_\lambda^G(z) = (1 - \overline{\lambda} z l_\lambda(z)) k_\lambda(z)$ for some positive definite sesquianalytic kernel $l_\lambda(z)$ defined on $\mathbb{D} \times \mathbb{D}$.

To complete the proof of the claim write

$$p(\overline{\lambda}, z) = \prod_{j=1}^{n-1} (z - A_j)(\overline{\lambda} - \overline{A}_j) - a p_2(\overline{\lambda}, z)$$

and in addition observe that $p(\overline{\lambda}, z)$ is symmetric with $\deg p \leqslant n$.

We also mention that for the case where $0 \notin \Lambda$ the extremal function is $G(z) = c P_M k_0(z)$ for some constant c, and an almost identical argument leads to the proof with $r = n$. Hence, the proof of the claim is complete.

If d denotes the degree of the symmetric polynomial p in the statement of Theorem 4.2, in light of the above claim, it remains to show that $d \geqslant n$. Furthermore, from the proof of the above claim, if l_λ is the associated kernel for M, then

$$\overline{\lambda} z l_\lambda(z) = \frac{p(\overline{\lambda}, z)}{\prod_{j=1}^{r} (z - A_j)(\overline{\lambda} - \overline{A}_j)},$$

where $r = n-1$, if $0 \in \Lambda$ and $r = n$, if $0 \notin \Lambda$. Now note that $p(\overline{\lambda}, z)$ is in addition a positive definite sesquianalytic polynomial of the form $p(\overline{\lambda}, z) = \sum_{n,m=1}^{d} a_{n,m}\overline{\lambda}^n z^m$. Furthermore, a standard linear algebra argument shows that $\{(a_{n,m})\}_{n=1,m=1}^{d}$ is a positive definite matrix, and thus, by using the spectrum theorem for positive definite matrices, it is elementary to show that there are analytic polynomials φ_i, $i = 1, \dots, d$, such that $p(\overline{\lambda}, z) = \sum_{j=1}^{d} \overline{\varphi_j(\lambda)}\varphi_j(z)$. Hence,

$$\overline{\lambda} z l_\lambda(z) = \sum_{j=1}^{d} \frac{\overline{\varphi_j(\lambda)}\varphi_j(z)}{\prod_{\beta=1}^{r} (z - A_\beta)(\overline{\lambda} - \overline{A}_\beta)}.$$

By recalling the definition of the rank of a positive sesquianalytic kernel we immediately get that $\operatorname{rank} l_\lambda(z) \leqslant d$. Now, since $\lambda_i \neq \lambda_j$, $i \neq j$, $i, j \in I$, Theorem 3.4 implies that $\operatorname{rank} l_\lambda(z) = n$ and hence $d \geqslant n$. This concludes the proof of the theorem. ∎

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GEORGE CHAILOS
University of Tennessee
Knoxville, TN 37920
USA

E-mail: chailos@math.utk.edu

Current address

Intercollege
Makedonitissas Ave
1700 Nicosia
CYPRUS

E-mail: chailos.g@intercollege.ac.cy

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