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INJECTIVE REAL FACTORS ARE HYPERFINITE

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ABSTRACT. Following the proof developed by Haagerup in the complex case, it is shown that all injective real factors on separable Hilbert spaces are hyperfinite.

KEYWORDS: *Real factor, injective, hyperfinite, involutory antiautomorphism.*

MSC (2000): Primary 46L35; Secondary 46L10.

The purpose of this brief note is to indicate how the proof given by Haagerup in [8] applies to the real situation, to show that all injective real factors on separable Hilbert spaces are hyperfinite. This contradicts Theorem 2 and Corollary 2 of [9] which claim the opposite.

It was shown in [6], [7] and [11] that there is a unique injective real factor in both the II_1 and II_∞ cases and that this factor is hyperfinite. Thus it remains only to consider the type III case, which is dealt with by slightly modifying the arguments in [8].

Let N be an injective factor with a faithful normal state ψ, let Φ be an involutory $*$-antiautomorphism of N, let $R = \{n \in N : \Phi(n) = n^*\}$ and let $\varphi = \frac{1}{2}(\psi + \psi \circ \Phi)$, which is faithful, normal and Φ-invariant. The GNS Hilbert space H_φ has an antilinear isometry K of order 2, defined by $K[n]_\varphi = [\Phi(n^*)]_\varphi$ for each $n \in N$, such that $\pi_\varphi(\Phi(n)) = K\pi_\varphi(n)^*K$ for each $n \in N$ and such that $s \mapsto Ks^*K$ gives an involutory antiautomorphism Φ' of $\pi_\varphi(N)'$. In the following lemma, which establishes the required version of semidiscreteness, $\Phi_{\mathbb{R}}$ will be used for the transpose map on a matrix algebra M_m.

Lemma 1. *There exist positive integers* $(m_\lambda)_{\lambda\in\Lambda}$ *and nets of completely positive maps* $S_\lambda : N \to M_{m_\lambda}$ *and* $T_\lambda : M_{m_\lambda} \to N$ *such that* $S_\lambda\Phi = \Phi_{\mathbb{R}}S_\lambda$, $\Phi T_\lambda = T_\lambda\Phi_{\mathbb{R}}$, $S_\lambda(1) = 1$, $T_\lambda(1) = 1$ *and, for each* $n \in N$, $(T_\lambda \circ S_\lambda)(n)$ *converges to* n *in the* σ*-strong* topology.*

Proof. Let N act on H_φ and recall that the mapping $n \otimes s \mapsto ns$ extends to a representation of $N \otimes_{\min} N'$ on H_φ. The corresponding state $k : n \otimes s \mapsto \langle ns\xi_\varphi, \xi_\varphi\rangle$ is $\Phi \otimes \Phi'$ invariant and, as in the proof of Lemma 2.2 of [3], is therefore approximated by states $p = \sum c_i\omega_{z_i}$ and by the corresponding states $p \circ (\Phi \otimes \Phi')$. Noting that $\frac{1}{2}(\omega_z + \omega_z \circ (\Phi \otimes \Phi')) = \omega_{(z+(K\otimes K)z)/2} + \omega_{i(z-(K\otimes K)z)/2}$, we can take $(K \otimes K)z_i = z_i$. The vectors $x_1, \ldots, x_m, y_1, \ldots, y_m \in H_\varphi$ considered at the top of page 68 of [3] can similarly be seen to be fixed by K, by noting that

$$(x \otimes y) + (Kx \otimes Ky) = \frac{1}{2}[(x + Kx) \otimes (y + Ky) - \mathrm{i}(x - Kx) \otimes \mathrm{i}(y - Ky)].$$

It follows that

$$\begin{aligned}(\sigma\Phi)(n) &= (\langle\Phi(n)x_j, x_i\rangle) = (\langle Kn^*Kx_j, x_i\rangle) = (\overline{\langle K^2n^*Kx_j, Kx_i\rangle}) \\ &= (\overline{\langle x_j, nx_i\rangle}) = (\langle nx_i, x_j\rangle) = \Phi_{\mathbb{R}}\sigma(n)\end{aligned}$$

(where σ and τ are defined on page 68 of [3]).

The image of τ is easily seen to consist of normal maps, so $\tau : M_m \to N'_*$. The map σ can be modified so that $\sigma(1) = 1$ (and still $\Phi_{\mathbb{R}}\sigma = \sigma\Phi$) by using the proof of Lemma 2.2 of [2] with x a unit vector with $Kx = x$. If b denotes the original value of $\sigma(1)$, which satisfies $\Phi_{\mathbb{R}}(b) = b^* = b$, then modifying τ to $a \mapsto \tau(b^{1/2}ab^{1/2})$ keeps $\tau\circ\sigma$ unchanged. Thus, as in Lemma 2.2 of [3], the map $\mathfrak{L}(k)$ can be approximated in the pointwise weak topology by a net of maps $\tau_\lambda \circ \sigma_\lambda$, where $\sigma_\lambda : N \to M_{m_\lambda}$ and $\tau_\lambda : M_{m_\lambda} \to N'_*$ for suitable integers m_λ, where $\sigma_\lambda(1) = 1$ and $\Phi_{\mathbb{R}}\sigma_\lambda = \sigma_\lambda\Phi$.

The argument outlined on pages 75 and 76 of [3] proceeds to modify τ_λ, using the method of Lemmas 4.3 and 4.4 of [4], so that $\tau_\lambda(1) = p$, where p (denoted by $\mathfrak{L}(k)(1)$ in [3]) satisfies $p(s) = \langle s\xi_\varphi, \xi_\varphi\rangle$ for $s \in N'$. It then uses the complete order isomorphism θ from N onto $[p]$ described on page 70 of [3] to produce completely positive maps $\tau'_\lambda = (\theta^{-1} \circ \tau_\lambda)$ so that the maps $\tau'_\lambda \circ \sigma_\lambda$ tend pointwise σ-weakly to the identity map on N. The convexity argument outlined on page 71 of [3] shows that the convergence is also σ-strong*. If we let $S_\lambda = \sigma_\lambda$ and $T_\lambda = \frac{1}{2}(\tau'_\lambda + \Phi\tau'_\lambda\Phi_{\mathbb{R}})$ then S_λ and T_λ have the required properties since $T_\lambda S_\lambda = \frac{1}{2}(\tau'_\lambda\sigma_\lambda + \Phi\tau'_\lambda\sigma_\lambda\Phi)$. (Note that if $\alpha_{\mathbb{R}} = \Phi_{\mathbb{R}} \circ *$ and $\alpha = \Phi \circ *$, then $\Phi\tau'_\lambda\Phi_{\mathbb{R}} = \alpha\tau'_\lambda\alpha_{\mathbb{R}}$, which is completely positive.) ∎

Lemma 2. *Let* N *be a properly infinite von Neumann algebra, let* Φ *be an involutory* $*$*-antiautomorphism of* N *and let* F *be a* Φ*-invariant finite dimensional subfactor of* N *with* $\{f \in F : \Phi(f) = f^*\}$ *isomorphic to* $M_m(\mathbb{R})$ *for some* m*. If* T *is a completely positive map from* F *to* N *with* $T\Phi = \Phi T$ *then there exists* $a \in N$ *with* $\Phi(a) = a^*$ *such that* $T(x) = a^*xa$ *for each* $x \in F$*.*

Proof. This is exactly the same as the proof of Proposition 2.1 of [8] if the initial system $(e_{ij})_{i,j=1,\ldots,m}$ of matrix units for F is chosen to satisfy $\Phi(e_{ij}) = e_{ji}$

for each i, j. Then, in turn,

$$(\Phi \otimes \Phi)\Big(\sum T(e_{ij}) \otimes e_{ij}\Big) = \Big(\sum T(e_{ij}) \otimes e_{ij}\Big)^*,$$
$$(\Phi \otimes \Phi)\Big(\sum b_{ij} \otimes e_{ij}\Big) = \Big(\sum b_{ij} \otimes e_{ij}\Big)^*$$

and

$$\Phi(a_{kl}) = a_{kl}^*,$$

where $b = \sum b_{ij} \otimes e_{ij}$ is the positive square root of $\sum T(e_{ij}) \otimes e_{ij}$ and $a_{kl} = \sum e_{il} b_{ki}$. By choosing isometries $v_{11}, \ldots, v_{mm}$ in $N \cap F'$ to satisfy $\Phi(v_{ij}) = v_{ij}^*$ it follows that

$$T(x) = \Big(\sum v_{ij} a_{ij}\Big)^* x \Big(\sum v_{ij} a_{ij}\Big)$$

with $\Phi\Big(\sum v_{ij} a_{ij}\Big) = \Big(\sum v_{ij} a_{ij}\Big)^*$. ∎

THEOREM 3. *Let N be an injective factor of type* III *on a separable Hilbert space, let Φ be an involutory $*$-antiautomorphism of N and let $R = \{n \in N : \Phi(n) = n^*\}$. Then R is hyperfinite.*

Proof. The proof of Theorem 2.2 of [8] applies, subject to suitable Φ-invariance, so the notation of that proof is maintained here. For any m the properly infinite real factor R contains a subfactor F_R isomorphic to $M_m(\mathbb{R})$ and hence N contains a Φ-invariant subfactor F isomorphic to M_m. Using Lemma 1, the maps S and T can be chosen to commute with Φ and, by Lemma 2, the isometry v can be taken to satisfy $\Phi(v) = v^*$. The unitary w which is strongly close to v can also be taken to satisfy $\Phi(w) = w^*$: to see this use, for example, the construction described in Lemma 1 of [1] and the spatial description of Φ obtained in Theorem 3.7 of [10]. Thus if $\Phi(u_k) = u_k^*$ then, with $y_k = w^* S(u_k) w$, $\Phi(y_k) = y_k^*$. Unlike the proof of Theorem 2.2 of [8], u_k itself cannot be taken to be unitary. However, it is a complex linear combination $\sum_i \lambda_{k,i} u_{k,i}$ of unitaries $u_{k,i}$ for each of which $y_{k,i} = w^* S(u_{k,i}) w \in w^* F w$ and $\|y_{k,i} - u_{k,i}\|_\varphi^\# < \frac{\varepsilon}{M}$, where $M = \sum |\lambda_{k,i}|$. Thus the corresponding linear combination $y_k = w^* S(u_k) w$ belongs to $w^* F_R w \cong M_m(\mathbb{R})$ with $\|y_k - u_k\|_\varphi^\# < \varepsilon$. The argument in Theorem 3 of [5] shows that this is sufficient to guarantee the existence of a dense union of increasing real matrix algebras in R. (The unitary u of Lemma 1 of [5] can be chosen with $\Phi(u) = u^*$, the matrix units e_{pq} in the proof of Theorem 3 can be chosen to satisfy $\Phi(e_{pq}) = e_{qp}$, the projections ε_{pp} for $1 \leqslant p \leqslant 2^k$ can be chosen with $\Phi(\varepsilon_{pp}) = \varepsilon_{pp}$ and hence the matrix units f_{pq} satisfy $\Phi(f_{pq}) = f_{qp}$; this shows that the real subfactor $w^* F_R w$ above can be taken to have size 2^k for some k and then the rest of the proof of Theorem 3 of [5] applies to the real setting to produce the required increasing sequence of subfactors.) ∎

It remains to reconcile the theorem above with Theorem 2 of [9] which claims that there is a unique conjugacy class of involutory antiautomorphisms of the hyperfinite factor M of type III_λ, for $0 < \lambda < 1$, for which the associated real algebra is hyperfinite. Since there are known to be two non conjugate involutory antiautomorphisms of the hyperfinite function of type III_λ when $0 < \lambda < 1$, that theorem implies that there is a real injective factor which is not hyperfinite. It is implicitly claimed in the proof that an isomorphism between two weakly dense unions of matrix subalgebras extends to an automorphism of M, which is not clearly the case (unlike the situation of Corollary 2.10 of [11], on which the proof of Theorem 2 of [9] is modelled).

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