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ON UNBOUNDED OPERATORS AFFILIATED WITH C^*-ALGEBRAS

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ABSTRACT. We show that the multipliers of Pedersen's ideal of a C^*-algebra A correspond to the densely defined operators on A which are affiliated with A, in the sense defined by Woronowicz, and whose domains contain Pedersen's ideal. We also extend the theory of q-continuity developed by Akemann to unbounded operators and show that these operators correspond to self-adjoint operators affiliated with A.

KEYWORDS: *Unbounded operators, multiplier algebras, Pedersen ideal, q-topology.*

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1. INTRODUCTION

One of the principal philosophies in the the study of C^*-algebras is that they are the non-commutative analogues of $C_0(X)$, the algebra of continuous functions on a topological space X which vanish at infinity. The multiplier algebra $M(A)$ of a C^*-algebra A, from this point of view, corresponds to the algebra $C_{\mathrm{b}}(X)$ of bounded functions on the same topological space.

This point of view is perhaps most clearly revealed in the works of Akemann([1], [2], [3]) and Giles and Kummer ([6]) on q-topologies, in which certain projections in the enveloping von Neumann algebra play a role analogous to that of open sets in a topology. Akemann, Pedersen and Tomiyama ([5]) showed that the analogue of real-valued continuous functions in this setting are precisely the self-adjoint elements of $M(A)$, and the ones which "vanish at infinity" correspond to the self-adjoint elements of A.

An obvious question, then, is what is the non-commutative analogue of the algebra $C(X)$ of continuous functions on a topological space? The corresponding question for von Neumann algebras is fairly well understood. The non-commutative analogue of unbounded measurable functions are the closed, unbounded operators affiliated with a von Neumann algebra.

If A is unital then this corresponds to the situation where X is compact, and so we would expect that the analogue of $C(X)$ would be A itself, so the real interest lies in non-unital C^*-algebras.

In the non-unital case, the problem can be approached by looking at appropriate spaces of multipliers. In the classical setting, $C(X)$ is the multiplier algebra of $C_c(X)$, the continuous functions of compact support on X. $C_c(X)$ is the minimal dense ideal of $C(X)$, and in the C^*-algebra setting this minimal dense ideal is called Pedersen's ideal of A. One can then study the multiplier algebra of this ideal as an analogue of $C(X)$, as was done by Lazar and Taylor ([8]). Phillips ([11]) showed that this multiplier algebra was an example of a pro-C^*-algebra, and was able to use that more general theory to obtain many results.

A second approach developed by Woronowicz ([13]) uses a particular transform which maps certain unbounded operators to bounded ones, where those which are mapped to the analogues of the bounded continuous functions must be the analogues of continuous functions. This method has been applied with some success to the study of quantum groups. Lance ([7]) showed that Woronowicz' approach has a natural expression in terms of unbounded operators on Hilbert C^*-modules.

A third approach, which we introduce in this paper, is to generalize the ideas of Akemann to unbounded operators. We define q-continuity for self-adjoint operators affiliated with an enveloping von Neumann algebra in a way that is precisely analogous to the definition given by Akemann for elements of the von Neumann algebra itself.

We will show that all three approaches are closely related. In particular, the multiplier algebra of the Pedersen ideal is exactly the set of operators from Woronowicz' approach whose domain includes the Pedersen ideal; and the operators from Woronowicz approach and the third approach are essentially the same.

2. C^*-ALGEBRAIC AFFILIATION

In [13] and [14] Woronowicz with Napiórkowiski considered a class of operators on a C^*-algebra A, defined as follows:

Definition 2.1. Let T be a densely defined linear operator on a C^*-algebra A with domain $\mathrm{D}(T)$, and let $M(A)$ be the multiplier algebra of A. T is *(C^*-algebraically) affiliated* with A, written $T \,\overline{\in}\, A$, if there exists a $z_T \in M(A)$ with $\|z_T\| \leqslant 1$ and

$$x \in \mathrm{D}(T) \iff \exists\, a \in A \text{ such that } x = (1 - z_T^* z_T)^{1/2} a \text{ and } Tx = z_T a.$$

z_T is called the *z-transform* of T.

Woronowicz showed that T is uniquely determined by the z-transform and, conversely, given any $z \in M(A)$, with $\|z\| \leqslant 1$ and $(1 - z^* z)^{1/2} A$ dense in A, there is an unbounded operator T on A with $z = z_T$.

The motivation behind this definition is that if we have a map $z : \mathbb{C} \to \mathbb{D}$ which is a homeomorphism onto its image, then $C(X)$ can be embedded (albeit, not $*$-homomorphically) into $C_b(X)$ by composing a function in $C(X)$ with z. Woronowicz chose the function

$$z(\xi) = \xi(1 + \overline{\xi}\xi)^{-1/2}$$

which has inverse

$$f(\xi) = \xi(1 - \overline{\xi}\xi)^{-1/2}.$$

The definition above essentially says that $Tx = f(z_T)x$, allowing for possible difficulties with the functional calculus.

Woronowicz was then able to prove the following facts:

THEOREM 2.2. *Let $T \,\overline{\in}\, A$. Then:*

(i) *T is a closed operator;*

(ii) $\mathrm{D}(T)$ *is a right ideal;*

(iii) *there is an adjoint T^* defined by $z_{T^*} = z_T^*$;*

(iv) *if B is another C^*-algebra and $\Phi : A \to M(B)$ is an essential $*$-homomorphism (i.e. Φ is a C^*-algebra morphism), then there is a $\Phi(T) \,\overline{\in}\, B$ where $z_{\Phi(T)} = \Phi(z_T)$;*

(v) *if $T = T^*$, let $\operatorname{sp} T$ be defined to be $f(\operatorname{sp} z_T)$ and $\zeta(\xi) = \xi$ for $\xi \in \operatorname{sp} T$. Then there is an essential $*$-homomorphism Φ from $C_0(\operatorname{sp} T)$ to $M(A)$, such that $\Phi(\zeta) = T$.*

The functional calculus given above completely justifies the motivation: z_T is precisely equal to $z(T)$, and $T = f(z_T)$.

Woronowicz was able to apply this theory to discover unbounded operators which corresponded to generating elements for certain purely algebraic quantum groups.

It should be noted, however, that the set of operators which are affiliated with a non-commutative C^*-algebra have no algebraic structure, in the same way that the collection of closed densely-defined operators on a Hilbert space have no algebraic structure.

3. THE MULTIPLIER ALGEBRA OF PEDERSEN'S IDEAL

Pedersen's ideal K_A of a C^*-algebra A is the minimal dense (two-sided) ideal of A (see [9] and [10]). Lazar and Taylor ([8]) investigated the multiplier algebra $\Gamma(K_A)$ of this ideal, and proved a number of significant theorems about it. Phillips ([11]) was able to simplify their work by observing that this algebra is in fact a pro-C^*-algebra — an inverse limit of C^*-algebras — and then using general theorems from that theory (see [12]) to reach many of the same results as Lazar and Taylor.

Phillips showed that $\Gamma(K_A)$ was the inverse limit

$$\Gamma(K_A) = \varprojlim_{x \in (K_A)_+} M_x$$

where $(K_A)_+$ is ordered by the C^*-algebra order, and M_x is the C^*-algebra of multipliers (S,T) where $S : \overline{Ax} \to \overline{Ax}$ and $T : \overline{xA} \to \overline{xA}$ are linear and $aT(b) = S(a)b$ for all $a \in \overline{xA}$ and $b \in \overline{Ax}$. Thus any element a of $\Gamma(K_A)$ can be represented by a coherent sequence $(a_x)_{x \in (K_A)_+}$ where $a_x \in M_x$.

Theorem 3.1. *Let A be a C^*-algebra, K_A its Pedersen's ideal and $\Gamma(K_A)$ the multiplier algebra of K_A. For any $a \in \Gamma(K_A)$ there is some $T_a \,\overline{\in}\, A$ with $K_A \subseteq \mathrm{D}(T_a)$ and $Tx = ax$ for all $x \in \mathrm{D}(T_a)$.*

Conversely, if $T \,\overline{\in}\, A$ and $K_A \subseteq \mathrm{D}(T)$, then there is an element $a \in \Gamma(K_A)$ such that $Tx = ax$ for all $x \in \mathrm{D}(T)$.

Proof. Given any $a \in \Gamma(K_A)$, the functional calculus on $\Gamma(K_A)$ tells us that there is some $z \in \Gamma(K_A)$ with

$$z = a(1 + a^*a)^{-1/2}.$$

In fact, $z_x = a_x(1 + a_x^*a_x)^{-1/2}$ for all $x \in (K_A)_+$, and since $\|z_x\| \leqslant 1$ for all $x \in (K_A)_+$, we have $\|z\| = \sup\limits_{x \in (K_A)_+} \|z_x\| \leqslant 1$. Hence $z \in M(A)$.

Let $x \in (I - z^*z)^{1/2}A$, so that there is some $b \in A$ with $x = (I - z^*z)^{1/2}b$. For such x we note that $ax \in A$, since

$$ax = a(I - z^*z)^{1/2}b = a(I - a^*a(I + a^*a)^{-1})^{1/2}b = a(I + a^*a)^{-1/2}b = zb \in A.$$

Furthermore the above calculation shows that the operator

$$T_a x = ax$$

defined on $\mathrm{D}(T_a) = (I - z^*z)^{1/2}A$ has z-transform z. So, if $x \in K_A$, then

$$(I - z^*z)^{-1/2}x = (I - a^*a(I + a^*a)^{-1})^{-1/2}x = (I + a^*a)^{1/2}x \in K_A$$

since, using the functional calculus, $(I + a^*a)^{1/2} \in \Gamma(K_A)$. Hence $K_A \subseteq \mathrm{D}(T_a)$ and T is densely defined, and we conclude that T_a is C^*-affiliated with A.

If $T \,\overline{\in}\, A$ and $K_A \subseteq \mathrm{D}(T)$, let $z_T \in M(A)$ be its z-transform. For each x, $y \in K_A$, y and $x^* \in \mathrm{D}(T)$, so $y = (I - z_T^*z_T)^{1/2}b$ for some $b \in A$ and $x^* = (I - z_T^*z_T)^{1/2}c$, and hence $x = c(I - z_T^*z_T)^{1/2}$, for some $c \in A$. Define a multiplier (S, T) on K_A using multiplication by $(I - z_T^*z_T)^{-1/2}$:

$$T(y) = (I - z_T^*z_T)^{-1/2}y = b$$

and

$$S(x) = x(I - z_T^*z_T)^{-1/2} = c$$

and thus

$$S(x)y = cy = c(I - z_T^*z_T)^{1/2}b = xb = xT(y).$$

Hence $(I - z_T^*z_T)^{-1/2} \in \Gamma(K_A)$, and as a consequence, so is $a = z_T(I - z_T^*z_T)^{-1/2}$.

Now for any $x \in \mathrm{D}(T)$, $x = (I - z_T^*z_T)^{1/2}b$ for some $b \in A$, and thus

$$ax = z_T(I - z_T^*z_T)^{-1/2}x = z_Tb = Tx,$$

and so we have found the required element $a \in \Gamma(K_A)$. ∎

Although these operators from the multiplier algebra of Pedersen's ideal do not give all the operators affiliated with the C^*-algebra, they do have the advantage over the more general setting in that they actually form a $*$-algebra.

4. q-TOPOLOGIES AND UNBOUNDED OPERATORS

Akemann ([1], [2], [3]) and, independently, Giles and Kummer ([6]) introduced the idea of a q-topology.

Definition 4.1. Let A be a C^*-algebra and A^{**} its enveloping von Neumann algebra. We say that a projection $p \in A^{**}$ is *q-open* if $A^{**}p \cap A$ is a closed left ideal of A. A projection $p \in A^{**}$ is *q-closed* if $1 - p$ is q-open, and $p \in A^{**}$ is *q-compact* if there is some $a \in A$ with $ap = p$.

Equivalently, $p \in A^{**}$ is closed if the set of quasi-states supported on p,

$$F(p) = \{\varphi \in Q(A) : \varphi(1 - p) = 0\},$$

is weak-$*$ closed in the quasistates $Q(A)$ of A.

Projections which are q-open behave like characteristic functions of open sets, in that given a family of q-open projections p_α indexed by α in I, $\bigvee_{\alpha \in I} p_\alpha$ is also q-open. Unlike characteristic functions, however, given two q-open projections, p_1 and p_2, $p_1 \wedge p_2$ may not be q-open. Despite this defect, q-open projections are sufficiently analogous to a topology that Akemann and Eilers ([4]) constructed a non-commutative end theory in this setting.

By analogy with the classical setting, we can make the following definitions:

Definition 4.2. Let A be a C^*-algebra. We say that a self-adjoint element $a \in A^{**}_{\mathrm{sa}}$ is *q-continuous* if spectral projections of relatively open sets in $\operatorname{sp} T$ are q-open in A^{**}. We say that a q-continuous element *vanishes at infinity* if the spectral projections of closed sets which do not contain the identity are q-compact.

This definition heavily relies on the spectral properties of self-adjoint elements, and so does not extend to a definition for general elements. Nevertheless, these q-continuous elements are precisely what they should be, as shown by the following result of Akemann, Pedersen and Tomiyama ([5], Theorem 2.2).

Theorem 4.3. *Let A be a C^*-algebra. $a \in A^{**}_{\mathrm{sa}}$ is q-continuous if and only if $a \in M(A)_{\mathrm{sa}} \subseteq A^{**}$.*

Given the success of this approach in the bounded setting, and the well-established nature of the unbounded theory for von Neumann algebras, we can reasonably ask if this theory can be extended to unbounded operators.

Let M be a von Neumann algebra in $B(H)$. Recall that a closed, self-adjoint, densely defined operator T on H is affiliated with M if every spectral projection of T lies in M. In this situation, we write $T \,\eta\, M$. Recall also that there is a Borel functional calculus for self-adjoint operators affiliated with M.

Definition 4.4. Let A be a C^*-algebra and A^{**} its enveloping von Neumann algebra. We say that a self-adjoint operator $T \,\eta\, A^{**}$ is *q-continuous* if the spectral projections of relatively open sets in $\operatorname{sp} T$ are q-open in A^{**}.

Given this definition, we see immediately that we have a functional calculus.

PROPOSITION 4.5. *Let A be a C^*-algebra. If $T \,\eta\, A^{**}$ is q-continuous and $g : \operatorname{sp} T \to \mathbb{R}$ is continuous, then $g(T) \,\eta\, A^{**}$ is q-continuous. Moreover, if g is bounded, then $g(T) \in M(A)$ with $\|g(T)\| \leqslant \|g\|$.*

Proof. First note that $g(T)$ is defined by the functional calculus for operators affiliated with a von Neumann algebra, since g is Borel.

Let X be a relatively open subset of $\operatorname{sp} g(T) = g(\operatorname{sp} T)$. Then $g^{-1}(X)$ is relatively open in $\operatorname{sp} T$. Therefore $\chi_{g^{-1}(X)}(T)$ is q-open, but $\chi_{g^{-1}(X)}(T) = \chi_X(g(T))$. Hence $g(T)$ is q-continuous.

If g is bounded, then from the Borel functional calculus, $\|g(T)\| \leqslant \|g\|$ and $g(T) \in A^{**}_{\mathrm{sa}}$. But then by Theorem 4.3, $g(T) \in M(A)$. ∎

And so we can now show:

THEOREM 4.6. *Let A be a C^*-algebra. Let T be a self-adjoint operator $T \,\overline{\in}\, A$, then there is a unique q-continuous operator $T' \,\eta\, A^{**}$ with $Tx = T'x$ for all $x \in \mathrm{D}(T)$. Conversely, if $T' \,\eta\, A^{**}$ is q-continuous, then there is a unique self-adjoint operator $T \,\overline{\in}\, A$ with $Tx = T'x$ for all $x \in \mathrm{D}(T)$.*

Proof. We can represent A^{**} in $B(H_{\mathrm{u}})$ where $\pi_{\mathrm{u}} : A \to B(H_{\mathrm{u}})$ is the universal representation of A.

Let $T \,\overline{\in}\, A$ be self-adjoint, and let z_T be its z-transform. $z_T \in M(A)$, so z_T is q-continuous. Moreover z_T is self-adjoint and $1 \notin \operatorname{sp} z_T$. Let

$$f(\xi) = \xi(1 - \overline{\xi}\xi)^{-1/2},$$

which is continuous for $|\xi| < 1$, so we have that $T' = f(z_T)$ is q-continuous and affiliated with A^{**}, and unique.

If $x \in \mathrm{D}(T)$, then for any $\xi \in H_{\mathrm{u}}$,

$$Tx\xi = z_T a\xi = z_T(1 - z_T^* z_T)^{-1/2} x\xi = T'x\xi,$$

so $Tx = T'x$.

Conversely, let $T' \,\eta\, A^{**}$ be q-continuous. Let

$$z(\xi) = \xi(1 + \overline{\xi}\xi)^{-1/2},$$

which is continuous on $\mathbb{C}$. By Proposition 4.5, $z_T = z(T)$ is q-continuous, $\|z_T\| \leqslant 1$, and $z_T \in M(A)$. Moreover, $1 \neq \operatorname{sp} z_T = z(\operatorname{sp} T)$, so by the functional calculus for C^*-algebraically affiliated operators, there is a unique $T = f(z_T) \,\overline{\in}\, A$.

Once again, if $x \in \mathrm{D}(T)$, then for any $\xi \in H$,

$$Tx\xi = z_T a\xi = z_T(1 - z_T^* z_T)^{-1/2} x\xi = T'x\xi,$$

so $Tx = T'x$. ∎

This analysis has so far dealt only with self-adjoint operators. If $T \mathrel{\overline{\in}} A$, then z_T has the polar decomposition in A^{**}, namely $z_T = v|z_T|$, where $|z_T| = (z_T^* z_T)^{1/2} \in M(A)_{\mathrm{sa}}$ and v is a partial isometry in A^{**}. Then for any $x \in \mathrm{D}(T)$, we have

$$Tx = v|z_T|(1 - |z_T|^2)^{1/2}x.$$

In other words, there is an operator $|T| = f(|z_T|) \mathrel{\overline{\in}} A$, with $z_{|T|} = |z_T|$, $\mathrm{D}(|T|) = \mathrm{D}(T)$, and $T = v|T|$. Theorem 4.6 then tells us that we can find a self-adjoint, q-continuous operator $|T|' \,\eta\, A^{**}$ which agrees with $|T|$ on $\mathrm{D}(T)$, and so for all $x \in \mathrm{D}(T)$,

$$Tx = v|T|'x.$$

The operator $T' = v|T|'$ is affiliated with A^{**}.

We can also recover T from T', since the uniqueness of polar decomposition of operators affiliated with A^{**} tells us that given T', $|T|' = |T'| = ((T')^*T)^{1/2}$ and v is the unique partial isometry such that $T' = v|T'|$. Theorem 4.6 then tells us how to find $|T|$ from $|T|'$, from which we have $T = v|T|$.

Thus, even in the non-selfadjoint case, q-continuity and C^*-algebraic affiliation are closely linked.

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