J. OPERATOR THEORY
51(2004), 255–279

CROSSED PRODUCTS OF C^*-ALGEBRAS BY GROUPOIDS AND INVERSE SEMIGROUPS

MAHMOOD KHOSHKAM and GEORGES SKANDALIS

Communicated by Şerban Strătilă

Abstract. We develop a new notion of action of inverse semigroups on C^*-algebras. The full and the reduced crossed product of a C^*-algebra by an inverse semigroup are developed. This construction unifies several notions of crossed product by inverse semigroups. Moreover, the relation between the crossed product by an inverse semigroup S and by its associated groupoid is investigated.

Keywords: *C^*-algebras, groupoids, inverse semigroups, crossed products, covariant representations, Hilbert modules.*

MSC (2000): 46L55, 46L08.

1. INTRODUCTION

This paper is, in some way, a continuation of our previous work ([9]), where we studied the regular representation of groupoids and used it for applications to inverse semigroups.

In the present paper, we include actions into our setting. We develop a new notion of action of an inverse semigroup on a C^*-algebra and construct associated full and reduced crossed products. This includes various forms of crossed products by inverse semigroups as special cases. We also investigate the relation between the crossed product of a C^*-algebra by an inverse semigroup, and the crossed product by the groupoid associated by A. Paterson (cf. [15]) with the inverse semigroup.

Actions of inverse semigroups on a C^*-algebra A and associated constructions of crossed products appear in the literature in several forms, including the work of Nica in terms of localizations ([14]), partial actions of discrete groups of McClanahan ([13]) and the construction of Sieben ([19]). The later two are based on the notion of *partial automorphisms* (i.e. isomorphisms between ideals of A) due to Exel ([6]).

If one wants to include the semi-direct product of inverse semigroups, a notion of crossed product starting with a homomorphism $S \to \mathrm{End}(A)$ needs to be developed. In that case, isomorphisms of *quotients* of A rather than *ideals* are involved. We define a notion of action of inverse semigroups on C^*-algebras for which both notions of partial automorphisms and endomorphisms are included as special cases. Our construction is based on *subquotients.* We introduce a set $\mathrm{EQ}(A)$, consisting of quotients J/I, where $I \subset J$ are ideals of A. The triples (B, α, C), where $A, B \in \mathrm{EQ}(A)$, and $\alpha : C \to B$ is a $*$-isomorphism, are then shown to form an inverse semigroup $\mathrm{SQ}(A)$. An action of an inverse semigroup S on A is now defined to be a semigroup homomorphism $S \to \mathrm{SQ}(A)$. Since ideals are subquotients, an action by partial automorphisms is a particular case of our definition.

We define the *full crossed product* in our case as being the enveloping C^*-algebra $A \rtimes S$ of a natural convolution algebra. We then compute the representations of this crossed product: we establish a one to one correspondence between the representations of $A \rtimes S$ and naturally defined *covariant representations* of the pair (S, A). In particular, we introduce a family $(L^e, \lambda^e)_{e \in E}$ of covariant representations and define the *reduced crossed product* $A \rtimes_{\mathrm{r}} S$ to be the quotient of $A \rtimes S$ under this family of representations. To the best of our knowledege, such a reduced crossed product by an inverse semigroup was not defined before.

In the case of an action by partial automorphisms, our construction differs from the one of McClanahan or Sieben ([13], [19]). However, our construction is related to Sieben's in a more intricate way: It turns out that if an inverse semigroup S with idempotent set E acts on a C^*-algebra A, then the crossed product $A \rtimes E$ is endowed with a new action of S by partial automorphisms. We establish a $*$-isomorphism between the crossed product in the sense of Sieben of $A \rtimes E$ by this new action, and the full crossed product $A \rtimes S$ (in our sense).

A natural correspondence between groupoids and inverse semigroups was constructed by J. Renault ([17]), who associated to each r-discrete groupoid G the inverse semigroup of open G-sets, and compared the representations of these objects. In [16], J. Quigg and N. Sieben establish a correspondence between actions of an r-discrete groupoid G on a C^*-algebra and actions by partial automorphisms of the associated inverse semigroup; they prove that the resulting (full) crossed products by the inverse semigroup (in the sense of [19]) and by the groupoid G are naturally isomorphic.

In the reverse direction, A. Paterson associates to each inverse semigroup S a locally compact r-discrete groupoid G_S, such that the corresponding groupoid and inverse semigroup C^*-algebras (both full and reduced) are $*$-isomorphic (cf. [15]; see also [9]). We investigate here the connection between crossed products by S and G_S: if the action of S is by isomorphisms of quotients, then G_S acts on $A \rtimes E$ (where $E \subset S$ is the set of idempotents of S). Thus, using the results of [16], we deduce an isomorphism $(A \rtimes E) \rtimes G_S \cong A \rtimes S$. Moreover, we establish a natural isomorphism $(A \rtimes E) \rtimes_{\mathrm{r}} G_S \cong A \rtimes_{\mathrm{r}} S$.

Here is a summary of the paper.

In Section 2, we collect basic facts and notation about groupoids, inverse semigroups and associated C^*-algebras.

In Section 3, we review groupoid actions and crossed products of C^*-algebras by actions of groupoids. In particular, we introduce the regular covariant representation (in a suitable Hilbert module) and reduced crossed products.

In Section 4, we study the inverse semigroup $\mathrm{SQ}(A)$ of subquotients of a C^*-algebra A.

The action of an inverse semigroup S is defined to be a semigroup homomorphism α from S into $\mathrm{SQ}(A)$. The corresponding *full* and *reduced crossed products* of A by the action α of S are defined in Section 5, where we also compute the representations of the full crossed product. We end this section by computing these crossed products in the case that $S(= E)$ consists of idempotents.

In Section 6 we establish the above mentionned connection between crossed products in our sense and the construction of Sieben; in the case of quotients, using this connection together with the work of Quigg and Sieben, we relate the crossed product by an inverse semigroup S with a crossed product with the groupoid G_S.

2. PRELIMINARIES

In this section we collect definitions, results, facts, and conventions to be used in subsequent sections.

2.1. Throughout the paper the word "ideal" will mean "closed two sided ideal", unless otherwise indicated. We denote by $\mathcal{M}(A)$ the multiplier algebra of a C^*-algebra A.

2.2. INVERSE SEMIGROUPS. We refer to [7], [5], [15], [11] for the basic definitions and properties of inverse semigroups and associated C^*-algebras.

A semigroup S is said to be an *inverse semigroup* if for each $u \in S$ there exists a unique element $u^* \in S$ such that $uu^*u = u$ and $u^*uu^* = u^*$. The set of idempotens of S, to be denoted by E_S (or simply E) is a commutative subsemigroup of S. It is a partially ordered set and a semilattice under the relation: $e \leqslant f$ if $ef = e$, and $e \wedge f = ef$. The partial ordering on S is given by $u \leqslant v$ if $u = ve$ for some $e \in E$.

The normed space $\ell^1(S)$ endowed with the operations

$$(f \star g)(w) = \sum_{uv=w} f(u)g(v), \quad f^*(u) = \overline{f(u^*)}$$

is a Banach $*$-algebra. The *full C^*-algebra* of S is the enveloping C^*-algebra of $\ell^1(S)$. It is denoted by $C^*(S)$.

The *reduced C^*-algebra* of S, denoted by $C^*_{\mathrm{r}}(S)$, is the image of $C^*(S)$ under the left regular representation $u \mapsto \lambda_u$ of S on $\ell^2(S)$ defined by

$$(\lambda_u \xi)(v) = \begin{cases} \xi(u^*v) & \text{if } vv^* \leqslant uu^*, \\ 0 & \text{otherwise.} \end{cases}$$

2.3. GROUPOIDS. We refer to [17], [2], [3], [4], [12], [1] and [9] for definitions and main properties of groupoids and associated C^*-algebras. Here we recall some notation. Let G be a groupoid, then:

– $G^{(0)}$ will denote its space of units;

– $s : G \to G^{(0)}$ and $r : G \to G^{(0)}$ denote respectively the source and range maps;

– $G^{(2)}$ denotes the set $\{(\gamma,\gamma') \in G \times G : s(\gamma) = r(\gamma')\}$ of composable elements;

– given $x \subset G^{(0)}$, we set $G_x = \{\gamma \in G : s(\gamma) = x\}$ and $G^x = \{\gamma \in G : r(\gamma) = x\}$.

A *locally compact groupoid* is a groupoid G endowed with a topology such that:

(a) the groupoid operations (composition, inversion, source and range maps) are continuous;

(b) the space of units $G^{(0)}$ is Hausdorff;

(c) each point of G has a compact (Hausdorff) neighborhood;

(d) the range and source maps are open (cf. [9], Definition 1.1).

2.4. GROUPOID C^*-ALGEBRAS. Let G be a locally compact groupoid. The full C^*-algebra of G is the enveloping C^*-algebra of (the completion of) a normed $*$-algebra $\mathcal{A}$. If G is Hausdorff, $\mathcal{A}$ is $C_c(G)$ the space of continuous complex valued functions with compact support on G.

If G is not Hausdorff the above definition must be modified. Following Connes (cf. [3], [4]), $\mathcal{A}$ is defined to be the space of complex valued functions on G spanned by functions which are continuous with compact support on an open Hausdorff set of G extended by 0 elsewhere. Note that such a function is generally *not continuous* on G.

In order to turn $\mathcal{A}$ into a normed algebra, we need a *Haar system* on G (cf. [17]), i.e. a collection $\nu = \{\nu_x\}_{x\in G^{(0)}}$ of positive regular Borel measures on G satisfying the following conditions:

(a) *Support:* for every $x \in G^{(0)}$, the support of ν_x is contained in G_x;

(b) *Invariance:* for all $\gamma_1 \in G$ and $f \in \mathcal{A}$, $\int f(\gamma\gamma_1)\,\mathrm{d}\nu_x(\gamma) = \int f(\gamma)\,\mathrm{d}\nu_y(\gamma)$, where $x = s(\gamma_1)$ and $y = r(\gamma_1)$;

(c) *Continuity:* for each $f \in \mathcal{A}$ the map $x \mapsto \int\limits_{G_x} f(\gamma)\,\mathrm{d}\nu_x(\gamma)$ is continuous.

For $x \in X$, we also note ν^x the measure on G defined by $\nu^x(f) = \nu_x(\widetilde{f})$, where $\widetilde{f}$ is the function $\gamma \mapsto f(\gamma^{-1})$.

For $f, g \in \mathcal{A}$, put

$$f^*(\gamma) = \overline{f(\gamma^{-1})} \quad \text{and} \quad (f \star g)(\gamma) = \int\limits_{G_x} f(\gamma\gamma_1^{-1})g(\gamma_1)\,\mathrm{d}\nu_x(\gamma_1),$$

where $x = s(\gamma)$. The norm on $\mathcal{A}$ is defined by

$$\|f\|_1 = \sup_{x\in G^{(0)}} \left\{ \max\Big(\int\limits_{G_x} |f(\gamma)|\,\mathrm{d}\nu_x(\gamma), \int\limits_{G_x} |f(\gamma^{-1})|\,\mathrm{d}\nu_x(\gamma) \Big) \right\}.$$

The *full groupoid C^*-algebra* $C^*(G,\nu)$ (or $C^*(G)$ when there is no ambiguity on the Haar system) is defined to be the enveloping C^*-algebra of the Banach $*$-algebra obtained by completion of $\mathcal{A}$ with respect to the norm $\|\cdot\|_1$.

2.5. REGULAR REPRESENATION OF GROUPOIDS. Recall some constructions from [9].

If G is a Hausdorff groupoid, then define a $C_0(X)$-valued scalar product on $\mathcal{A} = C_c(G)$ by letting $\langle \xi, \eta \rangle$ denote the restriction to X of $\xi^* \star \eta \in \mathcal{A}$; let the right action of $C_0(X)$ on $\mathcal{A}$ be given by $\xi f(\gamma) = \xi(\gamma) f(s(\gamma))$ (for $f \in C_0(X)$ and $\xi \in \mathcal{A}$). With these operations $\mathcal{A}$ is a pre-Hilbert $C_0(X)$-module. Let $L^2(G, \nu)$ be its Hilbert module completion. The formula $\lambda(f)\xi = f \star \xi$, where $f, \xi \in \mathcal{A}$, extends to a representation λ of $C^*(G)$ on $L^2(G, \nu)$ whose image is ($*$-isomorphic to) the reduced C^*-algebra of G, denoted by $C^*_r(G)$ (cf. [9], Theorem 2.3).

If G is not Hausdorff, this construction needs to be modified. One replaces X by the spectrum Y of the C^*-algebra $\mathcal{B}$ of Borel functions on X generated by restrictions to X of elements of $\mathcal{A}$. Since $\mathcal{B}$ contains the continuous functions on X vanishing at ∞, there exists a continuous map $p : Y \to X$ which is proper and onto (cf. [9], Proposition 2.6). The analogue of $L^2(G, \nu)$ is constructed as follows.

Given $\xi, \eta \in \mathcal{A}$, let $\langle \xi, \eta \rangle = (\xi^* \star \eta)|_X \in C_0(Y)$. The linear space $\mathcal{A} \otimes C_0(Y)$ is turned into a pre-Hilbert $C_0(Y)$-module and its (Hausdorff-)completion $\mathcal{E}$ is a Hilbert $C_0(Y)$-module (see 2.7 in [9]). The algebra $C^*(G)$ acts on $\mathcal{E}$ by $\lambda(f)(\xi \otimes g) = (f \star \xi) \otimes g$ for all $f, \xi \in \mathcal{A}$ and $g \in C_0(Y)$. The image is again $*$-isomorphic to $C^*_r(G)$ (cf. [9], Theorem 2.10).

2.6. THE GROUPOID ASSOCIATED WITH AN INVERSE SEMIGROUP. To each inverse semigroup S is associated a locally compact groupoid G_S (cf. [15]; see also [9]). To construct this groupoid, one first considers the spectrum X of the commutative C^*-algebra $C^*(E)$. An element e of E_S is then a $0,1$-valued continuous function on X, i.e. the characteristic function of a compact open subset F_e of X. Then G_S is the quotient of $\{(u, x) \in S \times X : x \in F_{u^*u}\}$ by the equivalence relation given by $(u, x) \sim (v, y)$ whenever $x = y$ and there exists $e \in E$ such that $x \in F_e$ and $ue = ve$.

Here are some important facts about G_S (cf. [15]; see [9], Proposition 3.2).

(a) The clopen sets F_e generate the topology of X.

(b) For $u \in S$, we denote by O_u the set of classes in G_S of $\{(u, x) : x \in F_{u^*u}\}$. The space G_S itself is covered by the compact open subsets O_u. The restrictions of the source (respectively range) map is a homeomorphism $s : O_u \to F_{u^*u}$ (respectively $r : O_u \to F_{uu^*}$).

(c) An element $e \in E$ defines a character on $C^*(E)$, whence an element $\varepsilon_e \in X$, by the formula

$$f(\varepsilon_e) = \begin{cases} 1 & \text{if } e \leqslant f, \\ 0 & \text{otherwise} \end{cases}$$

(for $f \in E$). Moreover, for $u \in S$, we let ε_u be the class of (u, ε_{u^*u}). We thus have an injection of S into G_S, which maps E into a dense subset of $X = G_S^{(0)}$.

THEOREM. *For any inverse semigroup S, we have the natural isomorphisms $C^*(S) \cong C^*(G_S)$ and $C^*_r(S) \cong C^*_r(G_S)$.*

2.7. $C(X)$-ALGEBRAS. (cf. [8]) (a) Let X be a locally compact (Hausdorff) space. A $C_0(X)$-algebra is a C^*-algebra B together with a morphism ρ from $C_0(X)$ into the center $Z(\mathcal{M}(B))$ of the multiplier algebra of B such that $\rho(C_0(X))B = B$. In what follows, the letter ρ will be often omitted: we will consider B as a $C_0(X)$-module, and write fb instead of $\rho(f)b$.

(b) Let B be a $C_0(X)$-algebra.

– If Ω is an open subset of X, one puts $B_\Omega = C_0(\Omega)B$. It is an ideal in B, and a $C_0(\Omega)$ algebra.

– If F is a closed subset of X, one puts $B_F = B/B_{X\setminus F}$.

– In particular, if $x \in X$, one writes B_x instead of $B_{\{x\}}$. The evaluation morphisms $p_x : B \to B_x$ define a morphism $B \to \prod_{x\in X} B_x$ which is injective.

– If $Y = F \cap \Omega$ is a locally closed subset of X, one easily see that $B_Y = (B_F)_\Omega = (B_\Omega)_F$ only depends on Y (up to a canonical isomorphism). The C^*-algebra B_Y is a $C_0(Y)$ algebra. For Z locally closed in Y, we have $(B_Y)_Z = B_Z$.

(c) The elements $b \in C_c(X)B \subset B$ are said to have *compact support* relative to the $C_0(X)$-structure on B. One may actually define the support of an element $b \in X$ to be the closure in X of the set of x such that $b_x \neq 0$.

(d) Let X, Y be locally compact spaces and $f : X \to Y$ a continuous map. To any $C_0(Y)$-algebra B is associated a $C_0(X)$-algebra $f^*(B)$: the graph $G_f \subset X \times Y$ of f is naturally homeomorphic to X. We consider the $C_0(X\times Y)$-algebra $C_0(X)\otimes B$ and put $f^*(B) = (C_0(X)\otimes B)_{G_f}$. We sometimes write $f^*(B) = C_0(X)\otimes_{C_0(Y)} B$. For $x \in X$, we have $(f^*(B))_x = B_{f(x)}$.

(e) A *morphism* $\varphi : A \to B$ of $C_0(X)$-algebras is a $C_0(X)$-linear morphism. It then defines a map $\varphi_x : A_x \to B_x$ for each $x \in X$. On the other hand, the family (φ_x) determines the morphism φ.

3. CROSSED PRODUCTS BY GROUPOIDS

In [18], Jean Renault defines actions of groupoids on C^*-algebras and associated crossed products. In [12], Section 3, Pierre-Yves Le Gall defines an action of a groupoid on a C^*-algebra D in a less restrictive sense than that of [18], in that the algebra D does not need to be a continuous field over the space of units X, but a $C_0(X)$-algebra in the sense of Kasparov ([8]) recalled above. An equivalent setting was also studied by Quigg and Sieben ([16], Section 3), who further constructed the full crossed product. We define here the full and reduced crossed product in the setting of [12] and [16].

We fix a locally compact groupoid G with a Haar system ν and denote by X its space $G^{(0)}$ of objects. We keep the notation recalled in 2.3 and 2.4 above.

Let us first recall Le Gall's definition of an action of a groupoid:

3.1. An action of G on a C^*-algebra D is given by a structure of $C_0(X)$-algebra on D and an isomorphism of $C_0(G)$-algebras $\alpha : s^*D \to r^*D$, such that, for each $(\gamma_1, \gamma_2) \in G^{(2)}$ we have $\alpha_{\gamma_1\gamma_2} = \alpha_{\gamma_1} \circ \alpha_{\gamma_2}$.

Note that for $\gamma \in G$, the map α_γ is by definition (cf. 2.7 (d) and (e)) a $*$-isomorphism $D_{s(\gamma)} \to D_{r(\gamma)}$.

3.2. If G is not Hausdorff, this definition has to be slightly modified by working with Hausdorff open subsets of G: An action of G on a C^*-algebra D is given by:

(a) a structure of $C_0(X)$-algebra on D with X and

(b) an isomorphism of $C_0(U)$-algebras $\alpha_U : s|_U^* D \to r|_U^* D$, for every open Hausdorff subset U of G, such that

(i) if $U \subset V$ are Hausdorff open subsets of G, then α_U is the restriction of α_V;

(ii) for each $(\gamma_1, \gamma_2) \in G^{(2)}$ we have $\alpha_{\gamma_1\gamma_2} = \alpha_{\gamma_1} \circ \alpha_{\gamma_2}$.

Condition (i) tells us that α_γ depends only on γ and not on the Hausdorff neighborhood U containing it. Thus (ii) makes sense.

3.3. FUNCTION ALGEBRA ASSOCIATED WITH A GROUPOID ACTION. Given a C^*-algebra D endowed with an action of a groupoid G, let $\mathcal{A}(D)$ be the function space defined as follows:

– *The Hausdorff case.* If G is Hausdorff, let s^*D be the $C_0(G)$-algebra corresponding to the source map $s : G \to X$ (cf. 2.7(d)). Let $\mathcal{A}(D) = C_c(s^*D) = C_c(G) \cdot s^*D$, i.e. continuous sections with compact support (cf. 2.7 (c)).

– *The non-Hausdorff case.* In this case the function space $\mathcal{A}(D) \subset \prod_{\gamma\in G} D_{s(\gamma)}$ is the set of linear combinations of elements with compact support in $s^*_{|U}D$ for some open Hausdorff subset U of G, where $s_{|U} : U \to X$ is the restriction to U of the source map.

– *The product and convolution* are defined in the following way.

Given $f, g \in \mathcal{A}(D)$, let

$$(f \star g)(\gamma) = \int \alpha_{\gamma_1}^{-1}(f(\gamma\gamma_1^{-1}))g(\gamma_1)\,\mathrm{d}\nu_{s(\gamma)}(\gamma_1) \quad \text{and} \quad f^*(\gamma) = \alpha_\gamma^{-1}(f(\gamma^{-1})^*).$$

It is easily seen, like in the case where $D = C_0(X)$, that these are well defined operations turning $\mathcal{A}(D)$ into a $*$-algebra (cf. [9], Section 1).

3.4. THE NORM $\|\cdot\|_1$. The norm on $\mathcal{A}(D)$ is defined by

$$\|f\|_1 = \sup_{x\in X}\left\{\max\left\{\int \|f(\gamma)\|\,\mathrm{d}\nu_x(\gamma), \int \|f(\gamma)\|\,\mathrm{d}\nu^x(\gamma)\right\}\right\}. \tag{3.1}$$

Let $(U_i)_{i\in I}$ be a covering of G by open Hausdorff subsets and set $\Omega = \coprod_{i\in I} U_i = \{(\gamma, i) \in G \times I : \gamma \in U_i\}$. It is a locally compact Hausdorff space. Let $s_\Omega : \Omega \to X$ be the (continuous) map $(\gamma, i) \mapsto s(\gamma)$. For $g \in C_c(\Omega)s^*_\Omega(D)$, we put

$$\|g\|_1 = \sup_{x\in X}\left\{\max\left\{\sum_{i\in I}\int \|g(\gamma, i)\|\,\mathrm{d}\nu_x(\gamma) : \sum_{i\in I}\int \|g(\gamma, i)\|\,\mathrm{d}\nu^x(\gamma)\right\}\right\}. \tag{3.2}$$

Moreover, we let $\varphi(g) \in \mathcal{A}(D)$ be the function $\gamma \mapsto \sum_i g(\gamma, i)$ (these are finite sums).

As in the case when $D = C(X)$ (cf. [9], Lemmas 1.3 and 1.4), the map φ is onto and, for $f \in \mathcal{A}(D)$, we have

$$\|f\|_1 = \inf\{\|g\|_1 : g \in C_c(\Omega)s^*_\Omega(D),\ \varphi(g) = f\}. \tag{3.3}$$

3.5. THE FULL CROSSED PRODUCT. With the product, involution, and norm $\|\cdot\|_1$ defined above, $\mathcal{A}(D)$ is a normed $*$-algebra exactly as in the case of a trivial action $D = C_0(X)$. The enveloping C^*-algebra of the Banach $*$-algebra obtained by completion of $\mathcal{A}(D)$ with respect to the norm $\|\cdot\|_1$ is called the *full crossed product of* D *by* G and is denoted by $D \rtimes_\alpha G$.

3.6. REGULAR REPRESENTATIONS AND THE REDUCED CROSSED PRODUCT. Let $x \in X$. Consider the Hilbert D_x-module $L^2(G_x, \nu_x) \otimes D_x$; it is the completion of the space $C_c(G_x; D_x)$ of continuous compactly supported functions on G_x with values in D_x with respect to the D_x valued inner product defined by $\langle g, h\rangle = (g^* \star h)(x) = \int g(\gamma)^* h(\gamma)\, \mathrm{d}\nu_x(\gamma)$. For $f \in \mathcal{A}(D)$ and $g \in C_c(G_x; D_x)$, put $\Lambda_x(f)g = f \star g$. For $f \in \mathcal{A}(D)$ and $g, h \in C_c(G_x; D_x)$ we have

$$\langle g, \Lambda_x(f)h\rangle = (g^* \star f \star h)(x) = \langle \Lambda_x(f^*)g, h\rangle. \tag{3.4}$$

Also,

$$\begin{aligned}\langle g, \Lambda_x(f)h\rangle &= \int g(\gamma)^*(f \star h)(\gamma)\, \mathrm{d}\nu_x(\gamma)\\ &= \iint g(\gamma)^* \alpha_{\gamma_1}^{-1}(f(\gamma\gamma_1^{-1}))h(\gamma_1)\, \mathrm{d}\nu_x(\gamma_1)\, \mathrm{d}\nu_x(\gamma).\end{aligned}$$

Let Ω be as in 3.4. Write $f = \varphi(f_0)$ and $f_0(\gamma, i) = f_1(\gamma, i)f_2(\gamma, i)$, where $f_0, f_1, f_2 \in C_c(\Omega)s^*(D)$ are such that, for all $(\gamma, i) \in \Omega$, we have $\|f_0(\gamma, i)\| = \|f_1(\gamma, i)\|^2 = \|f_2(\gamma, i)\|^2$. We find:

$$\begin{aligned}\langle g, \Lambda_x(f)h\rangle &= \sum_{i\in I} \iint g(\gamma)^* \alpha_{\gamma_1}^{-1}\big(f_1(\gamma\gamma_1^{-1}, i)f_2(\gamma\gamma_1^{-1}, i)\big)h(\gamma_1)\ \mathrm{d}\nu_x(\gamma_1)\, \mathrm{d}\nu_x(\gamma)\\ &= \sum_{i\in I} \iint k_1(\gamma, \gamma_1, i)^* k_2(\gamma, \gamma_1, i)\, \mathrm{d}\nu_x(\gamma_1)\, \mathrm{d}\nu_x(\gamma)\end{aligned}$$

where, for $i \in I$, $\gamma, \gamma_1 \in G_x$, we have put $k_1(\gamma, \gamma_1, i) = \alpha_{\gamma_1}^{-1}\big(f_1(\gamma\gamma_1^{-1}, i)\big)^* g(\gamma)$ and $k_2(\gamma, \gamma_1, i) = \alpha_{\gamma_1}^{-1}\big(f_2(\gamma\gamma_1^{-1}, i)\big)h(\gamma_1)$. Using the Cauchy-Schwarz inequality in the Hilbert D_x-module $L^2(G_x \times G_x \times I; D_x)$, we find

$$\|\langle g, \Lambda_x(f)h\rangle\| \leqslant \|k_1\|_2 \|k_2\|_2$$

where, for $j = 1, 2$, we put

$$\|k_j\|_2^2 = \Big\|\sum_{i\in I} \iint k_j(\gamma, \gamma_1, i)^* k_j(\gamma, \gamma_1, i)\, \mathrm{d}\nu_x(\gamma_1)\, \mathrm{d}\nu_x(\gamma)\Big\|.$$

Now

$$\begin{aligned}\|k_1\|_2^2 &= \Big\|\sum_{i\in I} \iint g(\gamma)^* \alpha_{\gamma_1}^{-1}\big(f_1(\gamma\gamma_1^{-1}, i)f_1(\gamma\gamma_1^{-1}, i)^*\big)g(\gamma)\, \mathrm{d}\nu_x(\gamma_1)\, \mathrm{d}\nu_x(\gamma)\Big\|\\ &\leqslant \Big\|\sum_{i\in I} \iint g(\gamma)^* \|f_1(\gamma\gamma_1^{-1}, i)f_1(\gamma\gamma_1^{-1}, i)^*\| g(\gamma)\, \mathrm{d}\nu_x(\gamma_1)\, \mathrm{d}\nu_x(\gamma)\Big\|\\ &= \Big\|\int g(\gamma)^* \Big(\sum_{i\in I} \int \|f_0(\gamma\gamma_1^{-1}, i)\|\, \mathrm{d}\nu_x(\gamma_1)\Big) g(\gamma)\, \mathrm{d}\nu_x(\gamma)\Big\|.\end{aligned}$$

Moreover, $\sum_{i\in I}\int \|f_0(\gamma\gamma_1^{-1},i)\|\,\mathrm{d}\nu_x(\gamma_1) = \sum_{i\in I}\int \|f_0(\gamma_2,i)\|\,\mathrm{d}\nu^{r(\gamma)}(\gamma_2) \leqslant \|f_0\|_1$ (where $\|f_0\|_1$ is given by formula (3.2)). It follows that $\|k_1\|^2 \leqslant \|g\|_2^2\|f_0\|_1$. In the same way, $\|k_2\|^2 \leqslant \|h\|_2^2\|f_0\|_1$. We deduce that

$$\|\langle g, \Lambda_x(f)h\rangle\| \leqslant \|f_0\|_1\,\|g\|_2\,\|h\|_2. \tag{3.5}$$

This is true for all f_0 such that $\varphi(f_0) = f$. Taking the infimum of the right hand side in formula (3.5), we find (using formula (3.3))

$$\|\langle g, \Lambda_x(f)h\rangle\| \leqslant \|f\|_1\,\|g\|_2\,\|h\|_2. \tag{3.6}$$

From formulas (3.4) and (3.6) we deduce that $\Lambda_x(f)$ extends to an element denoted by $\Lambda_x(f) \in \mathcal{L}(L^2(G_x,\nu_x)\otimes D_x)$ with adjoint $\Lambda_x(f)^*$. Finally, Λ_x yields a $*$-representation of $D \rtimes_\alpha G$.

DEFINITION. The *reduced crossed product* $D \rtimes_{\alpha,\mathrm{r}} G$ of D by G is the quotient of the full crossed product with respect to the family $(\Lambda_x)_{x\in X}$ of representations defined above.

3.7. Here is an equivalent construction of the reduced crossed product, analogous to the one of [9], Theorem 2.10 (outlined here in 2.5).

Let $\mathcal{D}$ be the set of bounded sections of $\prod_x D_x$. An element $f \in \mathcal{A}(D)$ defines by restriction to $X \subset G$ an element $f_{|X} \in \mathcal{D}$. Denote by D' the C^*-subalgebra of $\mathcal{D}$ generated by these elements. Note that, if G is Hausdorff, then $D' = D$. For $\xi,\eta \in \mathcal{A}(D)$ and $f, g \in D'$ we then put $\langle(\xi\otimes f),(\eta\otimes g)\rangle = f^*(\xi^*\star\eta)_{|X}g \in D'$.

In this way, the right D' module $\mathcal{A}(D)\otimes D'$ is turned into a pre-Hilbert D'-module. Its (Hausdorff-)completion is a Hilbert D'-module denoted $L^2(G,\nu;D')$.

Note that the crossed product $D \rtimes_\alpha G$ acts on $L^2(G,\nu;D')$ by $\Lambda(f)(\xi\otimes g) = (f\star\xi)\otimes g$ for all $f,\xi \in \mathcal{A}(D)$ and $g \in D'$. The image is again $*$-isomorphic to $D \rtimes_{\alpha,\mathrm{r}} G$. Indeed, for $x \in X$, let $p_x : D' \to D_x$ be the natural evaluation map. Since $D' \subset \mathcal{D}$, the family $(p_x)_x$ is a faithful family. Whence, for $T \in \mathcal{L}(L^2(G,\nu;D'))$, we have $\|T\| = \sup_x \|T\otimes_{p_x} 1\|$. For $x \in X$, we have $L^2(G,\nu;D')\otimes_{p_x} D_x \cong L^2(G_x,\nu_x)\otimes D_x$ and, under this identification, for every $f \in \mathcal{A}(D)$ we have $\Lambda(f)\otimes_{p_x} 1 \cong \Lambda_x(f)$. Whence $\|\Lambda(f)\| = \sup_x \|\Lambda_x(f)\| = \|f\|_{D \rtimes_{\alpha,\mathrm{r}} G}$.

REMARK. Let Y be the locally compact Hausdorff space associated with G, $p : Y \to X$ the continuous map and $L^2(G,\nu)$ the Hilbert $C_0(Y)$-module constructed in [9] Section 2 (see 2.5 above). Then, p^*D is a $C_0(Y)$-algebra. Consider the Hilbert $C_0(Y)$-module $L^2(G,\nu)$ defined in Section 2. Let

$$L^2(G,\nu;p^*(D)) = L^2(G,\nu)\otimes_{C_0(Y)} p^*D.$$

One may also construct a representation of $D \rtimes_\alpha G$ on the Hilbert $p^*(D)$-module $L^2(G,\nu;p^*(D))$ and thus give a third definition of the reduced crossed product.

To relate these constructions, one notes that there is a natural injection $j : D' \to p^*(D)$: one just needs to define $j(f\otimes b)$ where $f \in \mathcal{A}$, $b \in D$, since by definition of $\mathcal{A}(D)$ such elements generate a dense subspace; put then $j(f\otimes b) =$

$f|_X \otimes_{C(X)} b \in C_0(Y) \otimes_{C(X)} D = p^*(D)$, where $f|_X$ is, by definition of Y, an element of $C_0(Y)$.

In this way, $L^2(G,\nu;p^*(D)) = L^2(G,\nu;D') \otimes_{D'} p^*(D)$. We thus get an isomorphism $x \mapsto x \otimes 1$ between these reduced crossed products.

Note that, in general, $D' \neq p^*(D)$: let, for instance, Γ be a non trivial discrete group and put, $G = \Gamma \times [0,1]/\sim$, where $(g,s) \sim (h,t)$ iff $s = t = 0$ or $(g,s) \sim (h,t)$. Put $D = \mathbb{C}$, and let $X = [0,1]$ act by $f \cdot b = f(0)b$. In other words, $D_0 = \mathbb{C}$ and $D_t = \{0\}$ for $t \neq 0$. Obviously $D' = D$. On the other hand, there are two points of Y which map to 0 (cf. [9], Example 1.2), whence $p^*(D) = \mathbb{C} \oplus \mathbb{C}$.

4. THE INVERSE SEMIGROUP $SQ(A)$

In this section we construct an inverse semigroup of automorphisms between quotients of ideals of a C^*-algebra. Let us begin by recalling a few easy facts about quotients of ideals.

4.1. Let A be a C^*-algebra. We denote by $\mathrm{EQ}(A)$ the set of quotients of ideals of A. Such "ideal quotients" will be sometimes called *subquotients* somewhat unproperly. In other words, $B \in \mathrm{EQ}(A)$ if there exist closed two-sided ideals I, J in A such that $I \subset J$ and $B = J/I$. Note that we have a natural morphism $q_B : A \to \mathcal{M}(B)$ whose image contains B and that $B = J_B/I_B$ where $I_B = \ker(q_B)$ and $J_B = q_B^{-1}(B)$. Note however that this writing is far from being unique in general: one can write $B = J/I$ where J is any ideal $J \subset J_B$ such that $J + I_B = J_B$ and $I = J \cap I_B$.

REMARK. One may make things a little more formal and precise, by saying that $\mathrm{EQ}(A)$ is the quotient of the set of pairs (I,J) of ideals of A such that $I \subset J$ by the equivalence relation

$$(I,J) \sim (I',J') \iff I \cap J' = I' \cap J \quad \text{and} \quad I + J' = I' + J.$$

4.2. The set $\mathrm{EQ}(A)$ has a natural order: if $B, C \in \mathrm{EQ}(A)$, we write $B \prec C$ if B is a subquotient of C, i.e. the morphism $q_B : A \to \mathcal{M}(B)$ is written as a composition $\overline{q}_{B,C} \circ q_C$ where $\overline{q}_{B,C}$ is the extension to the multiplier algebra of a morphism $q_{B,C} : C \to \mathcal{M}(B)$ such that $B \subset q_{B,C}(C)$. This is equivalent to saying that B and C are written as quotients J/I and J'/I' with $I' \cap J \subset I$ and $J \subset I + J'$. Another equivalent condition is $I_C \subset I_B$ and $J_B \subset J_C + I_B$. In that case, B and C can be written as quotients J/I and J'/I' with $I' \subset I \subset J \subset J'$.

Moreover, if B is a subquotient of A, any subquotient of B is naturally a subquotient of A. In other words, we may identify $\mathrm{EQ}(B)$ with $\{C \in \mathrm{EQ}(A) : C \prec B\}$.

4.3. With this order, $\mathrm{EQ}(A)$ is a semi-lattice: for $B, C \in \mathrm{EQ}(A)$ we have

$$B \wedge C = \big((J_B \cap J_C) + (I_B + I_C)\big)/(I_B + I_C) = (J_B \cap J_C)/\big(J_B \cap J_C \cap (I_B + I_C)\big).$$

Indeed, it is obvious that $B \wedge C \prec B$ and $B \wedge C \prec C$; if $D \in \mathrm{EQ}(A)$ satisfies $D \prec B$ and $D \prec C$, then $I_D \supset I_B + I_C$ and $J_D \subset (J_B + I_D) \cap (J_C + I_D)$. It follows that $D \prec B \wedge C$.

4.4. In the commutative case, elements of $\mathrm{EQ}(C_0(X))$ are $C_0(Y)$ (seen as an ideal in the quotient $C_0(\overline{Y})$ of $C_0(X)$) where Y runs over locally closed subsets of X. We have $C_0(Y) \prec C_0(Z)$ if and only if $Y \subset Z$ and $C_0(Y) \wedge C_0(Z) = C_0(Y \cap Z)$.

4.5. We denote by $\mathrm{SQ}(A)$ the set of triples (B, α, C) where $B, C \in \mathrm{EQ}(A)$ and $\alpha : C \to B$ is a $*$-isomorphism.

If $\alpha : C \to B$ is a $*$-isomorphism, to any subquotient D of C there corresponds a subquotient $\alpha(D)$ and an isomorphism $\alpha_{:D} : D \to \alpha(D)$ with extension $\overline{\alpha}_{:D}$ to the multipliers, such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & B \\ \downarrow & & \downarrow \\ \mathcal{M}(D) & \xrightarrow{\overline{\alpha}_{:D}} & \mathcal{M}(\alpha(D)) \end{array}$$

commutes.

4.6. The composition of two elements (B, α, C) and (B', α', C') of $\mathrm{SQ}(A)$ is the triple

$$(B, \alpha, C) \cdot (B', \alpha', C') = (\alpha(B' \wedge C), \beta, (\alpha')^{-1}(B' \wedge C))$$

where β is the composition $\alpha_{:B' \wedge C} \circ \alpha'_{:(\alpha')^{-1}(B' \wedge C)}$.

Note that if $D, D' \in \mathrm{EQ}(A)$ are such that $(B' \wedge C) \prec D \prec C$ and $(B' \wedge C) \prec D' \prec B'$, then

$$(B, \alpha, C) \cdot (B', \alpha', C') = (\alpha(D), \alpha_{:D}, D) \cdot (D', \alpha'_{:(\alpha')^{-1}(D')}, (\alpha')^{-1}(D')). \tag{4.1}$$

Indeed, one checks immediately that $D \wedge D' = B' \wedge C$; also $\big(\alpha_{:D}\big)_{:B' \wedge C} = \alpha_{:B' \wedge C}$ and $\big(\alpha'_{:(\alpha')^{-1}(D')}\big)_{:(\alpha')^{-1}(B' \wedge C)} = \alpha'_{:(\alpha')^{-1}(B' \wedge C)}$.

Also write $(B, \alpha, C) \cdot (B', \alpha', C') = (B_1, \alpha_1, C_1)$ and let $D \in \mathrm{EQ}(A)$. We easily see that

$$(\alpha_1(C_1 \wedge D), (\alpha_1)_{:C_1 \wedge D}, C_1 \wedge D) = (B, \alpha, C) \cdot (\alpha'(C' \wedge D), (\alpha')_{:C' \wedge D}, C' \wedge D). \tag{4.2}$$

In the same way, setting $D_1 = \alpha_1^{-1}(B_1 \wedge D)$, we find

$$(B_1 \wedge D, (\alpha_1)_{:D_1}, D_1) = (B \wedge D, \alpha_{:\alpha^{-1}(B \wedge D)}, \alpha^{-1}(B \wedge D)) \cdot (B', \alpha', C'). \tag{4.3}$$

We have:

4.7. PROPOSITION. *With the above operations,* $\mathrm{SQ}(A)$ *is an inverse semigroup.*

Proof. The inverse of (B, α, C) is (C, α^{-1}, B). The only thing which has to be proved is associativity of the composition. Let (B, α, C), (B', α', C') and (B'', α'', C'') be elements of $\mathrm{SQ}(A)$.

Write $(B, \alpha, C) \cdot (B', \alpha', C') = (B_1, \alpha_1, C_1)$. We have (using formula (4.2)),

$$(B_1, \alpha_1, C_1) \cdot (B'', \alpha'', C'') = (\alpha_1(B'' \wedge C_1), (\alpha_1)_{:B'' \wedge C_1}, B'' \wedge C_1) \cdot (B'', \alpha'', C'').$$

It follows from formula (4.2) that both $((B, \alpha, C) \cdot (B', \alpha', C')) \cdot (B'', \alpha'', C'')$ and $(B, \alpha, C) \cdot ((B', \alpha', C') \cdot (B'', \alpha'', C''))$ remain unchanged if we replace (B', α', C') by

$$(\alpha'(C' \wedge B''), (\alpha')_{:C' \wedge B''}, C' \wedge B'').$$

We may therefore assume that $C' \prec B''$. In the same way, using formula (4.3), we may also assume that $B' \prec C$. But in this case, both products are equal to

$$(\alpha(B'), \alpha_{:B'} \circ \alpha' \circ \alpha''_{:(\alpha'')^{-1}(C')}, (\alpha'')^{-1}(C')). \quad \blacksquare$$

To $B \in \mathrm{EQ}(A)$ corresponds the idempotent (B, id_B, B) in $\mathrm{SQ}(A)$. In this way, the set of idempotents of $\mathrm{SQ}(A)$ identifies with $\mathrm{EQ}(A)$.

5. ACTIONS OF INVERSE SEMIGROUPS AND CROSSED PRODUCTS

The main purpose of this section is to describe a general construction of a crossed product by an action of an inverse semigroup. In the next section, we will investigate the connection between this crossed product, the one defined by Sieben in [19], and the crossed product by the associated groupoid.

ACTIONS.

5.1. DEFINITION. Let S be an inverse semigroup. An *action* of S on a C^*-algebra A is a semigroup homomorphism $\alpha : S \to \mathrm{SQ}(A)$.

In other words, an action of an inverse semigroup S with set of idempotents E is given by a pair (B, α) where $B = (B_e)_{e \in E}$ is a collection of elements of $\mathrm{EQ}(A)$ such that $B_{ef} = B_e \wedge B_f$, and for each $u \in S$, α_u is an isomorphism $\alpha_u : B_{u^*u} \to B_{uu^*}$ satisfying $(B_{uvv^*u^*}, \alpha_{uv}, B_{v^*u^*uv}) = (B_{uu^*}, \alpha_u, B_{u^*u}) \cdot (B_{vv^*}, \alpha_v, B_{v^*v})$ for all $u, v \in S$.

5.2. EXAMPLES. Actions by partial automorphisms, or by endomorphisms of an inverse semigroup are easily seen to be special cases of the above definition. They actually form two extreme cases of our definition.

(a) Recall that *a partial automorphism* of A is a triple (I, α, J), where I and J are closed ideals of A and $\alpha : J \to I$ is a $*$-isomorphism (cf. [6]). Partial automorphisms of A form an inverse semigroup $\mathrm{PAut}(A)$. Sieben (cf. [19]) defines an action of a unital inverse semigroup S on A to be a semigroup homomorphism $\beta : S \to \mathrm{PAut}(A)$. Note that $\mathrm{PAut}(A)$ is a sub-semigroup of $\mathrm{SQ}(A)$, therefore an action in Sieben's sense is a particular case of an action in the sense of Definition 5.1 when the B_e's are ideals.

(b) At the other end, we may also consider the case where all the B_e's are quotients.

A particular case is obtained by a semigroup homomorphism $\beta : S \to \mathrm{End}(A)$. Given such a morphism for each $e \in E$ let $B_e = A/\ker \beta_e$. If $e, f \in E$, we have $\beta_{ef} = \beta_e \circ \beta_f = \beta_f \circ \beta_e$, thus $\ker \beta_{ef} \supset \ker \beta_e + \ker \beta_f$; if $x \in \ker \beta_{ef}$, then $\beta_e(\beta_e(x)) = \beta_e(x)$. Hence, $x - \beta_e(x) \in \ker \beta_e$ and $\beta_f(\beta_e(x)) = 0$, whence $\beta_e(x) \in \ker \beta_f$; thus $x \in \ker \beta_e + \ker \beta_f$. Whence $B_{ef} = B_e \wedge B_f$.

Note that for each $u \in S$, since $\beta_{u^*u} = \beta_{u^*} \circ \beta_u$ and $\beta_u = \beta_u \circ \beta_{u^*u}$ we have $\ker \beta_u = \ker \beta_{u^*u}$, and β_u defines an isomorphism $\alpha_u : B_{u^*u} \to B_{uu^*}$ given by $\alpha_u \circ q_{u^*u} = q_{uu^*} \circ \beta_u$, where $q_{u^*u} : A \to B_{u^*u}$ and $q_{uu^*} : A \to B_{uu^*}$ are the quotient maps.

For instance, consider the group $\Gamma = \mathbb{Z}^{(\mathbb{N})}$ with the basis $(e_n)_{n \in \mathbb{N}}$. The endomorphisms β_s and β_{s^*} of Γ given by $\beta_s(e_n) = e_{n+1}$ and $\beta_{s^*}(e_n) = \begin{cases} e_{n-1} & \text{if } n \neq 0, \\ 0 & \text{if } n = 0, \end{cases}$

yield an action β of the bicyclic semigroup T (i.e. the semigroup generated by s, s^* with the property $s^*s = 1$) on $\mathbb{Z}^{(\mathbb{N})}$. Corresponding to this action is a morphism $T \to \mathrm{End}(C^*(\Gamma))$.

(c) Here is a simple example where not all of the B_e's are ideals or quotients.

The sub-semigroup $S = \{0, 1, e = e_{11}, u = e_{12}, u^* = e_{21}, f = e_{22}\}$ of $M_2(\mathbb{C})$ (for the matrix product) is an inverse semigroup.

Let A be an extension of $\mathcal{K}$ by $\mathcal{K}$ (i.e. $A = \widetilde{\mathcal{K}} \otimes \mathcal{K}$). Let $J \simeq \mathcal{K}$ be its non-trivial ideal and $B = A/J \simeq \mathcal{K}$ its nontrivial quotient.

Set $B_1 = A$, $B_e = J$, $B_f = B$ and $B_0 = \{0\}$ and let α_u be an isomorphism from J onto B. In this way, the semigroup S acts on A.

CONSTRUCTION OF THE CROSSED PRODUCT. We now define the crossed product constructions for actions in the sense of Definition 5.1. Let $B, C, D \in \mathrm{EQ}(A)$. If $D \prec B$, we have a natural morphism $q_{D,B} : C \to \mathcal{M}(D)$. Moreover, if $D = B \wedge C$, for every $x \in B$ and $y \in C$, $q_{D,B}(x)q_{D,C}(y) \in D$. In this way, we get a bilinear map $(x, y) \mapsto x \diamond y = q_{D,B}(x)q_{D,C}(y)$ from $B \times C \to B \wedge C$.

5.3. PROPOSITION. *Let S be an inverse semigroup and (B, α) an action of S on a C^*-algebra A. For $\varphi \in \prod_{u\in S} B_{u^*u}$, set $\|\varphi\|_1 = \sum_u \|\varphi(u)\|$ $(\in \mathbb{R}_+ \cup \{+\infty\})$. Put $\ell^1(S, B) = \Big\{\varphi \in \prod_{u\in S} B_{u^*u} : \|\varphi\|_1 < +\infty\Big\}$. For $\varphi, \psi \in \ell^1(S, B)$, set*

$$(\varphi \star \psi)(w) = \sum_{uv=w} \alpha_{v^*u^*u}(\varphi(u) \diamond \alpha_v(\psi(v))), \quad \varphi^*(u) = \alpha_{u^*}(\varphi(u^*)^*).$$

With these operations $\ell^1(S, B)$ is a Banach $*$-algebra.

An element $\varphi \in \ell^1(S, B)$ is formally written as a sum

$$\varphi = \sum_{u\in S} \delta_u \varphi(u) = \sum_{u\in S} \alpha_u(\varphi(u))\delta_u.$$

Proof. Let $u, v \in S$, $x \in B_{u^*u}$ and $y \in B_{v^*v}$. Then $x \diamond \alpha_v(y) \in B_{u^*uvv^*}$, whence $\alpha_{v^*u^*u}(x \diamond \alpha_v(y)) \in B_{v^*u^*uv}$. Moreover $\|\alpha_{v^*u^*u}(x \diamond \alpha_v(y))\| \leqslant \|x\|\|y\|$. It follows that $\varphi \star \psi \in \ell^1(S, B)$.

It is also easy to see that $\|\varphi^*\|_1 = \|\varphi\|_1$ and $(\varphi \star \psi)^* = \psi^* \star \varphi^*$, and hence the only thing that remains to be proved is the associativity of $\star$.

If $u \in S$, we denote by $\overline{\alpha}_u : \mathcal{M}(B_{u^*u}) \to \mathcal{M}(B_{uu^*})$ the extension of α_u to the multipliers. Let $e \in E$; we denote by the same symbol q_e, for every $f \in E$ such that $e \leqslant f$ the natural map $q_e : B_f \to \mathcal{M}(B_e)$ and by $\overline{q}_e$ its extension to multipliers. If $e \leqslant f \leqslant g$ we have an equality $\overline{q}_e \circ q_f = q_e : B_g \to \mathcal{M}(B_e)$.

Note that if $u \in S$ and $e \in E$, we have $q_{ueu^*} \circ \alpha_u = \overline{\alpha}_{ue} \circ q_{eu^*u}$ (by definition of the inverse semigroup $\mathrm{SQ}(A)$). Extending this equality to multipliers, we find

$$\overline{q}_{ueu^*} \circ \overline{\alpha}_u = \overline{\alpha}_{ue} \circ \overline{q}_{eu^*u} : \mathcal{M}(B_{u^*u}) \to \mathcal{M}(B_{ueu^*}).$$

Let $u, v, w \in S$, $x \in B_{u^*u}$, $y \in B_{v^*v}$ and $z \in B_{ww^*}$. We want to show that

$$\Big((\delta_u x)(\delta_v y)\Big)(\delta_w z) = (\delta_u x)\Big((\delta_v y)(\delta_w z)\Big).$$

We have

$$(\delta_u x)(\delta_v y) = \delta_{uv}\alpha_{v^*u^*u}\big(q_{u^*uvv^*}(x)q_{u^*uvv^*}(\alpha_v(y))\big)$$
$$= \delta_{uv}(\overline{\alpha}_{v^*u^*u} \circ q_{u^*uvv^*})(x)q_{v^*u^*uv}(y),$$

whence $((\delta_u x)(\delta_v y))(\delta_w z) = \delta_{uvw}x_1 y_1 z_1$, where

$x_1 = \overline{\alpha}_{w^*v^*u^*uv} \circ \overline{q}_{ww^*v^*u^*uv} \circ \overline{\alpha}_{v^*u^*u} \circ q_e(x)$,
$y_1 = \overline{\alpha}_{w^*v^*u^*uv} \circ \overline{q}_{ww^*v^*u^*uv} \circ q_{v^*u^*uv}(y) = \overline{\alpha}_{w^*v^*u^*uv} \circ q_{ww^*v^*u^*uv}(y)$, and
$z_1 = q_{w^*v^*u^*uvw}(z)$. Note that $\overline{q}_{ww^*v^*u^*uv} \circ \overline{\alpha}_{v^*u^*u} = \overline{\alpha}_{ww^*v^*u^*u} \circ \overline{q}_{vww^*v^*u^*u}$,
and hence
$x_1 = \overline{\alpha}_{w^*v^*u^*u} \circ q_{vww^*v^*u^*u}(x)$. On the other hand, $(\delta_v y)(z\delta_w) = \delta_{vw}\overline{\alpha}_{w^*v^*v} \circ q_{ww^*v^*v}(y)q_{w^*v^*vw}(z))$. Thus,

$$(\delta_u x)((\delta_v y)(z\delta_w)) = \delta_{uvw}x_2 y_2 z_2,$$

where

$x_2 = \overline{\alpha}_{w^*v^*u^*u} \circ q_{u^*uvww^*v^*}(x) = x_1$,
$y_2 = \overline{q}_{w^*v^*u^*uvw} \circ \overline{\alpha}_{w^*v^*v} \circ q_{ww^*v^*v}(y) = y_1$, and
$z_2 = \overline{q}_{w^*v^*u^*uvw} \circ q_{w^*v^*vw}(z) = z_1$. ∎

5.4. DEFINITION. The *full crossed product of A by the action α of S* is by definition the enveloping C^*-algebra of this Banach $*$-algebra, and is denoted by $A \rtimes_\alpha S$.

5.5. REMARKS. (a) Let $\widetilde{S}$ be the inverse semigroup obtained by adjoining a unit to S, i.e. $\widetilde{S} = S \cup \{1\}$, with operations $1^* = 1$ and $u1 = 1u = u$ for all $u \in \widetilde{S}$.

An action (B, α) of S on a C^*-algebra A can be extended to an action (still denoted by (B, α)) of $\widetilde{S}$ on A by setting $B_1 = A$ and $\alpha_1 = \mathrm{id}_A$. Then $\ell^1(S, B)$ is a closed two sided ideal in $\ell^1(\widetilde{S}, B)$.

Recall that if J is a closed, two sided, self-adjoint ideal in a Banach $*$-algebra D, then its enveloping C^*-algebra identifies with the closure of J in the enveloping C^*-algebra D. In other words, the homomorphism $C^*(J) \to C^*(D)$ is injective. Also see Lemma 2.3, [16].

Indeed, we just have to construct a homomorphism $D \to \mathcal{M}(C^*(J))$ extending the natural map $i_J : J \to C^*(J)$. Adjoining a unit to D, we may assume that D is unital. Let then $a \in D$ with $\|a\| < 1$. Put $b = \sqrt{1 - a^*a}$ (using holomorphic functionnal calculus). We have $b = b^*$ and $a^*a + b^2 = 1$. It follows that, for all $x \in J$, we have $(ax)^*(ax) + (bx)^*(bx) = x^*x$. Whence $\|i_J(ax)\| \leqslant \|i_J(x)\|$. In this way, we associate to a a left multiplier of $C^*(J)$. Using a similar construction, we get a right multiplier of $C^*(J)$, whence the result.

It follows that $A \rtimes_\alpha S$ is identified to a closed two-sided ideal of $A \rtimes_\alpha \widetilde{S}$.

Furthermore, the map $a \mapsto \delta_1 a$ is a $*$-homomorphism from A into $A \rtimes_\alpha \widetilde{S}$. It defines a homomorphism $A \to \mathcal{M}(A \rtimes_\alpha S)$.

(b) Let A' be a subquotient of A such that for all $e \in E$, $B_e \prec A'$. The algebra $\ell^1(S, B)$ is the same when considering the B_e as being in $\mathrm{EQ}(A)$ or in $\mathrm{EQ}(A')$. If S is unital, then we may take $A' = B_1$ (where 1 is the unit element of S).

COVARIANT REPRESENTATIONS. Until the end of the section, we fix an inverse semigroup S with the set of idempotents E and an action (B, α) of S on a C^*-algebra A, and examine the representations of the crossed product $A \rtimes_\alpha S$. For $e \in E$, we denote by $q_e : A \to \mathcal{M}(B_e)$ the natural map.

5.6. Definition. A *covariant representation* of (A, S, α) on a Hilbert module H (over some C^*-algebra C) is a pair (π, σ), where $\pi : A \to \mathcal{L}(H)$ and $\sigma : S \to \mathcal{L}(H)$ are $*$-representations such that, for all $u \in S$, $a, b \in A$ satisfying $q_{u^*u}(a) \in B_{u^*u}$ and $\alpha_u(q_{u^*u}(a)) = q_{uu^*}(b)$ we have

$$\pi(b)\sigma(u) = \sigma(u)\pi(a). \tag{5.1}$$

5.7. Theorem. (a) *Let* (π, σ) *be a covariant representation of* (A, S, α) *on a Hilbert module* H*. There is a unique representation* $\Pi : A \rtimes_\alpha S \to \mathcal{L}(H)$ *satisfying* $\Pi(\delta_u q_{u^*u}(a)) = \sigma(u)\pi(a)$*, for all* $u \in S$*,* $a \in A$ *such that* $q_{u^*u}(a) \in B_{u^*u}$*.*

(b) *Conversely, every representation* Π *of* $A \rtimes_\alpha S$ *on a Hilbert space* H *is of the above form.*

Proof. (a) Let $u, v \in S$, $x \in B_{u^*u}$, and $y \in B_{v^*v}$. Take $a, b, b' \in A$ such that $q_{u^*u}(a) = x$, $q_{v^*v}(b') = y$, and $q_{vv^*}(b) = \alpha_v(y)$. Note that, by property (5.1), $\Pi(\delta_v y) = \pi(b)\sigma(v)$.

In particular, $\Pi\big((\delta_v y)^*\big) = \sigma(v^*)\pi(b^*) = \Pi(\delta_v y)^*$.

Put $e = v^*u^*uv$. We have $x \diamond \alpha_v(y) = q_{u^*uvv^*}(ab)$; let also $c \in A$ be such that $q_{v^*u^*uv}(c) = \alpha_{v^*u^*u}(x \diamond \alpha_v(y))$. We have

$$\begin{aligned}
\Pi\big((\delta_u x)(\delta_v y)\big) &= \Pi\big(\delta_{uv} q_{v^*u^*uv}(c)\big) \\
&= \sigma(uv)\pi(c) \\
&= \sigma(u)\sigma(u^*uv)\pi(c) && \text{writing } uv = u(u^*uv) \\
&= \sigma(u)\pi(ab)\sigma(u^*uv) && \text{by property (5.1)} \\
&= \sigma(u)\pi(ab)\sigma(u^*u)\sigma(v) \\
&= \sigma(u)\sigma(u^*u)\pi(ab)\sigma(v) && \text{by property (5.1)} \\
&= (\sigma(u)\sigma(u^*u)\pi(a))(\pi(b)\sigma(v)) \\
&= \Pi(\delta_u x)\Pi(\delta_v y).
\end{aligned}$$

(b) Let now H be a Hilbert space and $\Pi : A \rtimes_\alpha S \to \mathcal{L}(H)$ be a representation. Up to replacing H by $\Pi(A \rtimes_\alpha S)H$ we may assume that Π is nondegenerate. It uniquely extends to a representation, denoted by $\widetilde{\Pi}$, of $A \rtimes_\alpha \widetilde{S}$, in which $A \rtimes_\alpha S$ is an ideal (Remark 5.5 (a)). Now $x \mapsto \delta_1 x$ is an embedding of A in $A \rtimes_\alpha \widetilde{S}$. Put $\pi(a) = \Pi(\delta_1 a)$. In this way, we get a representation of A.

The construction of σ is a consequence of the following lemma:

5.8. Lemma. *Let* $u \in S$*. For any approximate identity* (a^λ) *of* B_{u^*u}*, the net* $\Pi(\delta_u a^\lambda)$ *converges* $*$*-strongly in* $\mathcal{L}(H)$ *to a partial isometry.*

Proof. Let $(e_{i,j})_{1 \leqslant i,j \leqslant 2}$ denote the matrix unit of $M_2(\mathbb{C})$. The map $j_u : B_{u^*u} \otimes M_2(\mathbb{C}) \to A \otimes M_2(\mathbb{C})$ given by the formulae $j_u(x \otimes e_{1,1}) = \delta_{u^*u} x \otimes e_{1,1}$, $j_u(x \otimes e_{2,1}) = \delta_u x \otimes e_{2,1}$, $j_u(x \otimes e_{1,2}) = x\delta_{u^*} \otimes e_{1,2}$, and $j_u(x \otimes e_{2,2}) = \alpha_u(x)\delta_{uu^*} \otimes e_{2,2}$ is a $*$-homomorphism. We deduce a $*$-representation $\Pi'_u = \big(\Pi \otimes \mathrm{id}_{M_2(\mathbb{C})}\big) \circ j_u : B_{u^*u} \otimes M_2(\mathbb{C}) \to \mathcal{L}(H) \otimes M_2(\mathbb{C}) = \mathcal{L}(H \oplus H)$.

It is now obvious that the net $\Pi'_u(a^\lambda \otimes e_{2,1})$ converges $*$-strongly to a partial isometry, and the lemma follows. ∎

For $u \in S$, we let $\sigma(u)$ be the strong-$*$-limit of $\Pi(\delta_u a^\lambda)$, where (a^λ) is an approximate unit for B_{u^*u}.

For $u, v \in S$, let (a^λ) and (b^λ) be bounded approximate identities of B_{u^*u} and B_{vv^*} respectively. Put $e = u^*uvv^*$. Consider the natural maps $q : B_{u^*u} \to \mathcal{M}(B_e)$, and $q' : B_{vv^*} \to \mathcal{M}(B_e)$. For $x \in B_e$, the net $\big(q(a^\lambda)q'(b^\lambda)x\big)$ converges to x (since $B_e = B_{vv^*} \wedge B_{u^*u}$ and (a^λ) is bounded). In the same way, the net $(xq(a^\lambda)q'(b^\lambda))$ converges to x i.e. $q(a^\lambda)q'(b^\lambda) = a^\lambda \diamond b^\lambda$ is an approximate identity of B_e; finally $\big(\alpha_{v^*u^*u}\big(a^\lambda \diamond b^\lambda\big)\big)$ is an approximate identity of $B_{v^*u^*uv}$, and $\Pi\big((\delta_u a^\lambda)(b^\lambda \delta_v)\big)$ converges to $\sigma(uv)$ by Lemma 5.8. Using again Lemma 5.8 and the boundedness of the nets (a^λ) and (b^λ), we see that $\Pi\big((\delta_u a^\lambda)(b^\lambda \delta_v)\big)$ converges to $\sigma(u)\sigma(v)$, so that $\sigma(uv) = \sigma(u)\sigma(v)$, and σ is a representation of S. As the $\sigma(u)$'s are partial isometries, σ is a $*$-representation of S.

Let $u \in S$ and $a, b \in A$ with $q_{u^*u}(a) \in B_{u^*u}$ and $q_{uu^*}(b) = \alpha_u \circ q_{u^*u}(a)$. We have $\delta_u a^\lambda \delta_1 a = \delta_u a^\lambda q_{u^*u}(a)$, hence the net $(\delta_u a^\lambda \delta_1 a)$ converges in norm to $\delta_u q_{u^*u}(a)$ (in $A \rtimes_\alpha S$). Therefore,

$$\Pi(\delta_u q_{u^*u}(a)) = \lim \Pi(\delta_u a^\lambda \delta_1 a) = \sigma(u)\pi(a). \tag{5.2}$$

Also, $\delta_1 b \delta_u a^\lambda = \delta_u \alpha_{u^*} q_{uu^*}(b) a^\lambda = \delta_u q_{u^*u}(a) a^\lambda$, and hence the net $(\delta_1 b \delta_u a^\lambda)$ converges in norm to $\delta_u q_{u^*u}(a)$ (in $A \rtimes_\alpha S$), whence

$$\Pi(\delta_u q_{u^*u}(a)) = \lim \Pi(\delta_1 b \delta_u a^\lambda) = \pi(b)\sigma(u). \tag{5.3}$$

From formulas (5.2) and (5.3) it follows that (π, σ) is a covariant representation of (A, S, α). Moreover, the corresponding representation of $A \rtimes_\alpha S$ will be Π by formula (5.2). ∎

REMARK. Note that (b) needs not be true if H is a *Hilbert module* and not a *Hilbert space*: the δ_u's need not be multipliers of $A \rtimes_\alpha S$ (*e.g.* if A and S are unital and B_e is not unital), whence the representation of $A \rtimes_\alpha S$ on itself by left multiplication does not give rise to a representation of S.

Let (π, σ) be a covariant representation of (A, S, α) on a Hilbert module H. Note that, for all $e \in A$ and $a \in A$ such that $q_e(a) \in B_e$, we have $\sigma(e)\pi(a) = \pi(a)\sigma(e)$ (by formula (5.1)). If moreover $q_e(a) = 0$, we find $\pi(a)\sigma(e) = 0$. It follows that there is a representation $\pi_e : B_e \to \mathcal{L}(\sigma(e)H)$ satisfying $\pi_e \circ q_e(a) = \pi(a) :_{\sigma(e)H}$ for each $a \in A$ such that $q_e(a) \in B_e$.

5.9. PROPOSITION. *Let (π, σ) be a covariant representation of (A, S, α) on a Hilbert module H and let $\Pi : A \rtimes_\alpha S \to \mathcal{L}(H)$ be the associated representation.*

(a) *If Π is non degenerate, the representations π and σ are nondegenerate and π is uniquely determined by Π.*

(b) *Assume that the following two conditions are satisfied:*

(i) *the representation $\sigma : C^*(S) \to \mathcal{L}(H)$ is nondegenerate;*

(ii) *for each $e \in E$, the representation $\pi_e : B_e \to \mathcal{L}(\sigma(e)H)$ given by $\pi_e \circ q_e(a) = \pi(a) :_{\sigma(e)H}$ for each $a \in A$ such that $q_e(a) \in B_e$ is nondegenerate;* *then Π is nondegenerate.*

(c) *If H is a Hilbert space and Π is nondegenerate, one may choose uniquely σ in such a way that condition* (ii) *is satisfied.*

Proof. (a) The span of $\Pi(\delta_u q_{u^*u}(a))\xi$ with $u \in S$, $a \in A$ such that $q_{u^*u}(a) \in B_{u^*u}$ and $\xi \in H$ is dense in H. Since $\Pi(\delta_u q_{u^*u}(a))\xi = \sigma(u)\pi(a)\xi$, it follows immediately that σ is nondegenerate. Also $\Pi(\delta_u q_{u^*u}(a))\xi = \pi(b)\sigma(u)\xi$, where $b \in A$ is such that $q_{uu^*}(b) = \alpha_u(q_{u^*u}(a))$, so that π is nondegenerate.

Let (π', σ') be another covariant representation associated with Π. Extend σ' to $\widetilde{S}$ by setting $\sigma'(1) = 1$. The corresponding representation $\widetilde{\Pi}$ of $A \rtimes_\alpha \widetilde{S}$ is the unique extension of the non degenerate representation Π, since $A \rtimes_\alpha S$ is an ideal in $A \rtimes_\alpha \widetilde{S}$ (Remark 5.5 (a)). For $a \in A$, we have $\pi'(a) = \sigma'(1)\pi'(a) = \widetilde{\Pi}(\delta_1 a)$. Therefore, Π determines π.

(b) If conditions (i) and (ii) are satisfied then for all $e \in E$, $\Pi(\delta_e B_e)H = \sigma(e)H$ (by condition (ii)), whence $\Pi(A)H$ contains $\sigma(e)H$; as the representation σ is non degenerate, the Hilbert space spanned by the $\sigma(e)H$ is dense in H. It follows that the representation Π is non degenerate.

(c) One may choose σ to be given as in Lemma 5.8. Then for every $e \in E$, and every approximate identity (a^λ) of B_e, $\sigma(e)$ is the strong-$*$-limit of $\pi_e(a^\lambda)\sigma(e)$, i.e. the representation π_e is nondegenerate.

On the other hand, let $u \in S$, choose an approximate identity (b^λ) of $\{x \in A;\ q_{u^*u}(x) \in B_{u^*u}\}$. Then $a^\lambda = q_{u^*u}(b^\lambda)$ is an approximate identity of B_{u^*u}. If the representation π_{u^*u} is nondegenerate, then $\sigma(u^*u)$ is the strong-$*$-limit of $\pi_{u^*u}(a^\lambda)\sigma(u^*u) = \Pi(a^\lambda \delta_{u^*u}) = \Pi(\delta_{u^*u} a^\lambda)$. Therefore $\sigma(u)$ is the strong-$*$-limit of $\sigma(u)\Pi(\delta_{u^*u}a^\lambda) = \sigma(u)\sigma(u^*u)\pi(b^\lambda) = \Pi(\delta_u a^\lambda)$. In other words, $\sigma(u)$ is given by Lemma 5.8. ∎

REMARK. Note that, in general, the representation σ is not determined by Π: take for instance $S = E = \{1, e\}$, $B_1 = A$ and $B_e = \{0\}$. Then $\sigma(e)$ can be taken to be *any* projection in $\mathcal{L}(H)$!

REGULAR REPRESENTATIONS AND THE REDUCED CROSSED PRODUCT. Let $e \in E$, and $S_e = \{u \in S : u^*u = e\}$. Consider the Hilbert B_e-module $\ell^2(S_e; B_e) = B_e \otimes \ell^2(S_e)$. For $x \in A$, $u \in S$ and $\xi \in \ell^2(S_e; B_e)$, we put

$$L^e(a)\xi(u) = \overline{\alpha}_{u^*}(q_{uu^*}(a))\xi(u) \quad\text{and}\quad (\lambda^e(u)\xi)(v) = \begin{cases} \xi(u^*v) & \text{if } vv^* \leqslant uu^*, \\ 0 & \text{otherwise.} \end{cases}$$

5.10. PROPOSITION. *For every $e \in E$, the pair (L^e, λ^e) is a nondegenerate covariant representation of (A, S, α) on the Hilbert A-module $\ell^2(S_e; B_e)$.*

Proof. Let $u \in S$ and $a, b \in A$ be such that $q_{u^*u}(a) \in B_{u^*u}$ and $q_{uu^*}(b) = \alpha_u(q_{u^*u}(a))$. Let $\xi \in \ell^2(S_e; B_e)$ and $v \in S_e$.

If $vv^* \leqslant uu^*$, we have

$$(\lambda^e(u)L^e(a)\xi)(v) = (L^e(a)\xi)(u^*v) = \overline{\alpha}_{v^*u}(q_{u^*vv^*u}(a))\xi(u^*v)$$

and

$$(L^e(b)\lambda^e(u)\xi)(v) = \overline{\alpha}_{v^*}(q_{v^*v}(b))(\lambda^e(u)\xi)(v) = \overline{\alpha}_{v^*}(q_{vv^*}(b))\xi(u^*v).$$

With the notations q_f, $\overline{q}_f$ and $\overline{\alpha}_w$ that we already used in the proof of Proposition 5.3, we have

$$\begin{aligned}\overline{\alpha}_{v^*u}(q_{vv^*}(a)) &= \overline{\alpha}_{v^*} \circ \overline{\alpha}_{vv^*u} \circ \overline{q}_{u^*vv^*u} \circ q_{u^*u}(a) = \overline{\alpha}_{v^*} \circ \overline{q}_{vv^*}(\alpha_u(q_{u^*u}(a))) \\ &= \overline{\alpha}_{v^*} \circ \overline{q}_{vv^*}(q_{uu^*}(b)) = \overline{\alpha}_{v^*}(q_{vv^*}(b)).\end{aligned}$$

If $vv^* \not\leqslant uu^*$, then $(\lambda^e(u)L^e(a)\xi)(v) = (L^e(b)\lambda^e(u)\xi)(v) = 0$. ∎

5.11. DEFINITION. The *reduced crossed product* of the action α, denoted by $A \rtimes_{\alpha,\mathrm{r}} S$, is defined to be the quotient of $A \rtimes_\alpha S$ under the family of representations associated with the nondegenerate covariant representations (L^e, λ^e) for $e \in E$.

THE CROSSED PRODUCT $A \rtimes E$. Let us consider the case where $S = E$ consists only of idempotents.

5.12. PROPOSITION. *For any $e \in E$, the closed linear span J_e in $A \rtimes_\alpha E$ of $\{\delta_f x : f \in E,\ f \leqslant e,\ x \in B_f\}$ is a closed two sided ideal of $A \rtimes_\alpha E$. For $e, f \in E$, we have $J_{ef} = J_e \cap J_f$.*

Proof. By definition J_e is the closed linear span of $\{\delta_f x : f \in E,\ f \leqslant e,\ x \in B_f\}$. For $f, g \in E$ and $x \in B_f$, $y \in B_g$, we have $(\delta_f x)(\delta_g y) = \delta_{fg}(x \diamond y)$. It follows that if $f \leqslant e$ or $g \leqslant e$, then $(\delta_f x)(\delta_g y) \in J_e$, whence J_e is an ideal.

Let $e, f, e', f' \in E$ such that $e' \leqslant e$ and $f' \leqslant f$, $x \in B_{e'}$ and $y \in B_{f'}$. One has $e'f' \leqslant ef$, whence $(\delta_{e'} x)(\delta_{f'} y) = \delta_{e'f'}(x \diamond y) \in J_{ef}$. It follows that $J_e \cap J_f \subset J_{ef}$. The opposite inclusion is obvious. ∎

Note that, if (a^λ) is an approximate identity of B_e, then for all $f \in E$ such that $f \leqslant e$ and $x \in B_f$, the nets $(q_f(a^\lambda)x)$ and $(xq_f(a^\lambda))$ converge to x, whence $(\delta_e a^\lambda)$ is an approximate identity of J_e.

In the case when the B_e's are quotients, we can give a rather explicit description of $A \rtimes_\alpha E$. Recall that $C^*(E)$ is a commutative C^*-algebra that we identify with a $C_0(X)$, where $X = G_S^{(0)}$. Under this identification, the elements of E are $\{0, 1\}$-valued functions on X.

5.13. PROPOSITION. *We write $C^*(E) = C_0(X)$. For $x \in X$, let I_x be the ideal in A generated by $\ker q_e$ for $e \in E$ with $e(x) = 1$. Put $I = \{\varphi \in C_0(X; A) : \forall x \in X,\ \varphi(x) \in I_x\}$. There is a natural embedding*

$$\Psi : A \rtimes_\alpha E \to C_0(X; A)/I.$$

If the B_e's are quotients, i.e. if for all $e \in E$, $q_e(A) = B_e$, then Ψ is an isomorphism.

Proof. For $e \in E$ and $x \in X$ such that $e(x) = 1$, let $q_{x,e} : B_e \to A/I_x$ be the composition $B_e \hookrightarrow A/\ker q_e \to A/I_x$ (since $\ker q_e \subset I_x$).

Let $e \in E$ and $a \in A$ be such that $b = q_e(a) \in B$. Define $\Psi(\delta_e b)$ to be the image in $C_0(X;A)/I$ of $\delta_e \otimes a \in C^*(E) \otimes A = C_0(X;A)$. In other words, for $x \in X$, we have:

$$\big(\Psi(\delta_e b)\big)(x) = \begin{cases} 0 & \text{if } e(x) = 0, \\ q_{x,e}(b) & \text{if } e(x) = 1. \end{cases}$$

This writing shows that $\Psi(\delta_e b)$ only depends on b (and not on a). It follows also easily that Ψ extends to a $*$-homomorphism from $\ell^1(E,B)$ to $C_0(X;A)/I$, and thus defines a $*$-homomorphism $\Psi : A \rtimes_\alpha E \to C_0(X;A)/I$.

Let now Π be an irreducible representation of $A \rtimes_\alpha E$ on a Hilbert space H, and let (π, σ) be the corresponding covariant representation of (A, E, α) satisfying condition (b)(ii) of Proposition 5.9. For any $a \in A$ and $e \in E$, we have $\pi(a)\sigma(e) = \sigma(e)\pi(a)$. One deduces immediately that $\sigma(e)$ commutes with $\Pi(A \rtimes_\alpha E)$, whence it is a scalar. Therefore, it is a character of $C^*(E)$: there exists $x \in X$ such that $\sigma(e) = e(x)$. Let then $e \in E$ be such that $e(x) = 1$. By condition (b)(ii) of Proposition 5.9, there is a non degenerate representation π_e of B_e on $\sigma(e)H = H$ such that, for every $a \in A$ satisfying $q_e(a) \in B_e$, we have $\pi(a) = \pi_e(q_e(a))$. It follows that $\pi(a) = 0$ for all $a \in I_x$, whence there is a representation π' of A/I_x such that $\Pi = \pi' \circ h_x \circ \Psi$, where $h_x : C_0(X;A)/I \to A/I_x$ is the evaluation map.

It follows that Ψ is one to one.

Assume now that the B_e's are quotients. Let t be the trivial action of E on A given by $t(e) = (A, \mathrm{id}, A)$ for all $e \in E$. We obviously have $A \rtimes_t E = A \otimes C^*(E) = C_0(X, A)$. Moreover we have a natural surjective $*$-homomorphism $p : C_0(X, A) = A \rtimes_t E \to A \rtimes_\alpha E$ which maps $\delta_e a$ into $\delta_e q_e(a)$. Now $\Psi \circ p$ is the quotient map $C_0(X, A) \to C_0(X, A)/I$ (as checked on generators), whence Ψ is onto. ∎

Let $x \in X$. Denote by $h_x : C_0(X, A)/I \to A/I_x$ the evaluation map. It is easily seen that, for $e \in E$, the map $h_{\varepsilon_e} \circ \Psi : A \rtimes_\alpha E \to A/\ker q_e \subset \mathcal{M}(B_e)$ is the regular representation associated with e. For $x \in X$ and $f \in C_0(X;A)/I$, $h_x(f)$ is the limit of $h_{\varepsilon_e}(f)$ along the net $\mathcal{F}_x$. It follows that the regular representations form a faithful family of $A \rtimes_\alpha E$. In other words,

$$A \rtimes_\alpha E = A \rtimes_{\alpha,\mathrm{r}} E.$$

6. RELATION WITH CROSSED PRODUCTS IN THE SENSE OF SIEBEN AND WITH GROUPOID CROSSED PRODUCTS

Let us briefly recall Sieben's construction in [19]. As mentioned before, he defines an action of an inverse semigroup S on a C^*-algebra A to be a semigroup homomorphism $\beta : S \to \mathrm{PAut}(A)$.

On $L = \{\psi \in \ell^1(S, A) : \psi(u) \in I_{uu^*}\}$, where $\beta(u) = (I_{uu^*}, \beta_u, I_{u^*u})$, define the convolution by:

$$(\varphi \star \psi)(w) = \sum_{uv=w} \beta_u[\beta_{u^*}(\varphi(u))\psi(v)],$$

and the involution by

$$\varphi^*(u) = \beta_u(\varphi(u^*)^*).$$

The crossed product of A by S is the closure of a quotient of L under the norm induced by taking supremum over a family of representations (called covariant representations — cf. [19], Definition 3.4) of the pair (S, A). This condition is more restrictive than the one we used (Definition 5.6). In order not to mix these definitions, a covariant representation in Sieben's sense will be called hereafter *strictly covariant.*

6.1. DEFINITION. (cf. [19], Definition 3.4) A *strictly covariant representation* of (A, S, α) on a Hilbert space H is a pair (π, σ), where $\pi : A \to \mathcal{L}(H)$ and $\sigma : S \to \mathcal{L}(H)$ are $*$-representations such that:

(a) for all $u \in S$ and $a \in I_{u^*u}$ we have $\pi(\beta_u(a))\sigma(u) = \sigma(u)\pi(a)$;

(b) for all $e \in E$, $\sigma(e)$ is the projection onto $\pi(I_e)H$.

It is very easily seen that every strictly covariant representation is covariant in the sense of 5.6. On the other direction, every representation of L satisfies condition (a) in Definition 6.1, but not necessarily condition (b).

To see the exact relation between our notion of crossed product and the one defined by Sieben, note that when B and C are ideals and $D = B \wedge C = B \cap C$, we have $x \diamond y = q_{D,B}(x)q_{D,C}(y) = xy$, for $x \in B$ and $y \in C$. Since $\alpha_{v^*u^*u}$ is the restriction of α_{v^*}, our convolution formula in Proposition 5.3 reduces in this case to

$$(\varphi \star \psi)(w) = \sum_{uv=w} \alpha_{v^*}(\varphi(u)\alpha_v(\psi(v))).$$

It follows that the map $\varphi \mapsto \psi$ where $\psi(u) = \beta_u(\varphi(u))$ is a $*$-isomorphism from our $\ell^1(S, B)$ onto Sieben's L. However, his crossed product is a quotient of the one defined here. In particular, Sieben proves that: $A \rtimes E \cong A$ (cf. [19], Corrolary 4.6).

Here is a result which relates our construction to Sieben's.

6.2. THEOREM. *Let $\alpha : S \to \mathrm{SQ}(A)$ be an action of S on a C^*-algebra A. Associated to α is a natural action β of S on $A \rtimes_\alpha E$ by partial automorphisms. The crossed product $A \rtimes_\alpha S$ is naturally isomorphic to the crossed product in Sieben's sense of $A \rtimes_\alpha E$ by β.*

The action β is defined in the following obvious lemma, where the J_e's are those defined in Proposition 5.12.

6.3. LEMMA. (a) *For $u \in S$, there is a $*$-isomorphism $\beta_u : J_{u^*u} \to J_{uu^*}$ such that, for all $f \in E$ satisfying $f \leqslant u^*u$ and for all $x \in B_f$, we have* $\beta_u(\delta_f x) = \delta_{ufu^*}\alpha_{u:B_f}(x)$.

(b) *For all $u \in S$ and $e \in E$, such that $e \leqslant u^*u$, the morphism β_{ue} is the restriction to B_e of β_u.*

(c) *If $u, v \in S$ are such that $u^*u = vv^*$, we have $\beta_u \circ \beta_v = \beta_{uv}$.*

From Lemma 6.3, it follows that the map $\beta : u \mapsto (J_{uu^*}, \beta_u, J_{u^*u})$ is an action of S by partial automorphisms.

Let now Π be a nondegenerate faithful representation of $A \rtimes_\alpha S$. Associated to Π is a covariant representation (π, σ) of (A, S, α) satisfying the conditions (b)(i) and (b)(ii) of Proposition 5.9. Let $\overline{\pi}$ be the restriction of Π to $A \rtimes_\alpha E$.

6.4. LEMMA. *The pair $(\overline{\pi}, \sigma)$ is a strictly covariant representation of the triple $(A \rtimes_\alpha E, S, \beta)$.*

Proof. Let $u \in S$, $f \in E$ and $x \in B_f$. Assume that $f \leqslant u^*u$. Let $a, b \in A$ such that $q_f(a) = x$ and $q_{ufu^*}(b) = \alpha_{uf}(x)$. We have $\overline{\pi}(\delta_f x) = \Pi(\delta_f x) = \sigma(f)\pi(a)$. Moreover $\overline{\pi}(\beta_u(\delta_f x)) = \Pi(\delta_{ufu^*}\alpha_{uf}(x)) = \sigma(ufu^*)\pi(b) = \pi(b)\sigma(ufu^*)$. Therefore

$$\overline{\pi}(\beta_u(\delta_f x))\sigma(u) = \pi(b)\sigma(ufu^*)\sigma(u) = \pi(b)\sigma(uf) = \sigma(uf)\pi(a) = \sigma(u)\overline{\pi}(\delta_f x).$$

Moreover, by condition (b)(ii) of Proposition 5.9, the range of the projection $\sigma(e)$ is $\pi_e(B_e)\sigma(e)H$, which contains $\overline{\pi}(J_e)H$. The opposite inclusion holds for any covariant pair. ∎

Proof of Theorem 6.2. Let $\mathcal{A}$ denote the crossed product in the sense of Sieben of $A \rtimes_\alpha E$ by the action β. It follows from Lemma 6.4 that there is a representation Π' of $\mathcal{A}$ in H characterized by the formula $\Pi'(\delta_u z) = \sigma(u)\overline{\pi}(z)$ for $u \in S$ and $z \in J_{u^*u}$. In particular, $\Pi'(\delta_u(\delta_f x)) = \sigma(u)\sigma(f)x = \Pi(\delta_{uf}x)$ for $f \in E$ such that $f \leqslant u^*u$ and $x \in B_f$. Hence $\Pi'(\mathcal{A}) \subset \Pi(A \rtimes_\alpha S)$. Since Π is faithful, there exists a unique $*$-homomorphism $\chi : \mathcal{A} \to A \rtimes_\alpha S$ such that $\Pi' = \Pi \circ \chi$. Note that $\chi(\delta_u(\delta_f x)) = \delta_{uf}x$ for $u \in S$, $f \in E$ such that $f \leqslant u^*u$ and $x \in B_f$. Taking $f = u^*u$ we immediately see that χ is surjective.

Let now $(\overline{\pi}, \sigma)$ be a strictly covariant representation of $(A \rtimes_\alpha E, S, \beta)$. Denote by Π' the corresponding representation of $\mathcal{A}$. We may further assume that Π' is nondegenerate, whence $\overline{\pi}$ is nondegenerate. Corresponding to it is a covariant representation (π, τ) of (A, E, α), satisfying condition (b)(ii) in Proposition 5.9. Let $e \in E$ and (a^λ) an approximate identity of B_e. It follows from the construction of τ (cf. Lemma 5.8) that $\overline{\pi}(\delta_e a^\lambda)$ converges strongly to $\tau(e)$. On the other hand, $(\delta_e a^\lambda)$ is an approximate identity of B_e; whence by condition (b) in Definition 6.1, $\big(\overline{\pi}(\delta_e a^\lambda)\big)$ converges strongly to $\sigma(e)$.

Let $u \in S$ and $a, b \in A$ such that $q_{u^*u}(a) \in B_{u^*u}$ and $\alpha_u(q_{u^*u}(a)) = q_{uu^*}(b)$. Then $\delta_{uu^*}q_{uu^*}(b) = \beta_u(\delta_{u^*u}q_{u^*u}(a))$. Whence,

$$\begin{aligned}\pi(b)\sigma(u) &= \pi(b)\sigma(uu^*)\sigma(u) = \pi(b)\tau(uu^*)\sigma(u) = \overline{\pi}(\delta_{uu^*}q_{uu^*}(b))\sigma(u)\\ &= \sigma(u)\overline{\pi}(\delta_{u^*u}q_{u^*u}(a)) = \sigma(u)\pi(a).\end{aligned}$$

In other words, (π, σ) is a covariant representation of (A, S, α). For $u \in S$, $f \in E$ such that $f \leqslant u^*u$ and $x \in B_f$ we have $\Pi'(\delta_u(\delta_f x)) = \sigma(u)\overline{\pi}(\delta_f x) =$

$\sigma(u)\sigma(f)\pi(a) = \sigma(uf)\pi(a)$, for every $a \in A$ such that $q_f(a) = x$. In other words, we have $\Pi' = \Pi \circ \chi$, where Π is the representation of $A \rtimes_\alpha S$ associated with the covariant representation (π, σ).

This proves that every strictly covariant representation of $(A \rtimes_\alpha E, S, \beta)$ factors through χ, whence χ is faithful. ∎

THE CASE OF QUOTIENTS.

6.5. THEOREM. *Let S be an inverse semigroup with the set of idempotents E, and (B, α) an action of S on a C^*-algebra A. We assume the B_e's are quotients: in other words, we assume that for all $e \in E$, $q_e(A) = B_e$.*

(a) *The crossed product $A \rtimes_\alpha E$ is naturally endowed with an action of the groupoid G associated with S.*

(b) *We have natural isomorphisms*

$$A \rtimes_\alpha S \cong (A \rtimes_\alpha E) \rtimes G \quad \textit{and} \quad A \rtimes_{\alpha,\mathrm{r}} S \cong (A \rtimes_\alpha E) \rtimes_{\mathrm{r}} G.$$

This result is a consequence of Theorem 6.2 together with [16]. Before proceeding with the proof, let us recall some facts from [16]. If G is a locally compact r-discrete groupoid, to any action α of G on a C^*-algebra A there corresponds naturally an action by partial automorphisms of the inverse semigroup S_G of open G-sets ([16], Section 5). We then have an isomorphism of $A \rtimes_\alpha G$ with the crossed product of A by the action of S_G in the sense of [19] (cf. [16], Section 7). Moreover, an action by partial automorphisms of the inverse semigroup S_G on a C^*-algebra A comes from an action of G if and only if the algebra A is, in a compatible way, a $C_0(G^{(0)})$-algebra ([16], Section 6).

Note that all these are also true if one replaces S_G by any of its full sub-semigroups. In particular, this holds for any inverse semigroup S considered as a full sub-semigroup of the groupoid G_S. Indeed, the map $u \in S \mapsto O_u$ defines an embedding of S into S_{G_S} as a full sub-semigroup (see 2.6 (b)).

Proof. The algebra $C_0(X) = C^*(E)$ is generated by projections δ_e, $e \in E$. It acts on $A \rtimes_\alpha E$ by multiplication (Proposition 5.13). It follows now from [16] that there is an action of G on $A \rtimes_\alpha E$ such that the crossed product $(A \rtimes_\alpha E) \rtimes G$ is isomorphic to the crossed product in the sense of [19] of $A \rtimes_\alpha E$ by S; by Theorem 6.2, the latter is isomorphic to $A \rtimes_\alpha S$.

In the light of Proposition 5.13, it is quite easy to give the isomorphism

$$\Phi : A \rtimes_\alpha S \to (A \rtimes_\alpha E) \rtimes G$$

on the generators: let $u \in S$ and $b \in B_{u^*u}$. Then $\Phi(\delta_u b)$ is the image in $\mathcal{A} \subset (A \rtimes_\alpha E) \rtimes G$ of the function $\chi_u \varphi_b^s$, where χ_u is the characteristic function of the compact (Hausdorff) open set O_u (cf. 2.6 (b)) and φ_b^s is the function which to $\gamma \in O_u$ associates the class of b modulo $I_{s(\gamma)}$. Note that since $s(\gamma) \in F_{u^*u}$, we have $I_{s(\gamma)} \supset \ker q_{u^*u}$, whence the class of b modulo $I_{s(\gamma)}$ is well defined.

Let us now turn to the reduced crossed product. Consider the algebra D' associated with the action of $G = G_S$ in $D = A \rtimes_\alpha E$ introduced in 3.7. By definition, D' is a subalgebra of $\prod_{x \in X} D_x$; let $p_x : D' \to D_x$ be the natural evaluation map. The reduced crossed product $(A \rtimes_\alpha E) \rtimes_{\mathrm{r}} G$ is defined thanks to a faithful representation Λ on the Hilbert D'-module $L^2(G, \nu; D')$. For $x \in X$, we also put $\Lambda_x = \Lambda \otimes_{p_x} 1$. Let $e \in E$ and ε_e the corresponding element of X (2.6 (c)). Following the above identifications, one checks that $\Lambda_{\varepsilon_e} \circ \Phi$ is the regular representation of $A \rtimes_\alpha S$ associated with the covariant representation (L^e, λ^e) (Proposition 5.10). We have thus constructed a surjective homomorphism $\Psi : (A \rtimes_\alpha E) \rtimes_{\mathrm{r}} G \to A \rtimes_{\alpha,\mathrm{r}} S$.

We have to show that this homomorphism is an isomorphism, i.e. that the family $(\Lambda_{\varepsilon_e})_{e\in E}$ is a faithful family of representations of $(A \rtimes_\alpha E) \rtimes_{\mathrm{r}} G$.

Recall some facts from [9] (see also [15]). An element $x \in X$ is determined by the set $\mathcal{F}_x = \{e \in E : e(x) = 1\}$. Moreover, $\mathcal{F}_x$ is a directed ordered set by $p \ll q$ if $q \leqslant p$, and the net $(\varepsilon_e)_{e\in\mathcal{F}_x}$ converges to x.

By Proposition 5.13, $D_x = A/\Big(\overline{\bigcup_{e\in\mathcal{F}_x} \ker q_e}\Big)$. In particular, for $e \in E$ we have $D_{\varepsilon_e} = B_e$. For $x \in X$ and $e \in \mathcal{F}_x$ we denote by $q_{x,e} : B_e \to D_x$ the natural quotient map.

6.6. LEMMA. *For every $x \in X$ and $b \in D'$, we have* $p_x(b) = \lim_{e\in\mathcal{F}_x} q_{x,e}(p_{\varepsilon_e}(b))$.

Proof. The proof is similar to Lemma 3.4 of [9].

It is enough to check this equality on generators, i.e. if b is the restriction to X of some $\Phi(\delta_u a)$ for $u \in S$ and $a \in B_{u^*u}$. In that case, for $z \in X$, we have

$$p_z(b)(z) = \begin{cases} 0 & \text{if } z \notin O_{u^*u}, \\ 0 & \text{if } z \in O_{u^*u},\ \widetilde{(u,z)} \neq z, \\ q_{z,u^*u}(a) & \text{if } z \in O_{u^*u},\ \widetilde{(u,z)} = z, \end{cases}$$

where $\widetilde{(u,z)}$ denotes the class of (u,z) in G.

If $x \notin O_{u^*u}$, then as $u^*u \notin \mathcal{F}_x$. For every $e \in \mathcal{F}_x$ we have $e \nleqslant u^*u$, whence $(u^*u)(\varepsilon_e) = 0$, i.e. $\varepsilon_e \notin O_{u^*u}$; therefore $0 = q_{x,e}(p_{\varepsilon_e}(b))$ converges to $0 = p_x(b)$.

Second, take $x \in O_{u^*u}$, but $\widetilde{(u,x)} \neq x$. Then, for all $e \in \mathcal{F}_x$, we have $\varepsilon_p \notin O_{u^*u}$, for otherwise we get that $\widetilde{(u,x)} = \widetilde{(e,x)}$ which is in contradiction with $\widetilde{(u,x)} \neq x$; therefore $0 = q_{x,e}(p_{\varepsilon_e}(b))$ converges to $0 = p_x(b)$.

Finally, assume $\widetilde{(u,x)} = x$. Then, by definition of G, there exists $e_0 \in \mathcal{F}_x$ such that $ue_0 = e_0$. For all $e \in \mathcal{F}_x$ with $e_0 \ll e$, we have $ue = e$ whence $\widetilde{(u,\varepsilon_e)} = \varepsilon_e$. Therefore $p_{\varepsilon_e}(b) = q_{\varepsilon_e,u^*u}(a)$ and $p_x(b) = q_{x,u^*u}(a)$. It follows that for $e \in \mathcal{F}_x$ with $e_0 \ll e$, we have $q_{x,e}(p_{\varepsilon_e}(b)) = p_x(b)$. ∎

End of the proof of Theorem 6.5. By Lemma 6.6, the family of representations $(p_{\varepsilon_e})_{e\in E}$ of D' is faithful. It follows that the family of representations $(T \mapsto T \otimes_{p_{\varepsilon_e}} 1)_{e\in E}$ of $\Lambda((A \rtimes_\alpha E) \rtimes G) \subset \mathcal{L}(L^2(G,\nu;D'))$ is faithful (cf. Lemma 2.1, [9]). In other words, the family $(\Lambda_{\varepsilon_e})_{e\in E}$ is a faithful family of representations of $(A \rtimes_\alpha E) \rtimes_{\mathrm{r}} G$. ∎

6.7. EXAMPLE. We end by the computation of $A \rtimes S$ in the case of Example 5.2 (c). In this case, S is finite. By Property 2.3 (c), the points of G_S are closed: it follows that the injection with dense range from S into G_S given by 2.6 (c) is onto, i.e. $S \cong G_S$ and $E \cong S$. It follows immediately that we have an isomorphism

$$C^*(G_S) = C^*_{\mathrm{r}}(G_S) = C^*_{\mathrm{r}}(S) = C^*(S) \cong M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}.$$

Now consider the action of S on the C^*-algebra defined in Example 5.2 (c). In order to better understand the algebra $A \rtimes_\alpha S$, we will use the sub-semigroup $S_1 = S \setminus \{1\}$ with set of idempotents $E_1 = \{0, e, f\}$. First note that the map $(a, b) \mapsto a\delta_e + b\delta_f$ defines an isomorphism $J \oplus A/J \cong \mathcal{K} \oplus \mathcal{K} \cong A \rtimes_\alpha E_1$. It follows that we have an isomorphism $A \rtimes S_1 \cong M_2(\mathcal{K})$. Note now that $S = \widetilde{S_1}$ and $E = \widetilde{E_1}$. We thus get extensions

$$0 \to \mathcal{K} \oplus \mathcal{K} \to A \rtimes_\alpha E \to A \to 0 \quad \text{and} \quad 0 \to M_2(\mathcal{K}) \to A \rtimes_\alpha S \to A \to 0.$$

Put moreover $E_2 = \{1, e\}$. An easy check shows that

$$A \rtimes E_2 = \{(x, y) \in A \oplus A : x - y \in J\} \cong (\widetilde{\mathcal{K} \oplus \mathcal{K}}) \otimes \mathcal{K}.$$

Using the exact sequence $0 \to A \rtimes \{e\} \to A \rtimes E_2 \to A \to 0$, we find a commuting diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K} & \to & A \rtimes E_2 & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & M_2(\mathcal{K}) & \to & A \rtimes_\alpha S & \to & A & \to & 0. \end{array}$$

It follows that in the Busby invariant $A \to \mathcal{M}(M_2(\mathcal{K}))/M_2(\mathcal{K})$ associated to this exact sequence, the image of J is 0 and the image of any nonzero projection in A/J is nonzero, therefore lifts to an infinite dimensional projection in $\mathcal{M}(M_2(\mathcal{K}))$. We deduce an isomorphism

$$A \rtimes_\alpha S \cong (\widetilde{\mathcal{K} \oplus \mathcal{K}}) \otimes \mathcal{K}.$$

Acknowledgements. The first author was partially supported by an NSERC grant.

REFERENCES

1. C. Anantharaman-Delaroche, J. Renault, *Amenable Groupoids*, Monographies de L'Enseignement Mathématique, vol. 36, Geneva 2000.
2. A. Connes, Sur la théorie non commutative de l'intégration, *Lect. Notes in Math.*, vol. 725, Springer, New York 1979, pp. 19–143.
3. A. Connes, A survey of foliations and operator algebras, in *Operator Algebras and Applications*, Proc. Sympos. Pure Math., vol. 38, Amer. Math. Soc., Providence, RI, 1982, pp. 521-628.
4. A. Connes, *Non Commutative Geometry*, Academic Press, 1994.
5. J. Duncan, A. Paterson, C^*-algebras of inverse semigroups, *Proc. Edinburgh Math. Soc.* **28**(1985), 41–58.
6. R. Exel, Circle actions on C^*-algebras, partial auotomorphisms and a generalized Pimsner-Voiculescu exact sequence, *J. Func. Anal.* **122**(1994), 361–401.
7. J.W. Howie, *An Introduction to Semigroup Theory*, Academic Press, London 1976.
8. G.G. Kasparov, Equivariant KK-theory and the Novikov conjecture, *Invent. Math.* **91**(1988), 147–201.
9. M. Khoshkam, G. Skandalis, Regular representation of groupoids and applications to inverse semigroups, *J. Reine Angew. Math.* **546**(2002), 47–72.
10. A. Kumjian, On localizations and simple C^*-algebras, *Pacific J. Math.* **112**(1984), 141–192.
11. M. Lawson, *Inverse Semigroups: The Theory of Partial Symmetries*, World Sientific Publ. Co., Inc., River Edge, NJ, 1998.

12. P-Y. LE GALL, Théorie de Kasparov équivariante et groupoïdes. I, *K-Theory* **16** (1999), 361–390.
13. K. MCCLANAHAN, K-theory for partial crossed products by discrete groups, *J. Funct. Anal.* **130**(1995), 77–117.
14. A. NICA, On a groupoid construction for actions of certain inverse semigroups, *Internat. J. Math.* **5**(1994), 349–372.
15. A. PATERSON, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Birkhäuser, Boston 1998.
16. J. QUIGG, N. SIEBEN, C^*-actions of r-discrete groupoids and inverse semigroups, *J. Austral. Math. Soc. Ser. A* **66**(1999), 143–167.
17. J.N. RENAULT, *A Groupoid Approach to C^*-Algebras*, Lect. Notes in Math., vol. 793, Springer-Verlag, New York 1980.
18. J.N. RENAULT, Représentation des produits croisés d'algèbres de groupoïdes, *J. Operator Theory* **18**(1987), 67–97.
19. N. SIEBEN, C^*-crossed product by partial actions and actions of inverse semigroups, *J. Austral. Math. Soc. Ser. A* **63**(1997), 32–46.

MAHMOOD KHOSHKAM
Department of Mathematics
and Statistics
University of Saskatchewan
Saskatoon
CANADA

E-mail: khoshkam@snoopy.usask.ca

GEORGES SKANDALIS
Université Denis Diderot (Paris VII) - CNRS
UFR de Mathématiques - UMR 7586
Case Postale 7012
2, Place Jussieu
75251 Paris, Cedex 05
FRANCE

E-mail: skandal@math.jussieu.fr

Received March 16, 2002.

Note added in proof. In october 2003, Mahmood Khoshkam lost his fight against cancer. May this last joint article testify of our long-lasting friendship and mathematical collaboration.