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ON THE POLAR DECOMPOSITION OF THE ALUTHGE TRANSFORMATION AND RELATED RESULTS

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Dedicated to Professor Hisaharu Umegaki on his seventy-seventh birthday

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Abstract. Let $T = U|T|$ be the polar decomposition of a bounded linear operator T on a Hilbert space. The transformation $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called the Aluthge transformation and $\widetilde{T}_n$ means the n-th Aluthge transformation. In this paper, firstly, we show that $\widetilde{T} = VU|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$, where $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = V\ \big||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\big|$ is the polar decomposition. Secondly, we show that $\widetilde{T} = U|\widetilde{T}|$ if and only if T is binormal, i.e., $[|T|, |T^*|] = 0$, where $[A, B] = AB - BA$ for any operators A and B. Lastly, we show that $\widetilde{T}_n$ is binormal for all non-negative integer n if and only if T is centered, i.e., $\{T^n(T^n)^*,\ (T^m)^*T^m : n \text{ and } m \text{ are natural numbers}\}$ is commutative.

Keywords: *Aluthge transformation, polar decomposition, binormal operators, centered operators, weakly centered operators.*

MSC (2000): 47A05, 47B20.

1. INTRODUCTION

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be *positive* (denoted by $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in H$. Let $T = U|T|$ be the polar decomposition of T. In [1], Aluthge defined a transformation $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ which was later called the *Aluthge transformation*. Aluthge transformation is very useful, and many authors have obtained results by using it. Mainly, these results were on non-normal operators, for example [2], [8] and [12]. Moreover, for each non-negative integer n, Jung, Ko and Pearcy defined the *n-th Aluthge transformation* $\widetilde{T}_n$ in [8] as

$$\widetilde{T}_n = (\widetilde{\widetilde{T}_{n-1}}) \quad \text{and} \quad \widetilde{T}_0 = T.$$

One of the authors showed some properties of the n-th Aluthge transformation on operator norms as parallel results to those of powers of operators in [13], [14], [15] and [16]. On the other hand, the polar decomposition of Aluthge transformation was discussed in [1], but the complete solution of this problem has not been obtained. In Section 2, we will obtain the polar decomposition of the Aluthge transformation.

An operator T is said to be *binormal* if $[|T|, |T^*|] = 0$, where $|T| = (T^*T)^{\frac{1}{2}}$ and $[A, B] = AB - BA$ for operators A and B. Binormality of operators was defined by Campbell in [3], and he showed some properties of binormal operators in [4]. An operator T is said to be *centered* if the following sequence

$$\ldots, T^3(T^3)^*, T^2(T^2)^*, TT^*, T^*T, (T^2)^*T^2, (T^3)^*T^3, \ldots$$

is commutative, which is defined in [10]. Morrel and Muhly showed some properties of centered operators, and obtained a nice structure of centered operators. An operator T is said to be *quasinormal* if $T^*TT = TT^*T$. Relations among these operator classes are easily obtained as follows:

$$\text{quasinormal} \subset \text{centered} \subset \text{binormal}.$$

We remark that binormal operators are called *weakly centered operators* in [11].

In Section 3, we obtain a characterization of binormal operators via Aluthge transformation. Most results on $\widetilde{T}$ show that it generally has better properties than T. However, we have an example of a binormal operator T such that $\widetilde{T}$ is not binormal. In this section, we also obtain an equivalent condition to the binormality of $\widetilde{T_k}$ for all $k = 0, 1, 2, \ldots, n$.

In Section 4, we will show a characterization of centered operators.

2. POLAR DECOMPOSITION OF THE ALUTHGE TRANSFORMATION

In this section we show the polar decomposition of the Aluthge transformation as follows:

THEOREM 2.1. *Let $T = U|T|$ and*

$$|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = V\left||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right| \tag{2.1}$$

be the polar decompositions. Then $\widetilde{T} = VU|\widetilde{T}|$ is also the polar decomposition.

By Theorem 2.1, we can obtain the polar decomposition of the n-th Aluthge transformation for all natural number n, because the partial isometry which appears in the polar decomposition of $\widetilde{T}$ is only the product of two partial isometries.

Proof of Theorem 2.1. (i) Proof of $\widetilde{T} = VU|\widetilde{T}|$.

$$\begin{aligned}VU|\widetilde{T}| &= VU(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}}U^*U = V(|T^*|^{\frac{1}{2}}|T|\,|T^*|^{\frac{1}{2}})^{\frac{1}{2}}U \\ &= V\left||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right|U = |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U \quad \text{by (2.1)} \\ &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \widetilde{T}.\end{aligned}$$

(ii) We will show $N(\widetilde{T}) = N(VU)$.

$$\begin{aligned}VUx = 0 &\Leftrightarrow |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}Ux = 0 \quad \text{by } N(V) = N(|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}) \\ &\Leftrightarrow |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x = 0 \\ &\Leftrightarrow \widetilde{T}x = 0,\end{aligned}$$

that is, $N(VU) = N(\widetilde{T})$.

(iii) By (ii), we have $N(VU)^{\perp} = N(|\widetilde{T}|)^{\perp} = \overline{R(|\widetilde{T}|)}$. Then for any $x \in N(VU)^{\perp}$, there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset H$ such that $x = \lim_{n\to\infty} |\widetilde{T}|y_n$. Then we obtain

$$\begin{aligned}\|VUx\| &= \|VU\lim_{n\to\infty}|\widetilde{T}|y_n\| = \|\lim_{n\to\infty}VU|\widetilde{T}|y_n\| = \|\lim_{n\to\infty}\widetilde{T}y_n\| \quad \text{by (i)} \\ &= \lim_{n\to\infty}\|\widetilde{T}y_n\| = \lim_{n\to\infty}\|\,|\widetilde{T}|y_n\| = \|\lim_{n\to\infty}|\widetilde{T}|y_n\| = \|x\|,\end{aligned}$$

that is, VU is a partial isometry.

Therefore the proof is complete by (i), (ii) and (iii). ∎

3. APPLICATIONS TO BINORMAL OPERATORS

In this section we first show a characterization of binormal operators via Aluthge transformation.

THEOREM 3.1. *Let $T = U|T|$ be the polar decomposition. Then*

$$\widetilde{T} = U|\widetilde{T}| \iff T \text{ is binormal.}$$

REMARK 3.2. Usually, $\widetilde{T} = U|\widetilde{T}|$ in Theorem 3.1 is not the polar decomposition since $N(U) = N(\widetilde{T})$ does not hold (see Proposition 3.9).

Proof of Theorem 3.1. Proof of ($\Rightarrow$). By the assumption $\widetilde{T} = U|\widetilde{T}|$, we have

$$|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}U^* = \widetilde{T}U^* = U|\widetilde{T}|U^* \geqslant 0,$$

then $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = |T^*|^{\frac{1}{2}}|T|^{\frac{1}{2}}$, that is, T is binormal.

Proof of ($\Leftarrow$). If T is binormal, then we have $0 \leqslant |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = \left||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right|$. Then

$$\begin{aligned}\widetilde{T} &= |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}U = \left||T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}\right|U \\ &= UU^*(|T^*|^{\frac{1}{2}}|T|\,|T^*|^{\frac{1}{2}})^{\frac{1}{2}}U = U(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}} = U|\widetilde{T}|.\end{aligned}$$

Hence the proof is complete. ∎

For each $p > 0$, an operator T is said to be *p-hyponormal* if $(T^*T)^p \geqslant (TT^*)^p$. In particular, 1-hyponormality is called *hyponormality* and $\frac{1}{2}$-hyponormality is called *semi-hyponormality*. An operator T is said to be ∞*-hyponormal* if T is p-hyponormal for all $p > 0$ which is defined in [9]. An operator T is said to be *paranormal* if $\|T^2x\| \geqslant \|Tx\|^2$ for all unit vector $x \in H$. It is well known that the following relations among these classes of operators hold for $0 < q < p$:

$$\infty\text{-hyponormal} \subset p\text{-hyponormal} \subset q\text{-hyponormal} \subset \text{paranormal}.$$

As an application of Theorem 3.1, we have a result on hyponormality of paranormal operators as follows:

COROLLARY 3.3. *Let $T = U|T|$ be paranormal and $\widetilde{T} = U|\widetilde{T}|$. Then T is binormal and hyponormal.*

We note that T is ∞-hyponormal in Corollary 3.3 since binormality and hyponormality ensure ∞-hyponormality.

To prove Corollary 3.3, we need the following result:

THEOREM A. ([4]) *Let T be a binormal operator. If T is also paranormal, then T is hyponormal.*

Proof of Corollary 3.3. By Theorem 3.1 and $\widetilde{T} = U|\widetilde{T}|$, T is binormal. Hence T is binormal and paranormal, then T is hyponormal by Theorem A. ∎

Campbell obtained a binormal operator T such that T^2 is not binormal in [4], and Furuta obtained an equivalent condition for binormality of T^2 when T is binormal as follows:

THEOREM B. ([6]) *Let $T = U|T|$ be the polar decomposition of T. If T is binormal, then T^2 is binormal if and only if the following four properties hold:*

(i) $[(U^2)^*U^2, U^2(U^2)^*] = 0$;
(ii) $[U^2(U^2)^*, U^*|T|\,|T^*|U] = 0$;
(iii) $[(U^2)^*U^2, U|T|\,|T^*|U^*] = 0$;
(iv) $[U^*|T|\,|T^*|U, U|T|\,|T^*|U^*] = 0$.

On the other hand, as a nice application of Furuta inequality [7], Aluthge showed that if T is p-hyponormal for $0 < p \leqslant \frac{1}{2}$, then $\widetilde{T}$ is $(p+\frac{1}{2})$-hyponormal in [1]. This result states that $\widetilde{T}$ has a better property than T. Hence one might expect that $\widetilde{T}$ is also binormal if T is binormal. But there is a counterexample for this expectation as follows:

EXAMPLE 3.4. There exists a binormal operator T such that $\widetilde{T}$ is not binormal.

Let $T = \begin{pmatrix} 0 & 0 & 5 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \end{pmatrix}$ and $T = U|T|$ be the polar decomposition. Then T is binormal since

$$T^*T \cdot TT^* = TT^* \cdot T^*T = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 25 \end{pmatrix},$$

and also $|T| = (T^*T)^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, so that $U = T|T|^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \end{pmatrix}$.

Therefore $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & \sqrt{5} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{15}}{2} & \frac{-\sqrt{5}}{2} & 0 \end{pmatrix}$. We get that

$$(\widetilde{T})^*\widetilde{T} \cdot \widetilde{T}(\widetilde{T})^* = \begin{pmatrix} 20 & -\sqrt{3} & 0 \\ -5\sqrt{3} & 2 & 0 \\ 0 & 0 & 25 \end{pmatrix}$$

and

$$\widetilde{T}(\widetilde{T})^* \cdot (\widetilde{T})^*\widetilde{T} = \begin{pmatrix} 20 & -5\sqrt{3} & 0 \\ -\sqrt{3} & 2 & 0 \\ 0 & 0 & 25 \end{pmatrix}.$$

Hence $\widetilde{T}$ is not binormal.

Here we shall show an equivalent condition for binormality of $\widetilde{T}$ as follows:

Theorem 3.5. *Let $T = U|T|$ be the polar decomposition of a binormal operator T. Then the following assertions are equivalent:*

(i) *$\widetilde{T}$ is binormal;*

(ii) $[U^2|T|(U^2)^*, |T|] = 0$.

As a preparation of this discussion, we shall state the following lemma which is a modification of Theorem 2 of [5].

Lemma C ([5]). *Let $A, B \geqslant 0$ and $[A, B] = 0$. Then*

$$[P_{N(A)^\perp}, P_{N(B)^\perp}] = [P_{N(A)^\perp}, B] = [A, P_{N(B)^\perp}] = 0,$$

where $P_{\mathcal{M}}$ is the projection onto a closed subspace $\mathcal{M}$.

We remark that if T is binormal, then the following assertion holds by Lemma C.

$$[|T|, |T^*|] = [U^*U, U|T|U^*] = [|T|, UU^*] = [U^*U, UU^*] = 0. \tag{3.1}$$

Proof of Theorem 3.5. First, we note that T is binormal if and only if

$$[U|T|U^*, |T|] = 0. \tag{3.2}$$

Then we obtain

$$[U|T|U^*, U^2|T|(U^2)^*] = 0 \tag{3.3}$$

since

$$\begin{aligned} U^2|T|(U^2)^* \cdot U|T|U^* &= U \cdot U|T|U^* \cdot |T| \cdot U^* \\ &= U \cdot |T| \cdot U|T|U^* \cdot U^* \quad \text{by (3.2)} \\ &= U|T|U^* \cdot U^2|T|(U^2)^*. \end{aligned}$$

Therefore we have

$$
\begin{aligned}
|\widetilde{T}|^2|\widetilde{T}^*|^2 &= |T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}} \cdot |T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}} \\
&= U^* \cdot U|T|^{\frac{1}{2}}U^* \cdot |T| \cdot U|T|U^* \cdot U^2|T|(U^2)^* \cdot U|T|^{\frac{1}{2}}U^* \cdot U \\
&= U^*\{|T| \cdot U^2|T|(U^2)^*\}U|T|^2U^*U \quad \text{by (3.2) and (3.3)} \\
&= U^*\{|T| \cdot U^2|T|(U^2)^*\}U|T|^2
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
|\widetilde{T}^*|^2|\widetilde{T}|^2 &= |T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}} \cdot |T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}} \\
&= U^* \cdot U|T|^{\frac{1}{2}}U^* \cdot U^2|T|(U^2)^* \cdot U|T|U^* \cdot |T| \cdot U|T|^{\frac{1}{2}}U^* \cdot U \\
&= U^*\{U^2|T|(U^2)^* \cdot |T|\}U|T|^2U^*U \quad \text{by (3.2) and (3.3)} \\
&= U^*\{U^2|T|(U^2)^* \cdot |T|\}U|T|^2.
\end{aligned} \tag{3.5}
$$

Proof of (ii) $\Rightarrow$ (i). By (3.4) and (3.5), we have (i).

Proof of (i) $\Rightarrow$ (ii). Since $\widetilde{T}$ is binormal, we have

$$
\begin{aligned}
\{U^2|T|(U^2)^* \cdot |T|\}U|T|^2 &= UU^*\{U^2|T|(U^2)^* \cdot |T|\}U|T|^2 \\
&= UU^*\{|T| \cdot U^2|T|(U^2)^*\}U|T|^2 \quad \text{by (3.4) and (3.5)} \\
&= \{|T|UU^* \cdot U^2|T|(U^2)^*\}U|T|^2 \quad \text{by (3.1)} \\
&= \{|T| \cdot U^2|T|(U^2)^*\}U|T|^2,
\end{aligned}
$$

that is, $U^2|T|(U^2)^* \cdot |T| = |T| \cdot U^2|T|(U^2)^*$ on

$$\overline{R(U|T|^2)} = N(|T|^2U^*)^\perp = N(UU^*)^\perp = R(UU^*).$$

In other words,

$$U^2|T|(U^2)^* \cdot |T| \cdot UU^* = |T| \cdot U^2|T|(U^2)^* \cdot UU^* \tag{3.6}$$

holds. Hence, we have

$$
\begin{aligned}
U^2|T|(U^2)^* \cdot |T| &= U^2|T|(U^2)^* \cdot UU^* \cdot |T| \\
&= U^2|T|(U^2)^* \cdot |T| \cdot UU^* \quad \text{by (3.1)} \\
&= |T| \cdot U^2|T|(U^2)^* \cdot UU^* \quad \text{by (3.6)} \\
&= |T| \cdot U^2|T|(U^2)^*.
\end{aligned}
$$

Therefore the proof is complete. ∎

Next we show the following result on binormality of $\widetilde{T}_n$ for a non-negative integer n.

THEOREM 3.6. *Let $T = U|T|$ be the polar decomposition. Then for each non-negative integer n, the following assertions are equivalent:*

(i) *$\widetilde{T}_k$ is binormal for all $k = 0, 1, 2, \ldots, n$;*

(ii) *$[U^k|T|(U^k)^*, |T|] = 0$ for all $k = 1, 2, \ldots, n+1$.*

We prepare the following lemmas in order to prove Theorem 3.6.

Lemma 3.7. *Let T be the polar decomposition. For each natural number n, if*

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \text{for all } k = 1, 2, \ldots, n,$$

then the following properties hold:

(i) $U^k|T|^\alpha (U^k)^* = \left\{U^k|T|(U^k)^*\right\}^\alpha$ *for any* $\alpha > 0$ *and for all* $k = 1, 2, \ldots, \ldots, n+1$;

(ii) $[U^k|T|^\alpha (U^k)^*, |T|] = [U^k|T|^\alpha (U^k)^*, U^*U] = 0$ *for any* $\alpha > 0$ *and all* $k = 1, 2, \ldots, n$;

(iii) $U^s|T|^\alpha (U^s)^* U^t = U^s|T|^\alpha (U^{s-t})^*$ *and*

$$(U^t)^* U^s|T|^\alpha (U^s)^* = U^{s-t}|T|^\alpha (U^s)^*$$

for any $\alpha > 0$ *and all natural numbers* s *and* t *such that* $1 \leqslant t \leqslant s \leqslant n+1$;

(iv) $[U^s|T|^\alpha (U^s)^*, U^t|T|^\alpha (U^t)^*] = 0$ *for any* $\alpha > 0$ *and all natural numbers* s *and* t *such that* $s, t \in [1, n+1]$;

(v) $[(U^k)^*|T^*|^\alpha U^k, |T^*|] = [(U^{k-1})^*|T|^\alpha U^{k-1}, U|T|U^*] = 0$ *for any* $\alpha > 0$ *and all* $k = 1, 2, \ldots, n$;

(vi) $(U^s)^*|T|^\alpha U^s (U^t)^* = (U^s)^*|T|^\alpha U^{s-t}$ *and*

$$U^t (U^s)^*|T|^\alpha U^s = (U^{s-t})^*|T|^\alpha U^s$$

for any $\alpha > 0$ *and all natural numbers* s *and* t *such that* $1 \leqslant t \leqslant s \leqslant n$;

(vii) $U^{n+1}|\widetilde{T}|(U^{n+1})^* = U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot U^n|T|^{\frac{1}{2}}(U^n)^*$.

Proof. (i) We have only to prove the following: If $[U^k|T|(U^k)^*, |T|] = 0$ for all $k = 1, 2, \ldots, n$, then $U^{n+1}|T|^\alpha (U^{n+1})^* = \left\{U^{n+1}|T|(U^{n+1})^*\right\}^\alpha$. We prove this by induction on n, and also we remark that

$$[U^k|T|(U^k)^*, U^*U] = 0 \quad \text{for all } k = 1, 2, \ldots, n \tag{3.7}$$

by Lemma C and the assumption.

Case $n = 1$.

$$\begin{aligned} U^2|T|^\alpha (U^2)^* &= U(U|T|U^*)^\alpha U^* \\ &= U(U^*U)^{2\alpha}(U|T|U^*)^\alpha U^* \\ &= U(U^*U \cdot U|T|U^* \cdot U^*U)^\alpha U^* \quad \text{by (3.7)} \\ &= (UU^*UU|T|U^*U^*UU^*)^\alpha \\ &= \{U^2|T|(U^2)^*\}^\alpha. \end{aligned}$$

Assume that (i) holds for some natural number n. We show that it holds for $n+1$.

$$\begin{aligned} &U^{n+2}|T|^\alpha (U^{n+2})^* \\ &\quad = U\left\{U^{n+1}|T|^\alpha (U^{n+1})^*\right\} U^* \\ &\quad = U\left\{U^{n+1}|T|(U^{n+1})^*\right\}^\alpha U^* \qquad \text{by the inductive hypothesis} \\ &\quad = U(U^*U)^{2\alpha}\left\{U^{n+1}|T|(U^{n+1})^*\right\}^\alpha U^* \\ &\quad = U\left\{U^*U \cdot U^{n+1}|T|(U^{n+1})^* \cdot U^*U\right\}^\alpha U^* \quad \text{by (3.7)} \\ &\quad = \left\{UU^*UU^{n+1}|T|(U^{n+1})^*U^*UU^*\right\}^\alpha \\ &\quad = \left\{U^{n+2}|T|(U^{n+2})^*\right\}^\alpha. \end{aligned}$$

(ii) By the assumption, (i) and Lemma C, we have (ii).

(iii) By using (ii) repeatedly, we have

$$\begin{aligned}
U^s|T|^\alpha(U^s)^*U^t &= U\left\{U^{s-1}|T|^\alpha(U^{s-1})^*\cdot U^*U\right\}U^{t-1}\\
&= U\left\{U^*U\cdot U^{s-1}|T|^\alpha(U^{s-1})^*\right\}U^{t-1} && \text{by (ii)}\\
&= U^2\left\{U^{s-2}|T|^\alpha(U^{s-2})^*\cdot U^*U\right\}U^{t-2}\\
&= U^2\left\{U^*U\cdot U^{s-2}|T|^\alpha(U^{s-2})^*\right\}U^{t-2} && \text{by (ii)}\\
&= U^3\left\{U^{s-3}|T|^\alpha(U^{s-3})^*\cdot U^*U\right\}U^{t-3}\\
&= \cdots\\
&= U^t\left\{U^{s-t}|T|^\alpha(U^{s-t})^*\cdot U^*U\right\}\\
&= U^t\left\{U^*U\cdot U^{s-t}|T|^\alpha(U^{s-t})^*\right\} && \text{by (ii)}\\
&= U^t\cdot U^{s-t}|T|^\alpha(U^{s-t})^*\\
&= U^s|T|(U^{s-t})^*,
\end{aligned}$$

so that $U^s|T|^\alpha(U^s)^*U^t = U^s|T|^\alpha(U^{s-t})^*$ and $(U^t)^*U^s|T|^\alpha(U^s)^* = U^{s-t}|T|^\alpha(U^s)^*$.

(iv) We may assume $t < s$.

$$\begin{aligned}
U^s|T|^\alpha(U^s)^*\cdot U^t|T|^\alpha(U^t)^* &= U^s|T|^\alpha(U^{s-t})^*\cdot|T|^\alpha(U^t)^* && \text{by (iii)}\\
&= U^t\left\{U^{s-t}|T|^\alpha(U^{s-t})^*\cdot|T|^\alpha\right\}(U^t)^*\\
&= U^t\left\{|T|^\alpha\cdot U^{s-t}|T|^\alpha(U^{s-t})^*\right\}(U^t)^* && \text{by (ii)}\\
&= U^t|T|^\alpha\cdot U^{s-t}|T|^\alpha(U^s)^*\\
&= U^t|T|^\alpha(U^t)^*U^t\cdot U^{s-t}|T|^\alpha(U^s)^* && \text{by (iii)}\\
&= U^t|T|^\alpha(U^t)^*\cdot U^s|T|^\alpha(U^s)^*.
\end{aligned}$$

(v) Since $|T^*| = U|T|U^*$, we easily obtain

$$[(U^k)^*|T^*|^\alpha U^k, |T^*|] = [(U^{k-1})^*|T|^\alpha U^{k-1}, U|T|U^*]$$

for any $\alpha > 0$ and all $k = 1, 2, \ldots, n$, and also we have

$$\begin{aligned}
&(U^{k-1})^*|T|^\alpha U^{k-1}\cdot U|T|U^*\\
&\quad= (U^{k-1})^*|T|^\alpha\cdot U^k|T|U^*\\
&\quad= (U^{k-1})^*\left\{|T|^\alpha\cdot U^k|T|(U^k)^*\right\}U^{k-1} && \text{by (iii)}\\
&\quad= (U^{k-1})^*\left\{U^k|T|(U^k)^*\cdot|T|^\alpha\right\}U^{k-1} && \text{by the assumption}\\
&\quad= U|T|\cdot(U^k)^*|T|^\alpha U^{k-1} && \text{by (iii)}\\
&\quad= U|T|U^*\cdot(U^{k-1})^*|T|^\alpha U^{k-1}
\end{aligned}$$

for any $\alpha > 0$ and all $k = 1, 2, \ldots, n$.

(vi) Since $T^* = U^*|T^*|$ is polar decomposition of T^*, we have

$$(U^{s+1})^*|T^*|^\alpha U^{s+1}(U^t)^* = (U^{s+1})^*|T^*|^\alpha U^{s+1-t}$$

and

$$U^t(U^{s+1})^*|T^*|^\alpha U^{s+1} = (U^{s+1-t})^*|T^*|^\alpha U^{s+1}$$

for any $\alpha > 0$ and all natural numbers s and t such that $1 \leqslant t \leqslant s \leqslant n$ by (v) and (iii). So we get

$$(U^s)^*|T|^\alpha U^s(U^t)^* = (U^s)^*|T|^\alpha U^{s-t} \quad \text{and} \quad U^t(U^s)^*|T|^\alpha U^s = (U^{s-t})^*|T|^\alpha U^s.$$

(vii) By using (ii) and (iii), we have

$$\begin{aligned} U^{n+1}|\widetilde{T}|(U^{n+1})^* &= U^{n+1}(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}}(U^{n+1})^* \\ &= U^n(U|T|^{\frac{1}{2}}U^* \cdot |T| \cdot U|T|^{\frac{1}{2}}U^*)^{\frac{1}{2}}(U^n)^* \\ &= U^n \cdot U|T|^{\frac{1}{2}}U^* \cdot |T|^{\frac{1}{2}} \cdot (U^n)^* && \text{by (ii)} \\ &= U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot U^n|T|^{\frac{1}{2}}(U^n)^* && \text{by (iii).} \end{aligned}$$

Therefore the proof of Lemma 3.7 is complete. ∎

Lemma 3.8. *Let $T = U|T|$ be the polar decomposition and n be a natural number. If*

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \textit{for all } k = 1, 2, \ldots, n,$$

then the following assertions are equivalent:

(i) $[U^{n+1}|T|(U^{n+1})^*, |T|] = 0.$

(ii) $[U^n|\widetilde{T}|(U^n)^*, |\widetilde{T}|] = 0.$

Proof. At first, we remark that $[U|T|^{\frac{1}{2}}U^*, |T|] = 0$ by (ii) of Lemma 3.7, and also we have

$$\begin{aligned} |\widetilde{T}| &= (|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}} = U^*(U|T|^{\frac{1}{2}}U^* \cdot |T| \cdot U|T|^{\frac{1}{2}}U^*)^{\frac{1}{2}}U \\ &= U^* \cdot |T|^{\frac{1}{2}} \cdot U|T|^{\frac{1}{2}}U^* \cdot U = U^*|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}. \end{aligned} \tag{3.8}$$

Case $n = 1$. Since $[U|T|^{\frac{1}{2}}U^*, |T|] = 0$, we have

$$\begin{aligned} U|\widetilde{T}|U^* &= U(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}}U^* = (U|T|^{\frac{1}{2}}U^* \cdot |T| \cdot U|T|^{\frac{1}{2}}U^*)^{\frac{1}{2}} \\ &= (|T|^{\frac{1}{2}} \cdot U|T|U^* \cdot |T|^{\frac{1}{2}})^{\frac{1}{2}} = |(\widetilde{T})^*|. \end{aligned}$$

Hence $[U|\widetilde{T}|U^*, |\widetilde{T}|] = [|(\widetilde{T})^*|, |\widetilde{T}|]$, i.e. $\widetilde{T}$ is binormal, so that we can prove this case by Theorem 3.5.

Next, we shall prove that Lemma 3.8 holds for each natural number n such that $n \geqslant 2$.

Here, suppose that $[U^k|T|(U^k)^*, |T|] = 0$ for all $k = 1, 2, \ldots, n$. Then we have

$$
\begin{aligned}
&U^n|\widetilde{T}|(U^n)^* \cdot |\widetilde{T}| \\
&= \left\{U^n|T|^{\frac{1}{2}}(U^n)^* \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^*\right\} \cdot \left\{U^*|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by (3.8) and Lemma 3.7 (vii)} \\
&= U^n|T|^{\frac{1}{2}}(U^n)^* \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^* \cdot (U^*U)^2U^*|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\
&= U^*U \cdot U^n|T|^{\frac{1}{2}}(U^n)^* \cdot U^*U \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^* \cdot U^*|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 3.7 (ii)} \\
&= U^* \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot U^n|T|^{\frac{1}{2}}(U^n)^* \cdot |T|^{\frac{1}{2}} \cdot U|T|^{\frac{1}{2}} \\
&= U^* \left\{U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}}\right\} U^n|T|^{\frac{1}{2}}(U^n)^*U|T|^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 3.7 (ii)} \\
&= U^* \left\{U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}}\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}}
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
&|\widetilde{T}| \cdot U^n|\widetilde{T}|(U^n)^* \\
&= \left\{U^*|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right\} \cdot \left\{U^n|T|^{\frac{1}{2}}(U^n)^* \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^*\right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by (3.8) and Lemma 3.7 (vii)} \\
&= U^*|T|^{\frac{1}{2}}U \cdot U^n|T|^{\frac{1}{2}}(U^n)^* \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^* \cdot |T|^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 3.7 (ii)} \\
&= U^*|T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*U \cdot U^{n-1}|T|^{\frac{1}{2}}(U^{n-1})^* \cdot |T|^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 3.7 (iii)} \\
&= U^* \left\{|T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}}.
\end{aligned} \tag{3.10}
$$

Proof of (i) $\Rightarrow$ (ii). Since $[U^{n+1}|T|(U^{n+1})^*, |T|] = 0$, we have $[U^n|\widetilde{T}|(U^n)^*, |\widetilde{T}|] = 0$, that is, (ii) holds for n by (3.9) and (3.10).

Proof of (ii) $\Rightarrow$ (i). Assume $[U^n|\widetilde{T}|(U^n)^*, |\widetilde{T}|] = 0$. Then we have

$$
\begin{aligned}
&\left\{U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}}\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}} \\
&= UU^* \left\{U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}}\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}} \\
&= UU^* \left\{|T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}} \quad \text{by (3.9) and (3.10)} \\
&= \left\{|T|^{\frac{1}{2}} \cdot UU^* \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}} \quad \text{by (3.1)} \\
&= \left\{|T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*\right\} U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}}.
\end{aligned}
$$

It is equivalent to

$$U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}} = |T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*$$

on $\overline{R(U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}})}$. On the other hand, since $N(U) = N(|T|)$, we obtain

$$\begin{aligned}\overline{R(U^n|T|^{\frac{1}{2}}(U^{n-1})^*|T|^{\frac{1}{2}})} &= N(|T|^{\frac{1}{2}}U^{n-1}|T|^{\frac{1}{2}}(U^n)^*)^{\perp}\\ &= N(U^n|T|^{\frac{1}{2}}(U^n)^*)^{\perp}\\ &= N(|T|^{\frac{1}{4}}(U^n)^*)^{\perp}\\ &= N(U(U^n)^*)^{\perp}\\ &= \overline{R(U^nU^*)}.\end{aligned}$$

Therefore we have

$$U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}} \cdot U^n(U^n)^* = |T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot U^n(U^n)^*, \tag{3.11}$$

so that

$$\begin{aligned}&U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}}\\ &\quad= U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot |T|^{\frac{1}{2}} \cdot U^n(U^n)^* &&\text{by Lemma 3.7 (vi)}\\ &\quad= |T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^* \cdot U^n(U^n)^* &&\text{by (3.11)}\\ &\quad= |T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}U^* \cdot (U^n)^* &&\text{by Lemma 3.7 (iii)}\\ &\quad= |T|^{\frac{1}{2}} \cdot U^{n+1}|T|^{\frac{1}{2}}(U^{n+1})^*,\end{aligned}$$

that is, (i) holds for n.

Hence the proof is complete. ∎

In order to prove Theorem 3.6, we also use the following:

PROPOSITION 3.9. *Let $T = U|T|$ be the polar decomposition of a binormal operator T. Then $\widetilde{T} = U^*UU|\widetilde{T}|$ is also the polar decomposition of $\widetilde{T}$.*

The proof is easily obtained by applying the following result.

THEOREM D ([5]). *Let $T_1 = U_1|T_1|$ and $T_2 = U_2|T_2|$ be the polar decompositions of T_1 and T_2 respectively. If T_1 doubly commutes with T_2 (i.e., $[T_1, T_2] = 0$ and $[T_1, T_2^*] = 0$), then $T_1T_2 = U_1U_2|T_1|\,|T_2|$ is also the polar decomposition of T_1T_2, that is, U_1U_2 is a partial isometry with $N(U_1U_2) = N(|T_1|\,|T_2|)$ and $|T_1|\,|T_2| = |T_1T_2|$.*

Proof of Proposition 3.9. Since $|T|^{\frac{1}{2}} = U^*U|T|^{\frac{1}{2}}$ and $|T^*|^{\frac{1}{2}} = UU^*|T^*|^{\frac{1}{2}}$ are the polar decompositions of $|T|^{\frac{1}{2}}$ and $|T^*|^{\frac{1}{2}}$ respectively, then $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}} = U^*UUU^*|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}$ is the polar decomposition of $|T|^{\frac{1}{2}}|T^*|^{\frac{1}{2}}$ by Theorem D. Therefore we have that

$$\widetilde{T} = U^*UUU^* \cdot U|\widetilde{T}| = U^*UU|\widetilde{T}|$$

is also the polar decomposition of $\widetilde{T}$ by Theorem 2.1. ∎

Proof of Theorem 3.6. We shall prove Theorem 3.6 by induction on n. We remark that if $[U^k|T|(U^k)^*, |T|] = 0$ for all $k = 1, 2, \ldots, n+1$, then we have

$$[U^{n+1}|\widetilde{T}|(U^{n+1})^*, |T|] = [U^{n+1}|\widetilde{T}|(U^{n+1})^*, U^*U] = 0 \tag{3.12}$$

by (vii) of Lemma 3.7 and Lemma C.

Case $n = 1$. Was already shown in Theorem 3.5.

Suppose that Theorem 3.6 holds for some natural number n. (i) holds for $n+1$ if and only if

$$\widetilde{T}_k \text{ is binormal} \quad \text{for all } k = 0, 1, 2, \ldots, n+1.$$

By putting $S = \widetilde{T}$, it is equivalent to

$$T \text{ and } \widetilde{S}_k \text{ are binormal} \quad \text{for all } k = 0, 1, 2, \ldots, n. \tag{3.13}$$

Since $S = U^*UU|S|$ is the polar decomposition by Proposition 3.9, (3.13) holds if and only if

$$\begin{aligned} &T \text{ is binormal and} \\ &[(U^*UU)^k|S|\{(U^*UU)^k\}^*, |S|] = [U^*U \cdot U^k|\widetilde{T}|(U^k)^* \cdot U^*U, |\widetilde{T}|] = 0 \\ &\qquad\qquad\qquad\qquad \text{for all } k = 1, 2, \ldots, n+1 \end{aligned} \tag{3.14}$$

by the inductive hypothesis. On the other hand, if we assume (i) or (ii), then

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \text{for all } k = 1, 2, \ldots, n+1$$

by the inductive hypothesis, so that (3.14) is equivalent to

$$T \text{ is binormal and } [U^k|\widetilde{T}|(U^k)^*, |\widetilde{T}|] = 0 \quad \text{for all } k = 1, 2, \ldots, n+1 \tag{3.15}$$

by (3.12) and $U^*U|\widetilde{T}| = U^*U(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})^{\frac{1}{2}} = |\widetilde{T}|$. Moreover Lemma 3.8 assures that (3.15) is equivalent to

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \text{for all } k = 1, 2, \ldots, n+2,$$

i.e. (ii) holds for $n+1$.

Hence the proof is complete. ∎

4. CHARACTERIZATION OF CENTERED OPERATORS

In [10], Morrel and Muhly obtained properties of centered operators as follows:

THEOREM E. ([10]) *Let $T = U|T|$ be the polar decomposition of a centered operator T. Then the following assertions hold:*

(i) *U^n is a partial isometry for all natural number n;*

(ii) *the operators $\{(U^n)^*|T|U^n\}_{n=1}^{\infty}$ commute with one another;*

(iii) *$T^n = U^n\left\{|T| \cdot U^*|T|U \cdots (U^{n-1})^*|T|U^{n-1}\right\}$ is the polar decomposition for all natural number n.*

Moreover, they showed a characterization of centered operators as follows:

Theorem F. ([10]) *Let $T = U|T|$ be the polar decomposition and U be unitary. Then T is a centered operator if and only if the operators*

$$\{(U^n)^*|T|U^n\}_{n=-\infty}^{\infty}$$

commute with one another.

In this section, we show the following characterization of centered operators which is an extension of (ii) of Theorem E and Theorem F.

Theorem 4.1. *Let $T = U|T|$ be the polar decomposition. Then the following assertions are mutually equivalent:*

(i) *T is centered;*
(ii) *$[|T^n|, |(T^m)^*|] = 0$ for all natural numbers n and m;*
(iii) *$[|T^n|, |T^*|] = 0$ for all natural number n;*
(iv) *operators $\{(U^n)^*|T|U^n, U^n|T|(U^n)^*, |T|\}_{n=1}^{\infty}$ commute with one another;*
(v) *$[U^n|T|(U^n)^*, |T|] = 0$ for all natural number n;*
(vi) *$\widetilde{T}_n$ is binormal for all non-negative integer n.*

To prove Theorem 4.1, we will prepare the following lemmas.

Lemma 4.2. *Let T be the polar decomposition. For each natural numbers n and m, if*

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \text{for all } k = 0, 1, 2, \dots, m+n-2, \tag{4.1}$$

then the following assertions are equivalent:

(i) $[U^m|T|(U^m)^*, |T^n|] = 0$;
(ii) $[U^{m+n-1}|T|(U^{m+n-1})^*, |T|] = 0$.

Proof. We prove Lemma 4.2 by induction on n. The case $n = 1$ is obvious.

Assume that Lemma 4.2 holds for some natural number n and each natural number m. Then we prove that it holds for $n+1$ and each natural number m.

Here, let m be a natural number and suppose that (4.1) holds for $n+1$, i.e.,

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \text{for all } k = 0, 1, 2, \dots, m+n-1. \tag{4.2}$$

Then

$$[U|T|U^*, |T^n|] = [UU^*, |T^n|] = 0 \tag{4.3}$$

holds by the inductive assumption and Lemma C, and also we have

$$\begin{aligned} |T^{n+1}|^2 &= |T|U^*|T^n|^2U|T| \\ &= U^* \cdot U|T|U^* \cdot |T^n|^2 \cdot U|T| \\ &= U^* \cdot |T^n|^2 \cdot U|T|U^* \cdot U|T| \quad \text{by (4.3)} \\ &= U^*|T^n|^2U|T|^2. \end{aligned} \tag{4.4}$$

Therefore, we have

$$\begin{aligned} |T^{n+1}|^2 \cdot U^m|T|(U^m)^* &= U^*|T^n|^2U|T|^2 \cdot U^m|T|(U^m)^* && \text{by (4.4)} \\ &= U^*|T^n|^2U \cdot U^m|T|(U^m)^* \cdot |T|^2 && \text{by (4.2)} \\ &= U^*\left\{|T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^*\right\}U|T|^2 \end{aligned} \tag{4.5}$$

and

$$
\begin{aligned}
&U^m|T|(U^m)^* \cdot |T^{n+1}|^2 \\
&\quad = U^m|T|(U^m)^* \cdot U^*|T^n|^2U|T|^2 && \text{by (4.4)} \\
&\quad = U^* \left\{U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2\right\} U|T|^2 && \text{by Lemma 3.7 (iii).}
\end{aligned} \tag{4.6}
$$

Proof of (ii) $\Rightarrow$ (i). Assume that (ii) holds for $n+1$. Since

$$[U^{m+(n+1)-1}|T|(U^{m+(n+1)-1})^*, |T|] = [U^{(m+1)+n-1}|T|(U^{(m+1)+n-1})^*, |T|] = 0,$$

we have

$$[U^{m+1}|T|(U^{m+1})^*, |T^n|] = 0$$

by the inductive assumption. Hence we obtain

$$[|T^{n+1}|, U^m|T|(U^m)^*] = 0,$$

that is, (i) holds for $n+1$ by (4.5) and (4.6).

Proof of (i) $\Rightarrow$ (ii). Assume that (i) holds for $n+1$. Then we have

$$
\begin{aligned}
&U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2 \cdot U|T|^2 \\
&\quad = UU^* \left\{U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2\right\} U|T|^2 \\
&\quad = UU^* \left\{|T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^*\right\} U|T|^2 && \text{by (4.5) and (4.6)} \\
&\quad = |T^n|^2 \cdot UU^* \cdot U^{m+1}|T|(U^{m+1})^* \cdot U|T|^2 && \text{by (4.3)} \\
&\quad = |T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^* \cdot U|T|^2,
\end{aligned}
$$

that is,

$$U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2 = |T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^*$$

holds on

$$\overline{R(U|T|^2)} = N(|T|^2U^*)^\perp = N(UU^*)^\perp = R(UU^*).$$

Then we have

$$U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2 \cdot UU^* = |T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^* \cdot UU^*, \tag{4.7}$$

so we obtain

$$
\begin{aligned}
U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2 &= U^{m+1}|T|(U^{m+1})^* \cdot UU^* \cdot |T^n|^2 \\
&= U^{m+1}|T|(U^{m+1})^* \cdot |T^n|^2 \cdot UU^* && \text{by (4.3)} \\
&= |T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^* \cdot UU^* && \text{by (4.7)} \\
&= |T^n|^2 \cdot U^{m+1}|T|(U^{m+1})^*.
\end{aligned}
$$

Hence we have

$$[U^{(m+1)+n-1}|T|(U^{(m+1)+n-1})^*, |T|] = [U^{m+(n+1)-1}|T|(U^{m+(n+1)-1})^*, |T|] = 0,$$

that is, (ii) holds for $n+1$ by the inductive assumption.

Therefore the proof is complete. ∎

LEMMA 4.3. *Let $T = U|T|$ be the polar decomposition. For each natural number n, if*

$$[U^k|T|(U^k)^*, |T|] = 0 \quad \text{for all } k = 0, 1, 2, \dots, n-1,$$

then

$$|(T^n)^*| = U|T|U^* \cdot U^2|T|(U^2)^* \cdots U^n|T|(U^n)^*.$$

Proof. We prove Lemma 4.3 by induction on n.

The case $n = 1$ is obvious. Assume that Lemma 4.3 holds for some natural number n. Then we show that it holds for $n + 1$.

By the inductive assumption, we have

$$|(T^n)^*| = U|T|U^* \cdot U^2|T|(U^2)^* \cdots U^n|T|(U^n)^*. \tag{4.8}$$

Then we obtain

$$\begin{aligned}
&|(T^{n+1})^*| \\
&= (U|T| \cdot |(T^n)^*|^2 \cdot |T|U^*)^{\frac{1}{2}} \\
&= \left\{U|T| \left(U|T|U^* \cdot U^2|T|(U^2)^* \cdots U^n|T|(U^n)^*\right)^2 |T|U^*\right\}^{\frac{1}{2}} && \text{by (4.8)} \\
&= \left\{U|T| \cdot U|T|^2U^* \cdot U^2|T|^2(U^2)^* \cdots U^n|T|^2(U^n)^* \cdot |T|U^*\right\}^{\frac{1}{2}} && \text{by Lemma 3.7 (iv)} \\
&= \left\{U|T|(U^*U)^{n+1} \cdot U|T|^2U^* \cdot U^2|T|^2(U^2)^* \cdots U^n|T|^2(U^n)^* \cdot |T|U^*\right\}^{\frac{1}{2}} \\
&= \{U|T| \cdot U^*U \cdot U|T|^2U^* \cdot U^*U \cdot U^2|T|^2(U^2)^* \cdot U^*U \cdots U^*U \\
&\quad \cdot U^n|T|^2(U^n)^* \cdot U^*U \cdot |T|U^*\}^{\frac{1}{2}} && \text{by Lemma 3.7 (ii)} \\
&= \left\{U|T|U^* \cdot U^2|T|^2(U^2)^* \cdot U^3|T|^2(U^3)^* \cdots U^{n+1}|T|^2(U^{n+1})^* \cdot U|T|U^*\right\}^{\frac{1}{2}} \\
&= U|T|U^* \cdot U^2|T|(U^2)^* \cdot U^3|T|(U^3)^* \cdots U^{n+1}|T|(U^{n+1})^* && \text{by Lemma 3.7 (iv).}
\end{aligned}$$

Therefore the proof is complete. ∎

Proof of Theorem 4.1. Proofs of (i) ⇒ (ii), (ii) ⇒ (iii) and (iv) ⇒ (v) are obvious, and also the equivalence relation between (v) and (vi) was already proved in Theorem 3.6. Thus, we have only to prove (iii) ⇒ (v), (v) ⇒ (iv), (v) ⇒ (ii) and (ii) ⇒ (i).

Proof of (iii) ⇒ (v). Firstly $[U|T|U^*, |T|] = 0$ and $[U|T|U^*, |T^2|] = 0$ ensures $[U^2|T|(U^2)^*, |T|] = 0$ by Lemma 4.2. Secondly $[U^k|T|(U^k)^*, |T|] = 0$ for $k = 1, 2$ and $[U|T|U^*, |T^3|] = 0$ ensures $[U^3|T|(U^3)^*, |T|] = 0$ by Lemma 4.2. By repeating this method, we have (v).

Proof of (v) ⇒ (iv). By (v), $[U^n|T|(U^n)^*, |T|] = 0$ holds for all natural numbers n. Then we have

$$[(U^{n-1})^*|T|(U^{n-1}), U|T|U^*] = 0 \tag{4.9}$$

by (v) of Lemma 3.7. Hence

$$\begin{aligned}(U^n)^*|T|U^n \cdot |T| &= U^* \left\{(U^{n-1})^*|T|U^{n-1} \cdot U|T|U^*\right\} U \\ &= U^* \left\{U|T|U^* \cdot (U^{n-1})^*|T|U^{n-1}\right\} U \quad \text{by (4.9)} \\ &= |T| \cdot (U^n)^*|T|U^n,\end{aligned}$$

that is, $[(U^n)^*|T|U^n, |T|] = 0$ holds for all natural numbers n.

Moreover we obtain

$$\begin{aligned}&(U^n)^*|T|U^n \cdot U^m|T|(U^m)^* \\ &\quad = (U^n)^* \left\{|T| \cdot U^{n+m}|T|(U^{n+m})^*\right\} U^n \quad \text{by Lemma 3.7 (iii)} \\ &\quad = (U^n)^* \left\{U^{n+m}|T|(U^{n+m})^* \cdot |T|\right\} U^n \quad \text{by (v)} \\ &\quad = U^m|T|(U^m)^* \cdot (U^n)^*|T|U^n \quad \text{by Lemma 3.7 (iii)},\end{aligned}$$

that is, $[(U^n)^*|T|U^n, U^m|T|(U^m)^*] = 0$ holds for all natural numbers n and m.

Hence we have (iv).

Proof of (v) $\Rightarrow$ (ii). By (v) and Lemma 4.3, we have

$$\begin{aligned}&|(T^m)^*| \\ &\quad = U|T|U^* \cdot U^2|T|(U^2)^* \cdots U^m|T|(U^m)^* \quad \text{for all natural number } m,\end{aligned} \tag{4.10}$$

and also by (v) and Lemma 4.2, we have

$$[U^m|T|(U^m)^*, |T^n|] = 0 \quad \text{for all natural numbers } m \text{ and } n. \tag{4.11}$$

Hence we obtain (ii) from (4.10) and (4.11).

Finally, we show (ii) $\Rightarrow$ (i). For $s > t$, we have

$$\begin{aligned}|T^s|^2|T^t|^2 &= (T^t)^* \cdot |T^{s-t}|^2 \cdot |(T^t)^*|^2 \cdot T^t \\ &= (T^t)^* \cdot |(T^t)^*|^2 \cdot |T^{s-t}|^2 \cdot T^t \quad \text{by (ii)} \\ &= |T^t|^2|T^s|^2\end{aligned}$$

and

$$\begin{aligned}|(T^s)^*|^2|(T^t)^*|^2 &= T^t \cdot |(T^{s-t})^*|^2 \cdot |T^t|^2 \cdot (T^t)^* \\ &= T^t \cdot |T^t|^2 \cdot |(T^{s-t})^*|^2 \cdot (T^t)^* \quad \text{by (ii)} \\ &= |(T^t)^*|^2|(T^s)^*|^2,\end{aligned}$$

so that we have (i).

Therefore the proof is complete. ∎

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