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BACKWARD SHIFT INVARIANT SUBSPACES IN THE BIDISC. II

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ABSTRACT. For every invariant subspace M in the Hardy spaces $H^2(\Gamma^2)$, let V_z and V_w be multiplication operators on M. Then it is known that the condition $V_zV_w^* = V_w^*V_z$ on M holds if and only if M is a Beurling type invariant subspace. For a backward shift invariant subspace N in $H^2(\Gamma^2)$, two operators S_z and S_w on N are defined by $S_z = P_NL_zP_N$ and $S_w = P_NL_wP_N$, where P_N is the orthogonal projection from $L^2(\Gamma^2)$ onto N. It is given a characterization of N satisfying $S_zS_w^* = S_w^*S_z$ on N.

KEYWORDS: *Backward shift invariant subspaces, Hardy space in the bidisc.*

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1. INTRODUCTION

Let Γ^2 be the 2-dimensional unit torus. We denote by $(z,w) = (\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\varphi})$ the variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the usual Lebesgue space on Γ^2 with the norm $\|f\|_2 = \big(\int\limits_{\Gamma^2} |f(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\varphi})|^2 \mathrm{d}\theta\mathrm{d}\varphi/(2\pi)^2\big)^{1/2}$. The space L^2 is a Hilbert space with the usual inner product. For $f \in L^2$, the Fourier coefficients are given by

$$\widehat{f}(n,m) = \frac{\int\limits_{\Gamma^2} f(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\varphi})\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{e}^{-\mathrm{i}m\varphi}\,\mathrm{d}\theta\,\mathrm{d}\varphi}{(2\pi)^2} = \langle f, z^n w^m\rangle.$$

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space on Γ^2, that is,

$$H^2 = \{f \in L^2 : \widehat{f}(n,m) = 0 \text{ if } n < 0 \text{ or } m < 0\}.$$

For $f \in H^2$, we can write f as

$$f = \sum_{i,u=0}^{\infty}{}^{\oplus} a_{i,j} z^i w^j, \quad \text{where } \sum_{i,j=0}^{\infty} |a_{i,j}|^2 < \infty.$$

Let P be the orthogonal projection from L^2 onto H^2. For a closed subspace M of L^2, we denote by P_M the orthogonal projection from L^2 onto M. For a function $\psi \in L^\infty$, let $L_\psi f = \psi f$ for $f \in L^2$. The Toeplitz operator T_ψ on H^2 is defined by $T_\psi f = PL_\psi f$ for $f \in H^2$. It is well known that $T^*_\psi = T_{\overline{\psi}}$. It holds that $T^*_{z^n} T_{w^m} = T_{w^m} T^*_{z^n}$ for $n, m \geqslant 1$. A function $f \in H^2$ is called to be inner if $|f| = 1$ on Γ^2 almost everywhere. A closed subspace M of H^2 is called invariant if $zM \subset M$ and $wM \subset M$. In one variable case, an invariant subspace M of $H^2(\Gamma)$ has a form $M = qH^2(\Gamma)$, where q is inner. This is the well known Beurling theorem ([2]). In two variable case, the structure of invariant subspaces of H^2 is more complicated; see [1], [9], [11].

Let M be an invariant subspace of H^2. Then $T^*_z(H^2 \ominus M) \subset (H^2 \ominus M)$ and $T^*_w(H^2 \ominus M) \subset (H^2 \ominus M)$. We call a closed subspace N of H^2 to be backward shift invariant if $T^*_z N \subset N$ and $T^*_w N \subset N$. If N is a backward shift invariant subspace of H^2, then $H^2 \ominus N$ is invariant. There are studies of backward shift invariant subspaces of the unit circle Γ; see [3] and [12].

Let M be an invariant subspace of H^2 and $\psi \in L^\infty$. Let V_ψ be the operator on M defined by $V_\psi = P_M L_\varphi | M$. Then $V_z = T_z$ and $V^*_z = V_{\overline{z}}$ on M. In [8], Mandrekar proved that $V_z V^*_w = V^*_w V_z$ on M holds if and only if M is Beurling type, that is, $M = qH^2$ for some inner function q; see also [4], [9], [10].

In this paper, we study a similar type problem on a backward shift invariant subspace N of H^2. For $\psi \in L^\infty$, put

$$S_\psi = P_N L_\psi | N \quad \text{on } N.$$

Then we have $S^*_\psi = S_{\overline{\psi}}$ and $S^*_z = T^*_z$ on N. Our purpose is to characterize backward shift invariant subspaces N which satisfiy the condition $S_z S^*_w = S^*_w S_z$ on N. Recently, this problem was studied in [5] and [6]. Our theorem in this paper is the following complete characterization.

Theorem 2.1. *Let N be a backward shift invariant subspace of H^2 and $N \neq H^2$. Then $S_z S^*_w = S^*_w S_z$ on N holds if and only if N has one of the following forms:*

(i) $N = H^2 \ominus q_1(z)H^2$;
(ii) $N = H^2 \ominus q_2(w)H^2$;
(iii) $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2)$;

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

In Section 2, we prove our theorem as a continuation of the study of [6]. In Section 3, we study the above problem from another point of view.

Let $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ be the Hardy spaces on the unit circle Γ in variables z and w, respectively. We think that $H^2(\Gamma_z) \subset H^2$ and $H^2(\Gamma_w) \subset H^2$. In [6] it is proved that, if $S_z S^*_w = S^*_w S_z$ and $N \neq H^2$, then either $(H^2 \ominus N) \cap H^2(\Gamma_z) \neq \{0\}$ or $(H^2 \ominus N) \cap H^2(\Gamma_w) \neq \{0\}$ holds. We prove the following.

THEOREM 3.1. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) \neq \{0\}$. Put $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Put $\widetilde{M} = M \ominus q_1(z)H^2$. Then the following conditions are equivalent:*

(i) $S_z S_w^* = S_w^* S_z$;

(ii) $T_z^* \widetilde{M} \subset \widetilde{M}$;

(iii) *either* $\widetilde{M} = \{0\}$ *or* $\widetilde{M} = q_2(w)(H^2 \ominus q_1(z)H^2)$ *holds for some inner function* $q_2(w) \in H^2(\Gamma_w)$;

(iv) *either* $M = q_1(z)H^2$ *or* $M = q_1(z)H^2 + q_2(w)H^2$ *holds.*

Theorem 3.1 follows from Theorem 2.1 without difficulty. We also give a proof of Theorem 3.1 without using Theorem 2.1. Since Theorem 2.1 follows from Theorem 3.1, this means that we give two different proofs of Theorem 2.1. In the forthcoming paper [7], we study backward shift invariant subspaces N satisfying $S_z S_w^* \neq S_w^* S_z$ and $S_{z^2} S_w^* = S_w^* S_{z^2}^2$. In [7], both ideas will be used effectively.

2. PROOF OF THEOREM 2.1

To prove our theorem, we need some lemmas. The following two lemmas are proved in [6].

LEMMA 2.2. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Then the following conditions are equivalent:*

(i) $S_z S_w^* = S_w^* S_z$;

(ii) $S_w S_z^* = S_z^* S_w$;

(iii) $(M \ominus zM) \ominus (M \cap H^2(\Gamma_w)) \subset (M \cap H^2(\Gamma_z)) \oplus wM$;

(iv) $(M \ominus wM) \ominus (M \cap H^2(\Gamma_z)) \subset (M \cap H^2(\Gamma_w)) \oplus zM$.

LEMMA 2.3. *Let N be a backward shift invariant subspace of H^2 such that $N \neq H^2$. Let $M = H^2 \ominus N$. If $S_z S_w^* = S_w^* S_z$ holds, then either $M \cap H^2(\Gamma_z) \neq \{0\}$ or $M \cap H^2(\Gamma_w) \neq \{0\}$ holds.*

LEMMA 2.4. *Let $q_1(z)$ and $q_2(w)$ be one variable inner functions. Then $M = q_1(z)H^2 + q_2(w)H^2$ is an invariant subspace of H^2.*

Proof. We need to prove that M is closed. Since

$$H^2 \ominus q_2(w)H^2 = \sum_{j=0}^{\infty}{}^{\oplus} z^j (H^2(\Gamma_w) \ominus q_2(w)H^2(\Gamma_w)),$$

$H^2 \ominus q_2(w)H^2$ is z-invariant. Then $q_1(z)(H^2 \ominus q_2(w)H^2) \perp q_2(w)H^2$ and

$$\begin{aligned} M &= q_1(z)H^2 + q_2(w)H^2 \\ &= q_1(z)((H^2 \ominus q_2(w)H^2) \oplus q_2(w)H^2) + q_2(w)H^2 \\ &= (q_1(z)(H^2 \ominus q_2(w)H^2)) \oplus q_2(w)H^2. \end{aligned}$$

Hence M is closed. ∎

Proof of Theorem 2.1. Put $M = H^2 \ominus N$. Then M is an invariant subspace. Suppose that (i) holds. Then $M = q_1(z)H^2$, so that $M \ominus wM = q_1(z)H^2(\Gamma_z)$ and $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$. Hence $(M \ominus wM) \ominus (M \cap H^2(\Gamma_z)) = \{0\}$. By Lemma 2.2, $S_z S_w^* = S_w^* S_z$ holds. Similarly if (ii) holds, then $S_z S_w^* = S_w^* S_z$.

Suppose that (iii) holds. By Lemma 2.4, we have $M = q_1(z)H^2 + q_2(w)H^2$. Then we have

$$q_1(z), q_2(w) \in M. \tag{2.1}$$

If either $q_1(z)$ or $q_2(z)$ is constant, then we have $M = H^2$, so that $N = \{0\}$. In this case, trivially $S_z S_w^* = S_w^* S_z$ holds. Hence we may assume that both of $q_1(z)$ and $q_2(w)$ are not constant functions. We have $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, $M \cap H^2(\Gamma_w) = q_2(w)H^2(\Gamma_w)$, and

$$M \ominus zM \subset q_1(z)H^2(\Gamma_w) + q_2(w)H^2(\Gamma_w). \tag{2.2}$$

By Lemma 2.2, it is sufficient to prove

$$(M \ominus zM) \ominus q_2(w)H^2(\Gamma_w) \subset q_1(z)H^2(\Gamma_z) \oplus wM. \tag{2.3}$$

Let

$$f \in (M \ominus zM) \ominus q_2(w)H^2(\Gamma_w). \tag{2.4}$$

Then by (2.2),

$$f = q_1(z)h_1(w) + q_2(w)h_2(w), \quad h_1(w), h_2(w) \in H^2(\Gamma_w). \tag{2.5}$$

By (2.4), $f \perp zM$. Since $q_2(w)h_2(w) \perp zM$, we have

$$q_1(z)h_1(w) \perp z(q_1(z)H^2 + q_2(w)H^2).$$

Since $q_1(z)h_1(w) \perp zq_1(z)H^2$, we have $q_1(z)h_1(w) \perp zq_2(w)H^2$. Since $q_1(z)$ is not constant, $q_1(z) \not\perp z^n$ for some $n \geqslant 1$. Since $q_1(z)h_1(w) \perp z^n q_2(w)H^2(\Gamma_w)$, we get $h_1(w) \perp q_2(w)H^2(\Gamma_w)$. Hence $q_1(z)h_1(w) \perp q_2(w)H^2(\Gamma_w)$. By (2.4), $f \perp q_2(w)H^2(\Gamma_w)$. Therefore by (2.5), $q_2(w)h_2(w) \perp q_2(w)H^2(\Gamma_w)$. Thus we get $h_2(w) = 0$. Let $h_1(w) = \widehat{h}_1(0) + wh_1'(w)$, where $h_1'(w) \in H^2(\Gamma_w)$. By (2.1), $q_1(z)h_1'(w) \in M$. Hence we get

$$f = q_1(z)h_1(w) = \widehat{h}_1(0)q_1(z) + q_1(z)wh_1'(w) \in q_1(z)H^2(\Gamma_z) \oplus wM.$$

Thus (2.3) holds. Therefore $S_z S_w^* = S_w^* S_z$ holds.

Next, we prove the converse assertion. We may assume that $N \neq \{0\}$. Suppose that $S_z S_w^* = S_w^* S_z$. By Lemma 2.3, we may further assume that $M \cap H^2(\Gamma_w) \neq \{0\}$ holds. In this case, we shall prove that N has either the form (ii) or the form (iii). Similarly, if $M \cap H^2(\Gamma_z) \neq \{0\}$ holds, then we can prove that N has either the form either (i) or (iii).

By the Beurling theorem ([2]),

$$M \cap H^2(\Gamma_w) = q_2(w)H^2(\Gamma_w), \tag{2.6}$$

where $q_2(w)$ is an inner function. By Lemma 2.2,

$$(M \ominus zM) \ominus q_2(w)H^2(\Gamma_w) \subset (M \cap H^2(\Gamma_z)) \oplus wM.$$

Put

$$K_0 = (M \ominus zM) \ominus q_2(w)H^2(\Gamma_w). \tag{2.7}$$

Then

$$K_0 \subset (M \cap H^2(\Gamma_z)) \oplus wM \tag{2.8}$$

and

$$K_0 \perp (zM \oplus q_2(w)H^2(\Gamma_w)). \tag{2.9}$$

We have

$$q_2(w)H^2 = q_2(w)H^2(\Gamma_w) \oplus zq_2(w)H^2. \tag{2.10}$$

By (2.6), we have $q_2(w) \in M$. Then $zq_2(w)H^2 \subset zM$. Hence, by (2.9) and (2.10),

$$K_0 \perp q_2(w)H^2. \tag{2.11}$$

We also have

$$q_2(w)H^2 = \sum_{j=0}^{\infty}{}^{\oplus} z^j q_2(w)H^2(\Gamma_w). \tag{2.12}$$

Then

$$\begin{aligned} M &= \sum_{j=0}^{\infty}{}^{\oplus} z^j (M \ominus zM) \\ &= \sum_{j=0}^{\infty}{}^{\oplus} z^j (K_0 \oplus q_2(w)H^2(\Gamma_w)) && \text{by (2.7)} \\ &= \Big(\sum_{j=0}^{\infty}{}^{\oplus} z^j q_2(w)H^2(\Gamma_w)\Big) \oplus \Big(\sum_{j=0}^{\infty}{}^{\oplus} z^j K_0\Big) \\ &= q_2(w)H^2 \oplus \Big(\sum_{j=0}^{\infty}{}^{\oplus} z^j K_0\Big) && \text{by (2.12).} \end{aligned}$$

Hence

$$M = q_2(w)H^2 \oplus \Big(\sum_{j=0}^{\infty}{}^{\oplus} z^j K_0\Big). \tag{2.13}$$

Since (2.8) holds, it occurs one of the following three cases:

$$K_0 = \{0\}, \quad K_0 \subset wM, \quad \text{and} \quad K_0 \not\subset wM.$$

Case 1. $K_0 = \{0\}$.

In this case, by (2.13) it follows that $M = q_2(w)H^2$. Therefore $N = H^2 \ominus M = H^2 \ominus q_2(w)H^2$. Hence (ii) holds.

Case 2. $K_0 \subset wM$.

In this case, we shall prove that $K_0 = \{0\}$. Let $F \in K_0$. By our assumption of Case 2,

$$F = wf, \quad f \in M. \tag{2.14}$$

We shall prove that

$$f \in K_0. \tag{2.15}$$

We have

$$\begin{aligned} \Big\langle f, q_2(w)H^2 \oplus \sum_{j=1}^{\infty}{}^{\oplus} z^j K_0 \Big\rangle &= \Big\langle wf, w\Big(q_2(w)H^2 \oplus \sum_{j=1}^{\infty}{}^{\oplus} z^j K_0\Big)\Big\rangle \\ &= \Big\langle F, z\Big(\sum_{j=1}^{\infty}{}^{\oplus} z^{j-1} wK_0\Big)\Big\rangle \qquad \text{by (2.11) and (2.14)} \\ &= 0. \end{aligned}$$

The last equation follows from the facts

$$z\Big(\sum_{j=1}^{\infty}{}^{\oplus} z^{j-1} wK_0\Big) \subset zM, \quad F \in K_0, \quad \text{and} \quad K_0 \perp zM.$$

Then by (2.13), we have (2.15). Hence $F \in \bigcap_{n=1}^{\infty} w^n K_0$ holds, so that $F = 0$.

Case 3. $K_0 \not\subset wM$.

In this case, by (2.8) it follows that $M \cap H^2(\Gamma_z) \neq \{0\}$. By the Beurling theorem,

$$M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z), \quad \text{where } q_1(z) \text{ is inner.} \tag{2.16}$$

By (2.8) again, $K_0 \subset q_1(z)H^2(\Gamma_z) \oplus wM$ holds. Let $G \in K_0$. Then $G = q_1(z)h_0(z) \oplus wh_1$, where $h_0(z) \in H^2(\Gamma_z)$ and $h_1 \in M$. We have

$$G = \widehat{h}_0(0)q_1(z) \oplus zq_1(z)h_2(z) \oplus wh_1 \quad \text{for some } h_2(z) \in H^2(\Gamma_z). \tag{2.17}$$

By (2.16), we have $q_1(z) \in M$. Hence $zq_1(z)h_2(z) \in zM$. Then by (2.9), $G \perp zq_1(z)h_2(z)$ holds. Therefore by (2.17), $zq_1(z)h_2(z) = 0$, so that $G = \widehat{h}_0(0)q_1(z) \oplus wh_1$ holds. Thus we get

$$G = a_0 q_1(z) \oplus wh_1, \quad h_1 \in M. \tag{2.18}$$

Here we shall prove that

$$h_1 \in K_0. \tag{2.19}$$

Since $q_2(w) \in M$, $M = q_2(w)H^2 \oplus (M \ominus q_2(w)H^2)$. Put $h_1 = h_1' \oplus h_2'' \in q_2(w)H^2 \oplus (M \ominus q_2(w)H^2)$. Then we have $G = a_0 q_1(z) \oplus wh_1' \oplus wh_2''$. Since $wh_1' \in q_2(w)H^2$, by (2.11) $wh_1' \perp K_0$ holds. Since $G \in K_0$, we have $h_1' = 0$. Thus we get

$$h_1 \perp q_2(w)H^2. \tag{2.20}$$

We have

$$q_1(z) \perp w\Big(\sum_{j=1}^{\infty}{}^{\oplus} z^j K_0\Big). \tag{2.21}$$

Since $w\Big(\sum_{j=1}^{\infty}{}^{\oplus} z^j K_0\Big) \subset zM$, $G \in K_0$, and $K_0 \perp zM$, we have

$$G \perp w\Big(\sum_{j=1}^{\infty}{}^{\oplus} z^j K_0\Big). \tag{2.22}$$

Then we have

$$\begin{aligned}\Big\langle h_1, \sum_{j=1}^{\infty}{}^{\oplus} z^j K_0\Big\rangle &= \Big\langle wh_1, w\Big(\sum_{j=1}^{\infty}{}^{\oplus} z^j K_0\Big)\Big\rangle \\ &= \Big\langle G - a_0 q_1(z), w\Big(\sum_{j=1}^{\infty}{}^{\oplus} z^j K_0\Big)\Big\rangle \quad \text{by (2.18)} \\ &= 0 \quad \text{by (2.21) and (2.22).}\end{aligned}$$

Hence $h_1 \perp \sum_{j=1}^{\infty}{}^{\oplus} z^j K_0$. Therefore by (2.13) and (2.20), we get (2.19).

Applying (2.18) and (2.19) infinitely many times, we have

$$G = \sum_{j=0}^{\infty}{}^{\oplus} a_j q_1(z) w^j = q_1(z)\Big(\sum_{j=0}^{\infty}{}^{\oplus} a_j w^j\Big) \in q_1(z)H^2(\Gamma_w).$$

Hence $K_0 \subset q_1(z)H^2(\Gamma_w)$, so

$$\sum_{j=0}^{\infty}{}^{\oplus} z^j K_0 \subset q_1(z)H^2.$$

Therefore by (2.13), $M \subset q_1(z)H^2 + q_2(w)H^2$. By (2.6) and (2.16), we have $q_1(z), q_2(w) \in M$. Then $q_1(z)H^2 + q_2(w)H^2 \subset M$. Thus we get $M = q_1(z)H^2 + q_2(w)H^2$. Hence $N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2)$. ∎

COROLLARY 2.5. *Let N be a backward shift invariant subspace of H^2 and $N \neq H^2$. Let $M = H^2 \ominus N$. Then $S_z S_w^* = S_w^* S_z$ holds if and only if M has one of the following forms:*

(i) $M = q_1(z)H^2$;
(ii) $M = q_2(w)H^2$;
(iii) $M = q_1(z)H^2 + q_2(w)H^2$;

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

3. ANOTHER PROOF OF THEOREM 2.1

Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Then M is an invariant subspace. Let $q_1(z)$ be an inner function in $H^2(\Gamma_z)$. In this section, we assume that

$$q_1(z)H^2 \subset M \quad \text{and} \quad M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z). \tag{3.1}$$

Then $q_1(z)H^2 \subset M$. Put

$$\widetilde{M} = M \ominus q_1(z)H^2. \tag{3.2}$$

Then

$$H^2 \ominus q_1(z)H^2 = \widetilde{M} \oplus N \tag{3.3}$$

and $\widetilde{M}$ is w-invariant. The following is the main theorem in this section.

THEOREM 3.1. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) \neq \{0\}$. Put $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is an inner function. Put $\widetilde{M} = M \ominus q_1(z)H^2$. Then the following conditions are equivalent:*

(i) $S_z S_w^* = S_w^* S_z$;

(ii) $T_z^* \widetilde{M} \subset \widetilde{M}$;

(iii) *either $\widetilde{M} = \{0\}$ or $\widetilde{M} = q_2(w)(H^2 \ominus q_1(z)H^2)$ holds for some inner function $q_2(w) \in H^2(\Gamma_w)$;*

(iv) *either $M = q_1(z)H^2$ or $M = q_1(z)H^2 + q_2(w)H^2$ holds.*

To prove our theorem, we need to study the properties of $\widetilde{M}$.

LEMMA 3.2. *Let $f \in \widetilde{M}$. Then we have the following:*

(i) $T_w^* f \in \widetilde{M}$ *if and only if* $f \in w\widetilde{M}$;

(ii) $T_w^* f \perp \widetilde{M}$ *if and only if* $f \in \widetilde{M} \ominus w\widetilde{M}$.

Proof. (i) Suppose that $T_w^* f \in \widetilde{M}$. Put

$$f = \sum_{j=0}^{\infty}{}^{\oplus} w^j f_j(z), \quad f_j(z) \in H^2(\Gamma_z). \tag{3.4}$$

Then

$$\sum_{j=1}^{\infty}{}^{\oplus} w^{j-1} f_j(z) \in \widetilde{M}. \tag{3.5}$$

Since $w\widetilde{M} \subset \widetilde{M}$, it follows that $\sum_{j=1}^{\infty}{}^{\oplus} w^j f_j(z) \in \widetilde{M}$. By (3.4), we have $f_0(z) \in \widetilde{M}$. Then by (3.1),

$$f_0(z) \in \widetilde{M} \cap H^2(\Gamma_z) \subset M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z).$$

Then by (3.2), $f_0(z) \perp \widetilde{M}$. Thus we get $f_0(z) = 0$. Hence, by (3.4) and (3.5), $f \in w\widetilde{M}$ holds. The converse is trivial.

(ii) follows from the fact that $T_w^* f \perp \widetilde{M}$ if and only if $f \perp w\widetilde{M}$. ∎

We denote by $P_\perp$ the orthogonal projection from H^2 onto $H^2 \ominus q_1(z)H^2$. Then we have a Toeplitz type operator Q_z on $H^2 \ominus q_1(z)H^2$ such that

$$Q_z : H^2 \ominus q_1(z)H^2 \ni f \to P_\perp(T_z f) \in H^2 \ominus q_1(z)H^2. \tag{3.6}$$

Since $zM \subset M$, by (3.2) it follows that $Q_z\widetilde{M} \subset \widetilde{M}$. By (3.3), Q_z has the following matrix form:

$$Q_z = \begin{pmatrix} * & P_{\widetilde{M}} T_z|N \\ 0 & S_z \end{pmatrix} \quad \text{on } H^2 \ominus q_1(z)H^2 = \begin{pmatrix} \widetilde{M} \\ \oplus \\ N \end{pmatrix}. \tag{3.7}$$

Since $H^2 \ominus q_1(z)H^2$ is backward shift invariant, it follows that $T_w^*(H^2 \ominus q_1(z)H^2) \subset H^2 \ominus q_1(z)H^2$. Since $T_w^* N \subset N$, the operator T_w^* on $H^2 \ominus q_1(z)H^2$ has the following matrix form:

$$T_w^* = \begin{pmatrix} * & 0 \\ P_N T_w^*|\widetilde{M} & S_w^* \end{pmatrix} \quad \text{on } H^2 \ominus q_1(z)H^2 = \begin{pmatrix} \widetilde{M} \\ \oplus \\ N \end{pmatrix}. \tag{3.8}$$

Put

$$A = P_{\widetilde{M}} T_z|N \quad \text{and} \quad B = P_N T_w^*|\widetilde{M}. \tag{3.9}$$

LEMMA 3.3. *We have the following:*
(i) $T_w^* Q_z = Q_z T_w^*$ *on* $H^2 \ominus q_1(z)H^2$;
(ii) $T_w Q_z = Q_z T_w$ *on* $H^2 \ominus q_1(z)H^2$.

Proof. Let $f \in H^2 \ominus q_1(z)H^2$. Put

$$zf = f_1 \oplus f_2 \in (H^2 \ominus q_1(z)H^2) \oplus q_1(z)H^2. \tag{3.10}$$

Then $Q_z f = f_1$. Hence $T_w^* Q_z f = T_w^* f_1$. On the other hand, by (3.10) we have

$$zT_w^* f = T_w^* z f = T_w^* f_1 + T_w^* f_2.$$

Since $T_w^* q_1(z)H^2 \subset q_1(z)H^2$, then $T_w^* f_2 \in q_1(z)H^2$. Since $T_w^* f_1 \in H^2 \ominus q_1(z)H^2$, by the above we have $Q_z T_w^* f = T_w^* f_1$. Thus we get $T_w^* Q_z = Q_z T_w^*$.

Since $T_w(H^2 \ominus q_1(z)H^2) \subset H^2 \ominus q_1(z)H^2$, similarly we have $T_w Q_z = Q_z T_w$ on $H^2 \ominus q_1(z)H^2$. ∎

LEMMA 3.4. *$S_zS_w^* = S_w^*S_z$ holds if and only if $BA = 0$.*

Proof. By Lemma 3.3(i), $T_w^*Q_z = Q_zT_w^*$ on $H^2 \ominus q_1(z)H^2$. Then by (3.7) and (3.8), we have $BA + S_w^*S_z = S_zS_w^*$. Then $S_zS_w^* = S_w^*S_z$ if and only if $BA = 0$. ∎

THEOREM 3.5. *Let N be a backward shift invariant subspace of H^2 and $M = H^2 \ominus N$. Suppose that $M \cap H^2(\Gamma_z) \neq \{0\}$. Put $M \cap H^2(\Gamma_z) = q_1(z)H^2(\Gamma_z)$, where $q_1(z)$ is a one variable inner function. Put $\widetilde{M} = M \ominus q_1(z)H^2$. Then the following conditions are equivalent:*

(i) $S_zS_w^* = S_w^*S_z$;

(ii) $\widetilde{M} \ominus \{f \in \widetilde{M} : T_z^*f \in \widetilde{M}\} \subset w\widetilde{M}$;

(iii) $T_z^*\widetilde{M} \subset \widetilde{M}$.

Proof. (i) ⇔ (ii) By Lemma 3.4, condition (i) is equivalent to $BA = 0$. By (3.3), (3.9), and Lemma 3.2(i), we have that

$$\ker B = \{f \in \widetilde{M} : T_w^*f \in \widetilde{M}\} = w\widetilde{M}.$$

We denote by [ran A] the closed range of A. Let $A_1 = P_{\widetilde{M}}T_zP_N$ on $\widetilde{M} \oplus N$. Then we have [ran A] = [ran A_1]. Since $A_1^* = P_NT_z^*P_{\widetilde{M}}$, we have

$$\ker A_1^* = N \oplus \{f \in \widetilde{M} : T_z^*f \in \widetilde{M}\}.$$

Then

$$[\operatorname{ran} A] = [\operatorname{ran} A_1] = (\widetilde{M} \oplus N) \ominus \ker A_1^* = \widetilde{M} \ominus \{f \in \widetilde{M} : T_z^*f \in \widetilde{M}\}.$$

Therefore, it follows that $BA = 0$ if and only if

$$\widetilde{M} \ominus \{f \in \widetilde{M} : T_z^*f \in \widetilde{M}\} \subset w\widetilde{M}.$$

Thus we get (i) ⇔ (ii).

(ii) ⇒ (iii) Suppose that

$$\widetilde{M} \ominus \{f \in \widetilde{M} : T_z^*f \in \widetilde{M}\} \subset w\widetilde{M}. \tag{3.11}$$

Since $\{f \in \widetilde{M} : T_z^*f \in \widetilde{M}\}$ is a closed subspace, by (3.11) we have

$$\widetilde{M} \ominus w\widetilde{M} \subset \{f \in \widetilde{M} : T_z^*f \in \widetilde{M}\}. \tag{3.12}$$

Since $w\widetilde{M} \subset \widetilde{M}$, we have

$$\widetilde{M} = \sum_{j=0}^{\infty}{}^{\oplus} w^j(\widetilde{M} \ominus w\widetilde{M}). \tag{3.13}$$

To prove (iii), let $f \in \widetilde{M}$. Then by (3.13),

$$f = \sum_{j=0}^{\infty} w^jg_j, \quad \text{where } g_j \in \widetilde{M} \ominus w\widetilde{M}.$$

Since $T_z^*T_w = T_wT_z^*$ on H^2, by (3.12) we have

$$T_z^*f = \sum_{j=0}^{\infty} w^jT_z^*g_j \in \widetilde{M}.$$

(iii) ⇒ (ii) is trivial. ∎

For a one variable inner function $q(z)$, put $q^*(z) = \overline{z}(q(z) - \widehat{q}(0))$.

LEMMA 3.6. *Let $q_1(z)$ and $q_2(z)$ be inner functions. Then we have the following:*

(i) $T_z^* q_1(z) = q_1^*(z)$ *and* $q_1^*(z) \perp q_1(z)H^2(\Gamma_z)$;

(ii) *if* $q_1(z)H^2(\Gamma_z) \subsetneq q_2(z)H^2(\Gamma_z)$, *then the smallest closed* T_z^**-invariant subspace of* $H^2(\Gamma_z)$ *containing* $q_2(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$ *is equal to* $H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$;

(iii) *the closed subspace generated by* $T_z^{*n} q_1^*(z)$, $n = 0, 1, 2, \ldots$, *it is equal to* $H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$.

Proof. (i) Trivially $T_z^* q_1(z) = q_1^*(z)$ holds. For $h \in H^2(\Gamma_z)$, we have

$$\langle q_1^*(z), q_1(z)h \rangle = \langle T_z^* q_1(z), q_1(z)h \rangle = \langle q_1(z), z q_1(z)h \rangle = \langle 1, zh \rangle = 0.$$

Thus we get (i).

(ii) Let L be the smallest backward shift invariant subspace of $H^2(\Gamma_z)$ containing $q_2(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$. Then $L \subset H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$. Let $f \in H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$ such that $f \perp L$. Since $H^2(\Gamma_z) \ominus L$ is invariant, $z^k f \perp L$ for $k \geqslant 0$. Hence

$$z^k f \perp q_2(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z) \quad \text{for every } k \geqslant 0.$$

Since $q_2(z)H^2(\Gamma_z) \subset L \oplus q_1(z)H^2(\Gamma_z)$, we have $f \perp q_2(z)H^2(\Gamma_z)$. Hence

$$f \perp z^n (q_2(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)) \quad \text{for every } k \geqslant 0.$$

Since $q_2(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z) \neq \{0\}$, we have $f = 0$. Thus we get $L = H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$.

(iii) Let E be the closed subspace generated by $T_z^{*n} q_1^*(z), n \geqslant 0$. By (i), $E \subset H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$ and E is a backward shift invariant subspace of $H^2(\Gamma_z)$. Then $H^2(\Gamma_z) \ominus E = q_3(z)H^2(\Gamma_z)$ for some inner function $q_3(z)$ and $q_1(z)H^2(\Gamma_z) \subset q_3(z)H^2(\Gamma_z)$. If $q_1(z)H^2(\Gamma_z) = q_3(z)H^2(\Gamma_z)$ our assertion holds.

Suppose that $q_1(z)H^2(\Gamma_z) \subsetneq q_3(z)H^2(\Gamma_z)$. Put $q_4(z) = q_1(z)/q_3(z)$. Then $q_4(z)$ is a nonconstant inner function, and $q_1^*(z) = q_3(z)q_4^*(z) + \widehat{q}_4(0)q_3^*(z)$. We have $q_4^*(z) \neq 0$, so that $q_3(z)q_4^*(z) \not\perp q_3(z)H^2(\Gamma_z)$. By (i), $q_3^*(z) \perp q_3(z)H^2(\Gamma_z)$. Hence $q_1^*(z) \not\perp q_3(z)H^2(\Gamma_z)$. Since $q_1^*(z) \in E$, $E \not\perp q_3(z)H^2(\Gamma_z)$. This is a contradiction. Hence we get our assertion. ∎

Proof of Theorem 3.1. First, we shall prove our theorem using Corollary 2.5 and Theorem 3.5.

(i) $\Leftrightarrow$ (ii) follows from Theorem 3.5.

(i) $\Rightarrow$ (iv) follows from Corollary 2.5.

(iv) $\Leftrightarrow$ (iii) If $M = q_1(z)H^2$, then $\widetilde{M} = \{0\}$. Suppose that $M = q_1(z)H^2 + q_2(w)H^2$. Then

$$\begin{aligned} M &= q_1(z)H^2 + q_2(w)(q_1(z)H^2 \oplus (H^2 \ominus q_1(z)H^2)) \\ &= q_1(z)H^2 + q_2(w)(H^2 \ominus q_1(z)H^2). \end{aligned}$$

Since $H^2 \ominus q_1(z)H^2$ is w-invariant, we have

$$M = q_1(z)H^2 \oplus q_2(w)(H^2 \ominus q_1(z)H^2).$$

Thus we get $\widetilde{M} = q_2(w)(H^2 \ominus q_1(z)H^2)$.

The converse assertion is not difficult to prove.

(iii) $\Rightarrow$ (ii) is easy.

Here we give another proof of (ii) $\Rightarrow$ (iii) without using Corollary 2.5. We may assume that $\widetilde{M} \neq \{0\}$. By condition (ii), we have $T_z^*\widetilde{M} \subset \widetilde{M}$. Then $T_z^*\widetilde{M} \perp N$, so that $\widetilde{M} \perp zN$. Hence by (3.3) and (3.6),

$$Q_z N \subset N. \tag{3.14}$$

Since $\widetilde{M} \neq \{0\}$ and $w\widetilde{M} \subset \widetilde{M}$, $\widetilde{M} \ominus w\widetilde{M} \neq \{0\}$ holds. Let $f \in \widetilde{M} \ominus w\widetilde{M}$. Then by (3.3) and Lemma 3.2(ii), we have $T_w^* f \in N$. Hence $T_w^* T_z^* f = T_z^* T_w^* f \in N$. Since $T_z^*\widetilde{M} \subset \widetilde{M}$, $T_z^* f \in \widetilde{M}$ holds. Hence by Lemma 3.2(ii) again, $T_z^* f \in \widetilde{M} \ominus w\widetilde{M}$ holds. Thus we get

$$T_z^*(\widetilde{M} \ominus w\widetilde{M}) \subset \widetilde{M} \ominus w\widetilde{M}. \tag{3.15}$$

By (3.2), we have $f \in M$ and $zf = f_1 + f_2 \in \widetilde{M} \oplus q_1(z)H^2$. Then by (3.6), we have $Q_z f = f_1 \in \widetilde{M}$. Since $T_w^* f \in N$, by (3.14) and Lemma 3.3(i) we have $T_w^* Q_z f = Q_z T_w^* f \in N$. Then by (3.3) and Lemma 3.2(ii), $Q_z f \in \widetilde{M} \ominus w\widetilde{M}$ holds. Thus we get

$$Q_z(\widetilde{M} \ominus w\widetilde{M}) \subset \widetilde{M} \ominus w\widetilde{M}. \tag{3.16}$$

We define the operator W_z on $\widetilde{M}$ to $q_1(z)H^2$ by

$$W_z = P_{q_1(z)H^2} T_z = T_z - Q_z. \tag{3.17}$$

Then by Lemma 3.3(ii),

$$W_z T_w = T_w W_z \quad \text{on } \widetilde{M}. \tag{3.18}$$

Then $wW_z\widetilde{M} = W_z(w\widetilde{M}) \subset W_z\widetilde{M}$. Hence we get

$$w\overline{W_z\widetilde{M}} \subset \overline{W_z\widetilde{M}}, \tag{3.19}$$

where $\overline{W_z\widetilde{M}}$ is the norm closure of the space $W_z\widetilde{M}$. Since $\widetilde{M} \perp q_1(z)H^2$, $z\widetilde{M} \perp zq_1(z)H^2$ holds. Then by (3.17), we obtain

$$W_z\widetilde{M} \subset q_1(z)H^2 \ominus zq_1(z)H^2 = q_1(z)H^2(\Gamma_w).$$

Hence $\overline{q}_1(z)\overline{W_z\widetilde{M}} \subset H^2(\Gamma_w)$, so that by (3.19) and the Beurling theorem,

$$\overline{q}_1(z)\overline{W_z\widetilde{M}} = q_2(w)H^2(\Gamma_w) \tag{3.20}$$

for some inner function $q_2(w)$.

Let $f \in \widetilde{M} \ominus w\widetilde{M}$ and $g \in \widetilde{M}$. Since $Q_z\widetilde{M} \subset \widetilde{M}$, by Lemma 3.3(ii) we have $Q_z w\widetilde{M} \subset w\widetilde{M}$. Then by (3.16), $Q_z f \perp Q_z wg$ holds. Since $zf \perp zwg$, by (3.17) we have

$$0 = \langle zf, zwg\rangle = \langle Q_z f \oplus W_z f, Q_z wg \oplus W_z wg\rangle = \langle W_z f, W_z wg\rangle.$$

Then $W_z(\widetilde{M} \ominus w\widetilde{M}) \perp W_z(w\widetilde{M})$. Hence by (3.18), we get

$$W_z(\widetilde{M} \ominus w\widetilde{M}) \perp w\overline{W_z\widetilde{M}}.$$

Therefore by (3.20), we obtain

$$W_z(\widetilde{M} \ominus w\widetilde{M}) \subset \overline{W_z\widetilde{M}} \ominus w\overline{W_z\widetilde{M}} = [q_1(z)q_2(w)],$$

where $[q_1(z)q_2(w)]$ is the linear span of $q_1(z)q_2(w)$. If $W_z(\widetilde{M} \ominus w\widetilde{M}) = \{0\}$, by (3.16) and (3.17) it follows that $z(\widetilde{M} \ominus w\widetilde{M}) \subset \widetilde{M} \ominus w\widetilde{M}$. Then $z^n(\widetilde{M} \ominus w\widetilde{M}) \subset \widetilde{M} \ominus w\widetilde{M}$ for every positive integer n. Since $\widetilde{M} \ominus w\widetilde{M} \neq \{0\}$, we have that $z^n(\widetilde{M} \ominus w\widetilde{M}) \not\perp q_1(z)H^2$ for some n. These contradict with (3.2). Thus there exists f_0 in $\widetilde{M} \ominus w\widetilde{M}$ such that

$$W_z f_0 = a q_1(z)q_2(w) \quad \text{and} \quad a \neq 0. \tag{3.21}$$

Since $zf_0 = Q_z f_0 + W_z f_0$, we have

$$\begin{aligned} f_0 &= T_z^* Q_z f_0 + T_z^* W_z f_0 \\ &= T_z^* Q_z f_0 + a q_1^*(z) q_2(w) \quad \text{by (3.21) and Lemma 3.6(i).} \end{aligned}$$

Hence by (3.15) and (3.16), it follows that $q_1^*(z)q_2(w) \in \widetilde{M} \ominus w\widetilde{M}$, $n \geqslant 0$. By Lemma 3.6(iii), we obtain

$$q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)) \subset \widetilde{M} \ominus w\widetilde{M}. \tag{3.22}$$

We shall prove that

$$\widetilde{M} \ominus w\widetilde{M} = q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)). \tag{3.23}$$

Let

$$F \in (\widetilde{M} \ominus w\widetilde{M}) \ominus q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)). \tag{3.24}$$

Let i, j be nonnegative integers. Since $q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z))$ is invariant for the operator T_z^*,

$$T_z^{*i}(q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z))) \in q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)).$$

Since $\widetilde{M} \ominus w\widetilde{M} \perp w^n(\widetilde{M} \ominus w\widetilde{M})$ for every positive integer n, by (3.22) and (3.24) we have

$$w^j F \perp T_z^{*i}(q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)))$$

and

$$F \perp w^j T_z^{*i}(q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z))).$$

Hence

$$w^j F \perp \overline{z}^i q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)) \tag{3.25}$$

and

$$F \perp \overline{z}^i w^j q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)). \tag{3.26}$$

Since $q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z))$ is invariant for the operator Q_z, similarly we have

$$w^j F \perp Q_z^i(q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z))) \tag{3.27}$$

and

$$F \perp w^j Q_z^i(q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z))). \tag{3.28}$$

By (3.6),

$$Q_z^i(q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z))) = P_\perp(z^i q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z))).$$

Since $\widetilde{M} \perp q_1(z)H^2$ and $w^j F \in \widetilde{M}$, by (3.27) and (3.28) we have

$$w^j F \perp z^i q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)) \tag{3.29}$$

and

$$F \perp z^i w^j q_2(w)(H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)). \tag{3.30}$$

Since $\widetilde{M} \neq \{0\}$, by (3.2) $q_1(z)$ is not constant. Hence $H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z) \neq \{0\}$. Therefore by (3.25), (3.26), (3.29), and (3.30), we get $F = 0$. Thus we get (3.23).

By (3.23), we obtain

$$\widetilde{M} = \sum_{j=0}^{\infty}{}^{\oplus} w^j(\widetilde{M} \ominus w\widetilde{M}) = q_2(w)(H^2 \ominus q_1(z)H^2). \qquad \blacksquare$$

The following is a consequence sufficiently interesting in its own right.

COROLLARY 3.7. *Let $q_1(z)$ be a nonconstant inner function. Let L be a closed subspace of $H^2 \ominus q_1(z)H^2$ and $L \neq \{0\}$. Suppose that $wL \subset L$, $Q_zL \subset L$, and $Q_z^*L \subset L$. Then, there exists an inner function $q_2(w)$ such that $L = q_2(w)(H^2 \ominus q_1(z)H^2)$.*

Proof. We note that $Q_z^* = T_z^*$ on $H^2 \ominus q_1(z)H^2$. Put $M = L \oplus q_1(z)H^2$. Then, by our assumption, M is an invariant subspace and $q_1(z)H^2(\Gamma_z) \subset M \cap H^2(\Gamma_z)$. Put $M \cap H^2(\Gamma_z) = q_3(z)H^2(\Gamma_z)$, where $q_3(z)$ is inner. Then $q_1(z)H^2(\Gamma_z) \subset q_3(z)H^2(\Gamma_z)$.

Suppose that $q_1(z)H^2(\Gamma_z) \neq q_3(z)H^2(\Gamma_z)$. Let L_1 be the smallest closed subspace of $H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$ containing $q_3(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$ such that $T_z^*L_1 \subset L_1$. By Lemma 3.6(ii), $L_1 = H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z)$. Since $M \cap H^2(\Gamma_z) = q_3(z)H^2(\Gamma_z)$,

$$q_3(z)H^2(\Gamma_z) \ominus q_1(z)H^2(\Gamma_z) \subset L.$$

Since $T_z^*L = Q_z^*L \subset L$, we have $L_1 \subset L$. Hence we have

$$H^2 \ominus q_1(z)H^2 = \sum_{j=0}^{\infty}{}^{\oplus} w^j L_1 \subset L \subset H^2 \ominus q_1(z)H^2.$$

Therefore, $L = H^2 \ominus q_1(z)H^2$. Thus, we get our assertion.

Suppose that $q_1(z)H^2(\Gamma_z) = q_3(z)H^2(\Gamma_z)$. We have $L = M \ominus q_1(z)H^2$. By our assumption, $T_z^*L = Q_z^*L \subset L$. Then by Theorem 3.1, we have $L = q_2(w)(H^2 \ominus q_1(z)H^2)$ for an inner function $q_2(w)$. ∎

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REFERENCES

1. P.R. AHERN, D.N. CLARK, Invariant subspaces and analytic continuation in several variables, *J. Math. Mech.* **19**(1970), 963–969.
2. A. BEURLING, On two problems concerning linear transformations in Hilbert space, *Acta Math.* **81**(1949), 239–255.
3. J.A. CIMA, W.T. ROSS, *The Backward Shift on the Hardy Space*, Math. Surveys Monogr., vol. 79, Amer. Math. Soc., Providence, RI, 2000.
4. M. COTLAR, C. SADOSKY, A polydisk version of Beurling's characterization for invariant subspaces of finite multi-codimension, *Contemp. Math.* **212** (1998), 51–56.
5. R.G. DOUGLAS, R. YANG, Operator theory in the Hardy space over the bidisk. I, *Integral Equations Operator Theory* **38**(2000), 207–221.
6. K. IZUCHI, T. NAKAZI, Backward shift invariant subspaces in the bidisc, *Hokkaido Math. J.*, to appear.
7. K. IZUCHI, T. NAKAZI, M. SETO, Backward shift invariant subspaces in the bidisc. III, preprint.

8. V. Mandrekar, The validity of Beurling theorems in polydiscs, *Proc. Amer. Math. Soc.* **103**(1988), 145–148.
9. T. Nakazi, Certain invariant subspaces of H^2 and L^2 on a bidisc, *Canad. J. Math.* **40**(1988), 1272–1280.
10. T. Nakazi, Invariant subspaces in the bidisc and commutators, *J. Austral. Math. Soc. Ser. A* **56**(1994), 232–242.
11. W. Rudin, *Function Theory in Polydiscs*, Benjamin, New York 1969.
12. D. Sarason, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.* **127**(1967), 179–203.

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