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IRREDUCIBLE MULTIPLICATION OPERATORS ON SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let M_φ be the operator of multiplication by a bounded analytic function φ in a planar domain Ω on a Hilbert space of analytic functions in Ω. We give a sufficient condition for M_φ to be irreducible.

KEYWORDS: *Reducing subspace, commutant, Cowen-Douglas classes.*

MSC (2000): 47A15, 47B38.

1. INTRODUCTION

Let Ω be a bounded domain in the complex plane $\mathbb{C}$ and let $H^\infty(\Omega)$ be the space of all bounded analytic functions in Ω. We will consider Hilbert spaces of analytic functions in Ω and multiplication operators on such spaces induced by functions from $H^\infty(\Omega)$.

More specifically, we are interested in the structure of the reducing subspace lattice of the operator

$$M_\varphi : H \to H$$

for $\varphi \in H^\infty(\Omega)$. We mention some known results in this area that serve as a motivation for the present paper.

First, if H is the classical Hardy space of the unit disk $\mathbb{D}$, and if φ is an inner function on $\mathbb{D}$, then M_φ is a pure isometry and a shift operator on H, and so its reducing subspaces are in a one-to-one correspondence with the closed subspaces of $H \ominus (\varphi H)$. Therefore, the reducing subspace lattice of such an operator M_φ is isomorphic to the lattice of closed subspaces of $H \ominus (\varphi H)$. In particular, if φ is any inner function other than a Möbius map, then M_φ has infinitely many reducing subspaces. See [5] for this result and related references.

Second, if H is the Bergman space of $\mathbb{D}$ and φ is a Blaschke product of two zeros in $\mathbb{D}$, then $M_\varphi : H \to H$ has exactly two nontrivial reducing subspaces. See [11] for proof and a description of these reducing subspaces.

Finally, if H is the Hilbert space on $\mathbb{D}$ induced by a positive weight sequence $\{\omega_n\}$ (with $\omega_{n+1}/\omega_n \leqslant M$ for some constant M and all $n \geqslant 0$):

$$H = \Big\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \|f\|^2 = \sum_{n=0}^{\infty} \omega_n |a_n|^2 < \infty \Big\},$$

and if $\varphi(z) = z^N$ for some integer $N > 1$, then the operator $M_\varphi : H \to H$ has only a finite number of reducing subspaces if and only if there exist no integers i and j, $i \neq j$, such that

$$\frac{\omega_{i+kN}}{\omega_i} = \frac{\omega_{j+kN}}{\omega_j}$$

for all $k \geqslant 0$. See [9] for details. In particular, M_{z^N}, $N > 1$, has infinitely many reducing subspaces in the Hardy space, but it only has a finite number of reducing subspaces in the Bergman space.

The purpose of this paper is to show that, in general, reducing subspaces are hard to come by. For example, if $\Omega = \mathbb{D}$ and if φ is a "random" polynomial, then chances are that M_φ will have no non-trivial reducing subspace. See Section 5.

In the rest of the paper we assume that H is a Hilbert space of analytic functions in Ω with the following properties:

(1) Every point in Ω is a nonzero bounded linear functional on H, so that H possesses a reproducing kernel $K(z, w)$ with $K(z, z) > 0$ for all $z \in \Omega$.

(2) Every function $\varphi \in H^\infty(\Omega)$ is a pointwise multiplier of H, so that the operator of multiplication by φ, M_φ, is a bounded linear operator on H (by the closed-graph theorem).

(3) For every $\lambda \in \Omega$ the operator $M_{z-\lambda}$ is bounded below on H, so that the space $(z - \lambda)H$ is closed in H.

(4) For every $\lambda \in \Omega$ the space $H \ominus (z - \lambda)H$ is one dimensional.

Although our main theorem below is stated for any such Hilbert space, the spaces that we are most interested in are the classical Hardy and Bergman spaces on the open unit disk, which certainly satisfy all the conditions above.

THEOREM 1.1. *Suppose $\varphi \in H^\infty(\Omega)$. If there exists a nonempty open set $V \subset \varphi(\Omega)$ such that $\Omega \cap \varphi^{-1}(z)$ is a singleton for every $z \in V$, then the operator $M_\varphi : H \to H$ has no nontrivial reducing subspace, and its commutant consists of exactly the multiplication operators M_f with $f \in H^\infty(\Omega)$.*

Note that if Ω is the domain enclosed by a simple closed curve and φ is smooth up to the boundary, then φ maps $\partial\Omega$ to a closed curve γ, and the condition in the theorem above is equivalent to the following: there exists a point $z_0 \in \varphi(\Omega) - \gamma$ such that the winding number of γ about z_0 is equal to 1 or -1.

When $\Omega = \mathbb{D}$, no inner function φ satisfies the condition in the theorem above unless it is a Möbius map. Furthermore, if $\varphi = f(\psi)$, where f is a function in $H^\infty(\mathbb{D})$ and ψ is a non-Möbius inner function, then φ does not satisfy the condition in the theorem above. In the final section of the paper we shall explain that as long as the function φ is not a composition of another function with a non-Möbius inner function, then the condition in the theorem above very likely holds, at least in the case when φ is analytic in the closed disk.

2. PRELIMINARIES

For a positive integer n and a domain $U \subset \mathbb{C}$, the Cowen-Douglas class $B_n(U)$ consists of bounded linear operators T on any fixed separable infinite dimensional Hilbert space X with the following properties:

(a) $\operatorname{Ran}(\lambda - T) = X$ for every $\lambda \in U$.
(b) $\dim[\ker(\lambda - T)] = n$ for every $\lambda \in U$.
(c) $\operatorname{Span}\{\ker(\lambda - T) : \lambda \in U\} = X$.

Here Span denotes the closed linear span of a collection of sets in X.

The classes B_n were introduced by Cowen and Douglas; see [3]. Among the many subsequent contributions to the study of these classes we mention [4] and [10]. We will need the following result from the Cowen-Douglas theory ([3]).

Lemma 2.1. *Every operator T in $B_1(U)$ is irreducible.*

Fix any $w \in \Omega$. Since the point-evaluation at w is a bounded functional on H, there exists a unique function $K_w \in H$ such that

$$f(w) = \langle f, K_w \rangle$$

for all $f \in H$. The function

$$K : \Omega \times \Omega \to \mathbb{C}$$

defined by

$$K(z, w) = K_w(z), \quad z, w \in \Omega,$$

is called the reproducing kernel of H. It is well known that $K(z, w)$ is analytic in z and conjugate analytic in w. In fact, we always have

$$\overline{K(z, w)} = K(w, z)$$

for all $z, w \in \Omega$.

Lemma 2.2. *If V is a nonempty open set in Ω, then the functions K_z, where $z \in V$, span the whole space H.*

Proof. Let H_1 be the closed linear span of the functions K_z, $z \in V$. If $H_1 \neq H$, then there exists a unit vector $f \in H \ominus H_1$. In particular, for every $z \in V$ we have $K_z \in H_1$ and so

$$f(z) = \langle f, K_z \rangle = 0.$$

Since Ω is connected, the identity theorem tells us that $f = 0$, a contradiction to $\|f\| = 1$. ∎

Note that condition (4) is not needed in the above lemma, nor is it necessary for the following lemma.

LEMMA 2.3. *For any $z \in \Omega$ let X_z denote the subspace of H consisting of functions f with $f(z) = 0$, and let Y_z be the one-dimensional subspace of H spanned by the reproducing kernel K_z. Then we have*

$$H = X_z \oplus Y_z$$

for every $z \in \Omega$.

Proof. It is obvious that $X_z \perp Y_z$. Let P_z be the orthogonal projection from H onto Y_z. Given $f \in H$ we can write $f = g + P_z(f)$. Then

$$g(z) = \langle g, K_z \rangle = \langle f - P_z(f), K_z \rangle = \langle f, K_z \rangle - \langle P_z(f), K_z \rangle = \langle f, K_z \rangle - \langle f, K_z \rangle = 0,$$

so that $g \in X_z$. ∎

It will be clear that condition (4) is the most crucial one in our analysis. We will need the following consequences of this condition.

LEMMA 2.4. *If $f \in H$, $a \in \Omega$, and $f(a) = 0$, then $f = (z - a)g$ for some $g \in H$.*

Proof. See [7]. ∎

LEMMA 2.5. *If a and b are points in Ω, then*

$$\dim(H \ominus (z - a)(z - b)H) = 2.$$

Proof. If H satisfies condition (4), then so does the closed subspace $(z-a)H$ (see [7]). Thus the space

$$H \ominus (z - a)(z - b)H = (H \ominus (z - a)H) \oplus ((z - a)H \ominus (z - b)(z - a)H)$$

is two-dimensional. ∎

We will need to use the following well-known result from general functional analysis.

LEMMA 2.6. *Suppose S is a bounded linear operator on H. If S is bounded below, then S^* is onto.*

Proof. See Theorem 4.15 of [8]. ∎

3. IRREDUCIBLE MULTIPLICATION OPERATORS

In this section we show that a lot of multiplication operators on Hilbert spaces of analytic functions belong to the Cowen-Douglas class B_1 and hence are irreducible. Our analysis also reveals an interesting fact about the Cowen-Douglas classes: an operator may simultaneously belong to B_n for several different n; an example is included in Section 5.

PROPOSITION 3.1. *Suppose* $\varphi \in H^\infty(\Omega)$ *and there exists a domain* $V \subset \varphi(\Omega)$ *such that* $\Omega \cap \varphi^{-1}(z)$ *is a singleton for every* $z \in V$. *Then the adjoint of the operator* $M_\varphi : H \to H$ *belongs to the Cowen-Douglas class* $B_1(U)$, *where* $U = \{\overline{z} : z \in V\}$.

Proof. Let T be the adjoint of $M_\varphi : H \to H$. For $\lambda = \overline{w} \in U$, with $w \in V$, consider the operator

$$S_\lambda = \lambda - T = \overline{w} - M_\varphi^* = M_{w-\varphi}^*$$

on H. If z_0 is the pre-image in Ω of w under φ, then

$$w - \varphi(z) = \varphi(z_0) - \varphi(z) = (z - z_0)\psi(z), \quad z \in \Omega,$$

where ψ belongs to $H^\infty(\Omega)$ and is nonvanishing on Ω.

The function ψ is also bounded below on Ω. In fact, if we choose $\sigma > 0$ such that the closed disk D centered at w with radius σ is contained in V, then the pre-image in Ω of D under φ is a compact set C in Ω (since there is an analytic branch of φ^{-1} mapping V back into Ω), which must have a positive distance δ to $\partial\Omega$. Now, if ψ is not bounded below on Ω, then there exists a sequence $\{z_n\}$ in $\Omega - \{z_0\}$ such that $\psi(z_n) \to 0$ as $n \to \infty$. Since ψ is zero-free in Ω, we must have $z_n \to \partial\Omega$ as $n \to \infty$. On the other hand, the boundedness of Ω implies that $\varphi(z_n) \to \varphi(z_0)$, so there exists a positive integer N such that $\varphi(z_n) \in D$ for all $n > N$, or $z_n \in C$ for all $n > N$, a contradiction to $z_n \to \partial\Omega$.

Since $\psi(z)$ is bounded above and below on Ω, the operator M_ψ is invertible on H. Thus the operator

$$M_{w-\varphi} = M_{z-z_0} M_\psi$$

is bounded below on H. By Lemma 2.6, the operator $S_\lambda = M_{w-\varphi}^*$ must be onto.

With notation from the previous paragraphs, we also see that $f \in \ker(S_\lambda)$ if and only if

$$\langle M_{w-\varphi}^* f, g\rangle = 0, \quad g \in H,$$

if and only if

$$\langle f, (z - z_0)\psi g\rangle = 0, \quad g \in H.$$

By Lemma 2.4, the condition above is equivalent to

$$\langle f, g\rangle = 0, \quad g \in H, \ g(z_0) = 0.$$

If $K_{z_0}(w) = K(w, z_0)$ is the reproducing kernel of H at z_0, then according to Lemma 2.3, the condition above is equivalent to $f = cK_{z_0}$, where c is some constant. This shows that $\ker(S_\lambda) = \mathbb{C}K_{z_0}$, and so

$$\dim(\ker(S_\lambda)) = 1$$

for each $\lambda \in U$. Also, an application of Lemma 2.2 gives

$$\operatorname{Span}\{\ker(S_\lambda) : \lambda \in U\} = H. \quad \blacksquare$$

Combining Lemma 2.1 and Proposition 3.1, we have proved the first part of the main theorem.

4. THE COMMUTANT OF MULTIPLICATION OPERATORS

Recall that if T is a bounded linear operator on H, then the commutant of T, denoted $(T)'$, consists of all bounded linear operators S on H such that $TS = ST$.

Proposition 4.1. *Suppose $\varphi \in H^\infty(\Omega)$ and there exists a domain $V \subset \varphi(\Omega)$ such that $\Omega \cap \varphi^{-1}(z)$ is a singleton for every $z \in V$. Then*

$$(M_\varphi)' = \{M_h : h \in H^\infty(\Omega)\}.$$

Proof. It is obvious that every M_h, where $h \in H^\infty(\Omega)$, commutes with M_φ.

Let S be a bounded linear operator on H satisfying $SM_\varphi = M_\varphi S$. Then S also commutes with $M_{\lambda-\varphi}$ for every complex number λ. It follows that S^* commutes with $M^*_{\lambda-\varphi}$ for every constant λ.

Fix any $w \in V$, say, $w = \varphi(u)$ with u being the unique pre-image in Ω of w under the mapping φ. That S^* commutes with $M^*_{w-\varphi}$ implies that

$$S^* : \ker(M^*_{w-\varphi}) \to \ker(M^*_{w-\varphi}).$$

Since (see the proof of Proposition 3.1)

$$\ker(M^*_{w-\varphi}) = \mathbb{C}K_u,$$

there must exist a complex-valued function h defined on

$$U = \Omega \cap \varphi^{-1}(V)$$

such that

$$S^*K_u = \overline{h(u)}K_u$$

for all $u \in U$.

If g is any function in H, then

$$\langle g, S^*K_u\rangle = \langle g, \overline{h(u)}K_u\rangle = h(u)g(u)$$

for all $u \in U$. Since K_u is conjugate-analytic in u, and since for every $z \in \Omega$ there exists $g \in H$ such that $g(z) \neq 0$ (by condition (1)), the equation above shows that h is analytic in U and h has an analytic extension to Ω as well. Thus we extend h to the whole domain Ω and still use h to denote the resulting function. The identity theorem then gives

$$S^*K_z = \overline{h(z)}K_z$$

for all $z \in \Omega$. Since

$$|h(z)| = \left|\frac{\langle S^*K_z, K_z\rangle}{\langle K_z, K_z\rangle}\right| \leqslant \|S\|$$

for every $z \in \Omega$, we conclude that $h \in H^\infty(\Omega)$.

Finally, for any $f \in H$ and $z \in \Omega$, we have

$$Sf(z) = \langle Sf, K_z\rangle = \langle f, S^*K_z\rangle = \langle f, \overline{h(z)}K_z\rangle = h(z)f(z),$$

so that $S = M_h$. ∎

5. AN EXAMPLE AND SOME REMARKS

Consider the case $\Omega = \mathbb{D}$, the open unit disk, and the function

$$\varphi(z) = z(z+1), \quad z \in \mathbb{D}.$$

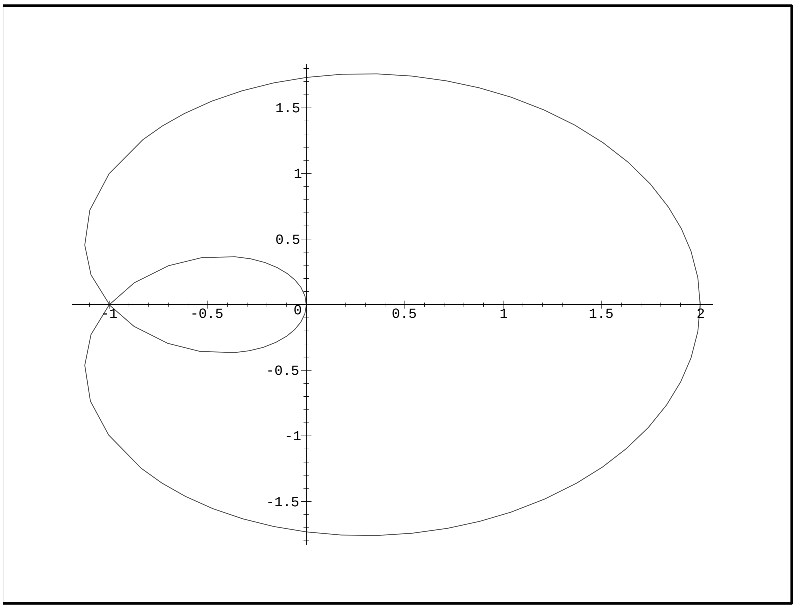

Figure 1. The image of the unit circle under $\varphi(z) = z(z+1)$

Let γ be the closed curve

$$\gamma(t) = \varphi(\mathrm{e}^{\mathrm{i}t}), \quad 0 \leqslant t \leqslant 2\pi;$$

γ is the image of the unit circle under the mapping φ. The image $\varphi(\mathbb{D})$ is then the region enclosed by γ; see Figure 1. Let V_1 be the part of $\varphi(\mathbb{D})$ wrapped around once by γ and let V_2 be the part of $\varphi(\mathbb{D})$ wrapped around twice by γ. Note that V_1 and V_2 are both invariant under complex conjugation.

It is clear that $\mathbb{D} \cap \varphi^{-1}(z)$ is a singleton for every $z \in V_1$. Thus the adjoint of the operator $M_{z(z+1)} : H \to H$ belongs to the Cowen-Douglas class $B_1(V_1)$, and hence $M_{z(z+1)}$ is irreducible on H.

Proposition 5.1. *The adjoint of the operator $M_{z(z+1)} : H \to H$ also belongs to the Cowen-Douglas class $B_2(V_2)$.*

Proof. For every $\lambda \in V_2$, the set $\mathbb{D} \cap \varphi^{-1}(\lambda)$ consists of two (not necessarily distinct) points a_1 and a_2 in $\mathbb{D}$. It follows that

$$z(z+1) - \lambda = (z - a_1)(z - a_2),$$

and so

$$M^*_{z(z+1)} - \overline{\lambda} = M^*_{(z-a_1)(z-a_2)}.$$

Since $M_{(z-a_1)(z-a_2)}$ is bounded below, Lemma 2.6 gives us

$$\mathrm{Ran}\,(M^*_{z(z+1)} - \overline{\lambda}) = H,$$

proving condition (a) in the definition of B_2. Also,

$$\ker(M^*_{z(z+1)} - \overline{\lambda}) = H \ominus \mathrm{Ran}\,(M_{(z-a_1)(z-a_2)}) = H \ominus [(z-a_1)(z-a_2)H]\,.$$

Thus Lemma 2.5 gives us

$$\dim(\ker(\overline{\lambda} - M^*_{z(z+1)})) = 2,$$

and we obtain condition (b) in the definition of B_2.

The above computation also shows that the kernel functions K_{a_1} and K_{a_2} both belong to $\ker(M^*_{z(z+1)} - \overline{\lambda})$. When λ varies in V_2, a_1 and a_2 cover an open set in $\mathbb{D}$. Condition (c) in the definition of B_2 then follows from Lemma 2.2. ∎

The arguments used in the proofs of Propositions 3.1 and 5.1 can easily be modified to yield the following generalization, whose proof is left to the interested reader.

PROPOSITION 5.2. *Suppose $\varphi \in H^\infty(\Omega)$ and V is a (nonempty) domain contained in $\varphi(\Omega)$. If there exists a positive integer n such that $\Omega \cap \varphi^{-1}(z)$ consists of n points (counting multiplicity) for every $z \in V$, then the adjoint of the operator $M_\varphi : H \to H$ belongs to the Cowen-Douglas class $B_n(U)$, where $U = \{\overline{z} : z \in V\}$.*

The function $\varphi(z) = z(z+1)$ is typical in a certain sense. In fact, if φ is analytic on the closed disk $\overline{\mathbb{D}}$, then the image of the unit circle under φ is a smooth closed curve γ which divides the complex plane into a finite number of regions $V_1, \ldots, V_n$, with the property that the winding number of γ about the points of any V_k stays constant (depending on k). Unless γ traverses itself more than once (such as the case of a non-Möbius finite Blaschke product, or the composition of a smooth function with such a Blaschke product), we suspect that one of these regions V_k will be wrapped around by γ only once, so that the condition in the main theorem should hold.

It was conjectured in [6] that M_φ, as an operator on the classical Hardy space of $\mathbb{D}$, has a nontrivial reducing subspace if and only if $\varphi = f(\psi)$, where $f \in H^\infty(\mathbb{D})$ and ψ is a non-Möbius inner function. This conjecture was shown to be true in [2] in the special case when φ is entire, but was then shown in [1] to be false in general.

In view of these remarks, we repeat (a special case of) a question in [2] here as the following.

CONJECTURE 5.3. *Suppose φ, not a constant, is analytic on the closed disk $\overline{\mathbb{D}}$ and γ is the image of the unit circle under φ. Then the following conditions are equivalent:*

(i) *There exists a point $z \in \varphi(\mathbb{D}) - \gamma$ such that the winding number of γ about z is* 1 *or* -1*, that is, the condition in the theorem holds.*

(ii) *The function φ is not of the form $\varphi = f(B)$, where f is analytic on $\overline{\mathbb{D}}$ and B is a finite Blaschke product with more than one zero in $\mathbb{D}$.*

It is obvious that condition (i) implies condition (ii). The main result of [2] implies that the conjecture above is true when φ is an entire function. In particular, a polynomial p satisfies the condition in the theorem (for $\Omega = \mathbb{D}$) if and only if $p \neq q(z^N)$, where q is a polynomial and $N > 1$.

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