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ELEMENTARY AND REFLEXIVE HYPERPLANES OF GENERALIZED TOEPLITZ OPERATORS

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ABSTRACT. Theorems of Azoff and Ptak on elementary and reflexive hyperplanes of classical Toeplitz operators are extended to hyperplanes of generalized Toeplitz operators associated with operators of regular norm-sequences.

KEYWORDS: *Elementary subspace, reflexive subspace, Toeplitz operator, residual set, spectral density.*

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1. INTRODUCTION

Let $\mathcal{H}$ be a (complex) Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the set of all (bounded, linear) operators acting on $\mathcal{H}$. It is known that $\mathcal{L}(\mathcal{H})$ can be identified with the dual of the Banach space $\tau(\mathcal{H})$ of trace class operators on $\mathcal{H}$. More precisely, the mapping $\Psi \in \mathcal{L}(\mathcal{L}(\mathcal{H}), \tau(\mathcal{H})^{\#})$, defined by $[A, \Psi(C)] := \operatorname{tr}(AC)$ with $A \in \tau(\mathcal{H})$, $C \in \mathcal{L}(\mathcal{H})$, is an invertible isometry (see e.g. [30]). We write C simply for $\Psi(C)$. This duality induces a weak* topology on $\mathcal{L}(\mathcal{H})$. As usual, for any $u, v \in \mathcal{H}$, let $u \otimes v \in \mathcal{L}(\mathcal{H})$ be defined by $(u \otimes v)(x) := \langle x, v\rangle u$ with $x \in \mathcal{H}$, and let $\mathcal{F}_1(\mathcal{H}) := \{u \otimes v : u, v \in \mathcal{H}\}$ be the set of operators of rank less than or equal to one.

Let $\mathcal{T}$ be an arbitrary weak* closed linear manifold in $\mathcal{L}(\mathcal{H})$. We recall that $\mathcal{T}$ is *elementary*, if every weak* continuous linear functional $\varphi : \mathcal{T} \to \mathbb{C}$ is induced by an operator $u \otimes v \in \mathcal{F}_1(\mathcal{H})$, that is $\varphi(C) = \operatorname{tr}(C(u \otimes v)) = \langle Cu, v\rangle$ holds for all $C \in \mathcal{T}$. Since $\mathcal{T}$ can be identified with the dual of the quotient space $\tau(\mathcal{H})/\mathcal{T}_{\perp}$, elementary means that each non-zero coset $A + \mathcal{T}_{\perp}$, $A \in \tau(\mathcal{H})$, contains a rank-one operator. This property played crucial role in the proof of S. Brown's theorem on the existence of proper invariant subspaces of subnormal operators, and was studied by many authors; see [7], [4] or [5].

The subspace $\mathcal{T}$ is called *reflexive*, if any operator $C \in \mathcal{L}(\mathcal{H})$, satisfying the condition $Ch \in (\mathcal{T}h)^{-}$ for all $h \in \mathcal{H}$, belongs to $\mathcal{T}$. By the Hahn–Banach Theorem this happens exactly when the set $\mathcal{F}_1(\mathcal{H}) \cap \mathcal{T}_{\perp}$ is total in the preannihilator $\mathcal{T}_{\perp}$.

Reflexivity means that the linear manifold $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ is determined by the action of the operators of $\mathcal{T}$ on $\mathcal{H}$; if $\mathcal{T}$ is a reflexive algebra then $\mathcal{T}$ is determined by its invariant subspace lattice. This property has attracted great attention too, see e.g. the articles [9] or [12] and the references given there. The subspace $\mathcal{T}$ is called *hereditarily reflexive*, if each weak* closed submanifold of $\mathcal{T}$ is reflexive. It is a remarkable fact that $\mathcal{T}$ *is hereditarily reflexive if and only if* $\mathcal{T}$ *is both elementary and reflexive* (see e.g. Proposition 1.7, [1]).

The subspace $\mathcal{T}$ is *transitive*, if $(\mathcal{T}h)^- = \mathcal{H}$ is true, for every non-zero vector $h \in \mathcal{H}$. Equivalently: $\mathcal{T}_\perp \cap \mathcal{F}_1(\mathcal{H}) = \{0\}$ holds, in contrast with the case of reflexivity. The manifold $\mathcal{T}$ is intransitive, precisely when there exist non-zero vectors $u, v \in \mathcal{H}$ such that $\langle Cu, v\rangle = 0$ for all $C \in \mathcal{T}$, that is when $\mathcal{T}$ is contained in the hyperplane $\mathcal{L}_{u,v}(\mathcal{H}) := (\mathbb{C}(u \otimes v))^\perp = \{C \in \mathcal{L}(\mathcal{H}) : \langle Cu, v\rangle = 0\}$ induced by the rank-one operator $u \otimes v$.

Let us consider now the Hardy space H^2 (see [15]), and the simple unilateral shift $S \in \mathcal{L}(H^2)$, $Sf := \chi f$, $f \in H^2$, where $\chi(z) = z$. We recall that $C \in \mathcal{L}(H^2)$ is a *Toeplitz operator*, if $S^*CS = C$; see Chapter 25, [13], and [25] for the basic facts concerning this important class of operators. It is easy to verify that the set $\mathcal{T}(S)$ of all Toeplitz operators is transitive (see Theorem 3.1, [1], or Theorem 2.5 of this paper). On the other hand, by a pioneering result of D. Sarason in [28], the commutant $\{S\}'$, which consists of the Toeplitz operators with analytic symbols, is a reflexive subspace of $\mathcal{T}(S)$. In their paper ([1]) E.A. Azoff and M. Ptak considered all weak* closed submanifolds of the set of Toeplitz operators. Taking any intransitive, weak* closed submanifold $\mathcal{T}$ of $\mathcal{T}(S)$, we can find non-zero vectors $u, v \in \mathcal{H}$ such that $\mathcal{T} \subset \mathcal{T}_{u,v}(S) := \mathcal{T}(S) \cap \mathcal{L}_{u,v}(\mathcal{H})$. Azoff and Ptak have shown that *the hyperplane* $\mathcal{T}_{u,v}(S)$ *is always elementary and reflexive.* Thus $\mathcal{T}_{u,v}(S)$ is hereditarily reflexive, and so the subspace $\mathcal{T} \subset \mathcal{T}_{u,v}(S)$ is also reflexive. Therefore, $\mathcal{T}(S)$ has the *dichotomy property* that any of its weak* closed subspaces is either transitive or reflexive. (Conversely, dichotomy in $\mathcal{T}(S)$ readily implies that every intransitive hyperplane is elementary and reflexive.)

Given an arbitrary $T \in \mathcal{L}(\mathcal{H})$, the *operator* $C \in \mathcal{L}(\mathcal{H})$ is called T*-Toeplitz*, if $T^*CT = r(T)^2C$, where $r(T)$ stands for the spectral radius of T. Let us consider the selfadjoint, weak* closed linear manifold $\mathcal{T}(T)$ of all T-Toeplitz operators. Our goal in this article is to examine how the aforementioned results can be extended from the classical setting $\mathcal{T}(S)$ to the generalized case $\mathcal{T}(T)$. In particular, we want to find conditions under which $\mathcal{T}(T)$ is transitive, and the hyperplane $\mathcal{T}_{u,v}(T) := \mathcal{T}(T) \cap \mathcal{L}_{u,v}(\mathcal{H})$ is elementary or reflexive. We shall assume that the operator T has regular norm-sequence, ensuring in this way the existence of a satisfactory symbolic calculus for $\mathcal{T}(T)$ (see [23]). We explore further properties of this calculus in Section 2, and show that the set $\mathcal{T}(T)$ of all T-Toeplitz operators is transitive, whenever the operator T is quasianalytic. We shall prove in Section 3, among others, that the hyperplane $\mathcal{T}_{u,v}(T)$ is elementary, provided the vectors u and v are analytically related to T in a certain sense, and if the cyclic operator T is full with analytic spectral densities. In Section 4 conditions are given for such analytic behaviour. It turns out that all non-zero vectors are analytic with respect to T, exactly when T is a quasiaffine transform of $r(T)S$. Finally, we show in Section 5 that the hyperplane $\mathcal{T}_{u,v}(T)$ is reflexive, whenever the vectors u and v are regular for T, which means the possibility of their inner-outer-type factorization in relation with T. This regularity property will be studied in more details in a separate paper.

Throughout the text several examples and remarks are given in order to illuminate the different facets of the statements to be proved.

In the remaining part of this section we summarize some of the fundamental results of [23], which will be needed in the sequel. We remind the reader that a sequence $\xi \in \ell^\infty = \ell^\infty(\mathbb{N},\mathbb{C})$ *almost converges* to the complex number c, in notation: $\text{a-}\lim_{n\to\infty} \xi(n) = c$, if the means $(n+1)^{-1}\sum_{k=m}^{m+n} \xi(k)$ converge to c uniformly in m, as n tends to infinity. The sequence ξ *almost converges* to c *in the strong sense*, and we use the notation $\text{as-}\lim_{n\to\infty} \xi(n) = c$, if $\text{a-}\lim_{n\to\infty} |\xi(n) - c| = 0$. The mapping $p : \mathbb{N} \to (0,\infty)$ is a *gauge function*, if the sequence $\{p(n+1)/p(n)\}_{n\in\mathbb{N}}$ almost converges to a positive number c_p in the strong sense, and $c_p^n \leqslant p(n)$ holds, for every $n \in \mathbb{N}$. If, in addition, the sequence $\{c_p^n/p(n)\}_{n\in\mathbb{N}}$ does not almost converge to zero, then p is called a *strict gauge function*. The sets of gauge functions and strict gauge functions are denoted by $\mathcal{P}$ and $\widetilde{\mathcal{P}}$, respectively. For any $p \in \widetilde{\mathcal{P}}$, we consider the positive number $\alpha_p := q(\{c_p^{2n}/p(n)^2\}_{n\in\mathbb{N}})$, where the functional q is defined by

$$q(\xi) := \inf\biggl\{\limsup_{k\to\infty} \frac{1}{m}\sum_{j=1}^{m} \xi(n_j + k) : m \in \mathbb{N}, n_1, \ldots, n_m \in \mathbb{N}\biggr\},$$

for bounded, real sequences ξ.

We say that the *norm-sequence* of the operator $T \in \mathcal{L}(\mathcal{H})$ is *compatible with the gauge function* $p \in \mathcal{P}$, and we write $T \in \mathcal{L}(p,\mathcal{H})$, if $\|T^n\| \leqslant p(n)$ is valid for every $n \in \mathbb{N}$, and if the sequence $\{\|T^n\|/p(n)\}_{n\in\mathbb{N}}$ does not almost converge to zero. We know that $c_p = r(T)$ is true in that case. The *operator* T is called *of regular norm-behaviour*, if $T \in \mathcal{L}(p,\mathcal{H})$ holds for some $p \in \mathcal{P}$. The class of operators with regular norm-sequences forms a large extension of the set of *power-bounded operators* T with $r(T) = 1$. Namely, the latter class is obtained by the special choice $p = M\mathbb{1}$ of the gauge function, where $M \in [1,\infty)$. For a detailed study of these operators we refer to [16], [18], [19], [20], and [24].

Let $\mathcal{B}$ denote the set of all Banach limits on ℓ^∞. Sometimes it is more convenient to write $L\text{-}\lim_{n\to\infty} \xi(n)$ instead of $L(\xi)$, where $L \in \mathcal{B}$, $\xi \in \ell^\infty$. For any $p \in \widetilde{\mathcal{P}}$, we shall consider the non-empty set $\mathcal{B}(p) := \bigl\{L \in \mathcal{B} : L\text{-}\lim_{n\to\infty} c_p^{2n}/p(n)^2 = \alpha_p\bigr\}$.

Let $T \in \mathcal{L}(p,\mathcal{H})$, $p \in \widetilde{\mathcal{P}}$ and $L \in \mathcal{B}(p)$ be given. We know by Theorem 3, [23], that the operator $A_{T,L} \in \mathcal{L}(\mathcal{H})$, defined by

$$\langle A_{T,L}x, y\rangle = L\text{-}\lim_{n\to\infty} p(n)^{-2}\langle T^{*n}T^n x, y\rangle, \quad x, y \in \mathcal{H},$$

is a positive, contractive T-Toeplitz operator, which is universal in the sense that for any selfadjoint T-Toeplitz operator B there exists a positive number β such that $-A_{T,L} \leqslant \beta B \leqslant A_{T,L}$. The equations $(A_{T,L}^{1/2}T)^*(A_{T,L}^{1/2}T) = T^*A_{T,L}T = r(T)^2 A_{T,L}$ imply that there exists a unique isometry $V_{T,L}$ on the subspace $(\operatorname{ran} A_{T,L})^-$ such that $r(T)V_{T,L}A_{T,L}^{1/2} = A_{T,L}^{1/2}T$. Let $U_{T,L} \in \mathcal{L}(\mathcal{K}_{T,L})$ be the minimal unitary extension of $V_{T,L}$ (determined uniquely up to an isomorphism), and let us consider the mapping $X_{T,L} \in \mathcal{L}(\mathcal{H}, \mathcal{K}_{T,L})$ defined by $X_{T,L}h := A_{T,L}^{1/2}h$, $h \in \mathcal{H}$. The transformation $X_{T,L}$ is a *universal T-unitary intertwiner*, that is,

$$X_{T,L} \in \mathcal{I}(T, r(T)U_{T,L}) := \{Y \in \mathcal{L}(\mathcal{H}, \mathcal{K}_{T,L}) : YT = r(T)U_{T,L}Y\},$$

and for every unitary operator $U \in \mathcal{L}(\mathcal{K})$ and for every T-unitary intertwiner $Y \in \mathcal{I}(T, r(T)U)$, there exists a (unique) transformation $Z \in \mathcal{I}(U_{T,L}, U)$ such that $Y = ZX_{T,L}$. Finally, it was shown in [23] that the mapping

$$\Phi_{T,L} : \{U_{T,L}\}' \to \mathcal{T}(T), \quad F \mapsto X_{T,L}^* F X_{T,L},$$

called a *symbolic calculus* for $\mathcal{T}(T)$, is a positive, linear bijection, such that

$$\alpha_p \|F\| \leqslant \|\Phi_{T,L}(F)\| \leqslant \|F\|$$

holds for every operator F in the commutant $\{U_{T,L}\}' := \mathcal{I}(U_{T,L}, U_{T,L})$ of $U_{T,L}$. We note that

$$\ker A_{T,L} = \mathcal{H}_0(T,p) := \big\{x \in \mathcal{H} : \text{a-}\lim_{n\to\infty} \|T^n x\|/p(n) = 0\big\},$$

and so $\mathcal{T}(T) \neq \{0\}$ is true if and only if $\mathcal{H}_0(T,p) \neq \mathcal{H}$, in which case the operator $T \in \mathcal{L}(p, \mathcal{H})$ is called *asymptotically non-vanishing with respect to* p, and the notation $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$ is used. (The dot in the index means that nothing is assumed on the asymptotic behaviour of the (Hilbert space) adjoint T^*.) In the contraction case $T \in \mathcal{L}(\mathbb{1}, \mathcal{H})$, the transformations $A_{T,L}, X_{T,L}, V_{T,L}, U_{T,L}$ and $\Phi_{T,L}$ are actually independent of the choice of $L \in \mathcal{B}(\mathbb{1}) = \mathcal{B}$, and so we can write A_T, X_T, V_T, U_T and Φ_T.

The classical Toeplitz operators and symbolic calculus are obtained by choosing T to be the simple unilateral shift S on the Hardy space H^2. It is clear that $S \in \mathcal{L}(\mathbb{1}, H^2)$, and U_S can be chosen to be the simple bilateral shift $\widetilde{S}$ on the space $L^2(\mathbb{T})$, that is $\widetilde{S}f := \chi f$, $f \in L^2(\mathbb{T})$, where the normalized Lebesgue measure μ is considered on the unit circle $\mathbb{T}$. Since X_S is the embedding of H^2 into $L^2(\mathbb{T})$, and since the commutant $\{\widetilde{S}\}'$ can be identified with $L^\infty(\mathbb{T})$ via the mapping $\Phi : L^\infty(\mathbb{T}) \to \{\widetilde{S}\}'$, $\varphi \mapsto M_\varphi$, where $M_\varphi f := \varphi f$, the *classical symbolic calculus* is given by the composition

$$\Phi_S \circ \Phi : L^\infty(\mathbb{T}) \to \mathcal{T}(S), \quad \varphi \mapsto T_\varphi := P_{H^2} M_\varphi | H^2.$$

2. SYMBOLIC CALCULUS AND TRANSITIVITY

Let us assume that $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$ and $L \in \mathcal{B}(p)$. Given any $C \in \{T\}'$, the operator $X_{T,L}C$ belongs to the set $\mathcal{I}(T, r(T)U_{T,L})$. Applying the universality of the T-unitary intertwiner $X_{T,L}$, we obtain that there exists a unique operator $D \in \{U_{T,L}\}'$ such that $DX_{T,L} = X_{T,L}C$. Let us consider the mapping $\gamma_{T,L} : \{T\}' \to \{U_{T,L}\}'$, $C \mapsto D$.

PROPOSITION 2.1. *If $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H}), p \in \widetilde{\mathcal{P}}, L \in \mathcal{B}(p)$, then the following statements are true:*

(i) *the transformation $\gamma_{T,L}$ is a contractive, unital algebra-homomorphism;*

(ii) *the composition $\Phi_{T,L} \circ \gamma_{T,L} : \{T\}' \to \mathcal{T}(T)$ maps any $C \in \{T\}'$ into $A_{T,L}C$;*

(iii) *for any $F \in \{U_{T,L}\}'$ and $C \in \{T\}'$, we have*

$$\Phi_{T,L}(F\gamma_{T,L}(C)) = \Phi_{T,L}(F)C \quad \text{and} \quad \Phi_{T,L}(\gamma_{T,L}(C)^* F) = C^*\Phi_{T,L}(F).$$

Proof. Let us show that $\gamma_{T,L}$ is contractive. If $C \in \{T\}'$ and $D = \gamma_{T,L}(C)$, then, for any $h \in \mathcal{H}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\|DU_{T,L}^{-n}X_{T,L}h\|^2 &= \|X_{T,L}Ch\|^2 = \langle A_{T,L}Ch, Ch\rangle = L\text{-}\lim_{n\to\infty} p(n)^{-2}\|T^nCh\|^2 \\ &= L\text{-}\lim_{n\to\infty} p(n)^{-2}\|CT^nh\|^2 \leqslant \|C\|^2\|X_{T,L}h\|^2 = \|C\|^2\|U_{T,L}^{-n}X_{T,L}h\|^2.\end{aligned}$$

Taking into account that $\bigvee_{n\in\mathbb{N}} U_{T,L}^{-n}X_{T,L}\mathcal{H} = \mathcal{K}_{T,L}$, we infer that $\|D\| \leqslant \|C\|$. It follows readily from the definition that $\gamma_{T,L}$ is a unital algebra-homomorphism.

The verification of (ii) and (iii) is a simple exercise and is left to the reader. ∎

Let us recall that the commutant of the unilateral shift $S \in \mathcal{L}(H^2)$ is the set of Toeplitz operators with analytic symbols: $\{S\}' = \{T_u : u \in H^\infty\}$, and clearly $\Phi_S \circ \gamma_S = I$. Thus, statement (iii) of the proposition is an extension of the well-known theorem claiming that $T_{uv} = T_uT_v$ holds, whenever v is analytic or u is coanalytic (see Problem 243, [13]).

It can be easily checked that the selfadjoint linear manifold $\mathcal{T}(T)$ of T-Toeplitz operators is weak* closed, it is closed even in the weak operator topology. Let us also note that $\mathcal{T}(U) = \{U\}'$ if U is unitary.

PROPOSITION 2.2. *Let $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$ and $L \in \mathcal{B}(p)$ be given;*

(i) *for any $F \in \{U_{T,L}\}'$ and $u, v \in \mathcal{H}$, we have*

$$[u \otimes v, \Phi_{T,L}(F)] = [X_{T,L}u \otimes X_{T,L}v, F]\,;$$

(ii) *the symbolic calculus $\Phi_{T,L} : \{U_{T,L}\}' \to \mathcal{T}(T)$ is a weak* homeomorphism.*

Proof. (i) It is immediate that

$$\begin{aligned}[u \otimes v, \Phi_{T,L}(F)] &= \operatorname{tr}((u \otimes v)\Phi_{T,L}(F)) = \langle \Phi_{T,L}(F)u, v\rangle \\ &= \langle X_{T,L}^* F X_{T,L}u, v\rangle = [X_{T,L}u \otimes X_{T,L}v, F].\end{aligned}$$

(ii) Let $\widetilde{\Phi}_{T,L} : \mathcal{L}(\mathcal{K}_{T,L}) \to \mathcal{L}(\mathcal{H}), F \mapsto X_{T,L}^* F X_{T,L}$ be the natural extension of $\Phi_{T,L}$ to the whole space $\mathcal{L}(\mathcal{K}_{T,L})$, and let us consider the linear mapping

$$\widetilde{\varphi}_{T,L} : \tau(\mathcal{H}) \to \tau(\mathcal{K}_{T,L}), \quad \sum_n u_n \otimes v_n \mapsto \sum_n X_{T,L}u_n \otimes X_{T,L}v_n,$$

where $\sum_n \|u_n\|\,\|v_n\| < \infty$ holds for the sequences $\{u_n\}, \{v_n\}$. We can see, as in (i), that $\Big[\widetilde{\varphi}_{T,L}\Big(\sum_n u_n \otimes v_n\Big), F\Big] = \Big[\sum_n u_n \otimes v_n, \widetilde{\Phi}_{T,L}(F)\Big]$, hence $\widetilde{\varphi}_{T,L}$ is well-defined. It is also clear that $\|\widetilde{\varphi}_{T,L}\| \leqslant \|X_{T,L}\|^2 = \|A_{T,L}\| \leqslant 1$, and that $\widetilde{\Phi}_{T,L}$ is the (Banach space) adjoint of $\widetilde{\varphi}_{T,L}$: $(\widetilde{\varphi}_{T,L})^{\#} = \widetilde{\Phi}_{T,L}$. Since $\widetilde{\Phi}_{T,L}$ maps the subspace $\{U_{T,L}\}'$ into $\mathcal{T}(T)$, it follows that $\widetilde{\varphi}_{T,L}$ transforms the preannihilator $\mathcal{T}(T)_\perp$ into the preannihilator $\{U_{T,L}\}'_\perp$. Let us consider the quotient spaces $\mathcal{Q}(T) := \tau(\mathcal{H})/\mathcal{T}(T)_\perp$, $\mathcal{Q}(U_{T,L}) := \tau(\mathcal{K}_{T,L})/\{U_{T,L}\}'_\perp$, and the canonical quotient transformations $\pi_T : \tau(\mathcal{H}) \to \mathcal{Q}(T), A \mapsto A + \mathcal{T}(T)_\perp, \widehat{\pi}_{T,L} : \tau(\mathcal{K}_{T,L}) \to \mathcal{Q}(U_{T,L}), A \mapsto A + \{U_{T,L}\}'_\perp$. We can form the quotient mapping $\varphi_{T,L} \in \mathcal{L}(\mathcal{Q}(T), \mathcal{Q}(U_{T,L}))$ defined by $\varphi_{T,L}(\pi_T(A)) := \widehat{\pi}_{T,L}(\widetilde{\varphi}_{T,L}(A))$, $A \in \tau(\mathcal{H})$. Then $(\varphi_{T,L})^{\#} = \Phi_{T,L}$, and since $\Phi_{T,L}$ is a bijection we infer that $\varphi_{T,L}$ is a bijection, as well. Taking into account that adjoint mappings are weak* continuous, we obtain that $\Phi_{T,L}$ is a weak* homeomorphism. ∎

In order to give a condition for transitivity, we consider cyclic vectors of the commutant of the associated unitary operators. The next lemma expresses independence of the choice of the Banach limit.

Lemma 2.3. *Let $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H}), p \in \widetilde{\mathcal{P}}$ be given, and let us assume that $L_1, L_2 \in \mathcal{B}(p)$. Then, for every $h \in \mathcal{H}$, the vector $X_{T,L_1}h$ is cyclic for $\{U_{T,L_1}\}'$ if and only if $X_{T,L_2}h$ is cyclic for $\{U_{T,L_2}\}'$.*

Proof. The universality of the transformation X_{T,L_1} implies that there exists a unique transformation $Z_{T,L_1,L_2} \in \mathcal{I}(U_{T,L_1}, U_{T,L_2})$ such that $Z_{T,L_1,L_2}X_{T,L_1} = X_{T,L_2}$. Since $Z_{T,L_1,L_2}U^{-n}_{T,L_1}X_{T,L_1} = U^{-n}_{T,L_2}X_{T,L_2}$ holds for every $n \in \mathbb{N}$, it follows that Z_{T,L_1,L_2} has dense range. Given any $F \in \{U_{T,L_1}\}'$, the transformation $Z_{T,L_1,L_2}FX_{T,L_1}$ belongs to $\mathcal{I}(T, r(T)U_{T,L_2})$. Hence, the universality of X_{T,L_2} yields that there exists a unique operator $G \in \{U_{T,L_2}\}'$ such that $Z_{T,L_1,L_2}FX_{T,L_1} = GX_{T,L_2}$. Let us consider the linear mapping $\gamma_{T,L_1,L_2} : \{U_{T,L_1}\}' \to \{U_{T,L_2}\}'$, $F \mapsto G$.

If the vector $X_{T,L_1}h$ is cyclic for the operator algebra $\{U_{T,L_1}\}'$, then the linear manifold $\{U_{T,L_1}\}'X_{T,L_1}h$ is dense in $\mathcal{K}_{T,L_1}$. Taking into account that Z_{T,L_1,L_2} has dense range, we infer that

$$\gamma_{T,L_1,L_2}(\{U_{T,L_1}\}')X_{T,L_2}h = Z_{T,L_1,L_2}\{U_{T,L_1}\}'X_{T,L_1}h$$

is dense in $\mathcal{K}_{T,L_2}$, and then so is the larger linear manifold $\{U_{T,L_2}\}'X_{T,L_2}h$ too. Thus, $X_{T,L_2}h$ is cyclic for $\{U_{T,L_2}\}'$. In virtue of symmetry, the proof is complete. ∎

Definition 2.4. The *vector* $h \in \mathcal{H}$ is *quasianalytic* for the operator $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, if $X_{T,L}h$ is cyclic for the commutant $\{U_{T,L}\}'$ for some (and so, by Lemma 2.3, for all) choice(s) of the Banach limit $L \in \mathcal{B}(p)$.

The operator $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, is called *quasianalytic*, if every non-zero vector $h \in \mathcal{H}$ is quasianalytic for T.

We note that the Toeplitz operator T_u is an asymptotically non-vanishing, quasianalytic contraction, if the analytic function $u \in H^\infty$ satisfies the conditions $\|u\|_\infty = 1$ and $\mu(\{z \in \mathbb{T} : |u(z)| = 1\}) > 0$. In particular, $S = T_\chi$ is quasianalytic. Further examples and a detailed study of the quasianalytic property can be found in [22].

Now, we are able to give a sufficient condition of transitivity of the set of all T-Toeplitz operators. Let us recall that the operator $T \in \mathcal{L}(p, \mathcal{H})$, $p \in \mathcal{P}$, is called of class $\mathcal{C}_{1\cdot ot}(p, \mathcal{H})$, and we say that T *is asymptotically strongly non-vanishing with respect to* p, if $\mathcal{H}_0(T, p) = \{0\}$. In that case the transformation

$X_{T,L}$ is injective, for any choice of $L \in \mathcal{B}(p)$. Let us note that every quasianalytic operator $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, is necessarily of class $\mathcal{C}_{1\cdot ot}(p, \mathcal{H})$.

Theorem 2.5. *Let $T \in \mathcal{C}_{1\cdot ot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$ be given;*

(i) *if the vector $h \in \mathcal{H}$ is quasianalytic for T, then $(\mathcal{T}(T)h)^{-} = \mathcal{H}$;*

(ii) *if the operator T is quasianalytic, then the operator space $\mathcal{T}(T)$ is transitive.*

Proof. Let us assume that $h \in \mathcal{H}$ is quasianalytic for T, and that $\langle Ch, k\rangle = 0$ holds for all $C \in \mathcal{T}(T)$, with some $k \in \mathcal{H}$. Let $L \in \mathcal{B}(p)$ be arbitrary. Then $\langle FX_{T,L}h, X_{T,L}k\rangle = \langle \Phi_{T,L}(F)h, k\rangle = 0$ is true, for every $F \in \{U_{T,L}\}'$. Thus $X_{T,L}k = 0$, whence $k = 0$ follows.

If T is quasianalytic, then $(\mathcal{T}(T)h)^{-} = \mathcal{H}$ is valid for every non-zero vector $h \in \mathcal{H}$, that is the operator space $\mathcal{T}(T)$ is transitive. ∎

3. ELEMENTARY HYPERPLANES

We are going to prove that, under some conditions, the intransitive hyperplanes $\mathcal{T}_{g,h}(T) = \{C \in \mathcal{T}(T) : \langle Cg, h\rangle = 0\}$ of T-Toeplitz operators are elementary.

Let $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H}), p \in \widetilde{\mathcal{P}}$ and $L_1, L_2 \in \mathcal{B}(p)$ be given. Let us consider the mappings $Z_{T,L_1,L_2} \in \mathcal{I}(U_{T,L_1}, U_{T,L_2})$ and $Z_{T,L_2,L_1} \in \mathcal{I}(U_{T,L_2}, U_{T,L_1})$ introduced in the proof of Lemma 2.3. The equations

$$Z_{T,L_1,L_2}U_{T,L_1}^{-n}X_{T,L_1}h = U_{T,L_2}^{-n}X_{T,L_2}h \quad \text{and} \quad Z_{T,L_2,L_1}U_{T,L_2}^{-n}X_{T,L_2}h = U_{T,L_1}^{-n}X_{T,L_1}h,$$

$n \in \mathbb{N}, h \in \mathcal{H}$, show that the transformations Z_{T,L_1,L_2} and Z_{T,L_2,L_1} are invertible, and $(Z_{T,L_1,L_2})^{-1} = Z_{T,L_2,L_1}$. Thus, the unitary operators U_{T,L_1} and U_{T,L_2} are similar: $U_{T,L_1} \approx U_{T,L_2}$, and so we infer by Proposition II.3.4, [26], that they are unitarily equivalent: $U_{T,L_1} \simeq U_{T,L_2}$.

Definition 3.1. The operator $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, will be called *absolutely continuous*, if the unitary operator $U_{T,L}$ is absolutely continuous (with respect to μ) for some (and so, for all) Banach limit(s) $L \in \mathcal{B}(p)$.

If $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, is absolutely continuous, then there exists a Borel subset $\rho(T)$ of $\mathbb{T}$, called the *residual set of* T, such that $\chi_{\rho(T)}\mathrm{d}\mu$ is a scalar spectral measure of the unitary operator $U_{T,L}$, that is $\chi_{\rho(T)}\mathrm{d}\mu$ is equivalent to the spectral measure of $U_{T,L}$, $L \in \mathcal{B}(p)$. (Here, and in the sequel, χ_α stands for the characteristic function of the set α.)

We note that $\rho(T)$ is uniquely determined up to sets of zero Lebesgue measure, and that $\rho(T)$ is independent of the choice of $L \in \mathcal{B}(p)$. A detailed study of residual sets of contractions was accomplished in [22].

Let us be given now an arbitrary absolutely continuous unitary operator $U \in \mathcal{L}(\mathcal{K})$, and let $E_U : \mathcal{B}_{\mathbb{T}} \to \mathcal{P}(\mathcal{K})$ be the spectral measure of U. (Here $\mathcal{B}_{\mathbb{T}}$ denotes the σ-algebra of Borel subsets of the unit circle $\mathbb{T}$, and $\mathcal{P}(\mathcal{K})$ stands for the set of (orthogonal) projections of $\mathcal{K}$.) Clearly, $U \in \mathcal{C}_{1\cdot}(\mathbb{1}, \mathcal{K})$ and $\chi_{\rho(U)}\mathrm{d}\mu$ is a scalar spectral measure of U. For any vectors $x, y \in \mathcal{K}$, let us consider the localization $E_{U,x,y}(\omega) := \langle E_U(\omega)x, y\rangle$, $\omega \in \mathcal{B}_{\mathbb{T}}$, what is a complex Borel measure on $\mathbb{T}$. Since $E_{U,x,y}$ is absolutely continuous with respect to $\chi_{\rho(U)}\mathrm{d}\mu$, there exists a unique function $\delta_{U,x,y} \in L^1(\rho(U)) := \chi_{\rho(U)} L^1(\mu)$ such that $E_{U,x,y} = \delta_{U,x,y} \mathrm{d}\mu$. The function $\delta_{U,x,y}$ can be called the *local spectral density function* of the absolutely continuous unitary operator U at the vectors x, y. It is clear that $\delta_{U,x} := \delta_{U,x,x}$ is non-negative, for any $x \in \mathcal{K}$.

If $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, is absolutely continuous, then we can introduce the *asymptotic local spectral density function* of T at $g, h \in \mathcal{H}$ with respect to $L \in \mathcal{B}(p)$ by

$$\widetilde{\delta}_{T,L,g,h} := \delta_{U_{T,L}, X_{T,L} g, X_{T,L} h} \in L^1(\rho(T)).$$

We shall write $\widetilde{\delta}_{T,L,g} := \widetilde{\delta}_{T,L,g,g}$ for short. Functions of this type were studied among others by Beauzamy and Cassier–Fack; see e.g. Chapter XII, [2], and [8]. We shall consider vector pairs which are related analytically, in a certain sense, to a unitary operator first, and then to an arbitrary operator of regular norm-behaviour.

Definition 3.2. We say that $(x, y) \in \mathcal{K} \times \mathcal{K}$ is an *analytic pair for the absolutely continuous unitary operator* $U \in \mathcal{L}(\mathcal{K})$, if $\int\limits_{\rho(U)} \log |\delta_{U,x,y}| \, \mathrm{d}\mu > -\infty$. *The vector* $x \in \mathcal{K}$ *is* called *analytic for* U, if the pair (x, x) is analytic with respect to U.

Given an absolutely continuous unitary operator $U \in \mathcal{L}(\mathcal{K})$, for any $x, y \in \mathcal{K}$, we define the measurable function $\gamma_{U,x,y}$ on $\rho(U)$ by

$$\gamma_{U,x,y}(z) := \begin{cases} \delta_{U,x,y}(z)(\delta_{U,x}(z))^{-1/2}(\delta_{U,y}(z))^{-1/2} & \text{if } \delta_{U,x}(z)\delta_{U,y}(z) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Considering the functional model of U (see e.g. [21]), it is immediate that $\delta_{U,x,y} = (\delta_{U,x})^{1/2}(\delta_{U,y})^{1/2}\gamma_{U,x,y}$ and that $|\gamma_{U,x,y}| \leqslant 1$. Thus, the pair (x, y) is analytic for U if and only if the vectors x, y are analytic for U and if $\int\limits_{\rho(U)} \log |\gamma_{U,x,y}| \, \mathrm{d}\mu > -\infty$. We note that $|\gamma_{U,x,y}| = \chi_{\rho(U)}$ is true for every $x, y \in \mathcal{K}$, provided *the operator* U *is cyclic*, that is when $\bigvee\limits_{n=0}^{\infty} U^n k = \mathcal{K}$ holds with some vector $k \in \mathcal{K}$.

This analytic property can be also described in terms of invariant subspaces of U. In order to formulate our statement we need some notations. As usual, $\operatorname{Lat} U$ stands for the invariant subspace lattice of U. For any vector set $\emptyset \neq \mathcal{M} \subset \mathcal{K}$, let $\mathcal{K}^+(U, \mathcal{M}) := \bigvee\limits_{n=0}^{\infty} U^n \mathcal{M} \in \operatorname{Lat} U$ be the invariant subspace, and let $\mathcal{K}(U, \mathcal{M}) := \bigvee\limits_{n=-\infty}^{\infty} U^n \mathcal{M} \in \operatorname{Lat} U \cap \operatorname{Lat} U^*$ be the reducing subspace induced by $\mathcal{M}$. Furthermore, let $\mathcal{K}^-(U, \mathcal{M}) := \mathcal{K}(U, \mathcal{M}) \ominus \mathcal{K}^+(U, \mathcal{M})$. For any Borel set $\alpha \in \mathcal{B}_{\mathbb{T}}$, let us consider the operator $M_\alpha \in \mathcal{L}(L^2(\alpha))$ defined by $M_\alpha f := \chi f$, and let $\alpha^{\mathrm{c}} := \mathbb{T} \setminus \alpha$. We introduce the absolutely continuous unitary operator $\widehat{U} := U \oplus M_{\rho(U)^{\mathrm{c}}}$ acting on the space $\widehat{\mathcal{K}} := \mathcal{K} \oplus L^2(\rho(U)^{\mathrm{c}})$. Finally, let $\mathcal{G}$ denote the set of all measurable, unimodular functions $q : \mathbb{T} \to \mathbb{T}$.

PROPOSITION 3.3. *Let $U \in \mathcal{L}(\mathcal{K})$ be an absolutely continuous unitary operator, and let $x, y \in \mathcal{K}$;*

(i) *the vector x is analytic for U if and only if the restriction of $\widehat{U}$ to the subspace $\widehat{\mathcal{K}}^+(\widehat{U}, x \oplus \chi_{\rho(U)^c})$ is unitarily equivalent to the unilateral shift S;*

(ii) *the pair (x, y) is analytic for U if and only if the vectors x, y are analytic for U, and if there exists a function $q \in \mathcal{G}$ such that*

$$q(\widehat{U})\widehat{\mathcal{K}}^+(\widehat{U}, x \oplus \chi_{\rho(U)^c}) \subset \widehat{\mathcal{K}}(\widehat{U}, \{x \oplus \chi_{\rho(U)^c}, y \oplus \chi_{\rho(U)^c}\}) \ominus \widehat{\mathcal{K}}^-(\widehat{U}, y \oplus \chi_{\rho(U)^c}).$$

Proof. In view of the functional model of U, the statement (i) is an immediate consequence of the Szegő Theorem and the well-known description of the non-reducing subspaces of cyclic unitary operators, see e.g. [15] and [14]. Statement (ii) follows from the refined characterization of the invariant subspace lattice of absolutely continuous unitary operators given in [21]; we refer in particular to the proof of Lemma 16, [21]. ∎

The previous proposition enables us to verify the following statement.

LEMMA 3.4. *Let $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, be absolutely continuous, and let $L_1, L_2 \in \mathcal{B}(p)$. Then, for any $h \in \mathcal{H}$, the vector $X_{T,L_1}h$ is analytic for U_{T,L_1} if and only if the vector $X_{T,L_2}h$ is analytic for U_{T,L_2}.*

Proof. Let us write $U_i = U_{T,L_i}$, $\mathcal{K}_i = \mathcal{K}_{T,L_i}$, $x_i = X_{T,L_i}h$ with $i = 1, 2$, $Z = Z_{T,L_1,L_2}$ and $\alpha = \rho(T)$ for short. We know that $Z \in \mathcal{I}(U_1, U_2)$ is invertible, and $Zx_1 = x_2$. Let us introduce also the notation $\widehat{x}_i = x_i \oplus \chi_{\alpha^c}$, $i = 1, 2$. It is clear that $\widehat{Z}\widehat{x}_1 = \widehat{x}_2$ holds for the intertwining mapping $\widehat{Z} := Z \oplus I \in \mathcal{I}(\widehat{U}_1, \widehat{U}_2)$. It follows that

$$\widehat{Z}\widehat{\mathcal{K}}_1^+(\widehat{U}_1, \widehat{x}_1) = \widehat{\mathcal{K}}_2^+(\widehat{U}_2, \widehat{x}_2) \quad \text{and} \quad \widehat{Z} \bigcap_{n=1}^{\infty} \widehat{U}_1^n \widehat{\mathcal{K}}_1^+(\widehat{U}_1, \widehat{x}_1) = \bigcap_{n=1}^{\infty} \widehat{U}_2^n \widehat{\mathcal{K}}_2^+(\widehat{U}_2, \widehat{x}_2).$$

Taking into account that $\widehat{U}_i|\widehat{\mathcal{K}}_i^+(\widehat{U}_i, \widehat{x}_i)$ is a cyclic unilateral shift (that is unitarily equivalent to S) if and only if $\bigcap_{n=1}^{\infty} \widehat{U}_i^n \widehat{\mathcal{K}}_i^+(\widehat{U}_i, \widehat{x}_i) = \{0\}$, $i = 1, 2$, we infer by Proposition 3.3 (i) that x_1 is analytic for U_1 precisely when x_2 is analytic for U_2. ∎

DEFINITION 3.5. *The vector $h \in \mathcal{H}$ is* called (asymptotically) *analytic for the absolutely continuous operator* $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, if $X_{T,L}h$ is analytic for the unitary operator $U_{T,L}$ for some (and so, by Lemma 3.4, for all) Banach limit(s) $L \in \mathcal{B}(p)$. If every non-zero vector $h \in \mathcal{H}$ is analytic for T, then *the operator T is* called *analytic*.

The pair $(g, h) \in \mathcal{H} \times \mathcal{H}$ is (asymptotically) *analytic for T with respect to the Banach limit $L \in \mathcal{B}(p)$*, if the pair $(X_{T,L}g, X_{T,L}h)$ is analytic for $U_{T,L}$.

Let $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H}), p \in \widetilde{\mathcal{P}}$ and $L \in \mathcal{B}(p)$ be given, and let us consider the bicommutant $\{U_{T,L}\}''$ of the unitary operator $U_{T,L}$. Since $\{U_{T,L}\}''$ is a weak* closed, selfadjoint subalgebra of $\{U_{T,L}\}'$, it follows by Proposition 2.2 that $\check{\mathcal{T}}_L(T) := \Phi_{T,L}(\{U_{T,L}\}'')$ is a weak* closed, selfadjoint subspace of $\mathcal{T}(T)$. We are going to show that the hyperplane of the form $\check{\mathcal{I}}_{L,g,h}(T) := \check{\mathcal{I}}_L(T) \cap \mathcal{L}_{g,h}(\mathcal{H})$ is elementary, if the pair (g,h) is analytic for T with respect to $L \in \mathcal{B}(p)$.

Let us note that *if T is cyclic, then so is $U_{T,L}$ too, but the reverse implication fails* (see Proposition 2, [27]). Furthermore, *$U_{T,L}$ is cyclic if and only if $\{U_{T,L}\}' = \{U_{T,L}\}''$, and the latter happens precisely when $\mathcal{T}(T) = \check{\mathcal{I}}_L(T)$.*

The following example shows that the class $\check{\mathcal{I}}_L(T)$ and the analyticity of a vector pair (g,h) do depend on the choice of the Banach limit $L \in \mathcal{B}(p)$.

Example 3.6. Let $\{\delta_j\}_{j=1}^\infty$ be a sequence of positive numbers and let $\{n_j\}_{j=1}^\infty$ be a sequence of positive integers such that $\delta_j > \delta_{j+1}$, $n_{j+1} > n_j + 1 > 2j+1$, $\delta_j^j < 1 + j^{-1}$, $\delta_j^{n_j - j} > 2 - j^{-1}$, $\delta_j^{n_j} < 2$ hold, for every $j \in \mathbb{N}$, and $\lim_{j\to\infty} \delta_j = 1$. Such sequences can be given recursively. Let us introduce the notation $\nu_k := \sum_{j=1}^k n_j$ with $k \in \mathbb{N}$, $\nu_0 := 0$, and let us define the sequence $\{w_k\}_{k=1}^\infty$ by

$$w_k := \begin{cases} \delta_j & \text{if } \nu_{j-1} < k < \nu_j, \\ \delta_j^{-n_j+1} & \text{if } k = \nu_j; \end{cases} \qquad \text{for } j \in \mathbb{N}.$$

Given an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ in the Hilbert space $\mathcal{H}$, let us consider the unilateral weighted shift $W \in \mathcal{L}(\mathcal{H})$, defined by $We_k := w_k e_{k+1}$, $k \in \mathbb{N}$. It is easy to verify that $\|W^k\| = \delta_j^k$, whenever $n_{j-1} \leqslant k < n_j$, $j \in \mathbb{N}$; $n_0 := 0$. It follows that $\sup_{k\in\mathbb{N}} \|W^k\| = \sup_{j\in\mathbb{N}} \delta_j^{n_j - 1} = 2$. Thus, W belongs to the class $\mathcal{C}_1.(2\cdot\mathbb{1}, \mathcal{H})$, where $2\cdot\mathbb{1} \in \widetilde{\mathcal{P}}, \mathcal{B}(2\cdot\mathbb{1}) = \mathcal{B}$, and $r(W) = 1$. A simple computation shows that

$$\beta_k := \|W^k e_1\| = \begin{cases} \delta_j^{k-\nu_{j-1}} & \text{if } \nu_{j-1} < k < \nu_j, \\ 1 & \text{if } k = \nu_j; \end{cases} \qquad \text{for } j \in \mathbb{N}.$$

We obtain that $q(\{4^{-1}\|W^k e_1\|^2\}_{k\in\mathbb{N}}) = 1$ and $q'(\{4^{-1}\|W^k e_1\|^2\}_{k\in\mathbb{N}}) = 4^{-1}$, where the functional q' is defined by

$$q'(\xi) := \sup\left\{ \liminf_{k\to\infty} \frac{1}{m} \sum_{j=1}^m \xi(n_j + k) : m \in \mathbb{N}, n_1, \ldots, n_m \in \mathbb{N} \right\},$$

for any bounded, real sequence ξ. It is known that $\{L(\xi) : L \in \mathcal{B}\} = [q'(\xi), q(\xi)]$. Hence, there exist Banach limits $L_1, L_2 \in \mathcal{B}$ such that

$$L_1\text{-}\lim_{k\to\infty} 4^{-1}\|W^k e_1\|^2 = 1 \quad \text{and} \quad L_2\text{-}\lim_{k\to\infty} 4^{-1}\|W^k e_1\|^2 = 4^{-1}.$$

It is easy to see that the operators $A_{W,L_i} \in \mathcal{L}(\mathcal{H})$, $i = 1,2$, are diagonal with respect to the basis $\{e_k\}_{k\in\mathbb{N}}$, and that $A_{W,L_1} e_1 = e_1$, $A_{W,L_2} e_1 = 4^{-1} e_1$. Short computation yields that, for $i = 1,2$, V_{W,L_i} is the unweighted unilateral shift $V_0 \in \mathcal{L}(\mathcal{H})$, defined by $V_0 e_k := e_{k+1}$, $k \in \mathbb{N}$. We may assume that $U_{W,L_i} = U_0$, $i = 1,2$, where U_0 is a bilateral shift on a larger space $\mathcal{K} \supset \mathcal{H}$, that is the basis $\{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}$ can be extended to an orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$ of $\mathcal{K}$ such that $U_0 e_k = e_{k+1}$ is true, for every $k \in \mathbb{Z}$. It turns out that X_{W,L_i}, $i = 1,2$ is bounded from below, and so W is similar to V_0. Furthermore, the intertwiner $Z_{W,L_1,L_2} \in \mathcal{I}(U_{W,L_1}, U_{W,L_2}) = \{U_0\}'$ is of the form $Z_{W,L_1,L_2} = 2^{-1} I$.

Let us introduce the operator $T := W \oplus V_0 \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. It is clear that $T \in \mathcal{C}_1.(2 \cdot \mathbb{1}, \mathcal{H} \oplus \mathcal{H})$ *is absolutely continuous,* $\rho(T) = \mathbb{T}$, *and* that T *is similar to the isometry* $V_0 \oplus V_0$. Taking the previous Banach limits L_1, L_2, we obtain that $U_{T,L_i} = U_0 \oplus U_0$ and $X_{T,L_i} = X_{W,L_i} \oplus 2^{-1} J_0$, $i = 1, 2$, where $J_0 \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is the natural embedding of $\mathcal{H}$ into $\mathcal{K}$. We conclude that $Z_{T,L_1,L_2} = 2^{-1} I \oplus I$.

Let us consider the vectors $g := 2e_1 \oplus 2e_1$ and $h := 2e_1 \oplus -2e_1$, and let us set $x_1 := X_{T,L_1} g = 2e_1 \oplus e_1$, $y_1 := X_{T,L_1} h = 2e_1 \oplus -e_1$, $x_2 := X_{T,L_2} g = e_1 \oplus e_1$ and $y_2 := X_{T,L_2} h = e_1 \oplus -e_1$. Since

$$\int_{\mathbb{T}} \chi^n \, \widetilde{\delta}_{T,L_1,g,h} \, \mathrm{d}\mu = \langle (U_0 \oplus U_0)^n x_1, y_1 \rangle = \begin{cases} 0 & \text{if } n \in \mathbb{Z} \setminus \{0\}, \\ 3 & \text{if } n = 0, \end{cases}$$

we infer that $\widetilde{\delta}_{T,L_1,g,h}$ is the constant function 3 on $\mathbb{T}$. On the other hand, we obtain that

$$\int_{\mathbb{T}} \chi^n \, \widetilde{\delta}_{T,L_2,g,h} \, \mathrm{d}\mu = \langle (U_0 \oplus U_0)^n x_2, y_2 \rangle = 0$$

is true for all $n \in \mathbb{Z}$, and so $\widetilde{\delta}_{T,L_2,g,h}$ is the zero function. Therefore, *the pair* (g, h) *is analytic for* T *with respect to* L_1, *but* (g, h) *is not analytic for* T *with respect to* L_2.

Let us consider now an arbitrary operator F in the bicommutant $\{U_{T,L_1}\}'' = \{U_{T,L_2}\}'' = \{U_0 \oplus U_0\}''$. Let us write $X_i = X_{T,L_i}$, $i = 1, 2$, and $Z = Z_{T,L_1,L_2}$ for short. We know that

$$\Phi_{T,L_2}(F) = X_2^* F X_2 = X_1^* Z^* F Z X_1 = \Phi_{T,L_1}(Z^* Z F),$$

and that $Z^* Z \in \{U_0 \oplus U_0\}'$ is invertible. Taking into account that Φ_{T,L_1} is a bijection, we obtain that $\breve{\mathcal{T}}_{L_2}(T) = \breve{\mathcal{T}}_{L_1}(T)$ is valid if and only if $Z^* Z \{U_0 \oplus U_0\}'' = \{U_0 \oplus U_0\}''$, and the latter condition holds exactly when $Z^* Z \in \{U_0 \oplus U_0\}''$. However, the operator $Z^* Z = 4^{-1} I \oplus I$ does not belong to the bicommutant of $U_0 \oplus U_0$, and so

$$\breve{\mathcal{T}}_{L_1}(T) \neq \breve{\mathcal{T}}_{L_2}(T).$$

For any Borel set $\alpha \in \mathcal{B}_{\mathbb{T}}$, let $L^1_{\mathrm{a}}(\alpha) := \big\{ f \in L^1(\alpha) : \int_\alpha \log |f| \, \mathrm{d}\mu > -\infty \big\}$. We recall that $\chi_\alpha H^1 \setminus \{0\} \subset L^1_{\mathrm{a}}(\alpha)$, and that by a well-known theorem of Bourgain we have $L^1_{\mathrm{a}}(\alpha) = (\chi_\alpha H^2 \overline{H^2}) \setminus \{0\}$; see [15] and [6].

DEFINITION 3.7. The absolutely continuous operator $T \in \mathcal{C}_{*}.(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, is *full with analytic spectral densities with respect to the Banach limit* $L \in \mathcal{B}(p)$, if

$$\{\widetilde{\delta}_{T,L,g,h} : g, h \in \mathcal{H}\} \supset L^1_{\mathrm{a}}(\rho(T)).$$

Now, we are ready to prove the main theorem of this section.

THEOREM 3.8. *Let us assume that the absolutely continuous operator* $T \in \mathcal{C}_{*}.(p, \mathcal{H}), p \in \widetilde{\mathcal{P}}$, *is full with analytic spectral densities with respect to the Banach limit* $L \in \mathcal{B}(p)$, *and let* $g, h \in \mathcal{H}$ *be given. If the pair* (g, h) *is analytic for* T *with respect to* L, *then the weak* closed subspace* $\breve{\mathcal{T}}_{L,g,h}(T)$ *of* T*-Toeplitz operators is elementary.*

Proof. Let us write $\alpha = \rho(T)$ for short. We know that the functional calculus $\Psi_{T,L} : L^\infty(\alpha) \to \{U_{T,L}\}'$, $f \mapsto f(U_{T,L})$ is a weak* continuous, isometric algebra-homomorphism with $\operatorname{ran}\Psi_{T,L} = \{U_{T,L}\}''$ (see e.g. Chapter IX, [10]). Let us consider the quotient spaces $\mathcal{Q}(T), \mathcal{Q}(U_{T,L})$ and the quotient mapping π_T introduced in the proof of Proposition 2.2. There exists a transformation $\psi_{T,L} \in \mathcal{L}(\mathcal{Q}(U_{T,L}), L^1(\alpha))$ such that $(\psi_{T,L})^\# = \Psi_{T,L}$. Then, the adjoint of $\lambda_{T,L} := \psi_{T,L} \circ \varphi_{T,L}$ is the composition $\Lambda_{T,L} = \Phi_{T,L} \circ \Psi_{T,L}$. Given any $u, v \in \mathcal{H}$, we infer by Proposition 2.2 (i) that

$$\begin{aligned}\big[\lambda_{T,L} \circ \pi_T(u \otimes v), f\big] &= \big[\pi_T(u \otimes v), \Lambda_{T,L}(f)\big] = \big[u \otimes v, \Phi_{T,L}(f(U_{T,L}))\big] \\ &= \big[X_{T,L}u \otimes X_{T,L}v, f(U_{T,L})\big] = \big\langle f(U_{T,L})X_{T,L}u, X_{T,L}v\big\rangle \\ &= \int_\alpha f\widetilde{\delta}_{T,L,u,v}\,\mathrm{d}\mu = \big[\widetilde{\delta}_{T,L,u,v}, f\big]\end{aligned}$$

is true for every $f \in L^\infty(\alpha)$, whence

$$\lambda_{T,L} \circ \pi_T(u \otimes v) = \widetilde{\delta}_{T,L,u,v}$$

follows.

Let us assume that the pair $(g, h) \in \mathcal{H}\times\mathcal{H}$ is analytic for T with respect to L, and let us consider an arbitrary element $\eta \in \mathcal{Q}(T)$. Since $\widetilde{\delta}_{T,L,g,h} \in L^1_{\mathrm{a}}(\alpha)$, there exists a number $w \in \mathbb{T}$ such that $f_0 + w\widetilde{\delta}_{T,L,g,h} \in L^1_{\mathrm{a}}(\alpha)$ holds for the function $f_0 := \lambda_{T,L}(\eta)$; actually, w can be any point of $\mathbb{T}$ with an exception of points in a set of measure zero (see the proof of Corollary 3.3 in [1]). Taking into account that T is full with analytic spectral densities with respect to L, we obtain that there exist vectors $g', h' \in \mathcal{H}$ such that $\widetilde{\delta}_{T,L,g',h'} = f_0 + w\widetilde{\delta}_{T,L,g,h}$. Now, given an arbitrary $B \in \breve{\mathcal{T}}_{L,g,h}(T)$, there exists a (unique) function $f \in L^\infty(\alpha)$ such that $B = \Lambda_{T,L}(f)$; furthermore $[g \otimes h, B] = 0$. Thus, we can write

$$\begin{aligned}[\eta, B] &= \big[\eta, \Lambda_{T,L}(f)\big] = \big[\lambda_{T,L}(\eta), f\big] = [f_0, f] \\ &= \big[\widetilde{\delta}_{T,L,g',h'} - w\widetilde{\delta}_{T,L,g,h}, f\big] = \big[\lambda_{T,L} \circ \pi_T(g' \otimes h' - wg \otimes h), f\big] \\ &= [g' \otimes h' - wg \otimes h, B] = [g' \otimes h', B].\end{aligned}$$

We have obtained that the functional η coincides with the functional $g' \otimes h'$ on the subspace $\breve{\mathcal{T}}_{L,g,h}(T)$. Taking into account that every weak* continuous linear functional defined on $\breve{\mathcal{T}}_{L,g,h}(T)$ can be extended to a weak* continuous linear functional on $\mathcal{T}(T)$, we conclude that the weak* closed subspace $\breve{\mathcal{T}}_{L,g,h}(T)$ is elementary. ∎

If the operator T is cyclic then the Banach limits in $\mathcal{B}(p)$ play equal role in connection with fullness of analytic spectral densities. Namely, the following statement holds.

Lemma 3.9. *If the cyclic, absolutely continuous operator $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, is full with analytic spectral densities with respect to a Banach limit $L \in \mathcal{B}(p)$, then so is with respect to any other Banach limit $L' \in \mathcal{B}(p)$ too.*

Proof. Let us consider the absolutely continuous unitary operators $U = U_{T,L}$ and $U' = U_{T,L'}$, the intertwining transformations $X = X_{T,L} \in \mathcal{I}(T, r(T)U)$ and

$X' = X_{T,L'} \in \mathcal{I}(T, r(T)U')$, and the invertible transformation $Z = Z_{T,L,L'} \in \mathcal{I}(U, U')$ satisfying the equation $ZX = X'$. Writing $\alpha = \rho(T)$ for short, we know that $\chi_\alpha \, \mathrm{d}\mu$ is a scalar spectral measure for U and U'. Given any vectors $g, h \in \mathcal{H}$ and an arbitrary function $f \in L^\infty(\alpha)$, we have

$$\int\limits_\alpha f\widetilde{\delta}_{T,L',g,h} \, \mathrm{d}\mu = \langle f(U')X'g, X'h\rangle = \langle f(U')ZXg, ZXh\rangle = \langle f(U)Xg, Z^*ZXh\rangle.$$

Since U is cyclic and $Z^*Z \in \{U\}'$, there exists a function $\zeta \in L^\infty(\alpha)$ such that $Z^*Z = \zeta(U)$. Taking into account that Z^*Z is positive and invertible, we infer that ζ is positive and bounded away from zero. The previous equations yield that

$$\int\limits_\alpha f\widetilde{\delta}_{T,L',g,h} \, \mathrm{d}\mu = \langle f(U)Xg, \zeta(U)Xh\rangle = \langle (f\zeta)(U)Xg, Xh\rangle = \int\limits_\alpha f\zeta\widetilde{\delta}_{T,L,g,h} \, \mathrm{d}\mu$$

holds for every $f \in L^\infty(\alpha)$, whence $\widetilde{\delta}_{T,L',g,h} = \zeta\widetilde{\delta}_{T,L,g,h}$ follows. Since $\zeta L^1_{\mathrm{a}}(\alpha) = L^1_{\mathrm{a}}(\alpha)$ is evidently true, we conclude that the set $\{\widetilde{\delta}_{T,L',g,h} : g, h \in \mathcal{H}\}$ also contains $L^1_{\mathrm{a}}(\alpha)$. ∎

In view of this lemma we can introduce the following concept.

DEFINITION 3.10. The cyclic, absolutely continuous operator $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, is said to be *full with analytic spectral densities*, if it is so with respect to some (and then to all) Banach limit(s) $L \in \mathcal{B}(p)$.

In the cyclic case Theorem 3.8 takes the following simpler form.

COROLLARY 3.11. *If the absolutely continuous operator $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, is cyclic and full with analytic spectral densities, then the hyperplane $\mathcal{T}_{g,h}(\mathcal{T})$ of T-Toeplitz operators is elementary, whenever the vectors $g, h \in \mathcal{H}$ are analytic for T.*

4. CONDITIONS OF ANALYTICITY

We are going to give conditions for the analytic properties, considered in Section 3, in terms of the existence of suitable invariant subspaces. To this end, we examine the asymptotic behaviour of restrictions of operators.

Let $\mathcal{M}$ be a non-zero invariant subspace of the operator $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$. Taking into account that $r(T) = c_p$, we obtain that the norm-sequence of $T|\mathcal{M}$ is compatible with p if and only if $r(T|\mathcal{M}) = r(T)$. We note that $T \in \mathcal{C}_{1\cdot}(p, \mathcal{H})$ readily implies $T|\mathcal{M} \in \mathcal{C}_{1\cdot}(p, \mathcal{M})$; however, $T|\mathcal{M} \in \mathcal{C}_{*\cdot}(p, \mathcal{M})$ can fail in the general case. Let us assume that $r(T|\mathcal{M}) = r(T)$ and $T|\mathcal{M} \in \mathcal{C}_{*\cdot}(p, \mathcal{M})$ hold. Setting any $L \in \mathcal{B}(p)$, let us consider the reducing subspace $\mathcal{K}_{T,L,\mathcal{M}} := \bigvee_{n\in\mathbb{N}} U^{-n}_{T,L} X_{T,L}\mathcal{M}$ of $U_{T,L}$, and the restriction $U_{T,L,\mathcal{M}} := U_{T,L}|\mathcal{K}_{T,L,\mathcal{M}}$. The transformation $X_{T,L,\mathcal{M}} \in \mathcal{L}(\mathcal{M}, \mathcal{K}_{T,L,\mathcal{M}})$, defined by $X_{T,L,\mathcal{M}}h := X_{T,L}h$, $h \in \mathcal{M}$, intertwines $T|\mathcal{M}$ with $r(T)U_{T,L,\mathcal{M}}$. Applying the universality of $X_{T|\mathcal{M},L} \in \mathcal{I}(T|\mathcal{M}, r(T)U_{T|\mathcal{M},L})$, we infer that there exists a unique transformation $Z_{T,L,\mathcal{M}} \in \mathcal{I}(U_{T|\mathcal{M},L}, U_{T,L,\mathcal{M}})$ such that $X_{T,L,\mathcal{M}} = Z_{T,L,\mathcal{M}}X_{T|\mathcal{M},L}$. It is easy to verify that $Z_{T,L,\mathcal{M}}$ is unitary. Thus, *if T is absolutely continuous, then so is the restriction $T|\mathcal{M}$, and the inclusion $\rho(T|\mathcal{M}) \subset \rho(T)$ is fulfilled* for the corresponding residual sets.

Theorem 4.1. *Let $T \in \mathcal{C}_{*\cdot}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, be absolutely continuous, and let us assume that there exists a non-zero invariant subspace $\mathcal{M} \in \operatorname{Lat} T$ such that $T|\mathcal{M} \in \mathcal{C}_{*\cdot}(p, \mathcal{M})$, $\rho(T|\mathcal{M}) = \rho(T)$, and the operator $r(T)^{-1}(T|\mathcal{M})$ is similar to a contraction;*

(i) *if $\rho(T) = \mathbb{T}$, then there exists a subspace $\mathcal{N} \in \operatorname{Lat} T$ such that $X_{T,L,\mathcal{N}}$ is bounded from below, and the restriction $U_{T,L,\mathcal{N}}|\operatorname{ran} X_{T,L,\mathcal{N}}$ is unitarily equivalent to the unilateral shift S, where $L \in \mathcal{B}(p)$ is arbitrary;*

(ii) *the operator T is full with analytic spectral densities;*

(iii) *if $\mathcal{M}$ is reducing for T, then the set of T-analytic vectors is total in $\mathcal{H}$.*

Proof. (i) Let us write $r = r(T)$ for short, let us fix $L \in \mathcal{B}(p)$, and let us assume that $r^{-1}(T|\mathcal{M})$ is similar to a contraction $Q \in \mathcal{L}(\mathcal{E}) : Br^{-1}(T|\mathcal{M}) = QB$ with an invertible $B \in \mathcal{L}(\mathcal{M}, \mathcal{E})$. Since $T|\mathcal{M} \in \mathcal{C}_{*\cdot}(p, \mathcal{M})$ is similar to rQ, it follows that $rQ \in \mathcal{C}_{*\cdot}(bp, \mathcal{E})$ holds with $b = \|B\|\,\|B^{-1}\|$, and that $U_{T|\mathcal{M},L}$ is unitarily equivalent to $U_{rQ,L}$. Thus, rQ is absolutely continuous and $\rho(rQ) = \rho(T|\mathcal{M}) = \rho(T) = \mathbb{T}$. We infer that $Q \in \mathcal{C}_{*\cdot}(p_0, \mathcal{E})$, where $p_0(n) = br^{-n}p(n)$, $n \in \mathbb{N}$, and $U_{Q,L} = U_{rQ,L}$, whence the absolute continuity of Q and the equation $\rho(Q) = \mathbb{T}$ follow. It is immediate that $p_0 \in \widetilde{\mathcal{P}}$ with $c_{p_0} = 1$, $\alpha_{p_0} = b^{-2}\alpha_p$, and $\mathcal{B}(p_0) = \mathcal{B}(p)$. Since Q is a contraction, the sequence $\{Q^{*n}Q^n\}_{n\in\mathbb{N}}$ converges to the positive operator $A_Q \in \mathcal{L}(\mathcal{E})$ in the strong operator topology. Hence

$$\langle A_{Q,L}x, y\rangle = L\text{-}\lim_{n\to\infty} p_0(n)^{-2}\langle Q^{*n}Q^n x, y\rangle = \alpha_{p_0}\langle A_Q x, y\rangle$$

is true, for every $x, y \in \mathcal{E}$. We obtain that $Q \in \mathcal{C}_{*\cdot}(\mathbb{1}, \mathcal{E})$ and that $U_Q = U_{Q,L}$. Thus Q is an absolutely continuous contraction with $\rho_{\mathbb{1}}(Q) = \mathbb{T}$, and so Theorem 3, [17], yields the existence of a subspace $\mathcal{F} \in \operatorname{Lat} Q$ such that $Q|\mathcal{F}$ is similar to S. Then $\mathcal{N} := B^{-1}\mathcal{F}$ will be invariant for T, and $T|\mathcal{N} \approx rS$. The universality of $X_{T|\mathcal{N},L}$ implies that $X_{T|\mathcal{N},L}$ is bounded from below, and that $U_{T|\mathcal{N},L}|\operatorname{ran} X_{T|\mathcal{N},L} \simeq S$. The equation $X_{T,L,\mathcal{N}} = Z_{T,L,\mathcal{N}}X_{T|\mathcal{N},L}$ yields that the same holds for the transformation $X_{T,L,\mathcal{N}}$ too.

(ii) Let us write $\alpha = \rho(T)$ for short, and let us consider the absolutely continuous operator $\widehat{T} = T \oplus rM_{\alpha^c} \in \mathcal{C}_{*\cdot}(p, \mathcal{H} \oplus L^2(\alpha^c))$, where $X_{\widehat{T},L} = X_{T,L} \oplus \alpha_p I$, $U_{\widehat{T},L} = U_{T,L} \oplus M_{\alpha^c}$ and $\rho(\widehat{T}) = \rho(T) \cup \alpha^c = \mathbb{T}$. Since the restriction of $r^{-1}\widehat{T}$ to its invariant subspace $\widehat{\mathcal{M}} = \mathcal{M} \oplus L^2(\alpha^c)$ is similar to a contraction and $\rho(\widehat{T}|\widehat{\mathcal{M}}) = \rho(T|\mathcal{M}) \cup \alpha^c = \mathbb{T}$, we infer by (a) that there exists $\widehat{\mathcal{N}} \in \operatorname{Lat} \widehat{T}$ such that $X_{\widehat{T},L,\widehat{\mathcal{N}}}$ is bounded from below and $U_{\widehat{T},L}|\operatorname{ran} X_{\widehat{T},L,\widehat{\mathcal{N}}} \simeq S$. In view of Bourgain's theorem, we obtain that $\widehat{T}$ is full with analytic spectral densities. Thus, given any $f \in L^1_{\mathrm{a}}(\alpha)$, we can find vectors $\widehat{g} = g \oplus g_0$, $\widehat{h} = h \oplus h_0 \in \mathcal{H} \oplus L^2(\alpha^c)$ such that $f + \chi_{\alpha^c} = \widetilde{\delta}_{\widehat{T},L,\widehat{g},\widehat{h}}$. Taking into account that $\mathcal{K}_{T,L} = \chi_\alpha(U_{\widehat{T},L})\mathcal{K}_{\widehat{T},L}$ and $L^2(\alpha^c) = \chi_{\alpha^c}(U_{\widehat{T},L})\mathcal{K}_{\widehat{T},L}$, we conclude that $f = \chi_\alpha \widetilde{\delta}_{\widehat{T},L,\widehat{g},\widehat{h}} = \widetilde{\delta}_{T,L,g,h}$, and so T is full with analytic spectral densities.

(iii) Exploiting the full power of Theorem 3, [17], we obtain that there exists a system $\{\widehat{\mathcal{N}}_\gamma\}_{\gamma\in\Gamma}$ of $\widehat{T}$-invariant subspaces satisfying the conditions $\bigvee\{\widehat{\mathcal{N}}_\gamma\}_{\gamma\in\Gamma} = \widehat{\mathcal{M}}$ and $\widehat{T}|\widehat{\mathcal{N}}_\gamma \approx rS$, $\gamma \in \Gamma$. Let $P \in \mathcal{L}(\mathcal{H} \oplus L^2(\alpha^c))$ be the orthogonal projection onto the component $\mathcal{H}$. For any non-zero vector $\widehat{h} \in \bigcup\{\widehat{\mathcal{N}}_\gamma\}_{\gamma\in\Gamma}$, we have $\widetilde{\delta}_{\widehat{T},L,\widehat{h}} \in$

$L^1_{\rm a}(\mathbb{T})$, whence $\widetilde{\delta}_{T,L,P\widehat{h}} = \chi_\alpha \widetilde{\delta}_{\widehat{T},L,\widehat{h}} \in L^1_{\rm a}(\alpha)$ follows. Hence, the vectors in the set $\mathcal{N}_0 := \big(\bigcup\{P\widehat{\mathcal{N}}_\gamma\}_{\gamma\in\Gamma}\big) \setminus \{0\}$, which is total in $\mathcal{M} = P\widehat{\mathcal{M}}$, are all analytic for T. The inequality $\widetilde{\delta}_{T,L,h\oplus k} = \widetilde{\delta}_{T,L,h} + \widetilde{\delta}_{T,L,k} \geqslant \widetilde{\delta}_{T,L,h}$ shows that every vector $h \oplus k$ in the total subset $\mathcal{N}_0 \oplus (\mathcal{H} \ominus \mathcal{M})$ of $\mathcal{H}$ is T-analytic. ∎

It can be seen from the proof that the general assumption of the previous theorem concerning $\mathcal{M}$ is equivalent to the existence of a vector $h \in \mathcal{H}$ such that h is quasianalytic for T, and the restriction of $r(T)^{-1}T$ to the subspace $\mathcal{H}^+(T,h) = \bigvee\{T^n h\}_{n=0}^\infty$ is similar to a contraction.

The following statement characterizes the analytic cyclic operators in the case, when the residual set is the whole circle. We recall that the operator $A \in \mathcal{L}(\mathcal{E})$ is a *quasiaffine transform of* the operator $B \in \mathcal{L}(\mathcal{F})$, in notation: $A \prec B$, if the set $\mathcal{I}(A,B)$ contains a *quasiaffinity*, that is an injective transformation with dense range. If an injection is included into $\mathcal{I}(A,B)$, then we say that A *can be injected into* B, and we use the notation: $A \overset{\mathrm{i}}{\prec} B$.

THEOREM 4.2. *Let $T \in \mathcal{C}_{1\cdot}(p,\mathcal{H}), p \in \widetilde{\mathcal{P}}$, be an absolutely continuous, cyclic operator with $\rho(T) = \mathbb{T}$. Then the following statements are equivalent:*

(i) *T is analytic;*
(ii) *a cyclic vector of T is analytic;*
(iii) *$T \prec r(T)S$.*

Proof. It is immediate that (i) implies (ii). If the cyclic vector $u \in \mathcal{H}$ is analytic for T, then the restriction $U^+ := U_{T,L}|\mathcal{K}^+_{T,L}(U_{T,L}, X_{T,L}u)$, $L \in \mathcal{B}(p)$, is unitarily equivalent to S by Proposition 3.3. Since the intertwining mapping $X^+ \in \mathcal{I}(T, r(T)U^+)$, defined by $X^+h := X_{T,L}h$, $h \in \mathcal{H}$, is a quasiaffinity, we obtain that $T \prec r(T)S$.

Let us assume now that $T \prec r(T)S$, and let $Y \in \mathcal{I}(T, r(T)S)$ be a quasiaffinity. There exists a unique mapping $Z \in \mathcal{I}(U_{T,L}, \widetilde{S})$ such that $\widetilde{Y} = ZX_{T,L}$, where $\widetilde{Y} \in \mathcal{I}(T, r(T)\widetilde{S})$ is the transformation defined by $\widetilde{Y}h := Yh$, $h \in \mathcal{H}$. Let $0 \neq h \in \mathcal{H}$ be arbitrary, and let us consider the cyclic subspaces $\mathcal{K}_1 := \mathcal{K}^+_{T,L}(U_{T,L}, X_{T,L}h)$ and $\mathcal{K}_2 := (L^2(\mathbb{T}))^+(\widetilde{S}, \widetilde{Y}h) = (H^2)^+(S, Yh)$. It is clear that the transformation $Z_0 := Z|\mathcal{K}_1 \in \mathcal{I}(U_{T,L}|\mathcal{K}_1, \widetilde{S}|\mathcal{K}_2)$ has dense range, and that $\widetilde{S}|\mathcal{K}_2 = S|\mathcal{K}_2 \simeq S$. Since $S^* \overset{\mathrm{i}}{\prec} (U_{T,L}|\mathcal{K}_1)^*$, it follows that $U_{T,L}|\mathcal{K}_1$ is not unitary. Taking into account that $U_{T,L}|\mathcal{K}_1$ is an absolutely continuous, cyclic isometry, we infer by the Wold decomposition that $U_{T,L}|\mathcal{K}_1 \simeq S$, and so h is analytic for T. ∎

REMARK 4.3. It can be shown in a similar way that the operator $T \in \mathcal{C}_{1\cdot}(p,\mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, is analytic, if $r(T)^{-1}T$ can be injected into the unilateral shift S_n of an arbitrary multiplicity n. (Here S_n denotes the orthogonal sum of n copies of S, acting on the orthogonal sum H^2_n of n copies of H^2.) It is immediate that in that case the operator $T \in \mathcal{L}(p,\mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, belongs to the class $\mathcal{C}_{10}(p,\mathcal{H})$, which means that $\mathcal{H}_0(T,p) = \{0\}$ and $\mathcal{H}_0(T^*,p) = \mathcal{H}$ hold. There are known conditions which guarantee fulfillment of the relation $T \overset{\mathrm{i}}{\prec} r(T)S_n$.

Let us assume that $T \in \mathcal{C}_{10}(\mathbb{1},\mathcal{H})$, and let us consider the Sz.-Nagy–Foiaş *characteristic function* Θ_T of T. We remind the reader that Θ_T is a contractive, analytic function defined on the open unit disc $\mathbb{D}$ and taking values in the set $\mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*})$, where $\mathcal{D}_T = (\operatorname{ran} D_T)^-, \mathcal{D}_{T^*} = (\operatorname{ran} D_{T^*})^-$ are the defect spaces, and $D_T = (I - T^*T)^{1/2}, D_{T^*} = (I - TT^*)^{1/2}$ are the defect operators of T. More precisely, Θ_T is given by the formula

$$\Theta_T(z) := (-T + zD_{T^*}(I - zT^*)^{-1}D_T)|\mathcal{D}_T, \quad z \in \mathbb{D},$$

see [26] for details. In view of Proposition 1, [31], and its proof, we know that *the contraction T can be injected into a unilateral shift S_n, if there exist a non-zero function $\delta \in H^\infty$ and a bounded, analytic mapping $\Psi : \mathbb{D} \to \mathcal{L}(\mathcal{D}_{T^*}, \mathcal{D}_T)$ such that $\Psi\Theta_T = \delta I$.* Furthermore, the latter condition is even necessary for $T \overset{\mathrm{i}}{\prec} S_n$, if n is finite.

In the following example we consider asymptotically non-vanishing, classical Toeplitz operators with analytic symbols, where the residual set is not the whole circle $\mathbb{T}$. It will be shown by the aid of these operators that *analyticity of T is not a necessary condition for the intransitive hyperplanes of $\mathcal{T}(T)$ to be elementary.*

Example 4.4. Let Ω be a simply connected domain included in the open unit disc $\mathbb{D}$, and bounded by a rectifiable Jordan curve Γ. Let us assume that the intersection $\gamma = \Gamma \cap \mathbb{T}$ is a Borel set of measure $0 < \mu(\gamma) < 1$, and let us consider a conformal mapping η of $\mathbb{D}$ onto Ω. The assumption on the boundary Γ implies that η can be extended to a homeomorphism $\eta : \mathbb{D}^- \to \Omega^-$, and that the restriction of η to the unit circle $\mathbb{T}$ is an absolutely continuous function of bounded variation (see e.g. Theorems 14.5.6 and 14.5.8, [11]). Furthermore, the derivative η' belongs to the Hardy class H^1, and $\eta'(\mathrm{e}^{\mathrm{i}t}) := \mathrm{d}\eta(\mathrm{e}^{\mathrm{i}t})/\mathrm{d}t = \mathrm{i}\mathrm{e}^{\mathrm{i}t} \lim_{r\to 1-0} \eta'(r\mathrm{e}^{\mathrm{i}t})$ is true a.e. for its boundary values (see Corollary 20.4.9, [11]).

Let us consider the Toeplitz operator $T_\eta \in \mathcal{L}(H^2)$, associated with the function η. It is clear that $\|T_\eta\| = 1$, the Borel set $\alpha := \eta^{-1}(\gamma) \subset \mathbb{T}$ is of measure $0 < \mu(\alpha) < 1$, and the operators $(T_\eta^*)^n T_\eta^n = T_{|\eta|^{2n}}$ converge to the operator $A_{T_\eta} = T_{\chi_\alpha}$ in the weak* topology, as n tends to infinity. We can identify the unitary operator U_{T_η} as follows. Let $M_{\alpha,\eta} \in \mathcal{L}(L^2(\alpha))$ be the multiplication operator given by $M_{\alpha,\eta} f := \eta f,\ f \in L^2(\alpha)$, and let us consider the intertwining mapping $X_\alpha \in \mathcal{I}(T_\eta, M_{\alpha,\eta})$ defined by $X_\alpha h := \chi_\alpha h,\ h \in H^2$. There exists a unique transformation $Z \in \mathcal{I}(U_{T_\eta}, M_{\alpha,\eta})$ such that $X_\alpha = ZX_{T_\eta}$. Since

$$\|ZU_{T_\eta}^{-n} X_{T_\eta} h\|^2 = \|M_{\alpha,\eta}^{-n} X_\alpha h\|^2 = \|\chi_\alpha h\|^2 = \langle A_{T_\eta} h, h\rangle = \|U_{T_\eta}^{-n} X_{T_\eta} h\|^2$$

holds for every $h \in H^2$ and $n \in \mathbb{N}$, and since X_α has dense range, it follows that Z is a unitary transformation. We obtain that T_η *is an absolutely continuous operator of class $\mathcal{C}_{1\cdot}(\mathbb{1}, H^2)$.*

In view of the Jordan Curve Theorem, we infer by Proposition 2, [29], that η is a (sequential) weak* generator of H^∞, and so T_η and S have the same invariant subspaces. Hence, the operator T_η *is cyclic.*

Let us determine now the asymptotic local spectral density functions $\widetilde{\delta}_{T_\eta,h}$, $h \in H^2$. To this end, let us consider the unitary transformation $W \in \mathcal{I}(M_\gamma, M_{\alpha,\eta})$

defined by $Wf := (f \circ (\eta|\alpha))|(\eta'|\alpha)|^{1/2}$. Since $W^*Z \in \mathcal{I}(U_{T_\eta}, M_\gamma)$ is unitary and $W^*ZX_{T_\eta} = W^*X_\alpha$, it follows that $\rho(T_\eta) = \gamma$ and that, for every $h \in H^2$, we have

$$\widetilde{\delta}_{T_\eta,h} = \delta_{U_{T_\eta},X_{T_\eta}h} = \delta_{M_\gamma,W^*X_\alpha h} = |W^*X_\alpha h|^2.$$

Let $0 \neq h \in H^2$ be arbitrary, and let us write $f := W^*X_\alpha h$ for short. The equations $\chi_\alpha h = X_\alpha h = Wf = (f \circ (\eta|\alpha))|(\eta'|\alpha)|^{1/2}$ yield that

$$\int_\gamma \log|f|\,\mathrm{d}\mu = \int_\alpha (\log|f \circ \eta|)|\eta'|\,\mathrm{d}\mu = \int_\alpha (\log|h| - (1/2)\log|\eta'|)|\eta'|\,\mathrm{d}\mu.$$

We obtain that *if* η' *is bounded* — what happens, in particular, when Γ is a C^∞-smooth curve (see [3]) —, *then* $\int_\gamma \log|f|\,\mathrm{d}\mu > -\infty$, and so T_η *is an analytic operator.*

However, the function η' is not bounded in general. For example, let η_0 be the conformal mapping of $\mathbb{D}$ onto the domain $\mathbb{D}^+ := \{z \in \mathbb{D} : \mathrm{Im}\, z > 0\}$, given as the inverse of the Riemann map $\psi : \mathbb{D}^+ \to \mathbb{D}$ defined by

$$\psi(z) := (z + z^{-1} + \mathrm{i})(z + z^{-1} - \mathrm{i})^{-1}.$$

Then $\gamma = \{\mathrm{e}^{\mathrm{i}t} : 0 \leqslant t \leqslant \pi\}$, $\alpha = \{\mathrm{e}^{\mathrm{i}t} : t_1 \leqslant t \leqslant t_2\}$ where $t_1 = 2\arctan(1/2)$, $t_2 = 2\pi - t_1$, and a short computation shows that $\eta_0(\mathrm{e}^{\mathrm{i}t}) = \exp(\mathrm{i}\arccos((1/2)\cot(t/2)))$ for $t_1 \leqslant t \leqslant t_2$. It is easy to check that $|(\eta_0'|\alpha)| \geqslant 1/4$, $\eta_0'|\alpha \notin L^\infty(\alpha)$ and $|(\eta_0'|\alpha)|\log|(\eta_0'|\alpha)| \in L^1(\alpha)$. It follows that $\int_\alpha (\log|h| - (1/2)\log|\eta_0'|)|\eta_0'|\,\mathrm{d}\mu = -\infty$ must hold for some $0 \neq h \in H^2$, and so *the operator* T_{η_0} *is not analytic.*

We can observe that in all cases $\mathcal{T}(T_\eta) = \{T_\varphi : \varphi \in L^\infty(\alpha)\}$. Indeed, the maximal abelian algebra $\{M_{\alpha,\varphi} : \varphi \in L^\infty(\alpha)\}$ is contained in the abelian commutant of the cyclic unitary operator $M_{\alpha,\eta}$, and so $\{M_{\alpha,\varphi} : \varphi \in L^\infty(\alpha)\} = \{M_{\alpha,\eta}\}'$. Since, for any function $\varphi \in L^\infty(\alpha)$, we have

$$\Phi_{T_\eta}(Z^*M_{\alpha,\varphi}Z) = X_{T_\eta}^* Z^* M_{\alpha,\varphi} Z X_{T_\eta} = X_\alpha^* M_{\alpha,\varphi} X_\alpha = T_\varphi,$$

it follows that $\mathcal{T}(T_\eta) = \mathrm{ran}\,\Phi_{T_\eta} = \{T_\varphi : \varphi \in L^\infty(\alpha)\}$. Given any non-zero vectors $g, h \in \mathcal{H}$, the hyperplane $\mathcal{T}_{g,h}(T_\eta)$ is an intransitive, weak* closed subspace of $\mathcal{T}(S)$. Thus, by the Azoff–Ptak Theorem $\mathcal{T}_{g,h}(T_\eta)$ is hereditarily reflexive, and so it is elementary. Therefore T_{η_0} *is an example for a non-analytic operator, where all intransitive hyperplanes of the set of* T_{η_0}*-Toeplitz operators are elementary.*

5. REFLEXIVE HYPERPLANES

We recall that for a linear manifold $\mathcal{S}$ of operators acting on the Hilbert space $\mathcal{H}$, $\mathrm{Ref}\,\mathcal{S}$ stands for the set of operators $C \in \mathcal{L}(\mathcal{H})$ satisfying the condition $Cx \in (\mathcal{S}x)^-$ for every $x \in \mathcal{H}$. Thus $\mathcal{S}$ is reflexive, precisely when $\mathcal{S} = \mathrm{Ref}\,\mathcal{S}$. We are going to show that the hyperplane $\mathcal{T}_{g,h}(T)$ is reflexive under certain conditions on the vectors $g, h \in \mathcal{H}$. The method applied is a distillation of the technique used in the fourth chapter of [1]. The symbolic calculus comes onto stage only at the end of the section, the majority of the results is valid in a very general setting. Crucial role is played by regular pairs of operators.

DEFINITION 5.1. Let $T \in \mathcal{L}(\mathcal{H})$ be arbitrary. *The pair* $(P, Q) \in \mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H})$ *is* called *regular for the set* $\mathcal{T}(T)$ of T-Toeplitz operators, and we write $(P, Q) \in \mathcal{R}(T)$, if $Q^* C P = C$ holds, for every $C \in \mathcal{T}(T)$.

We note that if $\mathcal{T}(T) \neq \{0\}$ and $(P, Q) \in \mathcal{R}(T)$, then $r(P)$ and $r(Q)$ are necessarily positive. Indeed, let C be a non-zero T-Toeplitz operator. Since $Q^{*n} C P^n = C$, we infer that $0 < \|C\|^{1/n} \leqslant \|Q^n\|^{1/n} \|C\|^{1/n} \|P^n\|^{1/n}$, $n \in \mathbb{N}$, and so $1 \leqslant r(Q) r(P)$.

The importance of such pairs is shown by the following lemma.

LEMMA 5.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be given, and let $g, h \in \mathcal{H}$ be arbitrary;*

(i) *if $(P, Q) \in \mathcal{R}(T)$ then $[g \otimes h, B] = [Pg \otimes Qh, B]$ holds, for all $B \in \mathcal{T}(T)$;*

(ii) *if $(Q, Q) \in \mathcal{R}(T)$ and $z \in r(Q)\mathbb{D}$, then*

$$[Qg \otimes h, B] = [(Q - zI)g \otimes (I - \overline{z}Q)^{-1} h, B]$$

is true, for all $B \in \mathcal{T}(T)$.

Proof. The verification of (i) is easy, and is left to the reader. To prove (ii), let us assume that $(Q, Q) \in \mathcal{R}(T)$ and $z \in r(Q)\mathbb{D}$. Given any $B \in \mathcal{T}(T)$, we can write

$$(I - zQ^*)^{-1} B (Q - zI) = \sum_{n=0}^{\infty} z^n Q^{*n} B Q - \sum_{n=0}^{\infty} z^{n+1} Q^{*n} B = BQ,$$

whence the equation of (ii) readily follows. ∎

The following lemma exhibits several examples for regular pairs. The easy proof is left to the reader.

LEMMA 5.3. *Let $T \in \mathcal{L}(\mathcal{H})$, and let us assume that $r(T) = 1$;*

(i) *the pair (T^k, T^k) is regular for $\mathcal{T}(T)$, for every non-negative integer k;*

(ii) *the pairs $(T - zI, T(I - \overline{z}T)^{-1})$ and $(T(I - zT)^{-1}, T - \overline{z}I)$ are regular for $\mathcal{T}(T)$, for all $z \in \mathbb{D}$;*

(iii) *if $(P_i, Q_i) \in \mathcal{R}(T)$ for $i = 1, 2$, then $(P_1 P_2, Q_1 Q_2) \in \mathcal{R}(T)$.*

Now, we are ready to verify the first reflexivity statement.

PROPOSITION 5.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be given with $r(T) > 0$. If the vectors $g, h \in \mathcal{H}$ are cyclic for T, then the hyperplane $\mathcal{T}_{g,h}(T)$ is reflexive.*

Proof. Taking into account that $\mathcal{T}(cT) = \mathcal{T}(T)$ and that cT and T have the same cyclic vectors, for any $c \in \mathbb{C} \setminus \{0\}$, we may assume that $r(T) = 1$. Let $g, h \in \mathcal{H}$ be arbitrary. We know by Lemma 5.2 (i) and Lemma 5.3 that the operator $H_k(z) := (T - zI)T^k g \otimes T(I - \overline{z}T)^{-1} T^k h$ belongs to $\mathcal{T}_{g,h}(T)_\perp$, for every non-negative integer k, and for every $z \in \mathbb{D}$. Considering the power series expansion

$$\begin{aligned} H_k(z) &= \big(T^{k+1} g - z T^k g\big) \otimes \sum_{n=0}^{\infty} \overline{z}^n T^{k+1+n} h \\ &= T^{k+1} g \otimes T^{k+1} h \; + \; \sum_{n=1}^{\infty} z^n \left(T^{k+1} g \otimes T^{k+1+n} h - T^k g \otimes T^{k+n} h\right), \end{aligned}$$

and applying Cauchy's formula, we obtain that $TT^i g \otimes TT^j h - T^i g \otimes T^j h$ is in $\mathcal{T}_{g,h}(T)_\perp$, whenever $0 \leqslant i \leqslant j$. Repeating the previous argument with the analytic function $G_k(z) := T(I - zT)^{-1}T^k g \otimes (T - \overline{z}I)T^k h$, we can see that the restriction $i \leqslant j$ can be dropped. Taking linear combinations, we infer that

$$Tp(T)g \otimes Tq(T)h - p(T)g \otimes q(T)h \in \mathcal{T}_{g,h}(T)_\perp$$

holds, for any polynomials p and q.

Assuming that $C \in \operatorname{Ref} \mathcal{T}_{g,h}(T)$, it follows that

$$0 = [Tp(T)g \otimes Tq(T)h - p(T)g \otimes q(T)h, C] = \langle (T^*CT - C)p(T)g, q(T)h \rangle$$

is true, for every polynomials p and q. If the vectors g and h are cyclic for T, then $T^*CT - C$ must be zero, that is $C \in \mathcal{T}(T)$. Since $C \in \operatorname{Ref} \mathcal{T}_{g,h}(T)$ and $g \otimes h \in \mathcal{T}_{g,h}(T)_\perp$, we conclude that $C \in \mathcal{T}_{g,h}(T)$, and so the hyperplane $\mathcal{T}_{g,h}(T)$ is reflexive. ∎

Our next aim is to extend the validity of Proposition 5.4 by relaxing the condition posed on the vectors g, h. To this end, we need the following lemma.

Lemma 5.5. *Let $T, Q \in \mathcal{L}(\mathcal{H})$ and $g, h \in \mathcal{H}$ be arbitrary;*
(i) *if $\mathcal{T}(T)Q \subset \mathcal{T}(T)$, then* $(\operatorname{Ref} \mathcal{T}_{Qg,h}(T))Q \subset \operatorname{Ref} \mathcal{T}_{g,h}(T)$;
(ii) *if $\mathcal{T}(T)Q \subset \mathcal{T}(T)$ and $(Q, Q) \in \mathcal{R}(T)$, then*

$$\{B \in \operatorname{Ref} \mathcal{T}_{Qg,h}(T) : BQ = 0\} \subset \operatorname{Ref} \mathcal{T}_{g,h}(T).$$

Proof. We may assume that $\mathcal{T}(T) \neq \{0\}$.

(i) Since, for any $Y \in \mathcal{T}_{Qg,h}(T)$, the operator YQ belongs to $\mathcal{T}(T)$ and $\langle (YQ)g, h \rangle = \langle Y(Qg), h \rangle = 0$, it follows that $YQ \in \mathcal{T}_{g,h}(T)$, and so $\mathcal{T}_{Qg,h}(T)Q \subset \mathcal{T}_{g,h}(T)$. We infer that $(\operatorname{Ref} \mathcal{T}_{Qg,h}(T))Q \subset \operatorname{Ref} (\mathcal{T}_{Qg,h}(T)Q) \subset \operatorname{Ref} \mathcal{T}_{g,h}(T)$ applying Lemma 4.5, [1].

(ii) Let $x \otimes y \in \mathcal{T}_{g,h}(T)_\perp \cap \mathcal{F}_1(\mathcal{H})$ be arbitrary. For any $Y \in \mathcal{T}_{Qg,h}(T)$, the operator YQ is in $\mathcal{T}_{g,h}(T)$ (as we have seen before), and $0 = \langle (YQ)x, y \rangle = \langle Y(Qx), y \rangle$, whence $Qx \otimes y \in \mathcal{T}_{Qg,h}(T)_\perp$ follows. We infer by Lemma 5.2 (ii) that

$$(Q - zI)x \otimes (I - \overline{z}Q)^{-1}y \in \mathcal{T}_{Qg,h}(T)_\perp$$

holds, for every $z \in r(Q)\mathbb{D}$, where $r(Q) > 0$ because of $\mathcal{T}(T) \neq \{0\}$. If $B \in \operatorname{Ref} \mathcal{T}_{Qg,h}(T)$ and $BQ = 0$, then

$$-z\langle Bx, (I - \overline{z}Q)^{-1}y \rangle = \langle B(Q - zI)x, (I - \overline{z}Q)^{-1}y \rangle = 0$$

is true, for every $z \in r(Q)\mathbb{D}$. Thus, we have

$$\langle Bx, y \rangle = \lim_{z \to 0} \langle Bx, (I - \overline{z}Q)^{-1}y \rangle = 0,$$

and so $B \in (\mathcal{T}_{g,h}(T)_\perp \cap \mathcal{F}_1(\mathcal{H}))^\perp = \operatorname{Ref} \mathcal{T}_{g,h}(T)$. ∎

Definition 5.6. *The operator $Q \in \mathcal{L}(\mathcal{H})$ is* called *T-inner*, and we write $Q \in \mathcal{R}_{\mathbf{i}}(T)$, if $\mathcal{T}(T)Q = \mathcal{T}(T)$ and $(Q, Q) \in \mathcal{R}(T)$.

We note that the conditions $\mathcal{T}(T)Q \subset \mathcal{T}(T)$ and $(Q, Q) \in \mathcal{R}(T)$ imply that $\mathcal{T}(T) = Q^*\mathcal{T}(T)Q \subset Q^*\mathcal{T}(T)$. Taking adjoints we obtain that $\mathcal{T}(T) = \mathcal{T}(T)^* \subset (Q^*\mathcal{T}(T))^* = \mathcal{T}(T)Q$, whence $\mathcal{T}(T)Q = \mathcal{T}(T)$ follows.

Proposition 5.7. *Let $T \in \mathcal{L}(\mathcal{H})$ and $g, h \in \mathcal{H}$ be given. If $\mathcal{T}_{g,h}(T)$ is reflexive and $P, Q \in \mathcal{R}_{\mathbf{i}}(T)$, then the hyperplane $\mathcal{T}_{Pg,Qh}(T)$ is also reflexive.*

Proof. Let $C \in \operatorname{Ref} \mathcal{T}_{Pg,h}(T)$ be arbitrary. We know by Lemma 5.5 (i) that $CP \in \operatorname{Ref} \mathcal{T}_{g,h}(T) = \mathcal{T}_{g,h}(T)$. The relation $\mathcal{T}(T)P = \mathcal{T}(T)$ implies that there exists an operator $A \in \mathcal{T}(T)$ such that $AP = CP$. Since $\langle A(Pg), h\rangle = \langle CPg, h\rangle = 0$, it follows that $A \in \mathcal{T}_{Pg,h}(T)$. Applying Lemma 5.5 (ii) with $B := C - A \in \operatorname{Ref} \mathcal{T}_{Pg,h}(T)$, we infer that $B \in \operatorname{Ref} \mathcal{T}_{g,h}(T) = \mathcal{T}_{g,h}(T)$. Thus, the operator $C = A + B$ is T-Toeplitz, and since $C \in \operatorname{Ref} \mathcal{T}_{Pg,h}(T)$, we conclude that $C \in \mathcal{T}_{Pg,h}(T)$. We have shown that the hyperplane $\mathcal{T}_{Pg,h}(T)$ is reflexive.

It is easy to check that, for any vectors $u, v \in \mathcal{H}$, the equations $\mathcal{T}_{v,u}(T) = (\mathcal{T}_{u,v}(T))^*$ and $\operatorname{Ref} \mathcal{T}_{v,u}(T) = (\operatorname{Ref} \mathcal{T}_{u,v}(T))^*$ hold. Hence, the hyperplane $\mathcal{T}_{v,u}(T)$ is reflexive if and only if so is the hyperplane $\mathcal{T}_{u,v}(T)$. Since $\mathcal{T}_{Pg,h}(T)$ is reflexive, we infer that $\mathcal{T}_{h,Pg}(T)$ is reflexive, too. Then, the first part of the proof yields that $\mathcal{T}_{Qh,Pg}(T)$ is reflexive, whence the reflexivity of $\mathcal{T}_{Pg,Qh}(T)$ immediately follows. ∎

In view of this proposition, and keeping in mind the classical case, it makes sense to introduce the following terminology.

Definition 5.8. Given any operator $T \in \mathcal{L}(\mathcal{H})$, *the vector $v \in \mathcal{H}$ is* called T-*outer*, in notation: $v \in \mathcal{R}_{\mathrm{o}}(T, \mathcal{H})$, if v is cyclic for T.

We say that *the vector $u \in \mathcal{H}$ is regular for the operator $T \in \mathcal{L}(\mathcal{H})$*, and we write $u \in \mathcal{R}(T, \mathcal{H})$, if there exist a T-outer vector $v \in \mathcal{R}_{\mathrm{o}}(T, \mathcal{H})$ and a T-inner operator $Q \in \mathcal{R}_{\mathbf{i}}(T)$ such that $u = Qv$.

The operator T is called *regular*, if every non-zero vector $u \in \mathcal{H}$ is regular for T.

Propositions 5.4 and 5.7 readily imply the following theorem.

Theorem 5.9. *If the vectors $u, v \in \mathcal{H}$ are regular for the cyclic operator $T \in \mathcal{L}(\mathcal{H})$, then the hyperplane $\mathcal{T}_{u,v}(T)$ of T-Toeplitz operators is reflexive.*

Combining this statement with the results of the previous sections we obtain the following

Corollary 5.10. *Let $T \in \mathcal{C}_{10}(p, \mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, be a cyclic, absolutely continuous operator, which is a quasiaffine transform of the operator $r(T)S$. Let us assume that there exists a non-zero subspace $\mathcal{M} \in \operatorname{Lat} T$, such that the restriction $r(T)^{-1}T|\mathcal{M}$ is similar to a contraction. Then the hyperplane $\mathcal{T}_{g,h}(T)$ is hereditarily reflexive, whenever the vectors $g, h \in \mathcal{H}$ are regular for T.*

Proof. Taking into account that $T|\mathcal{M} \prec r(T)S$ also holds, we infer that $\rho(T|\mathcal{M}) = \rho(T)$. Hence, Theorem 4.1 implies that T is full with analytic spectral densities. Since T is analytic by Theorem 4.2, Corollary 3.11 tells us that the hyperplane $\mathcal{T}_{g,h}(T)$ is elementary for any non-zero vectors $g, h \in \mathcal{H}$. If these vectors are regular for T, then $\mathcal{T}_{g,h}(T)$ is reflexive by Theorem 5.9, and so it is necessarily hereditarily reflexive. ∎

The following statement provides a characterization of T-inner operators, when T is an asymptotically strongly non-vanishing operator of regular norm-sequence.

PROPOSITION 5.11. *Let $T \in \mathcal{C}_1.(p,\mathcal{H}), p \in \widetilde{\mathcal{P}}$ and $L \in \mathcal{B}(p)$ be given. The operator $Q \in \mathcal{L}(\mathcal{H})$ is T-inner if and only if Q belongs to the commutant $\{T\}'$ of T and the operator $\gamma_{T,L}(Q) \in \{U_{T,L}\}''$ is unitary.*

Proof. Let us assume first that $Q \in \{T\}'$ and that $\gamma_{T,L}(Q) \in \{U_{T,L}\}''$ is unitary. Let $F \in \{U_{T,L}\}'$ be arbitrary. In view of Proposition 2.1, we know that

$$\Phi_{T,L}(F)Q = \Phi_{T,L}(F\gamma_{T,L}(Q)) \in \mathcal{T}(T),$$
$$\Phi_{T,L}(F) = \Phi_{T,L}(F\gamma_{T,L}(Q)^*\gamma_{T,L}(Q)) = \Phi_{T,L}(F\gamma_{T,L}(Q)^*)Q,$$

and

$$Q^*\Phi_{T,L}(F)Q = \Phi_{T,L}(\gamma_{T,L}(Q)^*F\gamma_{T,L}(Q)) = \Phi_{T,L}(\gamma_{T,L}(Q)^*\gamma_{T,L}(Q)F) = \Phi_{T,L}(F).$$

Taking into account that the symbolic calculus is a bijection, we conclude that $Q \in \mathcal{R}_{\mathbf{i}}(T)$.

Let us suppose now that Q is T-inner. Since $Q^*A_{T,L}Q = A_{T,L}$, it follows that $\|X_{T,L}Qh\| = \|X_{T,L}h\|$ holds for every $h \in \mathcal{H}$. Hence, there exists a unique isometry $W_Q \in \mathcal{L}((\operatorname{ran} X_{T,L})^-)$ such that $X_{T,L}Qh = W_QX_{T,L}h$, $h \in \mathcal{H}$. For any operator $F \in \{U_{T,L}\}'$, we have $Q^*X^*_{T,L}FX_{T,L}Q = Q^*\Phi_{T,L}(F)Q = \Phi_{T,L}(F) = X^*_{T,L}FX_{T,L}$, and so

$$\langle FW_QX_{T,L}h, W_QX_{T,L}k\rangle = \langle FX_{T,L}Qh, X_{T,L}Qk\rangle = \langle FX_{T,L}h, X_{T,L}k\rangle, \quad h,k \in \mathcal{H}.$$

Substituting $U_{T,L}$ in place of F, we infer that $W_Q^*V_{T,L}W_Q = V_{T,L}$, where $V_{T,L}$ is the restriction of $U_{T,L}$ to the subspace $(\operatorname{ran} X_{T,L})^-$. Since $V_{T,L}$ and W_Q are isometries, the range of $V_{T,L}W_Q$ must be contained in the range of W_Q. Thus, $V_{T,L}W_Q = W_QW_Q^*V_{T,L}W_Q = W_QV_{T,L}$. Then, for every $h \in \mathcal{H}$, we have

$$X_{T,L}QTh = r(T)W_QV_{T,L}X_{T,L}h = r(T)V_{T,L}W_QX_{T,L}h = X_{T,L}TQh.$$

Since $X_{T,L}$ is injective, we obtain that Q belongs to the commutant $\{T\}'$ of T.

For any operator $F \in \{U_{T,L}\}'$, the equations

$$\Phi_{T,L}(\gamma_{T,L}(Q)^*F\gamma_{T,L}(Q)) = Q^*\Phi_{T,L}(F)Q = \Phi_{T,L}(F)$$

yield that $\gamma_{T,L}(Q)^*F\gamma_{T,L}(Q) = F$. Setting $F = I$, we can see that $\gamma_{T,L}(Q)$ is an isometry. Since $\gamma_{T,L}(Q)$ commutes with $U_{T,L}$, and so with $U^*_{T,L}$ too, the orthogonal projection $E \in \mathcal{L}(\mathcal{K}_{T,L})$ onto the range of $\gamma_{T,L}(Q)$ belongs to the commutant $\{U_{T,L}\}'$. Hence, $I = \gamma_{T,L}(Q)^*E\gamma_{T,L}(Q) = E$, and so the operator $\gamma_{T,L}(Q)$ is unitary. Then, the equation $\gamma_{T,L}(Q)^*F\gamma_{T,L}(Q) = F$ implies that $F\gamma_{T,L}(Q) = \gamma_{T,L}(Q)F$, $F \in \{U_{T,L}\}'$, that is $\gamma_{T,L}(Q) \in \{U_{T,L}\}''$. ∎

We can see from the proof that, for an operator $T \in \mathcal{C}_1.(p,\mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, the assumption $(Q,Q) \in \mathcal{R}(T)$ solely implies that Q is T-inner, and so $\mathcal{T}(T)Q = \mathcal{T}(T)$.

We conclude this paper with the following comment.

REMARK 5.12. We have seen in Section 2 that, in the general setting, the set $A_{T,L}\{T\}'$ corresponds to the set $\{S\}' = \{T_u : u \in H^2\}$ of classical Toeplitz operators with analytic symbols $L \in \mathcal{B}(p)$. Sarason's theorem tells us that $\{S\}'$ is a reflexive subspace. In contrast, the operator space $A_{T,L}\{T\}'$ is transitive, whenever $T \in \mathcal{C}_1.(p,\mathcal{H})$, $p \in \widetilde{\mathcal{P}}$, is an absolutely continuous, cyclic, quasianalytic operator with $\mu(\rho(T)^{\mathrm{c}}) > 0$. Indeed, let us assume that $\langle A_{T,L}Cg, h\rangle = 0$ holds for every $C \in \{T\}'$, with some vectors $g, h \in \mathcal{H}$. Then $\langle U^n_{T,L}X_{T,L}g, X_{T,L}h\rangle =$

$r(T)^{-n}\langle X_{T,L}T^n g, X_{T,L}h\rangle = r(T)^{-n}\langle A_{T,L}T^n g, h\rangle = 0$ is true, for every $n \in \mathbb{N}$. Taking into account that $U_{T,L}$ is a cyclic, absolutely continuous unitary operator with scalar spectral measure $\chi_{\rho(T)}\mathrm{d}\mu$, and that $X_{T,L}g, X_{T,L}h$ are cyclic for $\{U_{T,L}\}'$ if g, h are non-zero vectors, we infer that g or h must be zero.

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