J. OPERATOR THEORY
51(2004), 411–433

FACTORIZATION THEORY FOR A CLASS OF TOEPLITZ + HANKEL OPERATORS

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Communicated by William B. Arveson

Abstract. In this paper we study operators of the form $M(\varphi) = T(\varphi) + H(\varphi)$ where $T(\varphi)$ and $H(\varphi)$ are the Toeplitz and Hankel operators acting on $H^p(\mathbb{T})$ with generating function $\varphi \in L^\infty(\mathbb{T})$. It turns out that $M(\varphi)$ is invertible if and only if the function φ admits a certain kind of generalized factorization.

Keywords: *Toeplitz operator, Hankel operator, Fredholm theory, factorization.*

MSC (2000): 47B35, 47A68.

1. INTRODUCTION

This paper is devoted to the study of operators of the form

$$M(\varphi) = T(\varphi) + H(\varphi) \tag{1.1}$$

acting on the Hardy space $H^p(\mathbb{T})$ where $1 < p < \infty$. Here $\varphi \in L^\infty(\mathbb{T})$ is a Lebesgue measurable and essentially bounded function on the unit circle $\mathbb{T}$. The Toeplitz and Hankel operators are defined by

$$T(\varphi) : f \mapsto P(\varphi f), \quad H(\varphi) : f \mapsto P(\varphi(Jf)), \quad f \in H^p(\mathbb{T}), \tag{1.2}$$

where J is the following flip operator,

$$J : f(t) \mapsto t^{-1} f(t^{-1}), \quad t \in \mathbb{T}, \tag{1.3}$$

acting on the Lebesgue space $L^p(\mathbb{T})$. The operator P stands for the Riesz projection,

$$P : \sum_{n=-\infty}^{\infty} f_n t^n \mapsto \sum_{n=0}^{\infty} f_n t^n, \quad t \in \mathbb{T}, \tag{1.4}$$

which is bounded on $L^p(\mathbb{T})$, $1 < p < \infty$, and whose image is $H^p(\mathbb{T})$. The complex conjugate Hardy space $\overline{H^p(\mathbb{T})}$ is the set of all functions f whose complex conjugate belongs to $H^p(\mathbb{T})$.

For a Banach algebra B (such as $L^\infty(\mathbb{T})$ or $H^\infty(\mathbb{T})$) we will denote by GB the group of all invertible elements.

It is well known that a necessary and sufficient condition for the Fredholmness and the invertibility of the Toeplitz operator $T(\varphi)$ with $\varphi \in L^\infty(\mathbb{T})$ can be given in terms of the Wiener-Hopf factorability of the generating function φ. Let us recall the underlying definitions and results [3], [5], [6] (see also [8] for the case $p = 2$).

A function $\varphi \in L^\infty(\mathbb{T})$ is said to admit a *factorization in* $L^p(\mathbb{T})$ if one can write

$$\varphi(t) = \varphi_-(t)t^\kappa\varphi_+(t), \quad t \in \mathbb{T}, \tag{1.5}$$

where κ is an integer and the factors φ_- and φ_+ satisfy the following conditions:

(i) $\varphi_- \in \overline{H^p(\mathbb{T})}$, $\varphi_-^{-1} \in \overline{H^q(\mathbb{T})}$,

(ii) $\varphi_+ \in H^q(\mathbb{T})$, $\varphi_+^{-1} \in H^p(\mathbb{T})$,

(iii) The linear operator $f \mapsto \varphi_+^{-1}P(\varphi_-^{-1}f)$, which is defined on the set of all trigonometric polynomials and takes values in $L^p(\mathbb{T})$, can be extended by continuity to a linear bounded operator acting from $L^p(\mathbb{T})$ into $L^p(\mathbb{T})$.

Here $p^{-1} + q^{-1} = 1$. A factorization where merely (i) and (ii) but not necessarily condition (iii) is fulfilled is called a *weak factorization in* $L^p(\mathbb{T})$. The number κ is called the index of the (weak) factorization and is uniquely determined. As far as the authors know, the term "weak factorization" is first used in this paper. The problem of weak factorization was studied by Litvinchuk and Spitkovsky in their book [6], but was called "factorization" (while our "factorization" was called "φ-factorization"). The term generalized factorization is sometimes used to indicate our factorization.

The crucial result is that for given $\varphi \in L^\infty(\mathbb{T})$ the operator $T(\varphi)$ is a Fredholm operator on the space $H^p(\mathbb{T})$ if and only if the function φ admits a factorization in $L^p(\mathbb{T})$. In this case the defect numbers are given by

$$\dim\ker T(\varphi) = \max\{0, -\kappa\}, \quad \dim\ker(T(\varphi))^* = \max\{0, \kappa\}. \tag{1.6}$$

Hence $T(\varphi)$ is invertible if and only if φ admits a factorization in $L^p(\mathbb{T})$ with index $\kappa = 0$.

The goal of the present paper is to obtain corresponding results for operators $M(\varphi)$ with $\varphi \in L^\infty(\mathbb{T})$. We will encounter another type of factorization of the function φ which is related to the Fredholm theory of these operators. The investigations taken up in this paper are motivated by the results obtained in [1] and [4].

2. BASIC PROPERTIES OF $M(\varphi)$

It is well known that for Toeplitz and Hankel operators the following relations hold:

$$T(\varphi\psi) = T(\varphi)T(\psi) + H(\varphi)H(\widetilde{\psi}), \tag{2.1}$$

$$H(\varphi\psi) = T(\varphi)H(\psi) + H(\varphi)T(\widetilde{\psi}). \tag{2.2}$$

Here $\widetilde{\psi}(t) = \psi(t^{-1})$. Adding both equations, it follows that

$$M(\varphi\psi) = T(\varphi)M(\psi) + H(\varphi)M(\widetilde{\psi}),$$

and hence

$$M(\varphi\psi) = M(\varphi)M(\psi) + H(\varphi)M(\widetilde{\psi} - \psi). \tag{2.3}$$

In some particular cases, this identity simplifies to a multiplicative relation:

$$M(\varphi\psi) = M(\varphi)M(\psi) \tag{2.4}$$

if $\varphi \in \overline{H^\infty(\mathbb{T})}$ or $\psi = \widetilde{\psi}$. Based on this identity one can establish a sufficient invertibility criteria for $M(\varphi)$, which anticipates to some extent the factorization result that we establish in this paper. Assume that $\varphi \in L^\infty(\mathbb{T})$ admits a factorization $\varphi = \varphi_-\varphi_0$, where $\varphi_- \in G\overline{H^\infty(\mathbb{T})}$ and $\varphi_0 \in GL^\infty(\mathbb{T})$ such that $\widetilde{\varphi}_0 = \varphi_0$. Then $M(\varphi)$ is invertible and its inverse is given by $M(\varphi_0^{-1})M(\varphi_-^{-1})$.

It is occasionally convenient to introduce the multiplication operator on $L^p(\mathbb{T})$,

$$L(\varphi) : L^p(\mathbb{T}) \to L^p(\mathbb{T}), \quad f \mapsto \varphi f, \tag{2.5}$$

and then consider Toeplitz and Hankel operators as restrictions of the following operators onto $H^p(\mathbb{T})$:

$$T(\varphi) = PL(\varphi)P|_{H^p(\mathbb{T})}, \quad H(\varphi) = PL(\varphi)JP|_{H^p(\mathbb{T})}. \tag{2.6}$$

The Toeplitz + Hankel operators $M(\varphi)$ are of the form

$$M(\varphi) = PL(\varphi)(I + J)P|_{H^p(\mathbb{T})}. \tag{2.7}$$

The following results gives estimates for the norm of $M(\varphi)$, where the constant C_p depends only on the parameter p.

Proposition 2.1. *Let $\varphi \in L^\infty(\mathbb{T})$. Then* $\|\varphi\|_{L^\infty(\mathbb{T})} \leqslant \|M(\varphi)\|_{\mathcal{L}(H^p(\mathbb{T}))} \leqslant C_p\|\varphi\|_{L^\infty(\mathbb{T})}$.

Proof. The upper estimate follows from (2.7). Note that the operators P and J are both bounded on $L^p(\mathbb{T})$. In order to prove the lower estimate, put $U_n = M(t^n)$. Since $U_nU_{-n} = I$ and $JU_n = U_{-n}J$, it is easy to see that

$$U_{-n}M(\varphi)U_n = (U_{-n}PU_n)L(\varphi)(U_{-n}PU_n) + (U_{-n}PU_{-n})L(\varphi)J(U_{-n}PU_n).$$

Using the fact that $U_{-n}PU_n \to I$ and $U_{-n}PU_{-n} \to 0$ strongly on $L^p(\mathbb{T})$ as $n \to \infty$, it follows that

$$U_{-n}M(\varphi)U_n \to L(\varphi) \tag{2.8}$$

strongly on $L^p(\mathbb{T})$ as $n \to \infty$. Since $U_{\pm n}$ are isometries on $L^p(\mathbb{T})$, this implies the lower estimate. ∎

Now we obtain a necessary condition for the Fredholmness of $M(\varphi)$.

PROPOSITION 2.2. *Suppose that $M(\varphi)$ is Fredholm on $H^p(\mathbb{T})$. Then $\varphi \in GL^\infty(\mathbb{T})$.*

Proof. The proof is based on standard arguments. If $M(\varphi)$ is a Fredholm operator, then there exist a $\delta > 0$ and a finite rank projection K on the kernel of $M(\varphi)$ such that

$$\|M(\varphi)f\|_{H^p(\mathbb{T})} + \|Kf\|_{H^p(\mathbb{T})} \geqslant \delta\|f\|_{H^p(\mathbb{T})}$$

for all $f \in H^p(\mathbb{T})$. Putting Pf instead of f, this implies that

$$\|M(\varphi)f\|_{L^p(\mathbb{T})} + \|KPf\|_{L^p(\mathbb{T})} + \delta\|(I-P)f\|_{L^p(\mathbb{T})} \geqslant \delta\|f\|_{L^p(\mathbb{T})}$$

for all $f \in L^p(\mathbb{T})$. Replacing f by U_nf and observing again that $U_{\pm n}$ are isometries on $L^p(\mathbb{T})$, it follows that

$$\|U_{-n}M(\varphi)U_nf\|_{L^p(\mathbb{T})} + \|KPU_nf\|_{L^p(\mathbb{T})} + \delta\|U_{-n}(I-P)U_nf\|_{L^p(\mathbb{T})} \geqslant \delta\|f\|_{L^p(\mathbb{T})}.$$

Now we take the limit $n \to \infty$. Because $U_n \to 0$ weakly, we have $KPU_n \to 0$ strongly. It remains to apply (2.8) and again the fact that $U_{-n}PU_n \to I$ strongly. We obtain

$$\|L(\varphi)f\|_{L^p(\mathbb{T})} \geqslant \delta\|f\|_{L^p(\mathbb{T})}.$$

This implies that $\varphi \in GL^\infty(\mathbb{T})$. ∎

The following lemma, which appears in slightly less general form as Lemma 2.9 in [2], turns out to be useful.

LEMMA 2.3. *Let X_1 and X_2 be linear spaces, $A : X_1 \to X_2$ be a linear and invertible operator, $P_1 : X_1 \to X_1$ and $P_2 : X_2 \to X_2$ be linear projections, and $Q_1 = I - P_1$ and $Q_2 = I - P_2$. Then $P_2AP_1 : \operatorname{Im} P_1 \to \operatorname{Im} P_2$ is invertible if and only if $Q_1A^{-1}Q_2 : \operatorname{Im} Q_2 \to \operatorname{Im} Q_1$ is invertible.*

Proof. The following formulas, which relate both inverses with each other, can be verified by a direct calculation:

$$\begin{aligned}(P_2AP_1)^{-1} &= P_1A^{-1}P_2 - P_1A^{-1}Q_2(Q_1A^{-1}Q_2)^{-1}Q_1A^{-1}P_2,\\ (Q_1A^{-1}Q_2)^{-1} &= Q_2AQ_1 - Q_2AP_1(P_2AP_1)^{-1}P_2AQ_1.\end{aligned}$$

The validity of these formulas implies the desired assertion. ∎

In what follows, we are going to relate $M(\varphi)$ with two other operators $\Phi(\varphi)$ and $\Psi(\varphi)$. First of all note that $J^2 = I$. Hence

$$P_J = \frac{I+J}{2}, \quad Q_J = \frac{I-J}{2} \tag{2.9}$$

are complementary projections acting on the space $L^p(\mathbb{T})$ and decompose this space into the direct sum $L^p(\mathbb{T}) = \operatorname{Im} P_J|_{L^p(\mathbb{T})} \dot{+} \operatorname{Im} Q_J|_{L^p(\mathbb{T})}$. Let $L^p_J(\mathbb{T}) = \operatorname{Im} P_J|_{L^p(\mathbb{T})}$. Given $\varphi \in L^\infty(\mathbb{T})$, define

$$\Phi(\varphi) = PL(\varphi)P_J, \quad \Psi(\varphi) = P_JL(\varphi)P, \tag{2.10}$$

where we consider these operators as acting between the following spaces:

$$\Phi(\varphi) : L^p_J(\mathbb{T}) \to H^p(\mathbb{T}), \quad \Psi(\varphi) : H^p(\mathbb{T}) \to L^p_J(\mathbb{T}), \tag{2.11}$$

PROPOSITION 2.4. *Let $\varphi \in GL^\infty(\mathbb{T})$. Then the following assertions are equivalent:*

(i) $M(\varphi)$ *is invertible in* $\mathcal{L}(H^p(\mathbb{T}))$;

(ii) $\Phi(\varphi)$ *is invertible in* $\mathcal{L}(L^p_J(\mathbb{T}), H^p(\mathbb{T}))$;

(iii) $\Psi(\psi)$ *is invertible in* $\mathcal{L}(H^p(\mathbb{T}), L^p_J(\mathbb{T}))$, *where* $\psi(t) = \varphi^{-1}(-t^{-1})$.

Proof. In order to prove the equivalence of (i) and (ii) we refer to the representation (2.7) and (2.10) of these operators. Moreover, note that $A = (I+J)P : H^p(\mathbb{T}) \to L^p_J(\mathbb{T})$ and $B = \frac{1}{2}P(I+J) : L^p_J(\mathbb{T}) \to H^p(\mathbb{T})$ are inverse to each other:

$$\begin{aligned} AB &= \frac{1}{2}(I+J)P(I+J) = \frac{P+JP+Q+JQ}{2} = \frac{I+J}{2}, \\ BA &= \frac{1}{2}P(I+J)^2P = P(I+J)P = P. \end{aligned}$$

Here we have used the basic relations, $J^2 = I$ and $JPJ = Q$, where $Q = I - P$.

In order to prove the equivalence of (ii) and (iii) we apply Lemma 2.3 with $P_1 = P_J$, $P_2 = P$, $Q_1 = Q_J$ and $Q_2 = Q$. We obtain that $\Phi(\varphi)$ is invertible if and only if the operator

$$Q_J L(\varphi^{-1})Q : \operatorname{Im} Q|_{L^p(\mathbb{T})} \to \operatorname{Im} Q_J|_{L^p(\mathbb{T})}$$

is invertible. Multiplying this operator from the left and right with J, we obtain the operator

$$Q_J J L(\varphi^{-1}) J P : H^p(\mathbb{T}) \to \operatorname{Im} Q_J|_{L^p(\mathbb{T})}$$

since $QJ = JP$ and $JQ_J = Q_J J$. Next we multiply the latter operator from the left and right with W, where $W : f(t) \mapsto f(-t)$. We obtain the operator

$$P_J W J L(\varphi^{-1}) J W P : H^p(\mathbb{T}) \to L^p_J(\mathbb{T})$$

since $WQ_J = P_J W$ (i.e., $WJ + JW = 0$) and $PW = WP$. This last operator is equal to $\Psi(\psi)$ since $WJL(\varphi^{-1})JW = L(\psi)$ as one can readily check. From this it follows that $\Phi(\varphi)$ is invertible if and only if so is $\Psi(\psi)$. ∎

3. WEAK ASYMMETRIC AND ANTISYMMETRIC FACTORIZATION

In what follows we introduce the notion of a weak asymmetric and a weak antisymmetric factorization of a matrix function $\varphi \in GL^\infty(\mathbb{T})$ in the space $L^p(\mathbb{T})$. Before we do so, however, let us for a moment try to anticipate the kind of factorization we should expect based on our notions from the theory of Toeplitz operators. If we suppose that $M(\varphi)$ is invertible on $H^p(\mathbb{T})$, then there exists a function $h \in H^p(\mathbb{T})$ such that $M(\varphi)h = 1$. Now this means, using the definition of J, that

$$\varphi(h + t^{-1}\tilde{h}) = g_-$$

where $g_- \in \overline{H^p(\mathbb{T})}$. Multiplying this equation with the factor $(1+t^{-1})^{-1}$ gives

$$\varphi(1+t^{-1})^{-1}(h + t^{-1}\tilde{h}) = (1+t^{-1})^{-1} g_-.$$

Introduce the functions

$$f_0 = (1+t^{-1})^{-1}(h + t^{-1}\tilde{h}) \quad \text{and} \quad f_- = (1+t^{-1})^{-1} g_-.$$

Thus $\varphi f_0 = f_-$, which gives upon assuming for a moment the invertibility of f_0 the factorization $\varphi = f_0^{-1} f_-$. On the other hand,

$$\tilde{f}_0 = (1+t)^{-1}(\tilde{h} + th) = (1+t^{-1})^{-1}(h + t^{-1}\tilde{h}) = f_0.$$

Hence f_0 is even. From the definition of f_- and f_0 it now follows that

$$(1+t^{-1})f_- \in \overline{H^p(\mathbb{T})} \quad \text{and} \quad |1+t|^{-1} f_0 \in L^p_{\text{even}}(\mathbb{T}).$$

Here and in what follows $L^p_{\text{even}}(\mathbb{T})$ stands for the set of all functions $\varphi_0 \in L^p(\mathbb{T})$ which are even, i.e., $\varphi_0 = \tilde{\varphi}_0$.

The above analysis shows that the factor $1 + t^{-1}$ plays a special role in the factorization theory of $M(\varphi)$ and hints of its presence in the following definitions. It turns out that the same is true for the factor $1 - t^{-1}$. This is not apparent from the heuristics that have just been presented, but will become clear later on.

A function $\varphi \in GL^\infty(\mathbb{T})$ is said to admit a *weak asymmetric factorization in* $L^p(\mathbb{T})$ if it admits a representation

$$\varphi(t) = \varphi_-(t) t^\kappa \varphi_0(t), \quad t \in \mathbb{T}, \tag{3.1}$$

such that $\kappa \in \mathbb{Z}$ and

(i) $(1+t^{-1})\varphi_- \in \overline{H^p(\mathbb{T})}$, $(1-t^{-1})\varphi_-^{-1} \in \overline{H^q(\mathbb{T})}$,

(ii) $|1-t|\varphi_0 \in L^q_{\text{even}}(\mathbb{T})$, $|1+t|\varphi_0^{-1} \in L^p_{\text{even}}(\mathbb{T})$,

where $p^{-1} + q^{-1} = 1$.

The uniqueness of a weak asymmetric factorization (up to a constant) is stated in the following proposition.

PROPOSITION 3.1. *Assume that φ admits two weak asymmetric factorizations in $L^p(\mathbb{T})$:*

$$\varphi(t) = \varphi_-^{(1)}(t) t^{\kappa_1} \varphi_0^{(1)}(t) = \varphi_-^{(2)}(t) t^{\kappa_2} \varphi_0^{(2)}(t), \quad t \in \mathbb{T}. \tag{3.2}$$

Then $\kappa_1 = \kappa_2$ and $\varphi_-^{(1)} = \gamma \varphi_-^{(2)}$, $\varphi_0^{(1)} = \gamma^{-1}\varphi_0^{(2)}$ with $\gamma \in \mathbb{C} \setminus \{0\}$.

Proof. Assume without loss of generality that $\kappa = \kappa_1 - \kappa_2 \leqslant 0$. From (3.2) it follows that

$$(\varphi_-^{(2)})^{-1} \varphi_-^{(1)} t^\kappa = \varphi_0^{(2)} (\varphi_0^{(1)})^{-1}. \tag{3.3}$$

Put $\psi = (1-t^{-2})(\varphi_-^{(2)})^{-1}\varphi_-^{(1)}$. From the conditions on the factors $(\varphi_-^{(2)})^{-1}$ and $\varphi_-^{(1)}$ stated in (i) of the definition of the weak asymmetric factorization, it follows that $\psi \in \overline{H^1(\mathbb{T})}$. Formula (3.2) gives

$$(1-t^{-2})^{-1}\psi(t) t^\kappa = \varphi_0^{(2)}(\varphi_0^{(1)})^{-1},$$

where the right hand side is an even function. Hence

$$(1-t^{-2})^{-1}\psi(t)t^\kappa = (1-t^2)^{-1}\psi(t^{-1})t^{-\kappa} \quad \text{and} \quad \psi(t)t^{2\kappa+2} = -\psi(t^{-1}).$$

Using the fact that $\psi \in \overline{H^1(\mathbb{T})}$, we obtain $\psi = 0$ if $\kappa \leqslant -1$ by inspecting the Fourier coefficients of ψ. This is a contradiction, since it would imply that $\varphi_-^{(1)} = 0$. Hence $\kappa = 0$. In this case it follows that $\psi(t) = \gamma(1-t^{-2})$, with $\gamma \neq 0$. Hence $\varphi_-^{(1)} = \gamma\varphi_-^{(2)}$. ∎

The above introduced weak asymmetric factorization is *not* a kind of Wiener-Hopf factorization. Although condition (i) on the factor φ_- implies that φ_- can be identified with a function that is analytic and nonzero on $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, the factor φ_0 is an even function defined on the unit circle.

However, it is possible to relate the weak asymmetric factorization to some kind of Wiener-Hopf factorization, which we will call weak antisymmetric factorization. In this factorization, the left and the right factor have a special dependence.

A function $F \in GL^\infty(\mathbb{T})$ is said to admit a *weak antisymmetric factorization in* $L^p(\mathbb{T})$ if it admits a representation

$$F(t) = \varphi_-(t) t^{2\kappa} \widetilde{\varphi}_-^{-1}(t), \quad t \in \mathbb{T}, \tag{3.4}$$

such that $\kappa \in \mathbb{Z}$ and

(i) $(1 + t^{-1})\varphi_- \in \overline{H^p(\mathbb{T})}$, $(1 - t^{-1})\varphi_-^{-1} \in \overline{H^q(\mathbb{T})}$,

where $p^{-1} + q^{-1} = 1$.

Obviously, condition (i) can be rephrased by the following equivalent condition given terms of the function $\widetilde{\varphi}_-$:

(ii) $(1 + t)\widetilde{\varphi}_- \in H^p(\mathbb{T})$, $(1 - t)\widetilde{\varphi}_-^{-1} \in H^q(\mathbb{T})$.

Hence the function φ_- can be identified with a function that is analytic and nonzero on $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$, while $\widetilde{\varphi}_-$ can be identified with a function that is analytic and nonzero on $\{z \in \mathbb{C} : |z| < 1\}$. In this sense, the weak antisymmetric factorization represents a kind of Wiener-Hopf factorization.

A necessary condition for the existence of a weak antisymmetric factorization is, of course, that $\widetilde{F}^{-1} = F$. The next result says that φ possesses a weak asymmetric factorization if and only if the function $F = \varphi\widetilde{\varphi}^{-1}$ possesses a weak antisymmetric factorization.

PROPOSITION 3.2. *Let $\varphi \in GL^\infty(\mathbb{T})$ and put $F = \varphi\widetilde{\varphi}^{-1}$.*

(i) *if φ admits a weak asymmetric factorization, $\varphi = \varphi_- t^\kappa \varphi_0$, then the function F admits a weak antisymmetric factorization with the same factor φ_- and the same index κ;*

(ii) *if F admits a weak antisymmetric factorization, $F = \varphi_- t^{2\kappa}\widetilde{\varphi}_-^{-1}$, then φ admits a weak asymmetric factorization with the same factor φ_-, the same index κ and the factor $\varphi_0 := t^{-\kappa}\varphi_-^{-1}\varphi$.*

Proof. (i) Starting from the weak asymmetric factorization of φ, we form $\widetilde{\varphi}^{-1} = \varphi_0^{-1} t^\kappa \widetilde{\varphi}_-^{-1}$ observing the fact that φ_0 is an even function, and obtain $\varphi\widetilde{\varphi}^{-1} = \varphi_- t^{2\kappa}\widetilde{\varphi}_-^{-1}$.

(ii) The definition of φ_0 implies that $\varphi = \varphi_- t^\kappa \varphi_0$. It remains to verify that φ_0 satisfies the required conditions. First of all,

$$\widetilde{\varphi}_0 = t^\kappa \widetilde{\varphi}_-^{-1}\widetilde{\varphi} = t^\kappa \widetilde{\varphi}_-^{-1} = t^{-\kappa}\varphi_-^{-1} F \widetilde{\varphi}_-^{-1} = t^{-\kappa}\varphi_-^{-1}\varphi = \varphi_0.$$

Hence φ_0 is an even function. From the conditions on φ_- and since $\varphi \in GL^\infty(\mathbb{T})$, it follows from equation $\varphi_0 = t^{-\kappa}\varphi_-^{-1}\varphi$ that $|1 - t|\varphi_0 \in L^q(\mathbb{T})$ and $|1 + t|\varphi_0^{-1} \in L^p(\mathbb{T})$. ∎

From the last two results it follows in particular that the index of a weak antisymmetric factorization is uniquely determined and also the factor φ_- is uniquely determined up to a nonzero complex constant.

4. THE FORMAL INVERSE OF $\Phi(\varphi)$

We proceed with establishing some auxiliary results that we will need later on. Let $\mathcal{R}$ stand for the linear space of all trigonometric polynomials. Suppose that we are given a weak asymmetric factorization of a function $\varphi \in GL^\infty(\mathbb{T})$ with the index $\kappa = 0$. Introduce

$$X_1 = \{(1-t^{-1})f(t) : f \in \mathcal{R}\},$$
$$X_2 = \{(1+t^{-1})\varphi_0^{-1}(t)f(t) : f \in \mathcal{R},\ f(t) = f(t^{-1})\}. \tag{4.1}$$

Note that X_1 is dense in the Banach spaces $L^p(\mathbb{T})$, $1 < p < \infty$, and that $X_2 \subseteq L^p_J(\mathbb{T})$.

LEMMA 4.1. *Assume $\varphi \in GL^\infty(\mathbb{T})$ admits a weak asymmetric factorization in $L^p(\mathbb{T})$ with the index $\kappa = 0$. Then the following assertions hold:*

(i) *the operator $B = L(\varphi_0^{-1})(I+J)PL(\varphi_-^{-1})$ is a well defined linear operator acting from X_1 into X_2;*

(ii) $\Phi(\varphi)B = P|_{X_1}$*;*

(iii) $\ker \Phi(\varphi) = \{0\}$.

Proof. (i) Let $f \in X_1$. We are going to compute Bf. First write $f(t) = (1-t^{-1})f_1(t)$ with some trigonometric polynomial f_1. Hence

$$\varphi_-^{-1}(t)f(t) = (1-t^{-1})\varphi_-^{-1}(t)f_1(t)$$

i.e., a function in $\overline{H^q(\mathbb{T})}$ times a trigonometric polynomial. We can uniquely decompose

$$\varphi_-^{-1}(t)f(t) = u_1(t) + p_1(t)$$

such that $u_1 \in t^{-1}\overline{H^q(\mathbb{T})}$ and p_1 is a polynomial. Applying the Riesz projection to $\varphi_-^{-1}f$ we obtain $P(\varphi_-^{-1}f) = p_1$. Hence

$$(Bf)(t) = \varphi_0^{-1}(t)(p_1(t) + t^{-1}p_1(t^{-1})). \tag{4.2}$$

Since $p_1(t) + t^{-1}p_1(t^{-1})$ vanishes at $t = -1$, this expression is $(1+t^{-1})$ times a trigonometric polynomial. Now it is easy to see that (4.2) is contained in X_2.

(ii) In order to prove (ii) we need more explicit expressions for the terms u_1 and p_1 defined in part (i). Given $f \in X_1$ write first $f(t) = (1-t^{-2})f_2(t)+\alpha(1-t^{-1})$ with some trigonometric polynomial f_2 and a constant α. Similar as before, it is possible to decompose

$$(1-t^{-1})\varphi_-^{-1}(t)f_2(t) = u_2(t) + p_2(t)$$

such that $u_2 \in t^{-1}\overline{H^q(\mathbb{T})}$ and p_2 is a polynomial. Multiplying by $(1+t^{-1})$ we obtain

$$\varphi_-^{-1}(t)(1-t^{-2})f_2(t) = \big((1+t^{-1})u_2(t) + t^{-1}p_2(0)\big) + \big((1+t^{-1})p_2(t) - t^{-1}p_2(0)\big).$$

Moreover, we decompose

$$(1-t^{-1})\varphi_-^{-1}(t) = \big((1-t^{-1})\varphi_-^{-1}(t) - \varphi_-^{-1}(\infty)\big) + \varphi_-^{-1}(\infty).$$

Combining these yield

$$\begin{aligned} u_1(t) &= (1+t^{-1})u_2(t) + t^{-1}p_2(0) + \alpha(1-t^{-1})\varphi_-^{-1}(t) - \alpha\varphi_-^{-1}(\infty) \\ p_1(t) &= (1+t^{-1})p_2(t) - t^{-1}p_2(0) + \alpha\varphi_-^{-1}(\infty). \end{aligned}$$

Indeed, $\varphi_-^{-1}f = u_1 + p_1$, where $u_1 \in t \in \overline{H^q(\mathbb{T})}$ and p_1 is a polynomial.

In order to show that $\Phi(\varphi)Bf = Pf$, we have to prove that $Pg = Pf$, where

$$g(t) = \varphi_-(t)(p_1(t) + t^{-1}p_1(t^{-1})).$$

It follows that

$$\begin{aligned} f(t) - g(t) &= \varphi_-(t)u_1(t) - \varphi_-(t)t^{-1}p_1(t^{-1}) \\ &= \varphi_-(t)\left((1+t^{-1})u_2(t) + t^{-1}p_2(0) + \alpha(1-t^{-1})\varphi_-^{-1}(t) - \alpha\varphi_-^{-1}(\infty)\right. \\ &\qquad \left. -(1+t^{-1})p_2(t^{-1}) + p_2(0) - t^{-1}\alpha\varphi_-^{-1}(\infty)\right) \\ &= (1+t^{-1})\varphi_-(t)\left(u_2(t) + p_2(0) - p_2(t^{-1})\right) - \alpha t^{-1} \\ &\qquad + \alpha - (1+t^{-1})\varphi_-(t)\alpha\varphi_-^{-1}(\infty). \end{aligned}$$

It can be verified easily that the expression on the right hand side belongs to $t^{-1}\overline{H^1(\mathbb{T})}$. Hence $Pf = Pg$.

(iii) Let $f \in \ker\Phi(\varphi)$. This means that $f \in L^p_J(\mathbb{T})$ and $P(\varphi f) = 0$. The latter can be rewritten in the form $f_- = \varphi f$ where $f_- \in t^{-1}\overline{H^p(\mathbb{T})}$. Multiplying with φ_-^{-1} it follows that $\varphi_-^{-1}f_- = \varphi_0 f$. Moreover,

$$t(1-t^{-1})\varphi_-^{-1}(t)f_-(t) = (t-1)\varphi_0(t)f(t) =: \psi(t).$$

Since $Jf = f$ and φ_0 is even, it is easily seen that $\widetilde{\psi} = -\psi$. Finally, we have $(1-t^{-1})\varphi_-^{-1}(t) \in \overline{H^q(\mathbb{T})}$. Hence $\psi \in \overline{H^1(\mathbb{T})}$. It follows that $\psi = 0$, thus $f_- = 0$. ∎

5. INVERTIBILITY AND ASYMMETRIC FACTORIZATION

In this section we derive necessary and sufficient conditions for the invertibility of $M(\varphi)$ in terms of an asymmetric factorization.

LEMMA 5.1. *Suppose that $\Phi(\varphi)$ is right invertible. Then there exist functions $f_- \neq 0$ and f_0 such that $f_-(t) = \varphi(t)f_0(t)$, $t \in \mathbb{T}$, and*

$$(5.1) \qquad (1+t^{-1})f_- \in \overline{H^p(\mathbb{T})}, \quad |1+t|f_0 \in L^p_{\mathrm{even}}(\mathbb{T}).$$

Proof. If $\Phi(\varphi)$ is right invertible, then $\operatorname{Im}\Phi(\varphi) = H^p(\mathbb{T})$. Let $h_0 \in L^p_J(\mathbb{T})$ such that $\Phi(\varphi)h_0 = 1$. Put $h_-(t) = \varphi(t)h_0(t)$. It follows that $h_- \in \overline{H^p(\mathbb{T})}$ and $h_- \neq 0$. Define $f_-(t) = (1+t^{-1})^{-1}h_-(t)$. Obviously, f_- satisfies the required conditions and

$$f_-(t) = (1+t^{-1})^{-1}\varphi(t)h_0(t).$$

On defining $f_0(t) = (1+t^{-1})^{-1}h_0(t)$, it is easily seen that $|1+t|f_0 \in L^p(\mathbb{T})$ and that f_0 is an even function. ∎

LEMMA 5.2. *Suppose that $\Psi(\psi)$ is left invertible, where $\psi(t) = \varphi^{-1}(-t^{-1})$. Then there exist functions $g_- \neq 0$ and g_0 such that $g_-(t) = g_0(t)\varphi^{-1}(t)$, $t \in \mathbb{T}$, and*

$$(1 - t^{-1})g_- \in \overline{H^q(\mathbb{T})}, \quad |1 - t|g_0 \in L^q_{\mathrm{even}}(\mathbb{T}). \tag{5.2}$$

Proof. If $\Psi(\psi)$ is left invertible, then $(\Psi(\psi))^*$ is right invertible. Identifying $(H^p(\mathbb{T}))^*$ with $H^q(\mathbb{T})$ and $(L^p_J(\mathbb{T}))^*$ with $L^q_J(\mathbb{T})$, it follows that $(\Psi(\psi))^* = \Phi(\overline{\psi})$, which is an operator acting from $L^q_J(\mathbb{T})$ into $H^q(\mathbb{T})$. It follows from the previous lemma that there exist $f_- \neq 0$ and f_0 such that

$$(1 + t^{-1})f_- \in \overline{H^q(\mathbb{T})}, \quad |1 + t|f_0 \in L^q_{\mathrm{even}}(\mathbb{T}), \quad \text{and} \quad f_-(t) = \overline{\varphi^{-1}(-t^{-1})}f_0(t).$$

Now we pass to the complex conjugate and make a substitution $t \mapsto -t^{-1}$. Putting $g_-(t) = \overline{f_-(-t^{-1})}$ and $g_0(t) = \overline{g_0(-t^{-1})}$, it follows that

$$(1 - t^{-1})g_- \in \overline{H^q(\mathbb{T})}, \quad |1 - t|g_0 \in L^q_{\mathrm{even}}(\mathbb{T}),$$

and $g_-(t) = \varphi^{-1}(t)g_0(t)$, as desired. ∎

In order to present the main result, we introduce the notion of an asymmetric factorization. A function $\varphi \in GL^\infty(\mathbb{T})$ is said to admit an *asymmetric factorization in $L^p(\mathbb{T})$* if it admits a representation

$$\varphi(t) = \varphi_-(t)t^\kappa\varphi_0(t), \quad t \in \mathbb{T}, \tag{5.3}$$

such that $\kappa \in \mathbb{Z}$ and

(i) $(1 + t^{-1})\varphi_- \in \overline{H^p(\mathbb{T})}$, $(1 - t^{-1})\varphi_-^{-1} \in \overline{H^q(\mathbb{T})}$,

(ii) $|1 - t|\varphi_0 \in L^q_{\mathrm{even}}(\mathbb{T})$, $|1 + t|\varphi_0^{-1} \in L^p_{\mathrm{even}}(\mathbb{T})$,

(iii) the linear operator $B = L(\varphi_0^{-1})(I + J)PL(\varphi_-^{-1})$ acting from X_1 into X_2 extends to a linear bounded operator $\widetilde{B}$ acting from $L^p(\mathbb{T})$ into $L^p_J(\mathbb{T})$.

Here $p^{-1} + q^{-1} = 1$, and X_1 and X_2 are the spaces (4.1).

An equivalent formulation of condition (iii) is the following:

(iii*) there exists a constant M such that $\|Bf\|_{L^p_J(\mathbb{T})} \leqslant M\|f\|_{L^p(\mathbb{T})}$ for all $f \in X_1$.

Notice here that X_1 is dense in $L^p(\mathbb{T})$ and $X_2 \subseteq L^p_J(\mathbb{T})$.

THEOREM 5.3. *Let $\varphi \in GL^\infty(\mathbb{T})$. The operator $M(\varphi)$ is invertible on $H^p(\mathbb{T})$ if and only if φ admits an asymmetric factorization in $L^p(\mathbb{T})$ with the index $\kappa = 0$.*

Proof. $\Rightarrow$ If $M(\varphi)$ is invertible, then by Proposition 2.4 the operators $\Phi(\varphi)$ and $\Psi(\psi)$ are also invertible. Applying Lemma 5.1 and Lemma 5.2 it follows that $f_- = \varphi f_0$ and $g_- = g_0\varphi^{-1}$ with the properties stated there. Combining this yields $g_-f_- = g_0f_0$. Now we are in a similar situation as in the proof of Proposition 3.1 (see also (3.3)). In the very same way one can prove that $g_-f_- = g_0f_0 =: \gamma$ is a constant. It must be nonzero since otherwise $g_- = 0$ or $f_- = 0$ (cf. the F. and M. Riesz Theorem). Now we put $\varphi_- = f_- = \gamma g_-^{-1}$ and $\varphi_0 = f_0^{-1} = g_0\gamma^{-1}$. From the conditions on the functions in Lemma 5.1 and Lemma 5.2, the conditions (i) and (ii) in the above definition of the asymmetric factorization follow. Hence we have shown that φ admits a weak asymmetric factorization with $\kappa = 0$.

From Lemma 4.1 (i) it follows that the operator B is well defined. Assertion (ii) of the same lemma implies that (since $\Phi(\varphi)$ is invertible)

$$B = \Phi(\varphi)^{-1} P|_{X_1}.$$

Both the left and right hand side in this equation represent operators defined on X_1, which is dense in $L^p(\mathbb{T})$. Obviously, the right hand side can be extended by continuity to a linear bounded operator acting from $L^p(\mathbb{T})$ into $H^p(\mathbb{T})$ (since it is the restriction of such an operator). Hence so can be the operator B.

$\Leftarrow$ Assume that conditions (i)–(iii) are satisfied. Denote the continuous extension of the operator B by $\widetilde{B}$. We can apply Lemma 4.1 (ii). Since X_1 is dense in $L^p(\mathbb{T})$ it follows that

$$\Phi(\varphi)\widetilde{B} = P,$$

where the operators are defined on $L^p(\mathbb{T})$. Hence $\Phi(\varphi)$ is right invertible, the right inverse being $\widetilde{B}|_{H^p(\mathbb{T})}$ (the restriction of $\widetilde{B}$ onto $H^p(\mathbb{T})$). Now we multiply the above equation from the right with $\Phi(\varphi)$ and obtain $\Phi(\varphi)\widetilde{B}|_{H^p(\mathbb{T})}\Phi(\varphi) = \Phi(\varphi)$, i.e.,

$$\Phi(\varphi)(\widetilde{B}|_{H^p(\mathbb{T})}\Phi(\varphi) - I) = 0.$$

From the triviality of the kernel of $\Phi(\varphi)$ (see Lemma 4.1 (iii)), we obtain $\widetilde{B}|_{H^p(\mathbb{T})}\Phi(\varphi) = I$. Hence $\Phi(\varphi)$ is invertible and its inverse is just $\widetilde{B}|_{H^p(\mathbb{T})}$. ∎

6. FREDHOLM THEORY AND ASYMMETRIC FACTORIZATION

In order to establish the Fredholm theory in terms of the above introduced asymmetric factorization, it is necessary to prove an assertion about the kernel and the cokernel of $M(\varphi)$. It is the analogue of Coburn's result for Toeplitz operators.

We begin with the following observation. Given $\varphi \in L^\infty(\mathbb{T})$, define

$$K = \{t \in \mathbb{T} : \varphi(t) = \varphi(t^{-1}) = 0\}. \tag{6.1}$$

This definition of course depends on the choice of representatives for φ, however the following remarks are independent of that choice. Because the characteristic function χ_K is an even function, it follows from (2.4) that $M(\chi_K)$ is a projection. Moreover, from the same relation we obtain that $M(\varphi)M(\chi_K) = M(\varphi\chi_K) = 0$ and thus

$$\operatorname{Im} M(\chi_K) \subseteq \ker M(\varphi). \tag{6.2}$$

If the set K has Lebesgue measure zero, then obviously $M(\chi_K) = 0$, whereas if K has a positive Lebesgue measure, then the image of $M(\chi_K)$ is infinite dimensional. The latter fact can be seen by decomposing K into pairwise disjoint and even sets $K_1, \dots, K_n$ with positive Lebesgue measure. Then $M(\chi_K) = M(\chi_{K_1}) + \cdots + M(\chi_{K_n})$, where $M(\chi_{K_1}), \dots, M(\chi_{K_n})$ are mutually orthogonal projections. By Proposition 2.1 all these projections are nonzero. Hence $\dim \operatorname{Im} M(\chi_K) \geqslant n$ for all n, i.e., $\dim \operatorname{Im} M(\chi_K) = \infty$.

PROPOSITION 6.1. *Let $\varphi \in L^\infty(\mathbb{T})$ and let K be as above. Then* $\ker M(\varphi) = \operatorname{Im} M(\chi_K)$ *or* $\ker M^*(\varphi) = \{0\}$.

Proof. Obviously, $M(\varphi)$ and its adjoint can be written in the form:

$$M(\varphi) = PL(\varphi)(I+J)P, \quad M^*(\varphi) = P(I+J)L(\overline{\varphi})P.$$

Suppose that we are given functions $f_+ \in H^p(\mathbb{T})$ and $g_+ \in H^q(\mathbb{T})$ such that $M(\varphi)f_+ = 0$ and $M^*(\varphi)g_+ = 0$ with $g_+ \neq 0$. We have to show that $f_+ \in \operatorname{Im} M(\chi_K)$. Introducing the functions

$$\begin{array}{ll} f(t) = f_+(t) + t^{-1}f_+(t^{-1}), & f_-(t) = \varphi(t)f(t), \\ g(t) = \overline{\varphi}(t)g_+(t), & g_-(t) = g(t) + t^{-1}g(t^{-1}), \end{array}$$

it follows that $f_- \in t^{-1}\overline{H^p(\mathbb{T})}$ and $g_- \in t^{-1}\overline{H^q(\mathbb{T})}$. From the definition of g_- we obtain that $t^{-1}g_-(t^{-1}) = g_-(t)$. Hence $g_- = 0$ by checking the Fourier coefficients. It follows that $t^{-1}g(t^{-1}) = -g(t)$. On the other hand, the definition of f says that $t^{-1}f(t^{-1}) = f(t)$. This implies that $(f\overline{g})(t^{-1}) = -(f\overline{g})(t)$. Also from the above relations we conclude that $f\overline{g} = f\varphi\overline{g}_+ = f_-\overline{g}_+$. Because $f_- \in t^{-1}\overline{H^p(\mathbb{T})}$ and $\overline{g}_+ \in \overline{H^q(\mathbb{T})}$, we have $f_-\overline{g}_+ \in t^{-1}\overline{H^1(\mathbb{T})}$. Considering again the Fourier coefficients, this shows $f\overline{g} = f_-\overline{g}_+ = 0$. Since $g_+ \in H^q(\mathbb{T})$ and $g_+ \neq 0$ the F. and M. Riesz Theorem says that $g_+(t) \neq 0$ almost everywhere on $\mathbb{T}$, and thus we obtain $f_- = 0$. Hence $\varphi f = 0$. Because $t^{-1}f(t^{-1}) = f(t)$, we have also $\widetilde{\varphi}f = 0$. The definition of the set K now implies that $(1-\chi_K)f = 0$. Noting that $M(1-\chi_K)f_+ = 0$, we finally arrive at $f_+ \in \operatorname{Im} M(\chi_K)$. ∎

Combining the previous proposition with Proposition 2.4 we obtain the following result.

COROLLARY 6.2. *Let $\varphi \in L^\infty(\mathbb{T})$. If $M(\varphi)$ is Fredholm on $H^p(\mathbb{T})$, then $M(\varphi)$ has a trivial kernel or a trivial cokernel.*

From this corollary we can obtain another interesting result, pertaining to the relation between invertibility and Fredholmness of $M(\varphi)$.

COROLLARY 6.3. *Let $\varphi \in L^\infty(\mathbb{T})$. Then $M(\varphi)$ is invertible on $H^p(\mathbb{T})$ if and only if $M(\varphi)$ is Fredholm on $H^p(\mathbb{T})$ and has index zero.* ∎

The desired Fredholm criteria in terms of the asymmetric factorization is established next.

THEOREM 6.4. *Let $\varphi \in GL^\infty(\mathbb{T})$. The operator $M(\varphi)$ is a Fredholm operator on $H^p(\mathbb{T})$ if and only if the function φ admits an asymmetric factorization* (5.3) *in $L^p(\mathbb{T})$. In this case the defect numbers are given by*

$$\dim\ker M(\varphi) = \max\{0,-\kappa\}, \quad \dim\ker(M(\varphi))^* = \max\{0,\kappa\}. \tag{6.3}$$

Proof. $\Rightarrow$ Assume that $M(\varphi)$ is Fredholm with index $-\kappa$. We consider the function $\psi(t) = t^{-\kappa}\varphi(t)$. Using the fact that a Hankel operator with a continuous generating function is compact and the identity (2.3), it follows that

$$M(\psi) = M(t^{-\kappa})M(\varphi) + \text{compact}.$$

Taking into account that $M(t^{-\kappa}) = T(t^{-\kappa}) + \text{compact}$ is Fredholm with index κ, it follows that $M(\psi)$ is a Fredholm operator with index zero. Now we are in a position to apply Corollary 6.3 in order to see that $M(\psi)$ is invertible. Theorem 5.3 implies that ψ possesses an asymmetric factorization with index zero. Hence φ possesses an asymmetric factorization with index κ.

$\Leftarrow$ Assume that φ possesses an asymmetric factorization with index κ. Then we consider again the function $\psi(t) = t^{-\kappa}\varphi(t)$, which thus possesses an asymmetric factorization with index zero. Theorem 5.3 implies that $M(\psi)$ is invertible. Very similar as above it follows that $M(\varphi)$ is Fredholm and has Fredholm index $-\kappa$.

As to the defect numbers, we know that $M(\varphi)$ has the Fredholm index $-\kappa$. On the other hand (by Corollary 6.2), the kernel or the cokernel of $M(\varphi)$ are trivial. This implies the formulas for the defect numbers. ∎

7. FREDHOLM AND INVERTIBILITY THEORY FOR PIECEWISE CONTINUOUS FUNCTIONS

The goal of this section is to find practical criteria that determine whether or not $M(\varphi)$ is an invertible operator on $H^p(\mathbb{T})$ when φ is a piecewise continuous function.

The first part of this section is devoted to the Fredholmness of $M(\varphi)$ for piecewise continuous functions φ. The results are immediate consequences of [7]. Let us start with the following definitions. The Mellin transform $\mathcal{M} : L^p(\mathbb{R}_+) \to L^p(\mathbb{R})$ is defined by

$$(\mathcal{M}f)(x) = \int_0^\infty \xi^{-1+1/p-\mathrm{i}x} f(\xi)\,\mathrm{d}\xi, \quad x \in \mathbb{R}. \tag{7.1}$$

For $\varphi \in L^\infty(\mathbb{R})$ the Mellin convolution operator $M^0(\varphi) \in \mathcal{L}(L^p(\mathbb{R}_+))$ is given by

$$M^0(\varphi)f = \mathcal{M}^{-1}(\varphi(\mathcal{M}f)). \tag{7.2}$$

Let S be the singular integral operator acting on the space $L^p(\mathbb{R}_+)$,

$$(Sf)(x) = \frac{1}{\pi\mathrm{i}} \int_0^\infty \frac{f(y)}{y-x}\,\mathrm{d}y, \quad x \in \mathbb{R}_+, \tag{7.3}$$

where the singular integral has to be understood as the Cauchy principal value, and let N stand for the integral operator acting on $L^p(\mathbb{R}_+)$ by

$$(Nf)(x) = \frac{1}{\pi\mathrm{i}} \int_0^\infty \frac{f(y)}{y+x}\,\mathrm{d}y, \quad x \in \mathbb{R}_+. \tag{7.4}$$

It is well known that S and N can be expressed as Mellin convolution operators

$$S = M^0(s), \quad N = M^0(n) \tag{7.5}$$

with the generating functions

$$s(z) = \coth\Big(\Big(z + \frac{\mathrm{i}}{p}\Big)\pi\Big), \quad n(z) = \sinh^{-1}\Big(\Big(z + \frac{\mathrm{i}}{p}\Big)\pi\Big), \quad z \in \mathbb{R} \tag{7.6}$$

(see, e.g., Section 2 of [7]). Notice that s and n are continuous on $\mathbb{R}$ and possess the limits $s(\pm\infty) = \pm 1$ and $n(\pm\infty) = 0$ for $z \to \pm\infty$. Moreover, $s^2 - n^2 = 1$.

Let $S_{\mathbb{T}}$ stand for the singular integral operator acting on $L^p(\mathbb{T})$,

$$(S_{\mathbb{T}} f)(t) = \frac{1}{\pi \mathrm{i}} \int_{\mathbb{T}} \frac{f(s)}{s-t}\,\mathrm{d}s, \quad t \in \mathbb{T}, \tag{7.7}$$

and let W stand for the "flip" operator

$$(Wf)(t) = f(t^{-1}) \tag{7.8}$$

acting on $L^p(\mathbb{T})$. Finally, for a piecewise continuous function $\varphi \in PC$ defined on the unit circle denote by

$$\varphi^{\pm}(t) = \lim_{\varepsilon \to \pm 0} \varphi(t e^{\mathrm{i}\varepsilon}) \tag{7.9}$$

the one-sided limits at a point $t \in \mathbb{T}$. As before let $L(\varphi)$ stand for the multiplication operator on $L^p(\mathbb{T})$ with the generating function φ.

The smallest closed subalgebra of $\mathcal{L}(L^p(\mathbb{T}))$, which contains the operators S, W and $L(\varphi)$ for $\varphi \in PC$ will be denoted by $\mathcal{S}^p(PC)$. Specializing Theorem 9.1 of [7] to the situation where we are interested in, we obtain the following result. Therein we put $\mathbb{T}_+ = \{\tau \in \mathbb{T} : \operatorname{Im} \tau > 0\}$.

THEOREM 7.1. (i) *For $\tau \in \{-1, 1\}$, there exists a homomorphism $H_\tau : \mathcal{S}^p(PC) \to \mathcal{L}((L^p(\mathbb{T}))^2)$, which acts on the generating elements as follows:*

$$H_\tau(S_{\mathbb{T}}) = \begin{pmatrix} S & -N \\ N & -S \end{pmatrix}, \quad H_\tau(W) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$
$$H_\tau(L(\varphi)) = \operatorname{diag}(\varphi^+(\tau) I, \varphi^-(\tau) I).$$

(ii) *For $\tau \in \mathbb{T}_+$, there exists a homomorphism $H_\tau : \mathcal{S}^p(PC) \to \mathcal{L}((L^p(\mathbb{T}))^4)$, which acts on the generating elements as follows:*

$$H_\tau(S_{\mathbb{T}}) = \begin{pmatrix} S & -N & 0 & 0 \\ N & -S & 0 & 0 \\ 0 & 0 & S & -N \\ 0 & 0 & N & -S \end{pmatrix}, \quad H_\tau(W) = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix},$$
$$H_\tau(L(\varphi)) = \operatorname{diag}(\varphi^+(\tau) I, \varphi^-(\tau) I, \varphi^+(\bar{\tau}) I, \varphi^-(\bar{\tau}) I).$$

(iii) *An operator $A \in \mathcal{S}^p(PC)$ is Fredholm if and only if the operators $H_\tau(A)$ are invertible for all $\tau \in \{-1, 1\} \cup \mathbb{T}_+$.*

Now we apply this theorem in order to study the Fredholmness of the operator $M(\varphi)$.

THEOREM 7.2. *Let $\varphi \in PC$. Then $M(\varphi)$ is Fredholm on $H^p(\mathbb{T})$ if and only if $\varphi^{\pm}(\tau) \neq 0$ for all $\tau \in \mathbb{T}$ and the following conditions are satisfied:*

$$\frac{1}{2\pi} \arg\left(\frac{\varphi^-(1)}{\varphi^+(1)}\right) \notin \frac{1}{2p} + \mathbb{Z}, \tag{7.10}$$

$$\frac{1}{2\pi} \arg\left(\frac{\varphi^-(-1)}{\varphi^+(-1)}\right) \notin \frac{1}{2} + \frac{1}{2p} + \mathbb{Z}, \tag{7.11}$$

$$\frac{1}{2\pi} \arg\left(\frac{\varphi^-(\tau)\varphi^-(\bar{\tau})}{\varphi^+(\tau)\varphi^+(\bar{\tau})}\right) \notin \frac{1}{p} + \mathbb{Z} \quad \text{for each } \tau \in \mathbb{T}_+. \tag{7.12}$$

Proof. Observe that $M(\varphi) = PL(\varphi)(I+J)P$ is defined on $H^p(\mathbb{T})$. Moreover, as has been shown in the proof of Proposition 2.4, the operators $(I+J)P : H^p(\mathbb{T}) \to L^p_J(\mathbb{T})$ and $(1/2)P(I+J) : L^p_J(\mathbb{T}) \to H^p(\mathbb{T})$ are inverse to each other. Hence $M(\varphi)$ is Fredholm on $H^p(\mathbb{T})$ if and only if the operator $A = (I+J)PM(\varphi)P(I+J)$ acting on $L^p_J(\mathbb{T})$ is Fredholm. Taking into account that $PJ = J(I-P)$ and $J^2 = I$, a simple computation gives

$$A = (I+J)PL(\varphi)(I+J).$$

Notice that $P = (I+S_{\mathbb{T}})/2$ and $J = L(t^{-1})W$. The space $L^p_J(\mathbb{T})$ is by definition the image of the operator $(I+J)$. The operator A belongs to $\mathcal{S}^p(PC)$ and thus we can apply Theorem 7.1 in order to examine the Fredholmness of the operator A. The result is that A is Fredholm on $L^p_J(\mathbb{T})$ if and only if for each $\tau \in \mathbb{T}_+ \cup \{-1, 1\}$ the operators $H_\tau(A)$ are invertible on the image of the operators $H_\tau(I+J)$.

For $\tau \in \{-1, 1\}$ we obtain that

$$H_\tau(I+J) = \begin{pmatrix} I & \tau I \\ \tau I & I \end{pmatrix} = \begin{pmatrix} I \\ \tau I \end{pmatrix} \begin{pmatrix} I & \tau I \end{pmatrix}$$

and

$$H_\tau(A) = \frac{1}{2} \begin{pmatrix} I & \tau I \\ \tau I & I \end{pmatrix} \begin{pmatrix} (I+S)\varphi^+(\tau) & -N\varphi^-(\tau) \\ N\varphi^+(\tau) & (I-S)\varphi^-(\tau) \end{pmatrix} \begin{pmatrix} I & \tau I \\ \tau I & I \end{pmatrix}.$$

Thus $H_\tau(A)$ is invertible on the image of $H_\tau(I+J)$ if and only if the operator

$$B_\tau = \begin{pmatrix} I & \tau I \end{pmatrix} \begin{pmatrix} (I+S)\varphi^+(\tau) & -N\varphi^-(\tau) \\ N\varphi^+(\tau) & (I-S)\varphi^-(\tau) \end{pmatrix} \begin{pmatrix} I \\ \tau I \end{pmatrix}$$

is invertible on $L^p(\mathbb{T})$. Obviously,

$$B_\tau = (I+S)\varphi^+(\tau) + (I-S)\varphi^-(\tau) + \tau N(\varphi^+(\tau) - \varphi^-(\tau)),$$

which is a Mellin convolution operator $B_\tau = M^0(b_\tau)$ with the generating function

$$b_\tau(z) = (1 + s(z) + \tau n(z))\varphi^+(\tau) + (1 - s(z) - \tau n(z))\varphi^-(\tau), \quad z \in \mathbb{R}.$$

Thus B_τ is invertible if and only if $b_\tau(\pm\infty) \neq 0$ and $b_\tau(z) \neq 0$ for all $z \in \mathbb{R}$. Obviously, $b_\tau(\pm\infty) = 2\varphi^{\pm}(\tau)$. The second condition, $b_\tau(z) \neq 0$ for all $z \in \mathbb{R}$, is equivalent to

$$\frac{\varphi^-(\tau)}{\varphi^+(\tau)} \neq \frac{s(z) + \tau n(z) + 1}{s(z) + \tau n(z) - 1} = \frac{\mathrm{e}^{(z+\mathrm{i}/p)\pi} + \tau}{\mathrm{e}^{-(z+\mathrm{i}/p)\pi} + \tau} = \tau \mathrm{e}^{(z+\mathrm{i}/p)\pi}.$$

This implies condition (7.10) and (7.11).

For $\tau \in \mathbb{T}_+$ we have

$$H_\tau(I+J) = \underbrace{\begin{pmatrix} I & 0 & 0 & \bar{\tau}I \\ 0 & I & \bar{\tau}I & 0 \\ 0 & \tau I & I & 0 \\ \tau I & 0 & 0 & I \end{pmatrix}}_{\mathbf{X}} = \underbrace{\begin{pmatrix} I & 0 \\ 0 & I \\ 0 & \tau I \\ \tau I & 0 \end{pmatrix}}_{\mathbf{Y}} \underbrace{\begin{pmatrix} I & 0 & 0 & \bar{\tau}I \\ 0 & I & \bar{\tau}I & 0 \end{pmatrix}}_{\mathbf{Z}}$$

and

$$\begin{aligned} &H_\tau(A) \\ &= \frac{1}{2}\mathbf{X}\begin{pmatrix} (I+S)\varphi^+(\tau) & -N\varphi^-(\tau) & 0 & 0 \\ N\varphi^+(\tau) & (I-S)\varphi^-(\tau) & 0 & 0 \\ 0 & 0 & (I+S)\varphi^+(\bar{\tau}) & -N\varphi^-(\bar{\tau}) \\ 0 & 0 & N\varphi^+(\bar{\tau}) & (I-S)\varphi^-(\bar{\tau}) \end{pmatrix}\mathbf{X}. \end{aligned}$$

Thus $H_\tau(A)$ is invertible on the image of $H_\tau(I+J)$ if and only if the operator B_τ given by

$$\mathbf{Z}\begin{pmatrix} (I+S)\varphi^+(\tau) & -N\varphi^-(\tau) & 0 & 0 \\ N\varphi^+(\tau) & (I-S)\varphi^-(\tau) & 0 & 0 \\ 0 & 0 & (I+S)\varphi^+(\bar{\tau}) & -N\varphi^-(\bar{\tau}) \\ 0 & 0 & N\varphi^+(\bar{\tau}) & (I-S)\varphi^-(\bar{\tau}) \end{pmatrix}\mathbf{Y}$$

is invertible. It follows that

$$B_\tau = \begin{pmatrix} (I+S)\varphi^+(\tau) & -N\varphi^-(\tau) \\ N\varphi^+(\tau) & (I-S)\varphi^-(\tau) \end{pmatrix} + \begin{pmatrix} (I-S)\varphi^-(\bar{\tau}) & N\varphi^+(\bar{\tau}) \\ -N\varphi^-(\bar{\tau}) & (I+S)\varphi^+(\bar{\tau}) \end{pmatrix}.$$

This operator is a Mellin convolution operator with the generating function

$$\begin{aligned} &b_\tau(z) \\ &= \begin{pmatrix} (1+s(z))\varphi^+(\tau) + (1-s(z))\varphi^-(\bar{\tau}) & n(z)(\varphi^+(\bar{\tau}) - \varphi^-(\tau)) \\ n(z)(\varphi^+(\tau) - \varphi^-(\bar{\tau})) & (1-s(z))\varphi^-(\tau) + (1+s(z))\varphi^+(\bar{\tau}) \end{pmatrix}. \end{aligned}$$

Note that

$$b_\tau(+\infty) = 2\begin{pmatrix} \varphi^+(\tau) & 0 \\ 0 & \varphi^+(\bar{\tau}) \end{pmatrix}, \quad b_\tau(-\infty) = 2\begin{pmatrix} \varphi^-(\bar{\tau}) & 0 \\ 0 & \varphi^-(\tau) \end{pmatrix}.$$

Finally,

$$\begin{aligned} \det b_\tau &= (1+s)^2\varphi^+(\tau)\varphi^+(\bar{\tau}) + (1-s)^2\varphi^-(\tau)\varphi^-(\bar{\tau}) \\ &\quad + (1-s^2)(\varphi^+(\tau)\varphi^-(\tau) + \varphi^+(\bar{\tau})\varphi^-(\bar{\tau})) \\ &\quad - n^2(\varphi^+(\tau) - \varphi^-(\bar{\tau}))(\varphi^+(\bar{\tau}) - \varphi^-(\tau)) \\ &= (1+s)^2\varphi^+(\tau)\varphi^+(\bar{\tau}) + (1-s)^2\varphi^-(\tau)\varphi^-(\bar{\tau}) \\ &\quad + (1-s^2)(\varphi^+(\tau)\varphi^+(\bar{\tau}) + \varphi^-(\tau)\varphi^-(\bar{\tau})) \\ &= 2(1+s)\varphi^+(\tau)\varphi^+(\bar{\tau}) + 2(1-s)\varphi^-(\tau)\varphi^-(\bar{\tau}). \end{aligned}$$

This expression is nonzero if and only if

$$\frac{\varphi^-(\tau)\varphi^-(\bar{\tau})}{\varphi^+(\tau)\varphi^+(\bar{\tau})} \neq \frac{s+1}{s-1} = \mathrm{e}^{2(z+\mathrm{i}/p)\pi}.$$

From this condition (7.12) follows. ∎

In the rest of this section we are going to establish the criteria for an operator $M(\varphi)$ to be invertible on $H^p(\mathbb{T})$ in the case where φ is a piecewise continuous function with a finite number of jumps. The proof is based partly on the Fredholm criteria that we have just established and partly on the notion of a weak asymmetric factorization.

It is well known that any piecewise continuous and nonvanishing function with a finite number of discontinuities at the points $t_1 = \mathrm{e}^{\mathrm{i}\theta_1}, \ldots, t_R = \mathrm{e}^{\theta_R}$ can be written as a product

$$\varphi(\mathrm{e}^{\mathrm{i}\theta}) = b(\mathrm{e}^{\mathrm{i}\theta}) \prod_{r=1}^{R} t_{\beta_r}(\mathrm{e}^{\mathrm{i}(\theta-\theta_r)}) \tag{7.13}$$

where b is a nonvanishing continuous function and

$$t_\beta(\mathrm{e}^{\mathrm{i}\theta}) = \exp(\mathrm{i}\beta(\theta-\pi)), \quad 0 < \theta < 2\pi. \tag{7.14}$$

The parameters in the formula are useful to decide invertibility. For example, it is well known ([2]), that the Toeplitz operator $T(\varphi)$ is invertible on $H^p(\mathbb{T})$ if and only if it can be represented in the above form with parameters satisfying

$$-\frac{1}{q} < \operatorname{Re}\beta_r < \frac{1}{p} \tag{7.15}$$

for all $1 \leqslant r \leqslant R$ and a continuous nonvanishing function b with winding number zero. For the corresponding result for the operators $M(\varphi)$ we prepare with the following proposition.

Proposition 7.3. *Let ψ be a function of the form*

$$\psi(\mathrm{e}^{\mathrm{i}\theta}) = t_{\beta^+}(\mathrm{e}^{\mathrm{i}\theta}) t_{\beta^-}(\mathrm{e}^{\mathrm{i}(\theta-\pi)}) \prod_{r=1}^{R} t_{\beta_r^+}(\mathrm{e}^{\mathrm{i}(\theta-\theta_r)}) t_{\beta_r^-}(\mathrm{e}^{\mathrm{i}(\theta+\theta_r)}) \tag{7.16}$$

where $\theta_1, \ldots, \theta_R \in (0,\pi)$ are distinct and
(i) $-1/q < \operatorname{Re}(\beta_r^+ + \beta_r^-) < 1/p$;
(ii) $-1/2 - 1/2q < \operatorname{Re}\beta^+ < 1/2p$ *and* $-1/2q < \operatorname{Re}\beta^- < 1/2 + 1/2p$.
Then ψ admits a weak asymmetric factorization in $L^p(\mathbb{T})$ with the index $\kappa = 0$.

Proof. First observe that

$$\begin{aligned} t_{\beta,\theta_r}(t) &= (1 - tt_r^{-1})^\beta (1 - t_r t^{-1})^{-\beta} \\ &= [(1 - t_r t^{-1})^{-\beta}(1 - t^{-1}t_r^{-1})^{-\beta}][(1 - tt_r^{-1})^\beta (1 - t^{-1}t_r^{-1})^\beta]. \end{aligned}$$

Hence we can factor $\psi = \psi_- \psi_0$ with

$$\psi_- = (1 - t^{-1})^{-2\beta^+}(1 + t^{-1})^{-2\beta^-} \prod_{r=1}^{R} (1 - t_r t^{-1})^{-\beta_r^+ - \beta_r^-}(1 - t^{-1}t_r^{-1})^{-\beta_r^+ - \beta_r^-},$$

$$\psi_0 = |1 - t|^{2\beta^+}|1 + t|^{2\beta^-} \prod_{r=1}^{R} (1 - tt_r^{-1})^{\beta_r^+}(1 - t^{-1}t_r^{-1})^{\beta_r^+}(1 - tt_r)^{\beta_r^-}(1 - t_r t^{-1})^{\beta_r^-}.$$

Since $-2\operatorname{Re}\beta^+ > -1/p$, $1 - 2\operatorname{Re}\beta^- > -1/p$ and $-\operatorname{Re}\beta_r^+ - \operatorname{Re}\beta_r^- > -1/p$ we have

$$(1 + t^{-1})\psi_- \in \overline{H^p(\mathbb{T})},$$

and since $1+2\mathrm{Re}\,\beta^+ > -1/q$, $2\mathrm{Re}\,\beta^- > -1/q$ and $\mathrm{Re}\,\beta_r^+ + \mathrm{Re}\,\beta_r^- > -1/q$ we have

$$(1-t^{-1})\psi_-^{-1} \in \overline{H^q(\mathbb{T})}.$$

Obviously, the function ψ_0 is even and thus it easily follows that $|1-t|\varphi_0 \in L^q_{\mathrm{even}}(\mathbb{T})$ and $|1+t|\varphi_0^{-1} \in L^p_{\mathrm{even}}(\mathbb{T})$. ∎

Now we are prepared to established the invertibility criteria for the operators $M(\varphi)$.

THEOREM 7.4. *Suppose that φ has finitely many jump discontinuities. Then $M(\varphi)$ is invertible on $H^p(\mathbb{T})$ if and only if φ can be written in the form*

$$\varphi(\mathrm{e}^{\mathrm{i}\theta}) = b(\mathrm{e}^{\mathrm{i}\theta})t_{\beta^+}(\mathrm{e}^{\mathrm{i}\theta})t_{\beta^-}(\mathrm{e}^{\mathrm{i}(\theta-\pi)})\prod_{r=1}^{R} t_{\beta_r^+}(\mathrm{e}^{\mathrm{i}(\theta-\theta_r)})t_{\beta_r^-}(\mathrm{e}^{\mathrm{i}(\theta+\theta_r)}) \tag{7.17}$$

where b is a continuous nonvanishing function with winding number zero, $\theta_1,\dots,\theta_R \in (0,\pi)$ are distinct, and

(i) $-1/q < \mathrm{Re}\,(\beta_r^+ + \beta_r^-) < 1/p$;

(ii) $-1/2-1/2q < \mathrm{Re}\,\beta^+ < 1/2p$ *and* $-1/2q < \mathrm{Re}\,\beta^- < 1/2+1/2p$.

Proof. We begin with the "if" part of the proof. Assume that the parameters satisfy the above inequalities, let b be a nonvanishing continuous function with winding number zero, and let ψ be the function (7.16). Observe that $\varphi = b\psi$.

It follows from Theorem 7.2 that $M(\psi)$ is Fredholm. Hence, by Theorem 6.4, the function ψ admits an asymmetric factorization in $L^p(\mathbb{T})$. Moreover, we know from Proposition 7.3 that ψ admits a weak asymmetric factorization in $L^p(\mathbb{T})$ with $\kappa = 0$. Due to the uniqueness of (weak) asymmetric factorizations in the sense of Proposition 3.1, it follows that ψ admits an asymmetric factorization in $L^p(\mathbb{T})$ with $\kappa = 0$. Thus, again by Theorem 6.4, $M(\psi)$ is Fredholm with index zero.

It is well known that under the above assumptions on the function b the Toeplitz operator $T(b)$ is Fredholm with index zero and the Hankel operator $H(b)$ is compact. Hence $M(b) = T(b) + H(b)$ is also a Fredholm operator with index zero. Taking (2.3) into account, we have

$$M(\varphi) = M(b)M(\psi) + H(b)M(\widetilde{\psi}-\psi), \tag{7.18}$$

where the last term is compact. Thus $M(\varphi)$ is Fredholm and has index zero. Corollary 6.3 now implies that $M(\varphi)$ is invertible.

In order to justify the other direction we first note that if $M(\varphi)$ is invertible then φ satisfies the conditions (7.10)–(7.12) of Theorem 7.2. Hence it is possible to choose parameters $\beta^+,\beta^-,\beta_r^+,\beta_r^-$ in such a way that fulfill the conditions (i) and (ii) and such that

$$\frac{\varphi^-(1)}{\varphi^+(1)} = \exp(2\pi\mathrm{i}\beta^+), \quad \frac{\varphi^-(-1)}{\varphi^+(-1)} = \exp(2\pi\mathrm{i}\beta^-),$$
$$\frac{\varphi^-(t_r)}{\varphi^+(t_r)} = \exp(2\pi\mathrm{i}\beta_r^+), \quad \frac{\varphi^-(t_r^{-1})}{\varphi^+(t_r^{-1})} = \exp(2\pi\mathrm{i}\beta_r^-).$$

With those parameters we can represent φ in the form (7.17), where it follows that b is a continuous nonvanishing function. It remains to show that the winding number of b is zero.

By what we have proved in the "if" part of the theorem, we know that for the function ψ defined by (7.16) the operator $M(\psi)$ is invertible. We rely on (7.18) and can conclude that the Fredholm index of $M(b)$ is zero. Thus b has winding number zero. ∎

We end this section with one more application of the factorization theorem and the results obtained above. This is an index theorem and was previously derived in the case of $p = 2$ in [1]. Since the index theorem derivation here is analogous to the one found in [1] most of the details are not included.

Let us first describe the essential spectrum of $M(\varphi)$, i.e. the set of all $z \in \mathbb{C}$ for which $M(\varphi) - zI$ is not a Fredholm operator on $H^p(\mathbb{T})$. For $a, b \in \mathbb{C}$ and $\theta \in (0,1)$ introduce the arc

$$\mathcal{C}_\theta(a;b) = \{a,b\} \cup \Big\{ z \in \mathbb{C} \setminus \{a,b\} : \frac{1}{2\pi} \arg\Big(\frac{z-b}{z-a}\Big) = \theta \Big\} \tag{7.19}$$

with endpoints a and b, which degenerates to the line segment $[a,b]$ in the case $\theta = 1/2$. Note that $\mathcal{C}_\theta(a;a) = \{a\}$. Moreover, for $\theta \in (0,1)$ we define the set

$$\begin{aligned} \mathcal{H}_\theta(a_1,a_2;b_1,b_2) &= \{a_1,a_2,b_1,b_2\} \\ &\cup \Big\{ z \in \mathbb{C} \setminus \{a_1,a_2,b_1,b_2\} : \frac{1}{2\pi} \arg\Big(\frac{(z-b_1)(z-b_2)}{(z-a_1)(z-a_2)}\Big) = \theta \Big\}. \end{aligned} \tag{7.20}$$

Note that $\mathcal{H}_\theta(a_1,a_2;b_1,b_2)$ is symmetric in a_1 and a_2 as well as in b_1 and b_2. In case of coinciding parameters we simply have $\mathcal{H}_\theta(a_1,c;b_1,c) = \mathcal{C}_\theta(a_1,b_1) \cup \{c\}$. In general $\mathcal{H}_\theta$ is more complicated. If one parameterizes $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, then $\mathcal{H}_\theta(a_1,a_2;b_1,b_2)$ is a certain subset of the following curve in x and y:

$$\operatorname{Im}\big(\mathrm{e}^{2\pi \mathrm{i}\theta} A(x+\mathrm{i}y)\overline{B(x+\mathrm{i}y)}\big) = 0, \tag{7.21}$$

where $A(z) = (z-a_1)(z-a_2)$ and $B(z) = (z-b_1)(z-b_2)$. The curve (7.20) is a (special) fourth degree curve except in the case $\theta = 1/2$, where it is a cubic curve. The reader can easily verify that the "geometric form" of $\mathcal{H}_\theta$ may be quite different.

COROLLARY 7.5. *Let $\varphi \in PC$. Then the essential spectrum of $M(\varphi)$ with respect to $H^p(\mathbb{T})$ is equal to*

$$\begin{aligned} &\bigcup_{\tau \in \mathbb{T}_+} \mathcal{H}_{1-1/p}(\varphi^-(\tau), \varphi^-(\overline{\tau}); \varphi^+(\tau), \varphi^+(\overline{\tau})) \\ &\qquad \cup \mathcal{C}_{1-1/2p}(\varphi^-(1); \varphi^+(1)) \ \cup\ \mathcal{C}_{1/2-1/2p}(\varphi^-(-1); \varphi^+(-1)). \end{aligned}$$

Since the essential range of a function $\varphi \in PC$, i.e. the spectrum of φ as an element of $L^\infty(\mathbb{T})$, is $\{\ \varphi^\pm(\tau)\ :\ \tau \in \mathbb{T}\ \}$, it is always contained in the essential spectrum of the operator $M(\varphi)$ with respect to $H^p(\mathbb{T})$. In general this inclusion is proper.

In what follows we give a geometric description of the spectrum and essential spectrum of $M(\varphi)$. We will assume for simplicity that φ has only finitely many jumps and is piecewise smooth. The general case could be treated with the same ideas, but would require more detailed explanations.

First of all, the values of $\varphi(t)$ as t runs along $\mathbb{T}$ (in positive orientation) describe a certain curve, which inherits the natural orientation induced from $\mathbb{T}$.

This curve is closed if φ is continuous, or consists of several components if φ has jumps.

Now we fill in certain additional pieces as follows. If φ is discontinuous at $\tau \in \{-1, 1\}$, then we add the arcs $\mathcal{C}_{1-1/2p}(\varphi^-(1); \varphi^+(1))$ and $\mathcal{C}_{1/2-1/2p}(\varphi^-(-1); \varphi^+(-1))$, respectively. If φ is discontinuous at $\tau \in \mathbb{T} \setminus \{-1, 1\}$, but is continuous at $\overline{\tau}$, then we fill in the arc (or line segment) $\mathcal{C}_{1-1/p}(\varphi^-(\tau); \varphi^+(\tau))$. Finally, if φ has jumps at both $\tau, \overline{\tau} \in \mathbb{T}\setminus\{-1,1\}$, then we add the curve $\mathcal{H}_{1-1/p}(\varphi^-(\tau), \varphi^-(\overline{\tau}); \varphi^+(\tau), \varphi^+(\overline{\tau}))$. These new pieces constitute themselves oriented curves.

Gluing together these pieces with the former curve we obtain an oriented curve $\varphi^{\#}$. By Corollary 7.5, the image of $\varphi^{\#}$ is exactly $\operatorname{sp}_{\mathrm{ess}} M(\varphi)$. We remark that the curve $\varphi^{\#}$ may consist of several component.

Because $\varphi^{\#}$ possesses an orientation, it is possible to associate to each point $z \notin \operatorname{Im} \varphi^{\#}$ a winding number $\operatorname{wind}(\varphi^{\#}, z)$. If $\varphi^{\#}$ consists of several components, this is just the sum of the usual winding numbers with respect to these components.

These considerations allow us to describe the spectrum of $M(\varphi)$:

$$\operatorname{sp} M(\varphi) = \operatorname{Im} \varphi^{\#} \cup \{\, z \notin \operatorname{Im} \varphi^{\#} : \operatorname{wind}(\varphi^{\#}, z) = 0\}. \tag{7.22}$$

The proof can be carried out by continuously deforming φ. We leave the details to the reader.

The following figures give an example of the image of a piecewise continuous curve φ and the (essential) spectrum of $M(\varphi)$ with respect to $H^p(\mathbb{T})$ for various values of p. The example comes from a product of four of the standard t_β functions, with jumps at $1, -1, \mathrm{i}$ and $-\mathrm{i}$ and values of β equal to $1/3, -3/4, 5/12$ and $1/3$ respectively and a normalizing factor of $-\mathrm{e}^{-\mathrm{i}\pi/24}$ for picture purposes only. Figures 2–6 show the curve $\varphi^{\#}$ for several values of p, or equivalently, the essential spectrum of $M(\varphi)$. Form this, the spectrum of $M(\varphi)$ can be determined.

In this example, the spectrum for the various values of p consists of the area bounded by the curves in Fig. 2–6. As one can see the spectrum as well as the essential spectrum of this operator is not always connected contrasted to the Toeplitz case.

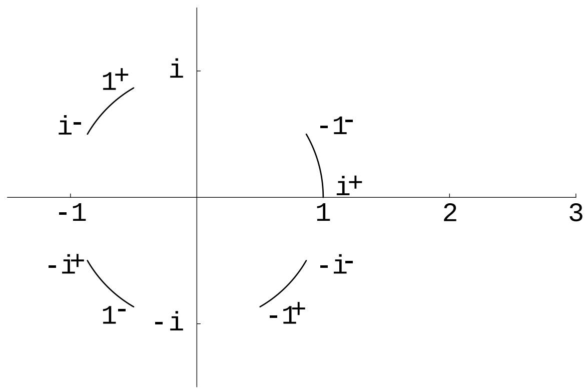

Fig. 1: Image of φ

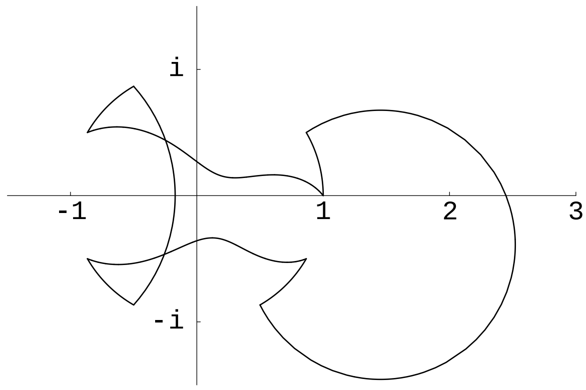

Fig. 2: $p = 1.3$

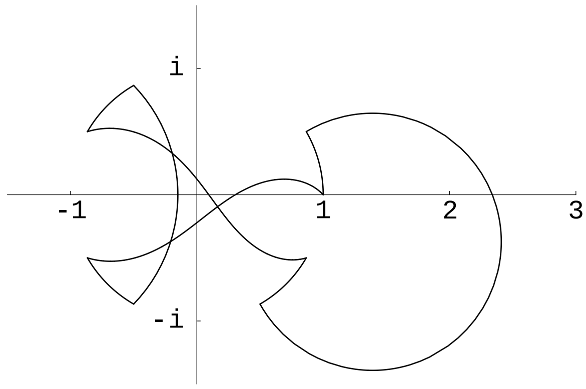

Fig. 3: $p = 1.32324434438365901$

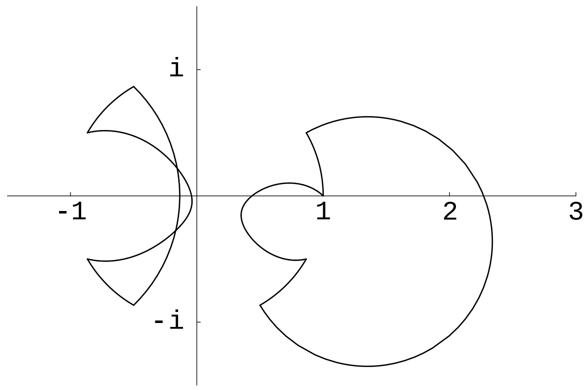

Fig. 4: $p = 1.34$

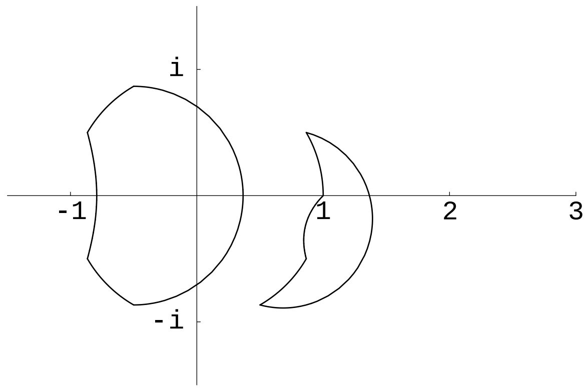

Fig. 5: $p = 2$

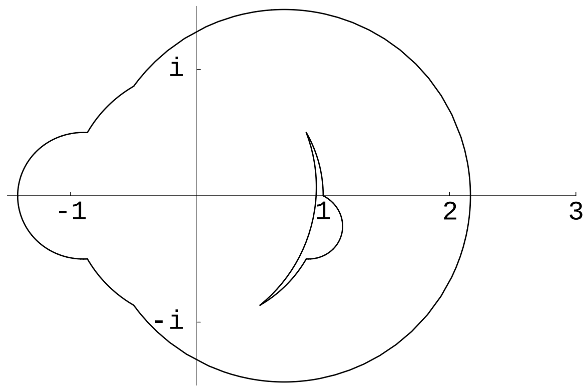

Fig. 6: $p = 5$

The authors have studied other equivalent conditions for invertibility of $M(\varphi)$ which generalize the idea of A_p-conditions for weighted spaces. These results are forth-coming and will be contained in the sequel to this paper.

Acknowledgements. The first author research was supported in part by NSF Grant DMS-9970879.

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Received June 17, 2002; revised February 24, 2003.