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A COMPARISON BETWEEN THE MAX AND MIN NORMS ON $C^*(F_n) \otimes C^*(F_n)$

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ABSTRACT. Let F_n, $n \geqslant 2$, be the free group on n generators, denoted by $U_1, U_2, \dots, U_n$. Let $C^*(F_n)$ be the full C^*-algebra of F_n. Let $\mathcal{X}$ be the vector subspace of the algebraic tensor product $C^*(F_n) \otimes C^*(F_n)$, spanned by $1 \otimes 1, U_1 \otimes 1, \dots, U_n \otimes 1, 1 \otimes U_1, \dots, 1 \otimes U_n$. Let $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ be the minimal and maximal C^* tensor norms on $C^*(F_n) \otimes C^*(F_n)$, and use the same notation for the corresponding (matrix) norms induced on $M_k(\mathbb{C}) \otimes \mathcal{X}$, $k \in \mathbb{N}$.

Identifying $\mathcal{X}$ with the subspace of $C^*(F_{2n})$ obtained by mapping $U_1 \otimes 1, \dots, 1 \otimes U_n$ into the $2n$ generators and the identity into the identity, we get a matrix norm $\|\cdot\|_{C^*(F_{2n})}$ which dominates the $\|\cdot\|_{\max}$ norm on $M_k(\mathbb{C}) \otimes \mathcal{X}$.

In this paper we prove that, with $N = 2n + 1 = \dim \mathcal{X}$, we have

$$\|X\|_{\max} \leqslant \|X\|_{C^*(F_{2n})} \leqslant (N^2 - N)^{1/2} \|X\|_{\min}, \quad X \in M_k(\mathbb{C}) \otimes \mathcal{X}.$$

KEYWORDS: *Connes's embedding conjecture, minimal tensor norm.*

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Let F_n be the free group on n generators, $n \geqslant 2$. Let $C^*(F_n)$ be the full C^*-algebra associated with F_n (see, e.g., [13]). As proved in [5] and [11], on the algebraic tensor product $C^*(F_n) \otimes C^*(F_n)$ there exist a maximal and a minimal C^*-algebra tensor norm, denoted by $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ respectively. Kirchberg, in [7], has revived the study of the C^*-tensor norms on $A \otimes A^{\mathrm{op}}$. One particular case of his very deep results shows that the equality of the two norms on $C^*(F_\infty) \otimes C^*(F_\infty)$ is equivalent to Connes's embedding problem ([4]).

In [10], it is proven that if E is a subspace of the algebraic tensor product $A_1 \otimes A_2$ of two C^*-algebras A_1 and A_2, which has a basis consisting of unitaries that generate (as an algebra) $A_1 \otimes A_2$, then the complete isometry of the operator-space structures induced on E by the max and min norms implies the equality of the $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ norms on $A_1 \otimes A_2$. This method is then used in [10] to re-prove (and generalize) Kirchberg's theorem that $C^*(F_n) \otimes_{\max} B(H) = C^*(F_n) \otimes_{\min} B(H)$.

In this paper we consider $\mathcal{X}$ the $N = 2n+1$-dimensional subspace of $C^*(F_n) \otimes C^*(F_n)$ generated by $\{1\otimes 1, U_1 \otimes 1, \dots, U_n \otimes 1, 1 \otimes U_1, \dots, 1 \otimes U_n\}$. This space inherits operator-space structures ([2], [6], [9]) corresponding to the two embeddings. We denote the corresponding norms on $\mathcal{X} \otimes M_k(\mathbb{C})$, for all k in $\mathbb{N}$, by $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$.

We prove that the norm $\|\cdot\|_{\min}$ dominates the $\|\cdot\|_{\max}$ norm, on all the tensor products in $\mathcal{X} \otimes M_k(\mathbb{C})$, $k \in \mathbb{N}$, by a factor $(N^2 - N)^{1/2}$, where $N = 2n+1$. More precisely, we prove that

$$\|X\|_{\max} \leqslant (N^2 - N)^{1/2} \|X\|_{\min}, \quad X \in M_k(\mathbb{C}) \otimes \mathcal{X}.$$

In particular, our result, in the terminology introduced by Pisier ([9]), also shows that the δ_{cb} (multiplicative) distance between the two N-dimensional operator spaces in $C^*(F_n) \otimes C^*(F_n)$, corresponding to the norms $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$, is at most $(N^2 - N)^{1/2}$ (in general ([9]) the δ_{cb} distance between two finite-dimensional operator spaces of dimension N is bounded by N).

1. DEFINITIONS

Let $k \in \mathbb{N}$ be a natural number and let $(e_{a,b})_{a,b=1}^k$ be a matrix unit in $M_k(\mathbb{C})$. Let $W_1, W_2, \dots, W_{2n}$ be the generators of F_{2n} and let $W_0 = \mathrm{Id}$. Let $\mathcal{X}$ be the subspace of $C^*(F_{2n})$ spanned by $W_0, W_1, \dots, W_{2n}$. Let X be an arbitrary element of $\mathcal{X}$. Then X^*X has the form

$$\sum_{a,b=1}^{k} \Big(\sum_{i \neq j} A_{ia,jb} W_i^* W_j + B_{a,b} \mathrm{Id} \Big) \otimes e_{a,b}.$$

The norm $\|X\|_{C^*(F_{2n})}$ for X in $C^*(F_{2n})$ is computed ([13], [2]) as the supremum over all Hilbert spaces H and all unitaries $U_1, U_2, \dots, U_{2n}$ acting on H, and all $\xi = \bigoplus_{a=1}^{k} \xi_a$, $\sum \|\xi_a\|^2 = 1$, in $H \oplus \cdots \oplus H$ (k times), of the quantity

$$\langle X^*X\xi, \xi\rangle = \sum_{a,b=1}^{k} \Big(\sum_{i \neq j} A_{ia,jb} \langle W_i^* W_j \xi_a, \xi_b\rangle + B_{a,b} \langle \xi_a, \xi_b\rangle \Big). \tag{1.1}$$

Since $C^*(F_{2n})$ is residually finite [3] (see also [13], [1]), it follows that the norm of X^*X might be computed using only finite-dimensional unitaries.

Let $\widetilde{V}_1, \dots, \widetilde{V}_n$ be the generators of a different copy of the free group F_n. We identify $\mathcal{X}$ with a subspace of the algebraic tensor product $C^*(F_n) \otimes C^*(F_n)$ by mapping 1 into $1 \otimes 1$, and W_i into $\widetilde{V}_i \otimes 1$ for $i = 1, 2, \dots, n$, W_{i+n} into $1 \otimes \widetilde{V}_i$ for $i = 1, 2, \dots, n$. With this identification, and by using again the fact that $C^*(F_n)$ is residually finite, it follows that the norm $\|X\|_{\max}$ viewed as an element of $(C^*(F_n) \otimes_{\max} C^*(F_n)) \otimes M_k(\mathbb{C})$ is computed by the same supremum as the one used for $\|X\|_{C^*(F_{2n})}$, with the additional restriction on the unitaries $U_1, \dots, U_{2n}$, that for $1 \leqslant i \leqslant n < j \leqslant 2n$, we have $[U_i, U_j] = 0$.

Clearly this gives (as in [2]) that $\|X\|_{C^*(F_{2n})} \geqslant \|X\|_{C^*(F_n)\otimes_{\max} C^*(F_n)}$. The norm $\|X\|_{\min}$ for X in $\mathcal{X} \otimes M_k(\mathbb{C})$, viewed as an element in $C^*(F_n) \otimes_{\min} C^*(F_n) \otimes M_k(\mathbb{C})$, is then computed by the same supremum formulas as for $\|X\|_{\max}$, by imposing the additional condition that the Hilbert space H splits as $K_1 \otimes K_2$ and there exist unitaries $\alpha_1, \dots, \alpha_n$ acting on K_1, and $\beta_1, \dots, \beta_n$ unitaries on K_2, such that $U_i = \alpha_i \otimes 1$ and $U_{i+n} = 1 \otimes \beta_i$ for $1 \leqslant i \leqslant n$ (see also [12]). Motivated by this we introduce the following definition:

Definition 1.1. A triplet $(H, (U_i)_{i=1}^{2n}, (\eta_a)_{a=1}^k)$ consisting of a Hilbert space H, unitaries $(U_i)_{i=1}^{2n}$ acting on H and vectors $(\eta_a)_{a=1}^k$ is called *in tensor position* if there exist a Hilbert space K, unitaries $\widetilde{U}_1, \dots, \widetilde{U}_n$, $\widetilde{V}_1, \dots, \widetilde{V}_n$ on K, vectors $(\widetilde{\eta}_a)_{a=1}^k$ in $K \otimes K$ with the following properties. Denote $\widetilde{W}_i = \widetilde{U}_i \otimes \mathrm{Id}_K$ for $1 \leqslant i \leqslant n$ and $\widetilde{W}_{i+n} = \mathrm{Id}_K \otimes \widetilde{V}_i$, $1 \leqslant i \leqslant n$. Also denote $U_0 = \mathrm{Id}_H$, $\widetilde{W}_0 = \mathrm{Id}_{K\otimes K}$. With these notations the following should hold true for $0 \leqslant i, j \leqslant n$, $1 \leqslant a, b \leqslant k$:

$$\langle U_i \eta_a, U_j \eta_b \rangle = \langle \widetilde{W}_i \widetilde{\eta}_a, \widetilde{W}_j \widetilde{\eta}_b \rangle.$$

2. Main result

Our main result gives a comparison between the norms $\|\cdot\|_{C^*(F_{2n})}$ and $\|\cdot\|_{\min}$ on the space $\mathcal{X}$ (and its tensor products $\mathcal{X} \otimes M_k(\mathbb{C})$). To do this we use the fact that, for any triplet $(H, (U_i)_{i=1}^{2n}, (\xi_a)_{a=1}^k)$, $U_0 = \mathrm{Id}$, the information contained in the matrix $\langle U_i \xi_a, U_j \xi_b \rangle$, $0 \leqslant i, j \leqslant 2n$, is unchanged (except for the Gram–Schmidt matrix of ξ_a) if we replace H, U_i and ξ_a by a direct sum and linear combinations of elementary triplets $(H^\alpha, (U_i^\alpha)_{i=1}^{2n}, (\xi_a^\alpha)_{a=1}^k)$ having the property that the vectors $\{U_i^\alpha \xi_a^\alpha\}_{i,a}$ are an orthonormal system (with the exception of some repetitions). The following lemma is an obvious property for triplets as in Definition 1.1:

Lemma 2.1. *Let Λ be a countable index set. Assume the following triplets $(H^\alpha, (U_i^\alpha)_{i=1}^{2n}, (\eta_a^\alpha)_{a=1}^k)_{\alpha\in\Lambda}$ are in tensor position. Let $(\mu_a^\alpha)_{a=1,\,\alpha\in\Lambda}^k$ be arbitrary complex numbers such that $\sum\limits_\alpha |\mu_a^\alpha|^2 \|\eta_a^\alpha\|^2 < \infty$ for all a. Let $H = \bigoplus\limits_{\alpha\in\Lambda} H^\alpha$, let $U_i = \bigoplus\limits_\alpha U_i^\alpha$ and $\eta_a = \bigoplus\limits_\alpha \mu_a^\alpha \eta_a^\alpha$.*

Then the triplet $(H, (U_i)_{i=1}^{2n}, (\eta_a)_{a=1}^k)$ is in tensor position.

Proof. For each $\alpha \in \Lambda$, use the definition of tensor position to find a Hilbert space K^α and unitaries $\widetilde{W}_i^\alpha = \widetilde{U}_i^\alpha \otimes \mathrm{Id}_{K^\alpha}$, $1 \leqslant i \leqslant n$, $W_{i+n}^\alpha = \mathrm{Id}_{K^\alpha} \otimes \widetilde{V}_i^\alpha$ as in Definition 1.1. Let $\widetilde{K} = \bigoplus\limits_\alpha K^\alpha$ and $\widetilde{H} = \widetilde{K} \otimes \widetilde{K} \supseteq \bigoplus\limits_\alpha K^\alpha \otimes K^\alpha$. Let $\widetilde{U}_i = \bigoplus\limits_\alpha \widetilde{U}_i^\alpha$, $\widetilde{V}_i = \bigoplus\limits_\alpha \widetilde{V}_i^\alpha$ and $\widetilde{W}_i = \widetilde{U}_i \otimes \mathrm{Id}_{\widetilde{K}}$, $1 \leqslant i \leqslant n$, $\widetilde{W}_{i+n} = \mathrm{Id}_{\widetilde{K}} \otimes \widetilde{V}_i$, $1 \leqslant i \leqslant n$, $\widetilde{\eta}_a = \bigoplus\limits_\alpha \mu_a^\alpha \widetilde{\eta}_a^\alpha$. Then the triplet $(\widetilde{H}, (\widetilde{W}_i)_{i=1}^{2n}, (\widetilde{\eta}_a)_{a=1}^k)$ has the property that

$$\langle U_i \eta_a, U_j \eta_b \rangle = \langle \widetilde{W}_i \widetilde{\eta}_a, \widetilde{W}_j \widetilde{\eta}_b \rangle$$

for all $0 \leqslant i, j \leqslant 2n$, $a, b = 1, 2, \dots, k$, and hence it is in tensor position. ∎

DEFINITION 2.2. For a triplet $(H,(U_i)_{i=1}^{2n},(\eta_a)_{a=1}^{k})$ with $U_0=\mathrm{Id}$, the *associated matrix* will be $X^U_{ia,jb}=X_{ia,jb}=\langle U_i\eta_a,U_j\eta_b\rangle$ for $0\leqslant i,j\leqslant 2n$, $a,b=1,2,\ldots,k$.

Clearly $X_{ia,ib}=\langle\eta_a,\eta_b\rangle$ for all i and all a,b. Also $X_{ia,jb}=\overline{X_{jb,ia}}$ by definition.

REMARK 2.3. The property in the definition of a triplet in tensor position is completely contained in the information the matrix X.

Moreover, with the notation in Lemma 2.1, if X is the matrix for the triplet $(H,(U_i)_{i=1}^{2n},(\eta_a)_{a=1}^{k})$ and X^α is the matrix for the triplets $(H^\alpha,(U_i^\alpha)_{i=1}^{2n},(\eta_a^\alpha)_{a=1}^{k})$, then we have

$$X_{ia,jb}=\sum_\alpha \mu_a^\alpha\,\overline{\mu_b^\alpha}\,X^\alpha_{ia,jb}.$$

It is easy to construct elementary triplets in tensor position.

LEMMA 2.4. *Let H be a separable Hilbert space. Let ε be a complex number of absolute value 1. Let n,k be strictly positive integers. Fix a vector η in H of length 1. Let $\eta_a=\eta$ for $a=1,\ldots,k$. Let $\alpha=(i_0,j_0)$, with $i_0,j_0\in\{0,1,\ldots,2n\}$, $i_0\neq j_0$. Assume $(U_i)_{i=1}^{2n}$ are unitaries such that*

$$\overline{\varepsilon}U_{i_0}\eta_a=\overline{\varepsilon}U_{i_0}\eta=U_{j_0}\eta_a=U_{j_0}\eta$$

and such that the vectors

$$\overline{\varepsilon}U_{i_0}\eta=U_{j_0}\eta,\quad \{U_k\eta : k\neq i_0,j_0\}$$

are pairwise orthogonal.

Then $(H,(U_i)_{i=1}^{2n},(\eta_a)_{a=1}^{k})$ is in tensor position, and the associated matrix is, for $0\leqslant i,j\leqslant 2n$, $1\leqslant a,b\leqslant k$,

$$\begin{aligned} &X^{\alpha,\varepsilon}_{ia,jb}=1 && \text{if } i=j,\\ &X^{\alpha,\varepsilon}_{i_0a,j_0b}=\varepsilon, && X^{\alpha,\varepsilon}_{j_0a,i_0b}=\overline{\varepsilon},\\ &X^{\alpha,\varepsilon}_{ia,jb}=0 && \text{if } i \text{ or } j \text{ are not in } \{i_0,j_0\} \text{ and } i\neq j. \end{aligned}$$

Proof. It is obvious that this should be the formula for the matrix $X^{\alpha,\varepsilon}$ associated to the triplet.

We need to construct a specific triplet in tensor position, which gives the matrix $X^{\alpha,\varepsilon}$. To do this we split into two cases.

First we analyze the case where $0\leqslant i_0\leqslant n$ and $n<j_0\leqslant 2n$. In this case consider a Hilbert space K of sufficiently large dimension. Let $e_0,e_1,\ldots$ be a basis for this Hilbert space and let η be the vector $e_0\otimes e_0$. With the notation from Definition 1.1, let $\widetilde{W}_{i_0}=\mathrm{Id}\otimes\mathrm{Id}$, $\widetilde{W}_{j_0}=\overline{\varepsilon}\mathrm{Id}\otimes\mathrm{Id}$ (which corresponds to the choice $\widetilde{U}_{i_0}=\mathrm{Id}$, $\widetilde{V}_{j_0-n}=\overline{\varepsilon}\mathrm{Id}$).

For $i\neq i_0$, $i=0,1,\ldots,n$, let $\widetilde{U}_i$ be a unitary on K, such that $\{\widetilde{U}_ie_0\}_{i\neq i_0}$ and e_0 is an orthonormal system in K. (For example we can send $\widetilde{U}_ie_0$ to other elements in the basis.) Likewise, we choose $\widetilde{V}_je_0$ such that $\{\widetilde{V}_je_0\}_{j\neq j_0-n}$ and e_0 is an orthonormal system. It is obvious now that the unitaries $(\widetilde{U}_i)_{i=1}^{n}$, $(\widetilde{V}_j)_{j=1}^{n}$ form a triplet in tensor position as in the statement of Lemma 2.4.

The case $0\leqslant i_0<j_0\leqslant n$ is easier and may be treated similarly. ∎

In the next lemma we provide a decomposition of an arbitrary triplet $(H,(U_i)_{i=1}^{2n},(\eta_a)_{a=1}^{k})$, with H finite-dimensional, into elementary triplets as in Lemma 2.4. The drawback out this construction is that in the decomposition of $(H,(U_i)_{i=1}^{2n},(\eta_a)_{a=1}^{k})$, the vectors in the triplet have greater length (by a factor of $(N^2-N)^{1/2}$, with $N=2n+1$).

Lemma 2.5. *Let H be a finite-dimensional vector space. Let n,k be strictly positive integer numbers. Let $U_0=\mathrm{Id}$, $U_1,\dots,U_{2n}$ be unitaries on H, and let $(\xi_a)_{a=1}^{k}$ be vectors in H.*

Then there exists a triplet $(\widetilde{K},(\widetilde{U}_i)_{i=1}^{2n},(\widetilde{\eta}_a)_{a=1}^{k})$ in tensor position, such that (with $N=2n+1$) we have:

(i) $\langle U_i\xi_a,U_j\xi_b\rangle=\langle \widetilde{U}_i\widetilde{\eta}_a,\widetilde{U}_j\widetilde{\eta}_b\rangle$, $i\neq j$;

(ii) $\langle \widetilde{\eta}_a,\widetilde{\eta}_b\rangle=(N^2-N)\langle\xi_a,\xi_b\rangle$;

for all $a,b=1,2,\dots,k$ and for all $i,j=0,1,\dots,2n$ (and $i\neq j$).

Proof. Let $(e_t)_{t\in T}$ be an orthonormal basis for H and let $\lambda_{i,a}^t$ be the components of the vector $U_i\xi_a$ in this basis for $i=0,1,\dots,2n$, $a=1,\dots,k$, $t\in T$. Then we have that

$$\langle U_i\xi_a,U_j\xi_b\rangle=\sum_t \lambda_{i,a}^t\,\overline{\lambda_{j,b}^t}\,,\quad i,j=0,1,\dots,2n,\ a,b=1,\dots,k. \tag{2.1}$$

The usual factorization formula ([8]) gives, with $\varepsilon=\sqrt{-1}$, that for all $i,j=0,1,\dots,2n$ and for all $a,b=1,\dots,k$ we have that

$$\lambda_{i,a}^t\,\overline{\lambda_{j,b}^t}=\frac14\sum_{s=0}^{3}\varepsilon^s(\lambda_{i,a}^t+\varepsilon^s\lambda_{j,a}^t)\,\overline{(\lambda_{i,b}^t+\varepsilon^s\lambda_{j,b}^t)}\,. \tag{2.2}$$

Note also that the following holds:

$$\frac14\sum_{s=0}^{3}(\lambda_{i,a}^t+\varepsilon^s\lambda_{j,a}^t)\,\overline{(\lambda_{i,b}^t+\varepsilon^s\lambda_{j,b}^t)}=\lambda_{i,a}^t\,\overline{\lambda_{i,b}^t}+\lambda_{j,a}^t\,\overline{\lambda_{j,b}^t}\,. \tag{2.3}$$

For a given pair $\alpha=(i,j)$, $0\leqslant i<j\leqslant n$, $a,b=1,\dots,k$, $t\in T$, and $s=0,1,2,3$, we let

$$\theta_{\alpha,a}^{t,s}=\lambda_{i,a}^t+\varepsilon^s\lambda_{j,a}^t.$$

With these notations the relations (2.2) and (2.3) become respectively

$$\langle U_i\xi_a,U_j\xi_b\rangle=\sum_t\lambda_{i,a}^t\lambda_{j,b}^t=\sum_{t,s}\varepsilon^s\theta_{\alpha,a}^{t,s}\,\overline{\theta_{\alpha,b}^{t,s}}, \tag{2.4}$$

$$\begin{aligned}\sum_{t,s}\theta_{\alpha,a}^{t,s}\,\overline{\theta_{\alpha,b}^{t,s}}&=\sum_t\lambda_{i,a}^t\,\overline{\lambda_{i,b}^t}+\sum_t\lambda_{j,a}^t\,\overline{\lambda_{j,b}^t}\\&=\langle U_i\xi_a,U_i\xi_b\rangle+\langle U_j\xi_a,U_j\xi_b\rangle=2\langle\xi_a,\xi_b\rangle.\end{aligned} \tag{2.5}$$

The relations (2.4) and (2.5) hold for all $0\leqslant i<j\leqslant 2n$, and all $a,b=1,2,\dots,k$.

For each fixed $t\in T$, $\alpha=(i_0,j_0)$, $0\leqslant i_0<j_0\leqslant 2n$, and each $s=0,1,2,3$, let $(H^{\alpha,s,t},(U_i^{\alpha,s,t})_{i=1}^{2n},(\eta_a^{\alpha,s,t})_{a=1}^{k})$ be the triplet constructed in Lemma 2.4 for

$\varepsilon = \varepsilon^s$. (This triplet does not depend on t, but for each t we consider one copy.) The matrix associated to this triplet is defined by

$$
(2.6)\qquad
\begin{aligned}
&X^{\alpha,s,t}_{ia,jb} = 0 \qquad \text{if } \{i,j\} \not\subseteq \{i_0, j_0\} \text{ and } i \neq j,\\
&X^{\alpha,s,t}_{ia,ib} = 1,\\
&X^{\alpha,s,t}_{i_0 a, j_0 b} = \varepsilon^s, \quad X^{\alpha,s,t}_{j_0 a, i_0 b} = \overline{\varepsilon^s} \quad \text{for all } a, b = 1, 2, \ldots, k.
\end{aligned}
$$

Let Λ be the set of pairs

$$\Lambda = \{(i,j) : 0 \leqslant i < j \leqslant 2n\}.$$

Let $\mu_a^{\alpha,s,t} = \theta^{s,t}_{\alpha,a}$ for all $\alpha \in \Lambda$, $s = 0,1,2,3$, $t \in T$. We apply Lemma 2.1 (and the Remark 2.3) to the direct sum of the triplets $(H^{\alpha,s,t}, (U_i^{\alpha,s,t})_{i=1}^{2n}, (\eta_a^{\alpha,s,t})_{a=1}^{k})$. In the direct sum $\widetilde{H} = \bigoplus\limits_{\alpha,s,t} H^{\alpha,s,t}$, $\widetilde{U}_i = \bigoplus\limits_{\alpha,s,t} U_i^{\alpha,s,t}$, $i = 1,2,\ldots,2n$, we consider the vectors $\widetilde{\eta}_a = \bigoplus\limits_{\alpha,s,t} \mu_a^{\alpha,s,t}\eta_a^{\alpha,s,t}$.

By Lemma 2.1, for fixed $i_0 < j_0$, $a, b = 1,2,\ldots,k$, we have

$$\langle \widetilde{U}_{i_0}\widetilde{\eta}_a, U_{j_0}\widetilde{\eta}_b\rangle = \sum_{\alpha,s,t} \mu_a^{\alpha,s,t}\mu_b^{\alpha,s,t} X^{\alpha,s,t}_{i_0 a, j_0 b}.$$

By the relation (2.6), and since $i_0 < j_0$, an entry in the matrix $X^{\alpha,s,t}_{i_0 a, j_0 b}$ is nonzero only when α is equal to (i_0, j_0), and is equal in this case to ε^s. Thus, with $\alpha_0 = (i_0, j_0)$ and using the relation (2.4), we obtain

$$
(2.7)\qquad \langle \widetilde{U}_{i_0}\eta_a, \widetilde{U}_{j_0}\eta_b\rangle = \sum_{s,t} \varepsilon^s \mu_a^{\alpha_0,s,t}\mu_b^{\alpha_0,s,t} = \sum_{s,t} \varepsilon^s \theta^{t,s}_{\alpha_0,a}\,\overline{\theta^{t,s}_{\alpha_0,b}} = \langle U_{i_0}\xi_a, U_{j_0}\xi_b\rangle
$$

for all $a, b = 1,\ldots,k$.

Since also $\langle U_{j_0}\xi_b, U_{i_0}\xi_a\rangle = \overline{\langle U_{i_0}\xi_a, U_{j_0}\xi_b\rangle}$ and similarly for $\widetilde{U}_i\widetilde{\eta}_a$, it follows that relation (2.7) holds for all $i_0 \neq j_0$, $0 \leqslant i_0, j_0 \leqslant 2n$.

Similar computations yield the value of $\langle \widetilde{\eta}_a, \widetilde{\eta}_b\rangle$. Indeed, by the relation (2.5) we have

$$
\begin{aligned}
\langle \widetilde{\eta}_a, \widetilde{\eta}_b\rangle &= \sum_{\alpha,s,t} \mu_a^{\alpha,s,t}\,\overline{\mu_b^{\alpha,s,t}} = \sum_{\alpha\in\Lambda}\sum_{s,t} \theta^{t,s}_{\alpha,a}\,\overline{\theta^{t,s}_{\alpha,b}}\\
&= \sum_{\alpha\in\Lambda} 2\langle \xi_a, \xi_b\rangle = \frac{N^2 - N}{2}\cdot 2\langle \xi_a, \xi_b\rangle = (N^2 - N)\langle \xi_a, \xi_b\rangle.
\end{aligned}
$$

By Lemmas 2.1 and 2.4, the triplet $(\widetilde{H}, (\widetilde{U}_i)_{i=1}^{2n}, (\widetilde{\eta}_a)_{a=1}^{k})$ is in tensor position. This completes the proof of Lemma 2.5. ∎

We now can prove the main result. We will show that on $\mathcal{X} = \mathrm{Sp}\{1\otimes 1, U_1 \otimes 1, \ldots, U_n \otimes 1, 1\otimes U_1, \ldots, 1\otimes U_n\}$, the matrix norm structures induced by the norms $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ on $C^*(F_n)\otimes C^*(F_n)$ are comparable by a factor $(N^2 - N)^{1/2}$.

In particular this shows (in the terminology introduced in [9]) that the δ_{cb} multiplicative distance between the two operator spaces is less than $(N^2 - N)^{1/2}$. (By [9], this distance is at most N.)

THEOREM 2.6. *Let n, k be integers, $n \geqslant 2$, $k \geqslant 1$. Let F_n be the free group on n generators $V_1, V_2, \ldots, V_n$. Consider the vector subspace $\mathcal{X}$ of $C^*(F_n) \otimes C^*(F_n)$ spanned by $\{1 \otimes 1, V_1 \otimes 1, \ldots, V_n \otimes 1, 1 \otimes V_1, \ldots, 1 \otimes V_n\}$. Clearly $\mathcal{X}$ has dimension $N = 2n + 1$.*

By embedding $\mathcal{X}$ into $C^*(F_n) \otimes_{\min} C^*(F_n)$ or $C^*(F_n) \otimes_{\max} C^*(F_n)$ respectively, we get two corresponding norms on $\mathcal{X} \otimes M_k(\mathbb{C})$, denoted by $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$.

Let F_{2n} be the free group on $2n$ generators $W_1, \ldots, W_{2n}$. We also identify $\mathcal{X}$ with a subspace of the full C^*-algebra $C^*(F_{2n})$ by mapping $1 \otimes 1$ into 1, and $V_1 \otimes 1, \ldots, V_n \otimes 1$ into $W_1, \ldots, W_n$, and $1 \otimes V_1, \ldots, 1 \otimes V_n$ into $W_{n+1}, \ldots, W_{2n}$, respectively. For X in $\mathcal{X} \otimes M_k(\mathbb{C})$ we denote the corresponding norm coming from this embedding by $\|X\|_{C^*(F_{2n})}$.

Then, for all X in $\mathcal{X} \otimes M_k(\mathbb{C})$, we have

$$\|X\|_{\min} \leqslant \|X\|_{\max} \leqslant \|X\|_{C^*(F_{2n})} \leqslant (N^2 - N)^{1/2} \|X\|_{\min}.$$

Proof. Let $(e_{a,b})_{a,b=1}^k$ be a matrix unit in $M_k(\mathbb{C})$ and let

$$X = \sum_{r,s=1}^{k} \sum_{i=0}^{2n} \lambda_{r,s}^i W_i \otimes e_{r,s}, \quad \lambda_{r,s}^i \in \mathbb{C}, \tag{2.8}$$

be an arbitrary element in $M_k(\mathbb{C}) \otimes \mathbb{C}$. (We denote by W_0 the identity.)

Then, obviously,

$$X^* X = \sum_{a,b=1}^{k} \Big(\sum_{\substack{i,j=0 \\ i \neq j}}^{2n} A_{ia,jb} W_i^* W_j + B_{a,b} \mathrm{Id} \Big) \otimes e_{a,b}, \tag{2.9}$$

where for $i \neq j$, $i, j = 0, \ldots, 2n$, $1 \leqslant a, b \leqslant k$, we have

$$A_{ia,jb} = \sum_{r=1}^{k} \lambda_{r,a}^i \overline{\lambda_{r,b}^i}, \tag{2.10}$$

$$B_{a,b} = \sum_{r=1}^{k} \sum_{i=0}^{2n} \lambda_{r,a}^i \, \overline{\lambda_{r,b}^i}. \tag{2.11}$$

Clearly the matrix $\sum_{a,b} B_{a,b} \otimes e_{a,b}$ is positive. By definition, the $C^*(F_{2n})$-norm of a noncommutative polynomial P in $\mathrm{Id}, W_1, \ldots, W_{2n}$ is computed by taking the supremum, over all unitaries $U_1, \ldots, U_{2n}$, of the norms of the operators obtained by replacing in P the unitaries W_i by U_i, $i = 1, 2, \ldots, 2n$.

By [3], $C^*(F_{2n})$ is residually finite ([13], [12], [1]), and hence we can restrict to a supremum over unitaries acting on finite-dimensional vector spaces.

As a consequence, the square of the $\|\cdot\|_{C^*(F_{2n})}$ norm of the element X is computed as the supremum, over all finite-dimensional Hilbert spaces H, all

$2n$-tuples of unitaries $U_1,\dots,U_{2n}$ acting on H, and all vectors $\xi=(\xi_a)_{a=1}^k$ in $H\oplus H\oplus\dots\oplus H$, $\sum\limits_{a=1}^k\|\xi_a\|^2=1$, of the quantities

$$\|X\xi\|^2=\langle X^*X\xi,\xi\rangle=\sum_{a,b}\Big(\sum_{\substack{i,j=0\\ i\neq j}}^{2n}A_{ia,jb}\langle U_j\xi_a,U_i\xi_b\rangle+B_{a,b}\langle\xi_a,\xi_b\rangle\Big).$$

Similarly, the norm $\|X\|_{\min}$ will be computed as the supremum of the same quantities, with the additional condition that the unitaries $U_1,\dots,U_{2n}$ are represented on a Hilbert space $H=K_1\otimes K_2$, and there are unitaries $\alpha_1,\dots,\alpha_n$, respectively $\beta_1,\dots,\beta_n$, on K_1, respectively K_2, such that $U_i=\alpha_i\otimes 1$, $U_{i+n}=1\otimes\beta_i$, $1\leqslant i\leqslant n$.

Hence, for every $\varepsilon>0$, there exists a triplet $(H,(U_i)_{i=1}^{2n},(\xi_a)_{a=1}^k)$ consisting of a finite-dimensional vector space, $2n$ unitaries on H and k vectors in H, such that, with $U_0=\mathrm{Id}$,

$$\|X^*X\|_{C^*(F_{2n})}-\varepsilon\leqslant\sum_{a,b=1}^k\Big(\sum_{\substack{i,j=0\\ i\neq j}}^{2n}A_{ia,jb}\langle U_j\xi_a,U_i\xi_b\rangle+B_{a,b}\langle\xi_a,\xi_b\rangle\Big).\tag{2.12}$$

By Lemma 2.5 we can find a triplet in tensor position, $(\widetilde H,(\widetilde U_i)_{i=1}^{2n},(\widetilde\eta_a)_{a=1}^k)$, consisting of unitaries $\widetilde U_i$ on $\widetilde H$, with $\widetilde U_0=\mathrm{Id}$, and vectors $\widetilde\eta_a\in\widetilde H$ such that for all a,b,

$$\langle U_j\xi_a,U_i\xi_b\rangle=\langle\widetilde U_j\widetilde\eta_a,\widetilde U_i\widetilde\eta_b\rangle,\qquad i\neq j,\ i,j=0,\dots,2n,\tag{2.13}$$

$$\langle\widetilde\eta_a,\widetilde\eta_b\rangle=(N^2-N)^{\frac12}\langle\xi_a,\xi_b\rangle.\tag{2.14}$$

The relation (2.14) implies that

$$\sum_a\|\widetilde\eta_a\|^2=\sum_a\langle\widetilde\eta_a,\widetilde\eta_a\rangle=(N^2-N)\sum_a\|\xi_a\|^2=(N^2-N).$$

Thus, by the definition of the norm $\|X\|_{\min}$, and since $(\widetilde U_i)_{i=1}^{2n}$ are in tensor position, it follows that

$$\sum_{a,b}^k\Big(\sum_{\substack{i,j=0\\ i\neq j}}^{2n}A_{ia,jb}\langle\widetilde U_j\widetilde\eta_a,\widetilde U_i\widetilde\eta_b\rangle+B_{a,b}\langle\widetilde\eta_a,\widetilde\eta_b\rangle\Big)\leqslant(N^2-N)\|X\|_{\min}^2.\tag{2.15}$$

Moreover, the relation (2.14) and the fact that the matrix $\sum\limits_{a,b}B_{a,b}\otimes e_{a,b}$ is positive imply that the right-hand side in the inequality (2.12) is less than the left-hand side in the inequality (2.15). Hence

$$\|X^*X\|_{C^*(F_{2n})}-\varepsilon\leqslant(N^2-N)\|X\|_{\min}^2.$$

Since ε is arbitrary, the result follows. ∎

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