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GAUSSIAN UPPER BOUNDS FOR HEAT KERNELS OF SECOND-ORDER ELLIPTIC OPERATORS WITH COMPLEX COEFFICIENTS ON ARBITRARY DOMAINS

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ABSTRACT. We consider second-order elliptic operators of the type $A = -\sum\limits_{k,j} D_j(a_{kj}D_k) + \sum\limits_{k} b_kD_k - D_k(c_k\,\cdot) + a_0$ acting on $L^2(\Omega)$ (Ω is a domain of $\mathbb{R}^d$, $d \geqslant 1$) and subject to various boundary conditions. We allow the coefficients a_{kj}, b_k, c_k and a_0 to be complex-valued bounded measurable functions. Under a suitable condition on the imaginary parts of the principal coefficients a_{kj}, we prove that for a wide class of boundary conditions, the semigroup $(\mathrm{e}^{-tA})_{t\geqslant 0}$ is quasi-L^p-contractive ($1 < p < \infty$). We show a pointwise domination of $(\mathrm{e}^{-tA})_{t\geqslant 0}$ by a semigroup generated by an operator with real-valued coefficients and prove a Gaussian upper bound for the associated heat kernel.

KEYWORDS: *Elliptic operators, boundary conditions, heat kernels, Gaussian bounds.*

MSC (2000): 35K20, 47D06.

1. INTRODUCTION

Behaviour of heat kernels has long been an active topic in functional analysis and partial differential equations. It is known that heat kernel bounds such as Gaussian bounds imply various interesting properties for the semigroup and its generator. Some of these properties are L^p-analyticity of the semigroup ([35], [4], [17], [23]), p-independence of the spectrum ([2], [17]), boundedness of spectral multipliers and functional calculi ([21], [19], [20]), maximal L^p-regularity ([24], [14]), boundedness on L^p, $1 < p \leqslant 2$, of Riesz transforms ([13]).

It is known that Gaussian upper bounds hold for self-adjoint uniformly elliptic operators with real coefficients, acting on domains and are subject to Dirichlet or Neumann boundary conditions (see Davies, [16]). Note that in the latter case, one needs a regularity assumption on the boundary of the domain. Arendt and ter Elst ([4]) extended these bounds to the case of non-symmetric operators of the type $A = -\sum_{k,j} D_j(a_{kj}D_k) + \sum_k b_k D_k - D_k(c_k \cdot) + a_0$, assuming some smoothness condition on the coefficients. The coefficients b_k, c_k are allowed to be complex (but smooth). This smoothness assumption was removed by Daners ([15]) but assuming that all the coefficients are real-valued. Related results on $\mathbb{R}^d$ for operators with real-valued coefficients can also be found in Norris and Stroock ([33]). The proof in [15] was extended recently by Karrmann ([25]) who proved a sub-Gaussian bound for some elliptic operators with unbounded coefficients. In [15], the coefficients are allowed to be time dependent, however, the proof seems to use that all the coefficients a_{kj}, b_k, c_k must be real. The situation for complex coefficients is very different. If the above operator A is acting in $L^2(\mathbb{R}^d)$, then the smoothness of the coefficients a_{kj} guaranties the validity of the Gaussian bound (see Auscher and Tchamitchian ([8]) and the references there). Note that such bound always holds if $d \leqslant 2$ and counterexamples are known for non-smooth coefficients when $d \geqslant 5$, see Auscher, Coulhon and Tchamitchian ([7]) and Davies ([18]). Similar results hold if A is acting in $L^2(\Omega)$, where Ω is a Lipschitz domain of $\mathbb{R}^d$ with small enough Lipschitz constant (see [9], Theorem 7). If the Lipschitz constant is large, the Gaussian property does not necessarily hold even when A has constant coefficients (see [9], Proposition 6; the example used there is taken from Maz'ya, Nazarov and Plamenevskii [31]). Considering elliptic operators with complex coefficients on arbitrary domains, one may ask for conditions on the coefficients that imply the Gaussian bound. The present paper deals with this question. We prove a Gaussian bound for operators subject to several boundary conditions. Our assumption on the coefficients concerns the imaginary parts $\operatorname{Im} a_{kj}$ of the principal coefficients (see Theorem 5.4 for the precise statement). We allow the coefficients b_k, c_k, a_0 to be bounded measurable and complex. Our result gives also more precise constants in the Gaussian bound. In order to prove this, we first prove that the semigroup generated by such an operator is pointwise dominated by the semigroup generated by an elliptic operator with real-valued coefficients (Theorems 3.1 and 3.3). This is achieved by using criteria for the domination of semigroups proved in [36] and [37]. We then study the quasi-L^p-contractivity of the semigroup $(\mathrm{e}^{-tA})_{t\geqslant 0}$, generated by $-A$ on $L^2(\Omega)$. It was already shown in [6] (Theorème 2.1 (ii)) (see also [34], Example 4.3) that if the coefficients c_k are not smooth, the semigroup $(\mathrm{e}^{-tA})_{t\geqslant 0}$ can not be quasi-L^∞-contractive. Here, we prove that it is quasi-L^p-contractive for all $p \in (1,\infty)$. We give an estimate for the constant w_p such that $\mathrm{e}^{-tw_p}\mathrm{e}^{-tA}$ is a contraction operator on $L^p(\Omega)$ for all $t \geqslant 0$. We prove that

$$w_p \leqslant \|(\operatorname{Re} a_0)^-\|_\infty + \frac{1}{\eta}\Big(\frac{1}{p} + \frac{1}{2}\Big)\sum_{k=1}^d \|b_k - c_k\|_\infty^2 + \frac{p}{\eta}\sum_{k=1}^d \|\operatorname{Re} c_k\|_\infty^2$$

for all $p \in [2,\infty)$ ($p \in (1,2)$ is obtained by duality), where $\eta > 0$ is the ellipticity constant of the matrix (a_{kj}). This extends related results in [15] and [25] in the sense that we consider more general boundary conditions and complex coefficients as well. Our strategy to prove quasi-L^p-contractivity is based on criteria for L^∞-contractivity of semigroups (cf. [34], [36], [6]) and interpolation arguments. It differs from the proof in [15] and is easier.

Our L^p-estimate together with a Sobolev inequality (assumption (5.1)) provide an estimate for the $L^1 - L^\infty$ norm of e^{-tA}, $t > 0$. It is important to know how this estimate depends on the coefficients of the operator in order to apply the well known Davies perturbation technique. This is the reason why we need to control the constant w_p above.

The Davies perturbation technique consists in showing $L^1 - L^\infty$ estimates for the perturbed semigroup $\mathrm{e}^{\lambda\phi}\mathrm{e}^{-tA}\mathrm{e}^{-\lambda\phi}$, where $\lambda \in \mathbb{R}$ and ϕ is any bounded smooth function on $\mathbb{R}^d$ such that $|\nabla\phi| \leqslant 1$. In order to prove such estimate, one needs that the operator defined by multiplication by $\mathrm{e}^{\lambda\phi}$ maps the domain of the sesquilinear form of A into itself. This property holds if A is subject to Dirichlet, Neumann or mixed boundary conditions, but it does not hold in general. For general boundary conditions, we prove a Gaussian bound which involves another metric. This metric coincides with the Euclidean one in the cases of Dirichlet, Neumann, mixed boundary conditions, e.g. (see Theorem 5.5).

Notation 1.1. We fix here some notation. The norm in $L^p(\Omega)$ of a function u will be denoted by $\|u\|_p$. The Lebesgue measure is written $\mathrm{d}x$. All the integrals will be taken with respect to $\mathrm{d}x$ (we may omit to write $\mathrm{d}x$ in some integrals). The inner product of L^2 is written $(\,\cdot\,,\cdot\,)$. By $W^{1,p}(\Omega)$, we denote the classical Sobolev spaces, $W^{1,2}(\Omega) = H^1(\Omega)$ and $H^1_0(\Omega)$ is the closure in $H^1(\Omega)$ of $C^\infty_{\mathrm{c}}(\Omega)$ (C^∞-functions with compact support in Ω). We will also use the notation $D_k = \frac{\partial}{\partial x_k}$, $\nabla u = (D_1 u, \ldots, D_d u)$ and $u^+ = \sup(u, 0)$, $u^- = \sup(-u, 0)$ (for every real-valued function u). For a measurable function u, the symbol $|u|$ denotes the function $x \to |u(x)|$, where the latter is the modulus of $u(x)$.

2. L^2-theory, positivity and irreducibility

In this section, we give the precise definition of our elliptic operators. They are defined by the sesquilinear form technique and thus generate strongly continuous semigroups on L^2. It is well understood when these semigroups map the cone of non-negative functions into itself. Here we prove that for several boundary conditions such semigroups are irreducible.

Let H be a Hilbert space and consider a sesquilinear form $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{C}$. The subspace $D(\mathfrak{a})$ is called the domain of $\mathfrak{a}$. We assume that $\mathfrak{a}$ is densely defined, nonnegative, continuous and closed. This means, respectively, that:

(i) $D(\mathfrak{a})$ is dense in H;

(ii) $\operatorname{Re}\mathfrak{a}(u, u) \geqslant 0 \;\; \forall u \in D(\mathfrak{a})$;

(iii) $|\mathfrak{a}(u, v)| \leqslant M\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}} \; \forall u, v \in D(\mathfrak{a})$,

(where M is a constant and $\|u\|_{\mathfrak{a}} := \sqrt{\operatorname{Re}\mathfrak{a}(u, u) + \|u\|^2}$)

(iv) $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$ is a complete space.

Under these assumptions, one can associate with $\mathfrak{a}$ an operator A defined as follows

$$D(A) = \{u \in D(\mathfrak{a}), \exists v \in H : \mathfrak{a}(u, \phi) = (v, \phi), \forall \phi \in D(\mathfrak{a})\}, \quad Au := v.$$

Here $(\cdot\,,\cdot)$ denotes the scalar product of H. It is well known (see [27], Chapter VI and IX) that $-A$ generates a holomorphic semigroup $(\mathrm{e}^{-tA})_{t\geqslant 0}$ on H.

Let now Ω be an open subset of $\mathbb{R}^d$, $d \geqslant 1$, and define on $L^2(\Omega) = L^2(\Omega, \mathbb{C})$ the sesquilinear form

(2.1)
$$\mathfrak{a}_V(u, v) = \int_\Omega \Big(\sum_{k,j=1}^{d} a_{kj}(x) D_k u \overline{D_j v} + \sum_{k=1}^{d} b_k(x) D_k u \overline{v} + c_k(x) u \overline{D_k v} + a_0(x) u \overline{v} \Big)$$

with domain $D(\mathfrak{a}_V) = V$, where V is a closed subspace of $H^1(\Omega)$ and $H^1_0(\Omega) \subseteq V \subseteq H^1(\Omega)$. We assume that

$$a_{kj}, b_k, c_k, a_0 \in L^\infty(\Omega, \mathbb{C}) \quad \text{for all } 1 \leqslant j, k \leqslant d, \tag{2.2}$$

with elliptic principal part, i.e., there exists a constant $\eta > 0$ such that

$$\operatorname{Re} \sum_{j,k=1}^{d} a_{kj}(x) \xi_j \overline{\xi_k} \geqslant \eta |\xi|^2 \quad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{C}^d. \tag{2.3}$$

Under these assumptions, there exists a constant w such that

$$\operatorname{Re} \mathfrak{a}_V(u, u) + w\|u\|_2^2 \geqslant \frac{\eta}{2} \|u\|_{H^1(\Omega)}, \quad \text{for all } u \in V. \tag{2.4}$$

Moreover, the form $(\mathfrak{a}_V + w)$ (defined by $(\mathfrak{a}_V + w)(u, v) = \mathfrak{a}_V(u, v) + w(u, v)$ for all $u, v \in V$) is continuous and closed. Indeed, let $u \in V$ and let $M_k := \||\operatorname{Re}(b_k + c_k)| + |\operatorname{Im}(b_k - c_k)|\|_\infty$. We have

$$\begin{aligned} \operatorname{Re} \mathfrak{a}_V(u, u) &= \operatorname{Re} \sum_{k,j} \int_\Omega a_{kj} D_k u \overline{D_j u} + \sum_k \int_\Omega \operatorname{Re}(b_k + c_k) \operatorname{Re}(D_k u \overline{u}) \\ &\quad + \sum_k \int_\Omega \operatorname{Im}(c_k - b_k) \operatorname{Im}(D_k u \overline{u}) + \int_\Omega \operatorname{Re} a_0 |u|^2 \\ &\geqslant \eta \sum_k \int_\Omega |D_k u|^2 - \sum_k M_k \int_\Omega |D_k u \overline{u}| - \int_\Omega (\operatorname{Re} a_0)^- |u|^2. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have for every $\varepsilon > 0$

$$\operatorname{Re} \mathfrak{a}_V(u, u) \geqslant \eta \sum_k \int_\Omega |D_k u|^2 - \varepsilon \sum_k \int_\Omega |D_k u|^2 - \left(\frac{\sum_k M_k^2}{\varepsilon} + \|(\operatorname{Re} a_0)^-\|_\infty \right) \int_\Omega |u|^2$$

which implies (2.4). The proof of the continuity follows from (2.3) and the inequality

$$|\mathfrak{a}_V(u, v)| \leqslant C \Big[\sum_k \|D_k u\|_2^2 + \|u\|_2^2 \Big]^{\frac{1}{2}} \Big[\sum_k \|D_k v\|_2^2 + \|v\|_2^2 \Big]^{\frac{1}{2}}$$

which holds for some constant C and all $u, v \in V$ (use the Cauchy-Schwarz inequality and the boundedness assumption of a_{kj}, b_k, c_k and a_0).

Finally, the fact that the form $(\mathfrak{a}_V + w)$ is closed follows from the assumption that V is a closed subspace of $H^1(\Omega)$. Note that the norm of $H^1(\Omega)$ and $\sqrt{\mathfrak{a}_V(\cdot,\cdot) + w(\cdot,\cdot)}$ are equivalent on V.

We can now associate with $\mathfrak{a}_V$ an operator A_V defined by

$$D(A_V) = \{u \in V : \exists v \in L^2(\Omega), \mathfrak{a}_V(u,\phi) = (v,\phi)\, \forall \phi \in V\}, \quad A_V u := v.$$

Formally, A_V is given by the expression

$$A_V u = -\sum_{k,j=1}^{d} D_j(a_{kj} D_k u) + \sum_{k=1}^{d} (b_k D_k u - D_k(c_k u)) + a_0 u. \tag{2.5}$$

It is subject to the boundary conditions that are determined by the space V. Here are some examples of such conditions:

(i) $V = H_0^1(\Omega)$: Dirichlet boundary conditions;
(ii) $V = H^1(\Omega)$: Neumann boundary conditions;
(iii) Mixed boundary conditions can be defined by taking

$$V = \overline{\{u_{|\Omega} : u \in C_c^\infty(\mathbb{R}^d \setminus \Gamma)\}}^{H^1(\Omega)} \tag{2.6}$$

where Γ is a closed subset of the boundary $\partial\Omega$ of Ω, $u_{|\Omega}$ denotes the restriction of u to Ω and $\overline{\{\dots\}}^{H^1(\Omega)}$ denotes the closure in $H^1(\Omega)$.

Roughly speaking, (i) corresponds to the condition $u = 0$ on the boundary $\partial\Omega$. (ii) corresponds to the condition

$$\sum_{j=1}^{d} \Big(\sum_{k=1}^{d} a_{kj} D_k u + c_j u\Big) n_j = 0 \quad \text{on } \partial\Omega,$$

where $\overrightarrow{n} = (n_1, \dots, n_d)$ denotes the outer unit normal on $\partial\Omega$. (iii) corresponds to the Dirichlet boundary condition on Γ and the Neumann on $\partial\Omega \setminus \Gamma$. All this can be done precisely by applying Green's formula if both $\partial\Omega$ and the coefficients a_{kj}, c_k are smooth enough.

We denote by $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ the semigroup generated by $-A_V$ on $L^2(\Omega)$. Note that

$$\|\mathrm{e}^{-tA_V}\|_{\mathcal{L}(L^2(\Omega))} \leqslant \mathrm{e}^{wt}, \quad \forall t \geqslant 0 \tag{2.7}$$

for some constant w and we may choose

$$w = \frac{1}{4\eta} \sum_{k=1}^{d} \|\,|\operatorname{Re}(b_k + c_k)| + |\operatorname{Im}(b_k - c_k)|\,\|_\infty^2 + \|(\operatorname{Re} a_0)^-\|_\infty. \tag{2.8}$$

This follows from the inequality $\operatorname{Re}\mathfrak{a}_V(u,u) + w(u,u) \geqslant 0$ (for all $u \in V$), which can be shown analogously to (2.4).

It is shown in [36] that the semigroup $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ is real (i.e. each operator e^{-tA_V} maps the subset of real-valued functions of $L^2(\Omega)$ into itself) if and only if $\operatorname{Re} u \in V$ for all $u \in V$ and $\mathfrak{a}_V(u,v) \in \mathbb{R}$ for all real $u, v \in V$. One checks easily that this is again equivalent to the fact that $\operatorname{Re} u \in V$ for all $u \in V$ and the form $\mathfrak{a}_V$ is given by

$$\mathfrak{a}_V(u,v) = \int_\Omega \Big(\sum_{j,k=1}^{d} \operatorname{Re}(a_{kj}) D_k u \overline{D_j v} + \sum_{k=1}^{d} \big(\operatorname{Re}(b_k) D_k u \overline{v} + \operatorname{Re}(c_k) u \overline{D_k v}\big) + \operatorname{Re}(a_0) u \overline{v} \Big) \tag{2.9}$$

for every $u, v \in V$.

This means that the form $\mathfrak{a}_V$ (and hence the operator A_V) is given by real-valued coefficients. In [34] and [36], it is shown that the semigroup $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ is positive (i.e. for each $t \geqslant 0$, e^{-tA_V} maps the subset of non-negative functions into itself) if and only if the form $\mathfrak{a}_V$ has real-valued coefficients and the space V satisfies the condition

$$u \in V \Rightarrow (\operatorname{Re} u)^+ \in V. \tag{2.10}$$

It is not difficult to check that in each case (i)–(iii) above, the space V satisfies the condition (2.10) (see also [34], [36], [4]). This implies that for these boundary conditions, if $\mathfrak{a}_V$ satisfies (2.9), then the semigroup generated by $-A_V$ is positive.

Now we prove a stronger result. We prove that if Ω is connected, then the semigroup is irreducible, that is, for every nontrivial $0 \leqslant u \in L^2(\Omega)$ and every $t > 0$

$$\mathrm{e}^{-tA_V} u(x) > 0 \quad \text{for a.e. } x \in \Omega.$$

More precisely, we have

Theorem 2.1. *Assume that* $(\operatorname{Re} u)^+ \in V$ *for all* $u \in V$ *and that the form* $\mathfrak{a}_V$ *satisfies* (2.9). *Consider the following assertions:*

(i) *the semigroup* $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ *is irreducible;*

(ii) *the open set* Ω *is connected.*

Then (ii) *implies* (i). *The converse holds if* A_V *is subject to Dirichlet, Neumann or mixed boundary conditions.*

The proof relies on the following criterion for irreducibility.

Let $\mathfrak{a}$ be a densely defined nonnegative continuous and closed sesquilinear form acting in $H = L^2(X,\mu)$ ((X,μ) is any σ-finite measure space). Denote by A the associated operator with $\mathfrak{a}$, and by $(\mathrm{e}^{-tA})_{t\geqslant 0}$ the semigroup generated by $-A$ on $L^2(X,\mu)$. We have

Theorem 2.2. *Assume that the semigroup* $(\mathrm{e}^{-tA})_{t\geqslant 0}$ *is positive. The following assertions are equivalent:*

(i) $(\mathrm{e}^{-tA})_{t\geqslant 0}$ *is irreducible;*

(ii) *if* G *is a measurable subset of* X *such that* $\chi_G \cdot u \in D(\mathfrak{a})$ *and* $\operatorname{Re} \mathfrak{a}(\chi_G \cdot u, \chi_{X\setminus G} \cdot u) \geqslant 0$ *for all* $u \in D(\mathfrak{a})$, *then either* $\mu(G) = 0$ *or* $\mu(X \setminus G) = 0$.

The symbol χ_G stands for the characteristic function of the subset G.

Note that a more general definition of irreducibility can be found in [32], but it coincides with the present one since the semigroup $(\mathrm{e}^{-tA})_{t\geqslant 0}$ is holomorphic on $L^2(X,\mu)$.

A different version of Theorem 2.2 for symmetric Dirichlet forms can be found in [22] (Theorem 1.6.1). Here we do not assume the symmetry of the form and we do not also assume that it is a Dirichlet form. We show that Theorem 2.2 is a direct consequence of the criterion for invariance of closed convex sets given in [36].

Proof. It is well known that irreducibility of a positive holomorphic semigroup is equivalent to the fact that it has only trivial invariant closed ideals (see [32], p. 306). On the other hand, every closed ideal of $L^2(X,\mu)$ is of the form $L^2(G,\mu)$ for some measurable subset G, see [39], p. 157 (here we look at $L^2(G,\mu)$ as the subspace of $L^2(X,\mu)$, of functions that vanish (μ-a.e.) on the complement of G). Thus, the irreducibility of the semigroup $(\mathrm{e}^{-tA})_{t\geqslant 0}$ is equivalent to non-invariance of $L^2(G,\mu)$ for any $G\subseteq X$ such that $\mu(G)\neq 0$ and $\mu(X\setminus G)\neq 0$. Note that $L^2(G,\mu)$ is a closed convex set of $L^2(X,\mu)$ and the projection onto $L^2(G,\mu)$ is given by $\mathcal{P}u=\chi_G u$ for all $u\in L^2(X,\mu)$. Theorem 2.2 follows now by applying criteria given in [36] (see also [37]) for invariance of closed convex sets in terms of the sesquilinear form. ∎

Note that if the form $\mathfrak{a}$ is *local*, i.e., $\mathfrak{a}(u,v)=0$ for every $u,v\in D(\mathfrak{a})$ that have disjoint supports, then the condition $\operatorname{Re}\mathfrak{a}(\chi_G u,\chi_{X\setminus G}u)\geqslant 0$ is automatically satisfied whenever $\chi_G u\in D(\mathfrak{a})$. Hence for local forms, the irreducibility criterion is reduced to the question to know whether or not characteristic functions operate on $D(\mathfrak{a})$. More precisely,

COROLLARY 2.3. *Let $\mathfrak{a}$ and $(\mathrm{e}^{-tA})_{t\geqslant 0}$ be as in the previous theorem and assume that the form $\mathfrak{a}$ is local. The following assertions are equivalent:*

(i) *$(\mathrm{e}^{-tA})_{t\geqslant 0}$ is irreducible;*

(ii) *if G is a measurable subset of X such that $\chi_G D(\mathfrak{a})\subseteq D(\mathfrak{a})$, then either $\mu(G)=0$ or $\mu(X\setminus G)=0$.*

We use this corollary to prove Theorem 2.1.

Proof of Theorem 2.1. As mentioned above, the assumption on V and (2.9) imply that the semigroup $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ is positive.

Assume that (ii) holds. Let us denote by $\lambda(G)$ the Lebesgue measure of G. Suppose for contradiction that G is a subset of Ω such that $\lambda(G)>0$, $\lambda(\Omega\setminus G)>0$ and such that

$$u\in V\Rightarrow \chi_G u\in V. \tag{2.11}$$

We have in particular,

$$u\in C_{\mathrm{c}}^{\infty}(\Omega)\Rightarrow \chi_G u\in H^1(\Omega). \tag{2.12}$$

Fix $u\in C_{\mathrm{c}}^{\infty}(\Omega)$. Since the operator D_k satisfies $D_k v\chi_{\{v=0\}}=0$ for all $v\in H^1(\Omega)$, it follows that $D_k(\chi_G u)=\chi_G D_k u$ (the equality is in the a.e. sense). Hence, $\chi_G u\in W_0^{1,p}(\mathcal{O})$ for all $p\in[1,\infty]$, where $\mathcal{O}$ is an open subset of Ω with smooth boundary and contains the support of u. (Note that there always exists an increasing sequence of open subsets Ω_n with smooth boundaries and such that $\bigcup_n \Omega_n=\Omega$. Using the fact that the support of u is compact, one obtains the existence of such an open set $\mathcal{O}$). Choosing p large enough, we deduce from Sobolev imbedding theorems that $\chi_G u = v$ a.e. on $\mathcal{O}$, with v being a continuous function on the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$.

There exists $x_0 \in \Omega$ such that for every $\eta > 0$

$$\lambda(B(x_0,\eta)\cap G) > 0 \quad \text{and} \quad \lambda(B(x_0,\eta)\cap \Omega_1) > 0 \tag{2.13}$$

where $\Omega_1 = \Omega \setminus G$ and $B(x_0,\eta)$ denotes the open Euclidean ball with center x_0 and radius η (such an x_0 satisfying (2.13) exists, see below). We take $\eta > 0$ small enough such that $B(x_0, 2\eta) \subseteq \Omega$ and consider $u \in C_c^\infty(\Omega)$ such that $u(x) = 1$ for all $x \in B(x_0,\eta)$. We have for a.e. $x \in B(x_0,\eta)\cap G$ and a.e. $y \in B(x_0,\eta)\cap\Omega_1$

$$1 = |\chi_G u(x) - \chi_G u(y)| = |v(x) - v(y)|.$$

Using the continuity of v, we see that such an equality can not hold.

Now we prove the existence of x_0 satisfying (2.13). Assume that for every $x \in \Omega$, there exists $\eta > 0$ such that either $\lambda(B(x,\eta)\cap G) = 0$ or $\lambda(B(x,\eta)\cap\Omega_1) = 0$. Define $\mathcal{O}_1$ (respectively $\mathcal{O}_2$) as the union of all balls $B(x,\eta)$, where x and η are such that $\lambda(B(x,\eta)\cap G) = 0$ (respectively $\lambda(B(x,\eta)\cap(X\setminus G)) = 0$). One checks easily that $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint open sets such that $\Omega \subseteq \mathcal{O}_1\cup\mathcal{O}_2$. In addition, if $\Omega \subseteq \mathcal{O}_1$, then we obtain $\lambda(G) = 0$. Similarly, if $\Omega \subseteq \mathcal{O}_2$ then $\lambda(X\setminus G) = 0$. Since we assumed that $\lambda(G) > 0$ and $\lambda(X\setminus G) > 0$, we obtain a contradiction with the fact that Ω is connected. This proves existence of x_0.

Assume now that we have one of the boundary conditions listed in the theorem. That is, $V = H_0^1(\Omega), V = H^1(\Omega)$ or V is as in (2.6). If Ω is not connected, then $\Omega = G\cup\Omega_1$ where G and Ω_1 are two disjoint open sets. It is not hard to see that G satisfies (2.11) in each of the three cases above. Corollary 2.3 shows that the semigroup can not be irreducible. ∎

In Theorem 2.1, we can assert that (i) implies (ii) for several other boundary conditions. However, this implication is not true for all boundary conditions as the following example shows.

Example 2.4. Let $\Omega = (0,1)\cup(2,3)$. Define the form

$$\mathfrak{a}_V(u,v) = \int_0^1 u'\overline{v'}\,\mathrm{d}x, \quad D(\mathfrak{a}_V) = V = \{u \in H^1(\Omega) : u(0) = u(3)\}.$$

This corresponds to periodic boundary conditions at 0 and 3 and Neumann at 1 and 2. This form is well defined, since $H^1(\Omega)$ can be embedded into $C(\overline{\Omega})$ (Sobolev embedding). The semigroup $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ associated with this form is irreducible. Indeed, since V satisfies (2.10), the semigroup is positive. In addition, if $G \subseteq \Omega$ has non zero measure and such that $\chi_G V \subseteq V$, then we obtain as in the proof of Theorem 2.1 that either $G = (0,1)$, $G = (2,3)$ or $G = \Omega$. Assume that $G = (0,1)$. Hence $(\chi_{(0,1)}u)(0) = (\chi_{(0,1)}u)(3)$ for all $u \in V$. We deduce that $u(0) = 0$ for all $u \in V$, which is not the case. The same conclusion holds if $\Omega = (2,3)$. Thus, $G = \Omega$ and we conclude by Theorem 2.2 that the semigroup is irreducible.

We mention that when $a_{kj} = a_{jk}$ and $b_k = c_k = 0$ for all $1 \leqslant k,j \leqslant d$, Theorem 2.1 is shown in [16], Theorem 3.3.5 and in [3]. In [16] it is obtained as a consequence of a lower bound of the heat kernel. In [3], the result is shown for the Laplacian on the space of continuous functions by using the classical maximum principle. Both proofs are different from ours, which can also be used in other circumstances.

3. Domination of semigroups associated with operators with complex-valued coefficients

As explained in the introduction, there are several problems and difficulties when dealing with complex-valued coefficients. In this section, we want to dominate (in the pointwise sense) semigroups generated by elliptic operators with complex-valued coefficients by those generated by their relatives with real-valued coefficients. Of course, this domination can not hold in general, otherwise one obtains that the semigroup generated by any second-order elliptic operator with complex coefficients act on all L^p-spaces which is not the case (cf. [9], [18]). We show, however, that this domination holds under an additional assumption on the coefficients a_{kj}.

Let $\mathfrak{a}_V$ be as in the previous section. We first concentrate on the case of Dirichlet boundary conditions and show then how the domination result extends to other boundary conditions.

Let us write

$$\mathfrak{a} = \mathfrak{a}_{H_0^1(\Omega)} \quad \text{and} \quad A = A_{H_0^1(\Omega)}.$$

We suppose in addition to (2.2) and (2.3) that

$$\operatorname{Im}(a_{kj} + a_{jk}) = 0 \quad \text{and} \quad f_k := \sum_{j=1}^{d} D_j(\operatorname{Im} a_{kj}) \in L^1_{\text{loc}}(\Omega) \tag{3.1}$$

for all $k, j \in \{1, \dots, d\}$, where $D_j \operatorname{Im} a_{kj}$ is taken in the distributional sense.

Let η be the constant in (2.3) and set

$$m(x) := \frac{\sum_{k=1}^{d} [f_k + \operatorname{Im}(c_k - b_k)]^2}{4\eta}. \tag{3.2}$$

Now we define the form

$$\begin{aligned} \mathfrak{b}(u,v) &= \int_\Omega \Big(\sum_{k,j=1}^{d} \operatorname{Re}(a_{kj}) D_k u \overline{D_j v} + \sum_{k=1}^{d} (\operatorname{Re}(b_k) D_k u \overline{v} + \operatorname{Re}(c_k) u \overline{D_k v}) \\ &\qquad + \operatorname{Re}(a_0) u\overline{v} - m u \overline{v} \Big) \\ &:= \mathcal{E}(u,v) - \int_\Omega m u \overline{v}. \end{aligned}$$

We assume that the potential function m is form-bounded with respect to the form $\mathcal{E}$, with relative bound < 1, that is, there exists $\alpha \in \mathbb{R}$ and $\beta < 1$ such that

$$\int_\Omega m(x)|u|^2 \leqslant \alpha \int_\Omega |u|^2 + \beta \operatorname{Re} \mathcal{E}(u,u), \quad \forall u \in H_0^1(\Omega). \tag{3.3}$$

It follows from [27] (Theorem 1.33, Chapter VI) that the form $\mathfrak{b}$, with domain

$$D(\mathfrak{b}) = H_0^1(\Omega) = D(\mathfrak{a})$$

is well defined and there exists a constant $w \in \mathbb{R}$ such $\mathfrak{b} + w$ is nonnegative, continuous and closed. Let us denote by B the associated operator with the form $\mathfrak{b}$. Now we can state our domination result.

Theorem 3.1. *Assume that* (2.2), (2.3), (3.1) *and* (3.3) *are fulfilled. For every* $t \geqslant 0$ *and every* $f \in L^2(\Omega)$

$$|\mathrm{e}^{-tA} f(x)| \leqslant \mathrm{e}^{-tB}|f|(x) \quad \text{for a.e. } x \in \Omega.$$

The proof is based on criteria for the domination of semigroups proved in [36] and [37]. These criteria involve the ideal property and inequalities between associated sesquilinear forms. Since $D(\mathfrak{a}) = D(\mathfrak{b})$, it follows from [36] (Proposition 3.2 and Corollary 3.4) that the inequality in Theorem 3.1 holds if and only if

$$\text{(3.4)} \qquad \operatorname{Re} \mathfrak{a}(u,v) \geqslant \mathfrak{b}(|u|,|v|) \quad \text{for all } u,v \in H_0^1(\Omega) \text{ such that } u\overline{v} \geqslant 0.$$

Here $\overline{v} := \operatorname{Re} v - \mathrm{i} \operatorname{Im} v$ is the conjugate function of v. Note that $(\mathrm{e}^{-tB})_{t\geqslant 0}$ is a positive semigroup. This follows from [34] or [36], since $H_0^1(\Omega)$ satisfies (2.10) and $\mathfrak{b}(u^+,u^-) \leqslant 0$ for all real-valued $u \in D(\mathfrak{b})$.

We will need the following lemma.

Lemma 3.2. *Let* $u,v \in H^1(\Omega)$ *be such that* $u(x)\overline{v(x)} \geqslant 0$ *(for a.e.* $x \in \Omega$*). We have for each* $k \in \{1,\dots,d\}$*:*

(i) $\operatorname{Im}(D_k u\overline{v}) = |v| \operatorname{Im}(D_k u \operatorname{sign} \overline{u})$;

(ii) $|v| \operatorname{Im}(D_k u \operatorname{sign} \overline{u}) = |u| \operatorname{Im}(D_k v \operatorname{sign} \overline{v})$;

where $\operatorname{sign} f(x) = \frac{f(x)}{|f(x)|}$ *if* $f(x) \neq 0$ *and* 0 *otherwise.*

Proof. Let u and v be as in the lemma. Since $D_k u \chi_{\{u=0\}} = 0$, we have

$$D_k u\overline{v} = D_k u\overline{v} \frac{v\overline{u}}{|u|\,|v|}\chi_{\{u\neq 0\}}\chi_{\{v\neq 0\}} = |v| D_k u \frac{\overline{u}}{|u|}\chi_{\{u\neq 0\}}.$$

Assertion (i) follows by taking the imaginary parts.

In order to prove the second assertion we write

$$|v|u = |v|u \frac{v\overline{u}}{|u|\,|v|}\chi_{\{u\neq 0\}}\chi_{\{v\neq 0\}} = |u|v.$$

Hence $D_k|v|u + |v|D_k u = D_k|u|v + |u|D_k v$. We multiply each term with $\operatorname{sign} \overline{u} = \frac{\overline{u}}{|u|}\chi_{\{u\neq 0\}}$ and take the imaginary parts to obtain

$$|v| \operatorname{Im}(D_k u \operatorname{sign} \overline{u}) = \operatorname{Im}(D_k v\overline{u}\chi_{\{u\neq 0\}}) = \operatorname{Im}(D_k v\overline{u}).$$

This, together with assertion (i) (with u in place of v and vice-versa), give (ii). ∎

Proof of Theorem 3.1. Let $u,v \in H_0^1(\Omega)$ be such that $u\overline{v} \geqslant 0$. We have

$$D_k u\overline{D_j v} = D_k u \frac{\overline{u}}{|u|}\chi_{\{u\neq 0\}}\overline{D_j v}\frac{v}{|v|}\chi_{\{v\neq 0\}}.$$

Hence

$$\begin{aligned}
I_1 &:= \operatorname{Re} \sum_{k,j=1}^{d} \int_\Omega \operatorname{Re}(a_{kj}) D_k u \overline{D_j v} \\
&= \sum_{k,j=1}^{d} \int_\Omega \operatorname{Re}(a_{kj}) \operatorname{Re}(D_k u \operatorname{sign} \overline{u}) \operatorname{Re}(D_j v \operatorname{sign} \overline{v}) \\
&\quad + \sum_{k,j=1}^{d} \int_\Omega \operatorname{Re}(a_{kj}) \operatorname{Im}(D_k u \operatorname{sign} \overline{u}) \operatorname{Im}(D_j v \operatorname{sign} \overline{v}) \\
&= \sum_{k,j=1}^{d} \int_\Omega \operatorname{Re}(a_{kj}) \operatorname{Re}(D_k u \operatorname{sign} \overline{u}) \operatorname{Re}(D_j v \operatorname{sign} \overline{v}) \\
&\quad + \sum_{k,j=1}^{d} \int_\Omega \operatorname{Re}(a_{kj}) \operatorname{Im}(D_k u \operatorname{sign} \overline{u}) \operatorname{Im}(D_j u \operatorname{sign} \overline{u}) \frac{|v|}{|u|} \chi_{\{u \neq 0\}}
\end{aligned}$$

where we used the previous lemma in order to write the last equality. Recall now that $D_k|u| = \operatorname{Re}(D_k u \operatorname{sign} \overline{u})\ \ \forall u \in H^1(\Omega)$. Thus,

$$\begin{aligned}
I_1 = {} & \sum_{k,j=1}^{d} \int_\Omega \operatorname{Re}(a_{kj}) D_k|u| D_j|v| \\
& + \sum_{k,j=1}^{d} \int_\Omega \operatorname{Re}(a_{kj}) \operatorname{Im}(D_k u \operatorname{sign} \overline{u}) \operatorname{Im}(D_j u \operatorname{sign} \overline{u}) \frac{|v|}{|u|} \chi_{\{u \neq 0\}}.
\end{aligned} \tag{3.5}$$

We have now to handle the imaginary part. For arbitrary $u, v \in C_c^\infty(\Omega)$,

$$\begin{aligned}
I_2 &:= \operatorname{Re} \sum_{k,j=1}^{d} \int_\Omega \mathrm{i} \operatorname{Im}(a_{kj}) D_k u \overline{D_j v} = -\operatorname{Im} \sum_{k,j=1}^{d} \int_\Omega \operatorname{Im}(a_{kj}) D_k u \overline{D_j v} \\
&= \operatorname{Im} \sum_{k,j=1}^{d} \int_\Omega D_j \operatorname{Im}(a_{kj}) D_k u \overline{v} + \operatorname{Im} \sum_{k,j=1}^{d} \int_\Omega \operatorname{Im}(a_{kj}) D_k D_j u \overline{v}.
\end{aligned}$$

Since by assumptions $\operatorname{Im}(a_{kj} + a_{jk}) = 0$, we obtain

$$I_2 = \sum_{k=1}^{d} \int_\Omega f_k \operatorname{Im}(D_k u \overline{v}). \tag{3.6}$$

Using the Cauchy-Schwarz inequality and assumption (3.3), we see that (3.6) extends to all $u, v \in H_0^1(\Omega)$.

Assume again that $u\overline{v} \geqslant 0$. By Lemma 3.2, (3.6) becomes

$$I_2 = \sum_{k=1}^{d} \int_\Omega f_k \operatorname{Im}(D_k u \operatorname{sign} \overline{u}) |v|. \tag{3.7}$$

We come now to the terms of order 1. We have for $u, v \in H_0^1(\Omega)$ with $u\overline{v} \geqslant 0$,

$$
\begin{aligned}
I_3 &= \operatorname{Re} \sum_{k=1}^{d} \int_\Omega b_k D_k u \overline{v} + c_k u \overline{D_k v} \\
&= \sum_{k=1}^{d} \int_\Omega \operatorname{Re}(b_k) \operatorname{Re}(D_k u \overline{v}) - \operatorname{Im}(b_k) \operatorname{Im}(D_k u \overline{v}) + \operatorname{Re}(c_k) \operatorname{Re}(\overline{u} D_k v) \\
&\qquad + \operatorname{Im}(c_k) \operatorname{Im}(\overline{u} D_k v).
\end{aligned}
$$

As above, we write

$$
D_k u \overline{v} = D_k u \frac{v \overline{u}}{|u|\,|v|} \overline{v} \chi_{\{u \neq 0\}} \chi_{\{v \neq 0\}} = D_k u \overline{u} \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \chi_{\{v \neq 0\}}
$$

and thus,

$$
\operatorname{Re}(D_k u \overline{v}) = \operatorname{Re}(D_k u \operatorname{sign} \overline{u}) |v| = D_k |u|\,|v|.
$$

Using this and Lemma 3.2, we can rewrite I_3 as

$$
\begin{aligned}
I_3 &= \sum_{k=1}^{d} \int_\Omega \operatorname{Re}(b_k) D_k |u|\,|v| + \operatorname{Re}(c_k) |u| D_k |v| \\
&\qquad + \sum_{k=1}^{d} \int_\Omega (\operatorname{Im} c_k - \operatorname{Im} b_k) \operatorname{Im}(D_k u \operatorname{sign} \overline{u}) |v|.
\end{aligned} \tag{3.8}
$$

Concerning the term a_0, we have

$$
I_4 := \operatorname{Re} \int_\Omega a_0 u \overline{v} = \int_\Omega \operatorname{Re} a_0 |u|\,|v| \tag{3.9}
$$

for all $u, v \in H_0^1(\Omega)$ such that $u\overline{v} \geqslant 0$.

Since $\operatorname{Re} \mathfrak{a}(u,v) = I_1 + I_2 + I_3 + I_4$, we obtain from (3.5), (3.7), (3.8) and (3.9),

$$
\begin{aligned}
\operatorname{Re} \mathfrak{a}(u,v) = \int_\Omega \Big[& \sum_{k,j=1}^{d} \operatorname{Re}(a_{kj}) D_k |u| D_j |v| + \sum_{k=1}^{d} (\operatorname{Re}(b_k) D_k |u|\,|v| \\
& + \operatorname{Re}(c_k) |u| D_k |v| + \operatorname{Re}(a_0) |u|\,|v| \Big] \\
& + \sum_{k,j=1}^{d} \int_\Omega \operatorname{Re}(a_{kj}) \operatorname{Im}(D_k u \operatorname{sign} \overline{u}) \operatorname{Im}(D_j u \operatorname{sign} \overline{u}) \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \\
& + \sum_{k=1}^{d} \int_\Omega [f_k + \operatorname{Im}(c_k - b_k)] \operatorname{Im}(D_k u \operatorname{sign} \overline{u}) |v|.
\end{aligned}
$$

Using the ellipticity assumption (2.3) we obtain

$$\sum_{k,j=1}^{d}\int_{\Omega}\operatorname{Re}(a_{kj})\operatorname{Im}(D_k u\ \operatorname{sign}\overline{u})\operatorname{Im}(D_j u\ \operatorname{sign}\overline{u})\frac{|v|}{|u|}\chi_{\{u\neq 0\}}$$
$$+\sum_{k=1}^{d}\int_{\Omega}[f_k+\operatorname{Im}(c_k-b_k)]\operatorname{Im}(D_k u\ \operatorname{sign}\overline{u})|v|\geqslant -\int_{\Omega}m(x)|u|\,|v|,$$

which gives (3.4). This finishes the proof. ∎

It is clear from this proof that we may replace $\operatorname{Re}a_0$ in the last theorem by $(\operatorname{Re}a_0)^-$.

We want now to consider other boundary conditions in the previous theorem. Let $V, \mathfrak{a}_V$ and A_V be as in the last section. We assume that V satisfies (2.10) and that (2.2), (2.3) hold. Let $\mathcal{E}_V$ be the form given by the same expression as $\mathcal{E}$ but with domain $D(\mathcal{E}_V)=V$. Assume that (3.3) holds with $\mathcal{E}$ replaced by $\mathcal{E}_V$ and $H_0^1(\Omega)$ replaced by V, i.e.

$$\int_{\Omega}m(x)|u|^2\leqslant\alpha\int_{\Omega}|u|^2+\beta\operatorname{Re}\mathcal{E}_V(u,u),\quad\forall u\in V\tag{3.10}$$

with some constants $\alpha\in\mathbb{R}$ and $\beta<1$. Define now

$$\mathfrak{b}_V(u,v)=\mathcal{E}_V(u,v)-\int_{\Omega}m(x)u\overline{v},\quad\forall u,v\in D(\mathfrak{b}_V)=V.$$

We denote by B_V the associated operator with the form $\mathfrak{b}_V$.

We can extend the above theorem to the boundary conditions given by V if we have in addition

$$\sum_{k,j=1}^{d}\int_{\Omega}\operatorname{Im}(a_{kj})\phi\overline{D_j v}=-\sum_{k,j=1}^{d}\int_{\Omega}D_j\operatorname{Im}(a_{kj})\phi\overline{v}-\sum_{k,j=1}^{d}\int_{\Omega}\operatorname{Im}(a_{kj})D_j\phi\overline{v}\tag{3.11}$$

for all $v\in V$, $\phi\in C^\infty(\Omega)\cap H^1(\Omega)$ (this means that we assume that $D_j\operatorname{Im}a_{kj}$ exist as functions). Roughly, this assumption means that $\operatorname{Im}(a_{kj})$ are smooth and are $=0$ on parts of the boundary of Ω where functions in V do not necessarily vanish.

We have:

Theorem 3.3. *Assume that* (2.2), (2.3), (2.10), (3.1), (3.10) *and* (3.11) *are satisfied. For every* $t\geqslant 0$ *and every* $f\in L^2(\Omega)$

$$|\mathrm{e}^{-tA_V}f(x)|\leqslant\mathrm{e}^{-tB_V}|f|(x)\quad\textit{for a.e. } x\in\Omega.$$

Proof. As for Dirichlet boundary conditions, this theorem holds if and only if

$$\operatorname{Re}a_V(u,v)\geqslant\mathfrak{b}_V(|u|,|v|)\quad\text{for every } u,v\in V \text{ such that } u\overline{v}\geqslant 0.\tag{3.12}$$

The proof is the same as the one of Theorem 3.1. The only place where we used the fact that $V=H_0^1(\Omega)$ is in the proof of (3.6). Now, in order to prove (3.6) for $u,v\in V$ we proceed as above by taking first $u\in C^\infty(\Omega)\cap H^1(\Omega)$ and $v\in V$, then use (3.11) to integrate by parts. This gives then (3.6) for u and v as above. The Meyers-Serrin theorem ([1], p. 52) and the assumption that m is $\mathcal{E}_V$-bounded show that (3.6) holds for all $u,v\in V$. ∎

4. QUASI-CONTRACTIVITY ON $L^p(\Omega)$

Let $\mathfrak{a}_V$ be as in (2.1) and V be any closed subspace of $H^1(\Omega)$ that contains $H_0^1(\Omega)$. Recall that if (2.2) and (2.3) hold, then the semigroup $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ is defined on $L^2(\Omega)$ and satisfies the estimate (2.7). It is the aim of this section to examine such estimate in other L^p-spaces.

We say that $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ is quasi-L^∞-contractive if $(\mathrm{e}^{-t(A_V+w_\infty)})_{t\geqslant 0}$ is L^∞-contractive for some constant w_∞, i.e., $\|\mathrm{e}^{-t(A_V+w_\infty)}f\|_\infty \leqslant \|f\|_\infty$ for every $f \in L^2(\Omega)\cap L^\infty(\Omega)$ and $t \geqslant 0$. If $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ is quasi-L^∞-contractive, then an application of the Riesz-Thorin interpolation theorem, yields that $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ is quasi-L^p-contractive (that is, $(\mathrm{e}^{-t(A_V+w_p)})_{t\geqslant 0}$ is a contraction semigroup on $L^p(\Omega)$, for some constant w_p and $2 \leqslant p \leqslant \infty$). Let us recall the following criterion for L^∞-contractivity (cf. [34], [36], [10] and [6] Lemme 1.2).

Theorem 4.1. *Let $\mathfrak{a}$ be a densely defined form acting in $L^2(X,\mu)$ ((X,μ) is any σ-finite measure space). Assume that there exists a constant $w \in \mathbb{R}$ such that the form*

$$(\mathfrak{a}+w)(u,v) = \mathfrak{a}(u,v) + w(u,v), \quad u,v \in D(\mathfrak{a})$$

is non-negative, continuous and closed. Then the semigroup $(\mathrm{e}^{-tA})_{t\geqslant 0}$ is L^∞-contractive if and only if the following two conditions hold:

(i) $u \in D(\mathfrak{a})$ *implies* $(1\wedge|u|)\operatorname{sign} u \in D(\mathfrak{a})$;

(ii) $\operatorname{Re}\mathfrak{a}((1\wedge|u|)\operatorname{sign} u, (|u|-1)^+\operatorname{sign} u) \geqslant 0 \quad \forall u \in D(\mathfrak{a})$.

Here, $1\wedge|u| = \inf(1,|u|)$ and $\operatorname{sign} u = \frac{u}{|u|}\chi_{\{u\neq 0\}}$. Thus, independently of the coefficients of the form $\mathfrak{a}_V$, the condition

$$u \in V \Rightarrow (1\wedge|u|)\operatorname{sign} u \in V \tag{4.1}$$

is necessary for quasi-L^∞-contractivity of $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$. This restricts the range of boundary conditions for which we can hope for quasi-L^∞-contractivity. The above result applied to $\mathfrak{a} = \mathfrak{a}_V$ gives the following (see [6]).

Theorem 4.2. *The semigroup $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ is L^∞-contractive if and only if the following two conditions are satisfied:*

(i) $u \in V$ *implies* $(1\wedge|u|)\operatorname{sign} u \in V$;

(ii) *for all $u \in V$ such that $r\varphi_k\varphi_j \in L^1(\Omega,\mathbb{C})$ and $\varphi_k D_j r \in L^1(\Omega,\mathbb{C})$ for every $j,k = 1,\dots,d$ where $r = |u|$ and $\varphi_j = \varphi_j(u) := \frac{\operatorname{Im}(D_k u\cdot \operatorname{sign}\overline{u})}{r}\chi_{\{u\neq 0\}}$, we have*

$$\int_\Omega \Big(\sum_{j,k=1}^d (\operatorname{Re} a_{kj})\varphi_k\varphi_j r - \sum_{j,k=1}^d (\operatorname{Im} a_{kj})\varphi_k D_j r + \sum_{j=1}^d \operatorname{Im}(c_j - b_j)\varphi_j r + \sum_{j=1}^d (\operatorname{Re} c_j) D_j r + (\operatorname{Re} a_0) r \Big) \geqslant 0.$$

It is not true in general that $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ is quasi-L^∞-contractive, even when $V = H_0^1(\Omega)$ and the coefficients are real. One needs in particular some smoothness of the coefficients c_k (cf. [34], see also [6], Theorème 2.1 (ii)). Our next result shows, however, that quasi-L^p-contractivity holds. More precisely, THEOREM 4.3. *Assume that* (2.2), (2.3) *hold and the coefficients* $a_{kj}, 1 \leqslant k, j \leqslant d$, *are real-valued functions. Suppose also that* V *satisfies* (4.1). *Then the semigroup* $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ *extends boundedly to* $L^p(\Omega)$, *for every* $p \in (1, +\infty)$. *In addition,*

$$\|\mathrm{e}^{-tA_V}\|_{\mathcal{L}(L^p(\Omega))} \leqslant \mathrm{e}^{w_p t} \quad \forall t \geqslant 0 \tag{4.2}$$

where

$$w_p = \begin{cases} \|(\operatorname{Re} a_0)^-\|_\infty + \frac{1}{\eta}\left(\frac{1}{p}+\frac{1}{2}\right)\sum\limits_{k=1}^d \|b_k - c_k\|_\infty^2 + \frac{p}{\eta}\sum\limits_{k=1}^d \|\operatorname{Re} c_k\|_\infty^2, & p\in[2,\infty), \\ \|(\operatorname{Re} a_0)^-\|_\infty + \frac{1}{\eta}\left(\frac{1}{2}+\frac{p-1}{p}\right)\sum\limits_{k=1}^d \|b_k - c_k\|_\infty^2 + \frac{p}{\eta(p-1)}\sum\limits_{k=1}^d \|\operatorname{Re} b_k\|_\infty^2, & p\in(1,2]. \end{cases}$$

We point out that our aim here is not to give the best possible w_p. The value of w_p given here may be sharpened but we do not investigate this in the present paper. Note that if $V = H_0^1(\Omega)$ and $\operatorname{Re} c_k \in W^{1,\infty}(\Omega)$ for $1 \leqslant k \leqslant d$, then after an integration by parts,

$$\int_\Omega b_k D_k u\overline{v} + (\operatorname{Re} c_k)u\overline{D_k v} = \int_\Omega (b_k - \operatorname{Re} c_k)D_k u\overline{v} - \int_\Omega [D_k \operatorname{Re} c_k]u\overline{v}.$$

Thus, the term $\frac{4p}{\eta}\sum\limits_{k=1}^d \|\operatorname{Re} c_k\|_\infty^2$ does not appear anymore in the expression of w_p, when $p \geqslant 2$. However, the case of non smooth first order coefficients is of interest. In particular, we have to deal with such coefficients in order to obtain Gaussian upper bounds when a_{kj} are not smooth (see the next section).

Proof of Theorem 4.3. We define for each $z \in \mathbb{C}$ the form $\mathfrak{a}_V(z)$ by

$$\mathfrak{a}_V(z)(u,v) := \frac{1}{2}\sum_{k,j=1}^d \int_\Omega a_{kj} D_k u\overline{D_j v} + z\sum_{k=1}^d \int_\Omega \big((\operatorname{Re} c_k)D_k u\overline{v} + (\operatorname{Re} c_k)u\overline{D_k v}\big), \quad D(\mathfrak{a}_V(z)) = V.$$

We denote by $(T_z(t))_{t\geqslant 0}$ the semigroup generated by (minus) the associated operator with $\mathfrak{a}_V(z)$. Applying Theorem 4.2, we see that the semigroup $(T_{\mathrm{i}s})_{t\geqslant 0}$ and its adjoint are both L^∞-contractive for every $s \in \mathbb{R}$.

For $z = 1 + \mathrm{i}s$, $s \in \mathbb{R}$, the estimate (2.7) (applied to the semigroup of the form $\mathfrak{a}_V(1 + \mathrm{i}s)$) gives

$$\|T_{1+\mathrm{i}s}(t)\|_{\mathcal{L}(L^2(\Omega))} \leqslant \mathrm{e}^{w_2' t}$$

where $w_2' = \frac{2}{\eta}\sum\limits_{k=1}^d \|\operatorname{Re} c_k\|_\infty^2$.

On the other hand, it follows from [27], Theorems VII-4.2 and IX-2.6 that $T_z(t)$ depends analytically in z (for each $t \geqslant 0$). The Stein interpolation theorem allows to interpolate between the previous L^2 and L^∞-estimates. Thus, for every $p \in [2, \infty]$

$$\|T_{2/p}(t)\|_{\mathcal{L}(L^p(\Omega))} \leqslant \mathrm{e}^{\frac{2}{p}w_2' t}.$$

Applying this estimate to the semigroup associated with the form, where c_k is changed into $\frac{p}{2}c_k$, yields

$$\|T_1(t)\|_{\mathcal{L}(L^p(\Omega))} \leqslant \mathrm{e}^{\frac{p}{2}w_2' t} = \mathrm{e}^{\frac{p}{\eta}\sum\limits_{k=1}^{d}\|\operatorname{Re} c_k\|_\infty^2 t}. \tag{4.3}$$

Define now the form

$$\mathfrak{b}_V(u,v) := \frac{1}{2}\sum_{k,j=1}^{d}\int_\Omega a_{kj}D_k u\overline{D_j v} + \sum_{k=1}^{d}\int_\Omega (b_k - \operatorname{Re} c_k)D_k u\overline{v} + \mathrm{i}\operatorname{Im} c_k u\overline{D_k v} + \int_\Omega a_0 u\overline{v}$$

(with domain $D(\mathfrak{b}_V) = V$) and denote by $(S(t))_{t\geqslant 0}$ the semigroup generated by (minus) its associated operator. Applying Theorem 4.2, we obtain that the semigroup $(\mathrm{e}^{-wt}S(t))_{t\geqslant 0}$ is L^∞-contractive for every w such that

$$w + \operatorname{Re} a_0 - \frac{1}{2\eta}\sum_{k=1}^{d}|\operatorname{Im}(c_k - b_k)|^2 \geqslant 0 \quad \text{on } \Omega.$$

In particular, this holds for

$$w = \|(\operatorname{Re} a_0)^- + \frac{1}{2\eta}\sum_{k=1}^{d}|\operatorname{Im}(c_k - b_k)|^2\|_\infty.$$

This L^∞-estimate and (2.7) (applied with the coefficients of $\mathfrak{b}_V$) imply that for every $p \in [2,+\infty]$

$$\|S(t)\|_{\mathcal{L}(L^p(\Omega))} \leqslant \mathrm{e}^{[\frac{2}{p}w_2'' + w(1-\frac{2}{p})]t} \tag{4.4}$$

where $w_2'' = \frac{1}{2\eta}\sum\limits_{k=1}^{d}\|\,|\operatorname{Re}(b_k - c_k)| + |\operatorname{Im}(b_k - c_k)|\,\|_\infty^2 + \|(\operatorname{Re} a_0)^-\|_\infty$.

Since the form $\mathfrak{a}_V$ is the sum

$$\mathfrak{a}_V = \mathfrak{a}_V(1) + \mathfrak{b}_V,$$

it follows from the Trotter-Kato product formula (cf. [26]) that

$$\mathrm{e}^{-tA_V}f = \lim_{n\to+\infty}\Big(T_1\Big(\frac{t}{n}\Big)S\Big(\frac{t}{n}\Big)\Big)^n f \tag{4.5}$$

for every $f \in L^2(\Omega)$. This, together with (4.3) and (4.4), give

$$\|\mathrm{e}^{-tA_V}\|_{\mathcal{L}(L^p(\Omega))} \leqslant \mathrm{e}^{\left[\frac{2}{p}w_2'' + w\left(1-\frac{2}{p}\right) + \frac{p}{2}w_2'\right]t}. \tag{4.6}$$

This shows the desired estimate on $L^p(\Omega)$ for $p \in [2,\infty)$. The estimate on $L^p(\Omega)$ for $p \in (1,2]$ is obtained by applying the previous one to the adjoint semigroup $(\mathrm{e}^{-tA_V^*})_{t\geqslant 0}$ and arguing by duality. ∎

For complex-valued coefficients a_{kj}, we have the following

Theorem 4.4. *Assume that* $f_k = \sum_{j=1}^{d} D_j \operatorname{Im} a_{kj} \in L^\infty(\Omega)$ *for* $1 \leqslant k \leqslant d$ *and let* $m(x)$ *be as in* (3.2). *Assume that* $\operatorname{Im}(a_{kj} + a_{jk}) = 0$ *for all* $1 \leqslant k, j \leqslant d$ *(respectively, that the hypotheses of Theorem* 3.3 *are satisfied if* $V \neq H_0^1(\Omega)$*). Then the conclusion of the above theorem holds for the semigroup* $(\mathrm{e}^{-tA_{H_0^1}})_{t\geqslant 0}$ *(respectively, for* $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$*) with* $\|(\operatorname{Re} a_0)^-\|_\infty$ *replaced by* $\|(\operatorname{Re} a_0 - m)^-\|_\infty$ *in the expression of* w_p.

Proof. Apply Theorem 3.1 (respectively Theorem 3.3) and the previous result. ∎

Remark 4.5. Theorem 4.3 is known in some situations. In [15] (Theorem 5.1) a similar result is shown for Dirichlet, Neumann or Robin boundary conditions. The coefficients are allowed to be time dependent. However, it is used there that all the coefficients a_{kj}, b_k, c_k are real-valued. We point out that our proof is different from that of [15] and our results are valid for operators with more general coefficients and boundary conditions.

Following similar technique as in [15], Karrmann ([25]) extended the previously mentioned result of [15] to operators with unbounded coefficients (but still real) in the case of Dirichlet boundary conditions. Our results are given for operators with bounded coefficients b_k, c_k, but the method is enough flexible and can be adapted to some operators with unbounded coefficients. We shall not develop this in details. Our aim now is to use the previous results in order to prove Gaussian upper bounds for the corresponding heat kernels.

5. Gaussian upper bounds

In this section we prove Gaussian upper bounds for heat kernels of operators A_V. To this end, we first need to show the boundedness $L^1 - L^\infty$ of e^{-tA_V} for each $t > 0$. We need in addition to control the norm $\|\mathrm{e}^{-tA_V}\|_{\mathcal{L}(L^1,L^\infty)}$ in terms of the coefficients of the operator.

It is well known that the $L^1 - L^\infty$ boundedness of the semigroup holds once we have a Sobolev inequality. We assume that

$$V \text{ is continuously embedded into } L^{2^*}(\Omega), \tag{5.1}$$

where $2^* = \frac{2d}{d-2}$ if $d \geqslant 3$, $2^* = \infty$ if $d = 1$ and 2^* is any number in $(2, \infty)$ if $d = 2$. If $V = H_0^1(\Omega)$, where Ω is an arbitrary domain of $\mathbb{R}^d$, this embedding holds and one has for $d \geqslant 3$

$$\int_\Omega |\nabla u|^2 \geqslant c\|u\|_{2^*}^2 \quad \forall u \in H_0^1(\Omega). \tag{5.2}$$

If $V = H^1(\Omega)$, then (5.1) holds if Ω has smooth boundary. For example, if Ω has the extension property (i.e., there exists a bounded linear operator $P : H^1(\Omega) \to H^1(\mathbb{R}^d)$ such that Pu is an extension of u from Ω to $\mathbb{R}^d$), then (5.1) follows from the embedding of $H^1(\mathbb{R}^d)$ into $L^{2^*}(\mathbb{R}^d)$. In that case, every closed subspace V

of $H^1(\Omega)$ satisfies (5.1), too. There are several geometrical conditions on Ω that imply (5.1) (see [30], Section 4.9).

Note that (5.1) means that

$$\int_\Omega |\nabla u|^2 + \int_\Omega |u|^2 \geqslant c\|u\|_{2^*}^2 \quad \forall u \in V, \tag{5.3}$$

where $c > 0$ is a constant.

If $d \leqslant 2$, it is more convenient to work with Nash or Gagliardo-Nirenberg inequalities. The latter can be written on the form

$$c\|u\|_q \leqslant \|u\|_2^{1-d\frac{q-2}{2q}} (\|\nabla u\|_2 + \|u\|_2)^{d\frac{q-2}{2q}} \quad \forall u \in V \tag{5.4}$$

for all $q \in (2, \infty]$ such that $d\frac{q-2}{2q} < 1$. Note that (5.4) holds if and only if (5.3) holds (see [12], Section II). As in (5.2), if $V = H_0^1(\Omega)$, then (5.4) can be written

$$c\|u\|_q \leqslant \|u\|_2^{1-d\frac{q-2}{2q}} \|\nabla u\|_2^{d\frac{q-2}{2q}} \quad \forall u \in H_0^1(\Omega). \tag{5.5}$$

In the sequel, w_p will denote the same constant as in Theorem 4.3.

Lemma 5.1. *Suppose that V satisfies* (5.3) *and that the assumptions of Theorem* 4.3 *hold.*

(i) *If $d \geqslant 3$ then for every $t > 0$, e^{-tA_V} is bounded from $L^2(\Omega)$ into $L^{2^*}(\Omega)$. Moreover, for every $\varepsilon > 0$*

$$\|\mathrm{e}^{-tA_V}\|_{\mathcal{L}(L^2, L^{2^*})} \leqslant C_\varepsilon \mathrm{e}^{w_{2^*}t}\mathrm{e}^{\varepsilon t} t^{-\frac{1}{2}} \quad \forall t > 0$$

where C_ε is a positive constant depending only on η, d, ε, and the constant c in (5.3).

(ii) *If $d \leqslant 2$ then for every $q \in (2, \infty)$ and every $\varepsilon > 0$*

$$\|\mathrm{e}^{-tA_V}\|_{\mathcal{L}(L^2, L^q)} \leqslant C_\varepsilon \mathrm{e}^{w_q t}\mathrm{e}^{\varepsilon t} t^{-d\frac{q-2}{4q}} \quad \forall t > 0$$

where C_ε is a positive constant depending only on η, d, ε, and the constant c in (5.4).

If $V = H_0^1(\Omega)$, the estimates in both assertions hold with $\varepsilon = 0$.

Proof. We assume that $d \geqslant 3$. Note that (5.3) implies that for every $\varepsilon > 0$, there exists a constant c_ε such that

$$\int_\Omega |\nabla u|^2 + \varepsilon \int_\Omega |u|^2 \geqslant c_\varepsilon \|u\|_{2^*}^2 \quad \forall u \in V. \tag{5.6}$$

The assumptions (2.2) and (2.3) imply (see the proof of (2.4))

$$\begin{aligned}
\eta\|\nabla u\|_2^2 &\leqslant \operatorname{Re} \sum_{k,j} \int_\Omega a_{kj} D_k u \overline{D_j u} \\
&= \operatorname{Re}\left[\mathfrak{a}_V(u,u) - \sum_k \int_\Omega b_k D_k u \overline{u} + c_k u \overline{D_k u} - \int_\Omega a_0 |u|^2\right] \\
&\leqslant \operatorname{Re}\mathfrak{a}_V(u,u) + \sum_k \int_\Omega (|\operatorname{Re}(b_k + c_k)| + |\operatorname{Im}(c_k - b_k)|\, |D_k u|\, |u| + \int_\Omega (\operatorname{Re} a_0)^- |u|^2 \\
&\leqslant \operatorname{Re}\mathfrak{a}_V(u,u) + \frac{\eta}{2}\sum_k \|D_k u\|_2^2 + w\|u\|_2^2
\end{aligned}$$

where $w = \frac{1}{2\eta} \sum_{k=1}^{d} \| |\operatorname{Re}(b_k + c_k)| + |\operatorname{Im}(c_k - b_k)| \|_\infty^2 + \|(\operatorname{Re} a_0)^-\|_\infty$. Using this and (5.6), we obtain for every $\varepsilon > 0$

$$\operatorname{Re} \mathfrak{a}_V(u, u) + (w_{2^*} + \varepsilon) \int_\Omega |u|^2 \geqslant c'_\varepsilon \|u\|_{2^*}^2 \quad \forall u \in V, \tag{5.7}$$

where c'_ε is a constant depending only on η, d, ε, and the constant c in (5.3). Note that (5.7) holds with $\varepsilon = 0$ if $V = H_0^1(\Omega)$ (apply (5.2) instead of (5.3), in the proof).

We define the semigroup $T(t) := \mathrm{e}^{-tA_V} \mathrm{e}^{-w_{2^*} t} \mathrm{e}^{-\varepsilon t}$. By Theorem 4.3, the operator $\mathrm{e}^{-tA_V} \mathrm{e}^{-w_{2^*} t}$ is a contraction on $L^{2^*}(\Omega)$ (and so is $T(t)$). This and (5.7) imply that for every $f \in L^2(\Omega) \cap L^{2^*}(\Omega)$ and $t > 0$

$$\begin{aligned} c'_\varepsilon t \|T(t)f\|_{2^*}^2 &\leqslant c'_\varepsilon \int_0^t \|T(s)f\|_{2^*}^2 \, \mathrm{d}s \\ &\leqslant \int_0^t \operatorname{Re}\big[\mathfrak{a}_V(T(s)f, T(s)f) + (w_{2^*} + \varepsilon)(T(s)f, T(s)f)\big] \, \mathrm{d}s \\ &= \int_0^t -\frac{\mathrm{d}}{\mathrm{d}s} \|T(s)f\|_2^2 \, \mathrm{d}s = \|f\|_2^2 - \|T(t)f\|_2^2 \leqslant \|f\|_2^2. \end{aligned}$$

Hence, we have proved

$$\|\mathrm{e}^{-tA_V} f\|_{2^*} \leqslant \frac{1}{c'_\varepsilon} t^{-\frac{1}{2}} \mathrm{e}^{w_{2^*} t} \mathrm{e}^{\varepsilon t} \|f\|_2$$

and if $V = H_0^1(\Omega)$, this estimate holds without the extra term $\mathrm{e}^{\varepsilon t}$.

The proof of assertion (ii) is similar, one uses (5.4) instead of (5.3) (or (5.5) instead of (5.2), if $V = H_0^1(\Omega)$). The analog of (5.7) is

$$[\operatorname{Re} \mathfrak{a}_V(u, u) + (w_q + \varepsilon) \|u\|_2^2]^{d\frac{q-2}{4q}} \|u\|_2^{1 - d\frac{q-2}{2q}} \geqslant c'_\varepsilon \|u\|_q \quad \forall u \in V. \tag{5.8}$$

Thus, as previously, if $T(t) := \mathrm{e}^{-tA_V} \mathrm{e}^{-w_q t} \mathrm{e}^{-\varepsilon t}$ then

$$\begin{aligned} c'_\varepsilon t \|T(t)f\|_q^{\frac{4q}{d(q-2)}} &\leqslant c'_\varepsilon \int_0^t \|T(s)f\|_q^{\frac{4q}{d(q-2)}} \, \mathrm{d}s \\ \leqslant \int_0^t \|T(s)f\|_2^{\frac{4q}{d(q-2)} - 2} &\operatorname{Re}[\mathfrak{a}_V(T(s)f, T(s)f) + (w_q + \varepsilon)(T(s)f; T(s)f)] \, \mathrm{d}s \end{aligned}$$

$$\leqslant \int_0^t \|f\|_2^{\frac{4q}{d(q-2)}-2} \operatorname{Re}[\mathfrak{a}_V(T(s)f, T(s)f) + (w_q+\varepsilon)(T(s)f; T(s)f)]\,\mathrm{d}s$$

$$= \|f\|_2^{\frac{4q}{d(q-2)}-2} \int_0^t -\frac{\mathrm{d}}{\mathrm{d}s}\|T(s)f\|_2^2\,\mathrm{d}s$$

$$= \|f\|_2^{\frac{4q}{d(q-2)}-2}[\|f\|_2^2 - \|T(t)f\|_2^2] \leqslant \|f\|_2^{\frac{4q}{d(q-2)}}.$$

This proves the lemma. ∎

Theorem 5.2. *Suppose that V satisfies* (5.3) *and that the assumptions of Theorem* 4.3 *hold. For every $t>0$, e^{-tA_V} is bounded from $L^2(\Omega)$ into $L^\infty(\Omega)$, and*

$$\|\mathrm{e}^{-tA_V}\|_{\mathcal{L}(L^2,L^\infty)} \leqslant C_\varepsilon t^{-\frac{d}{4}} \mathrm{e}^{\alpha_1 t}\mathrm{e}^{\alpha_2 c' t}\mathrm{e}^{\varepsilon t} \quad \forall t>0,\ \forall \varepsilon>0$$

where C_ε is a positive constant depending only on η, ε, d, and the constant c in (5.3)*, c' is a constant depending only on d. The constants α_1 and α_2 are given by*

$$\alpha_1 := \|(\operatorname{Re} a_0)^-\|_\infty + \frac{1}{\eta}\sum_{k=1}^d \|b_k - c_k\|_\infty^2, \quad \alpha_2 := \frac{1}{\eta}\sum_{k=1}^d \|\operatorname{Re} c_k\|_\infty^2.$$

If $V = H_0^1(\Omega)$, the above estimate holds with $\varepsilon = 0$.

Proof. We first assume that $d \geqslant 3$.

For every $r \geqslant 2$, we have (cf. Theorem 4.3)

$$\|\mathrm{e}^{-tA_V}\|_{\mathcal{L}(L^r)} \leqslant \mathrm{e}^{w_r t} \quad \forall t \geqslant 0.$$

Using this and Lemma 5.1, we obtain by interpolation

$$\|\mathrm{e}^{-tA_V}\|_{\mathcal{L}(L^{p_\theta},L^{q_\theta})} \leqslant C_\varepsilon^\theta \mathrm{e}^{w_r(1-\theta)t}\mathrm{e}^{w_{2^*}\theta t}\mathrm{e}^{\varepsilon\theta t} t^{-\frac{\theta}{2}} \quad \forall t>0 \tag{5.9}$$

where $\frac{1}{p_\theta} = \frac{\theta}{2} + \frac{1-\theta}{r}$, $\frac{1}{q_\theta} = \frac{\theta}{2^*} + \frac{1-\theta}{r}$ for $\theta \in [0,1]$.

Fix $p \in (2,\infty)$ and choose $\theta = \frac{1}{p}$ and $r = 2(p-1)$. We obtain $p_\theta = p$ and $q_\theta = p \cdot \frac{d}{d-1}$. In addition,

$$(1-\theta)w_r + \theta w_{2^*} \leqslant \alpha_1 + \alpha_2 \frac{2(p-1)^2 + \frac{2d}{d-2}}{p} := \alpha_1 + \alpha_2\gamma_p.$$

Inserting this in (5.9), we obtain

$$\|\mathrm{e}^{-tA_V}\|_{\mathcal{L}\left(L^p, L^{\frac{pd}{d-1}}\right)} \leqslant C_\varepsilon^{\frac{1}{p}} \mathrm{e}^{\alpha_1 t}\mathrm{e}^{\alpha_2\gamma_p t}\mathrm{e}^{\frac{t\varepsilon}{p}} t^{-\frac{1}{2p}} \quad \forall t>0. \tag{5.10}$$

This estimate holds with $\varepsilon = 0$ if $V = H_0^1(\Omega)$.

In the rest of the proof, we follow similar arguments as in [12]. Set $R := \frac{d}{d-1}$, $t_k := \frac{d+1}{2d}(2R)^{-k}$ and $p_k = 2R^k$ for all integer $k \geqslant 0$. We have

$$\sum_{k\geqslant 0} t_k = 1, \quad \sum_{k\geqslant 0}\frac{1}{p_k} = \frac{d}{2}, \quad \sum_{k\geqslant 0} t_k\gamma_{p_k} = c',$$

where c' is a positive constant depending only on d. Applying now (5.10) (with p_k in place of p), yields for all $t > 0$

$$\begin{aligned}\|e^{-tA_V}\|_{\mathcal{L}(L^2,L^\infty)} &\leqslant \prod_{k\geqslant 0}\|e^{-tt_kA_V}\|_{\mathcal{L}(L^{p_k},L^{p_{k+1}})}\\ &\leqslant \prod_{k\geqslant 0} C_\varepsilon^{\frac{1}{p_k}} e^{\alpha_1 tt_k} e^{\alpha_2\gamma_{p_k}tt_k} e^{\frac{t\varepsilon}{p_k}} t^{-\frac{1}{2p_k}} t_k^{-\frac{1}{2p_k}} = C'_\varepsilon t^{-\frac{d}{4}} e^{\frac{\varepsilon t d}{2}} e^{\alpha_1 t} e^{c'\alpha_2 t}\end{aligned}$$

which is the desired estimate.

Assume now that $d \leqslant 2$. We use assertion (ii) of the previous lemma with $q = \frac{2n}{n-2}$ where n is any constant in $(2,\infty)$. We are in a position to apply the same proof as in the previous case with n in place of d. We obtain

$$\|e^{-tA_V}\|_{\mathcal{L}(L^2,L^\infty)} \leqslant C_\varepsilon t^{-\frac{n}{4}} t^{\left(\frac{1}{2}-d\frac{q-2}{4q}\right)\frac{n}{2}} e^{\frac{\varepsilon t n}{2}} e^{\alpha_1 t} e^{c'\alpha_2 t} = C_\varepsilon t^{-\frac{d}{4}} e^{\frac{\varepsilon t n}{2}} e^{\alpha_1 t} e^{c'\alpha_2 t}.$$

This proves the theorem. ∎

As a consequence of the above theorem and Theorem 4.4, we obtain for complex coefficients a_{kj}:

Corollary 5.3. *Assume that* $\sum_{j=1}^{d} D_j \operatorname{Im} a_{kj} \in L^\infty(\Omega), \operatorname{Im}(a_{kj}+a_{jk}) = 0$ *for all* $1 \leqslant k,j \leqslant d$ *and* (3.11) *holds (if* $V \neq H_0^1(\Omega)$*). Assume that* V *satisfies* (2.10), (4.1) *and* (5.1). *Then the conclusions of the above theorem hold with* $\|(\operatorname{Re} a_0)^-\|_\infty$ *replaced by* $\|(\operatorname{Re} a_0 - m)^-\|_\infty$ *(*m *is given by* (3.2)*).*

If we apply the last results to the adjoint semigroup $(e^{-tA_V^*})_{t\geqslant 0}$, we obtain an estimate for the $L^1 - L^2$ norm of $(e^{-tA_V})_{t\geqslant 0}$. Thus, under the assumptions of the last corollary or of Theorem 5.2 (if a_{kj} are real), we have

$$\text{(5.11)} \qquad \|e^{-tA_V}\|_{\mathcal{L}(L^1,L^\infty)} \leqslant C_\varepsilon t^{-\frac{d}{2}} e^{\varepsilon t} e^{c_0\alpha t}, \quad \forall t > 0, \forall \varepsilon > 0$$

where C_ε is a positive constant depending only on η, ε, d, and the constant c in (5.3), c_0 is a constant depending only on d and η (the precise value of c_0 in terms of d and η can be easily obtained from the proof of the previous theorem). The constant α is given by

$$\text{(5.12)} \quad \alpha = \|(\operatorname{Re} a_0 - m)^-\|_\infty + \sum_{k=1}^{d}\|b_k - c_k\|_\infty^2 + \sum_{k=1}^{d}(\|\operatorname{Re} b_k\|_\infty^2 + \|\operatorname{Re} c_k\|_\infty^2).$$

In particular, $\alpha = \|m\|_\infty$ if $\operatorname{Re} b_k = \operatorname{Re} c_k = \operatorname{Im}(b_k - c_k) = (\operatorname{Re} a_0)^- = 0,\ 1 \leqslant k \leqslant d$. Finally, (5.1) holds with $\varepsilon = 0$ if $V = H_0^1(\Omega)$.

The estimate (5.1) implies that e^{-tA_V} is given by a kernel $p_V(t,x,y)$, i.e., a measurable function on $(0,+\infty)\times\Omega\times\Omega$ such that

$$e^{-tA_V} f(x) = \int_\Omega p_V(t,x,y) f(y)\,\mathrm{d}y \quad \text{for a.e. } x \in \Omega,\ \forall t > 0,\ f \in L^2(\Omega).$$

In addition,

$$\text{(5.13)} \qquad |p_V(t,x,y)| \leqslant C_\varepsilon t^{-\frac{d}{2}} e^{\varepsilon t} e^{c_0\alpha t} \quad \forall t > 0, \forall \varepsilon > 0,$$

where the constant are the same as in (5.11). Using a well known perturbation technique due to E.B. Davies, we can convert (5.13) into a Gaussian upper bound. This is possible because of our good control of the constants involved in (5.13).

Let $\lambda \in \mathbb{R}$ and ϕ be a real-valued bounded C^∞-function on $\mathbb{R}^d$ such that $|\nabla \phi| \leqslant 1$ on $\mathbb{R}^d$. We need the following assumption on V:

$$u \in V \Rightarrow \mathrm{e}^{\lambda\phi} u \in V \tag{5.14}$$

for all $\lambda \in \mathbb{R}$ and all ϕ as above.

Under this assumption, we can define the form

$$\mathfrak{b}_V(u, v) := \mathfrak{a}_V(\mathrm{e}^{\lambda\phi} u, \mathrm{e}^{-\lambda\phi} v), \quad u, v \in V.$$

The form $\mathfrak{b}_V$ has a similar expression as $\mathfrak{a}_V$, but with terms $b_k - \lambda \sum_{j=1}^d a_{kj} D_j \phi$ in place of b_k, $c_k + \lambda \sum_{j=1}^d a_{jk} D_j \phi$ in place of c_k and $a_0 - \lambda^2 \sum_{k,j=1}^d a_{kj} D_k \phi D_j \phi + \lambda \sum_{j=1}^d (b_k - c_k) D_j \phi$ in place of a_0. Hence, if the assumptions of Theorem 5.2 or those of Corollary 5.3 are satisfied, we can apply (5.11) to the kernel of the semigroup $T(t) := \mathrm{e}^{-\lambda\phi} \mathrm{e}^{-tA_V} \mathrm{e}^{\lambda\phi}$ (generated by (minus) the associated operator with $\mathfrak{b}_V$). This and the assumption $|\nabla\phi| \leqslant 1$, give

$$\|\mathrm{e}^{\lambda\phi} \mathrm{e}^{-tA_V} \mathrm{e}^{-\lambda\phi}\|_{\mathcal{L}(L^1, L^\infty)} \leqslant C_\varepsilon t^{-\frac{d}{2}} \mathrm{e}^{\varepsilon t} \mathrm{e}^{\delta(\alpha+\lambda^2)t}, \quad \forall t > 0, \forall \varepsilon > 0 \tag{5.15}$$

where C_ε is as in (5.11), δ is a constant depending only on d, η and $\|a_{kj}\|_\infty$, and α is as in (5.12). This implies that

$$|p_V(t, x, y)| \leqslant C_\varepsilon t^{-\frac{d}{2}} \mathrm{e}^{\varepsilon t} \mathrm{e}^{\delta\alpha t} \mathrm{e}^{\lambda(\phi(x) - \phi(y)) + \delta\lambda^2 t}, \quad \forall t > 0, \forall \varepsilon > 0. \tag{5.16}$$

Taking $\lambda = \frac{\phi(y) - \phi(x)}{2\delta t}$ and optimizing over ϕ, yields the Gaussian upper bound

$$|p_V(t, x, y)| \leqslant C_\varepsilon t^{-\frac{d}{2}} \mathrm{e}^{\varepsilon t + \delta\alpha t} \mathrm{e}^{-\frac{|x-y|^2}{4\delta t}}, \quad \forall t > 0, \text{ a.e. } x, y \in \Omega. \tag{5.17}$$

We have proved:

Theorem 5.4. *Assume that* (2.2), (2.3), (4.1), (5.14) *and* (5.1) *hold. Assume in addition that one of the following conditions is satisfied:*

(i) *the coefficients a_{kj} are real-valued for $1 \leqslant k, j \leqslant d$;*

(ii) $\sum_{j=1}^d D_j \operatorname{Im} a_{kj} \in L^\infty(\Omega)$, $\operatorname{Im}(a_{kj} + a_{jk}) = 0$ *for* $1 \leqslant j, k \leqslant d$, (2.10) *and* (3.11) *hold.*

Then, the semigroup e^{-tA_V} is given by a kernel $p_V(t, x, y)$ that satisfies the Gaussian upper bound

$$|p_V(t, x, y)| \leqslant C_\varepsilon t^{-\frac{d}{2}} \mathrm{e}^{\varepsilon t + \delta\alpha t} \mathrm{e}^{-\frac{|x-y|^2}{4\delta t}}$$

for all $t > 0, \varepsilon > 0$ and a.e. $(x, y) \in \Omega \times \Omega$. Here α is as in (5.12), *C_ε is a positive constant depending only on η, ε, d and the constant c in* (5.3), *δ is a constant depending only on η, d and $\|a_{kj}\|_\infty$.*

If $V = H_0^1(\Omega)$, this estimate holds with $\varepsilon = 0$.

Remarks 5.5. (1) It is interesting to notice that when we apply (5.11) to the semigroup $(\mathrm{e}^{\lambda\phi}\mathrm{e}^{-tA_V}\mathrm{e}^{-\lambda\phi})_{t\geqslant 0}$, we can actually obtain a better estimate in (5.15). Simple calculations show that we can replace the term $\mathrm{e}^{\delta\alpha t}$ by $\mathrm{e}^{(1+\varepsilon')\alpha t}$ for every $\varepsilon' > 0$ (δ will depend then on ε'). Using this, the Gaussian upper bound in the previous theorem can be replaced by

$$|p_V(t,x,y)| \leqslant C_\varepsilon t^{-\frac{d}{2}}\mathrm{e}^{\varepsilon t}\mathrm{e}^{(\varepsilon'+1)\alpha t}\mathrm{e}^{-\frac{|x-y|^2}{4\delta t}}$$

for all $\varepsilon > 0, \varepsilon' > 0$ and all $t > 0$. The constants $C_\varepsilon, \alpha, \delta$ are as in the theorem, with the additional condition that δ depends on ε'.

(2) In case (ii) of the previous theorem, the assumption $\sum\limits_{j=1}^{d} D_j \operatorname{Im} a_{kj} \in L^\infty(\Omega)$ means that the potential m is bounded. By adapting the arguments developed for Schrödinger operators $-\Delta + m$ in [40], one can include some situations where m is not bounded (of course, the value of α must be changed).

As mentioned in the Introduction, related results on Gaussian upper bounds for operators with real-valued coefficients can be found in [5], [16], [4], [15]. For complex-valued coefficients, Gaussian upper bounds are proved in [9] in the case where Ω has Lipschitz boundary with a small Lipschitz constant and the coefficients a_{kj} are smooth (under these conditions, the authors obtain also Hölder continuity with respect to the space variables for the heat kernel).

We observe that the assumptions on V in Theorem 5.4 are satisfied for $V = H_0^1(\Omega)$ with arbitrary open set Ω and for $V = H^1(\Omega)$ or V as in (2.6), if Ω satisfies the extension property for example.

As mentioned in the beginning of the Introduction, there are several consequences of Gaussian upper bounds. We recall some of them here.

We assume that the assumptions of Theorem 5.4 hold. Then, the semigroup $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ extends to a holomorphic semigroup on $L^p(\Omega), 1 \leqslant p < \infty$, and the sector of holomorphy is the same as in $L^2(\Omega)$ (see [35], [23], [4]). In particular, if

$$a_{kj} = a_{jk}, b_k = \overline{c_k} \quad \text{and} \quad a_{kj}, a_0 \text{ are real} \tag{5.18}$$

for all $k, j \in \{1, \dots, d\}$, then $(\mathrm{e}^{-tA_V})_{t\geqslant 0}$ extends to a holomorphic semigroup on the open right half-plane on $L^p(\Omega)$, $1 \leqslant p < \infty$. Note also that under this condition, the spectrum of the corresponding generator in $L^p(\Omega)$ is independent of $p \in [1,\infty]$ (cf. [2], [17]). Assuming again (5.18), we obtain from the results on multipliers in [20] that for some constant w, the imaginary powers $(A+w)^{\mathrm{i}s}$, $s \in \mathbb{R}$, are bounded operators on $L^p(\Omega)$, $1 < p < \infty$, with norms estimated by $C_\varepsilon(1+|s|)^{d|\frac{1}{2}-\frac{1}{p}|+\varepsilon}$, for every $\varepsilon > 0$.

We turn now to another problem. We have assumed in Theorem 5.4 that the space V satisfies (5.14). This plays an important rôle in the proof of the Gaussian bound. Unfortunately, this assumption is not satisfied by some spaces V; and this reduces the range of boundary conditions to which the previous theorem applies. If V does not satisfy (5.14), we can prove a Gaussian upper bound by using another metric that takes into account the boundary conditions.

Let us say that a real-valued function $\phi \in W^{1,\infty}(\mathbb{R}^d)$ is V-admissible if

$$u \in V \Rightarrow \mathrm{e}^{\lambda\phi}u \in V, \quad \forall \lambda \in \mathbb{R}.$$

Set
$$W = \{\phi \in W^{1,\infty}(\mathbb{R}^d),\ \phi \text{ is } V\text{-admissible and } |\nabla u| \leqslant 1\}.$$
Now we define a metric ϱ_V on Ω by
$$\varrho_V(x,y) := \sup\{\phi(x) - \phi(y), \phi \in W\}.$$
We can repeat the same proof as above by taking ϕ in W. Thus, we obtain (5.16) with $\phi \in W$. We optimize over ϕ and obtain (5.17) with $\varrho_V(x,y)^2$ instead of $|x-y|^2$. Thus, we have proved:

Theorem 5.6. *Assume that* (2.2), (2.3), (4.1) *and* (5.1) *hold. Assume, in addition, that one of the following conditions are satisfied:*

(i) *the coefficients a_{kj} are real-valued for $1 \leqslant k, j \leqslant d$;*

(ii) $\sum\limits_{j=1}^{d} D_j \operatorname{Im} a_{kj} \in L^\infty(\Omega)$*,* $\operatorname{Im}(a_{kj} + a_{jk}) = 0$ *for* $1 \leqslant j, k \leqslant d$*,* (2.10) *and* (3.11) *hold.*

Then, the semigroup e^{-tA_V} *is given by a kernel* $p_V(t,x,y)$ *that satisfies the Gaussian upper bound*
$$|p_V(t,x,y)| \leqslant C_\varepsilon t^{-\frac{d}{2}} \mathrm{e}^{\varepsilon t + \delta \alpha t} \mathrm{e}^{-\frac{\varrho_V(x,y)^2}{4\delta t}}$$
for all $t > 0, \varepsilon > 0$ *and a.e.* $(x,y) \in \Omega \times \Omega$*. Here* α *is as in* (5.12)*,* C_ε *is a positive constant depending only on* η, ε, d *and the constant* c *in* (5.3)*,* δ *is a constant depending only on* η, d *and* $\|a_{kj}\|_\infty$*.*

Remark 5.7. We observe that in the case where $d = 1$, the assumption (4.1) can be relaxed in the previous theorem. Indeed, using (5.4) or (5.5) with $q = \infty$ we obtain an estimate for the $L^2 - L^\infty$ norm of e^{-tA_V} which is the main step in the proof. Note also that if Ω is an interval, then (5.4) is satisfied for every closed subspace V of $H^1(\Omega)$. Thus, if $d = 1$ and Ω is an interval, the above theorem holds for every elliptic operator A_V with complex-valued coefficients.

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