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FROBENIUS DUALITY IN C^*-TENSOR CATEGORIES

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Abstract. Rigidity in C^*-tensor categories is investigated from the viewpoint of Frobenius duality. Motivated by our previous studies on Jones index theory in bimodules, the existence and uniqueness are established for Frobenius duality. Similar results are proved for fiber functors, as well, with some applications to compact quantum groups.

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1. INTRODUCTION

In this paper, we shall investigate the renormalization of rigidity in C^*-tensor categories, which was inspired during studies of Jones index theory on subfactors.

Our idea goes back to [19], where fine structures of rigidity (called Frobenius duality in what follows) are found in the tensor category of bimodules of finite Jones index, i.e., the balanced renormalization of left and right dimensions (or traces) gives rise to a transparent formulation of algebraic structures behind Jones index theory (so called paragroups). See [10] and [12] for more sophisticated treatment of dimension or Jones index based on similar ideas.

The notion of Frobenius duality is tightly connected with the accompanied operation of taking conjugate objects and transposed intertwiners. The existence of such operations is obvious when one deals with bimodules based on von Neumann algebras due to the self-duality of Hilbert spaces.

On the other hand, in sector theory, another version of algebraic formulations of Jones index theory and C^*-tensor categories is possible as endomorphisms of infinite factors, where conjugates are only defined up to inner automorphisms and there are no canonical choices.

In spite of different appearances, they provide the equivalent information on algebraic data if one appeals to explicit computations. Seeking for the reason of such coincidences is one of major motivation in the present work and we shall give a definite solution to this problem.

Leaving detailed descriptions to later sections, the result is that, we can extract a systematic choice of rigidity pairings (i.e., Frobenius duality) in C^*-tensor categories, which turns out to be unique up to natural monoidal equivalences.

The existence of Frobenius duality has been effectively used in [8],[22] to construct random walk models of bimodules associated to C^*-tensor categories, whereas the uniqueness answers to the above question on equivalences of combinatorial data derived from C^*-tensor categories in various methods.

In the first section, we shall review formulations and results on (abstract properties of) conjugations in C^*-tensor categories.

The notion of conjugation is especially meaningful when it is related with the rigidity as investigated by many researchers ([14], [4] and so on) and, in Section 2, we shall further develop the idea of balanced renormalization in rigid C^*-tensor categories.

The notion of Frobenius duality (a fine structure of rigidity) is then formulated as a functorial choice of balanced rigidity pairings together with a compatible choice of conjugation.

All these are coupled, in Section 3, to show the existence and the uniqueness of (positive) Frobenius duality in rigid C^*-tensor categories of simple unit objects.

When the tensor category is realized as linear maps among finite-dimensional vector spaces, the rigidity turns out be the source of antipodes of Hopf algebras in Tannaka-Krein duality ([16]).

If the positivity comes into, i.e., if one deals with C^*-tensor categories of finite-dimensional Hilbert spaces, a kind of rigidity is utilized in [17] to reconstruct compact quantum groups.

However, one more step in renormalization of rigidity in the case of Tannaka duals gives rise to a strong form of positivity, which enables us to apply the reconstruction arguments in [20] and easily recover the Woronowicz's result on Tannaka-Krein duality (in a slightly generalized way).

2. INVOLUTIONS IN C^*-TENSOR CATEGORIES

A linear category $\mathcal{C}$ is called a *C^*-category* if hom-sets are complex Banach spaces with $*$-operation satisfying $\|f^*f\| = \|f\|^2$ for $f \in \mathrm{Hom}(X,Y)$. A monoidal category $\mathcal{C}$ is a *C^*-tensor category* if it is a C^*-category with unitary associativity and unit constraints and the $*$-operation is compatible with the tensor product $(f \otimes g)^* = f^* \otimes g^*$.

A monoidal functor F from a C^*-tensor category $\mathcal{C}$ into another C^*-tensor category $\mathcal{D}$ with multiplicativity $m_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ is *unitary* if it satisfies $F(f)^* = F(f^*)$ for $f \in \mathrm{Hom}(X,Y)$ and $m_{X,Y}$ is unitary for objects X, Y in $\mathcal{C}$.

An *involution* (or a conjugation) in a C^*-tensor category $\mathcal{C}$ is, by definition, a contravariant C^*-functor $X \mapsto X^*$, $\mathrm{Hom}(X,Y) \ni f \mapsto {}^tf \in \mathrm{Hom}(Y^*,X^*)$ (${}^t(f^*) = ({}^tf)^*$ particularly) with natural families of unitaries $\{c_{X,Y} : Y^* \otimes X^* \to (X \otimes Y)^*\}$ (conjugate multiplicativity) and $\{d_X : X \to (X^*)^*\}$ (duality) satisfying

$$\begin{array}{ccccc}
(X^* \otimes Y^*) \otimes Z^* & \xrightarrow{c\otimes 1} & (Y \otimes X)^* \otimes Z^* & \xrightarrow{c} & (Z \otimes (Y \otimes X))^* \\
{\scriptstyle a}\downarrow & & & & \downarrow{\scriptstyle {}^ta} \\
X^* \otimes (Y^* \otimes Z^*) & \xrightarrow[1\otimes c]{} & X^* \otimes (Z \otimes Y)^* & \xrightarrow[c]{} & ((Z \otimes Y) \otimes X)^*
\end{array}$$

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{d\otimes d} & X^{**} \otimes Y^{**} \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle c} \\ (X \otimes Y)^{**} & \xrightarrow[{}^t c]{} & (Y^* \otimes X^*)^* \end{array}$$

and ${}^t d_X = d_{X^*}^{-1} : X^{***} \to X^*$. (The naturality means that ${}^t(f \otimes g) \overset{c}{\sim} {}^t g \otimes {}^t f$ and $f \overset{d}{\sim} {}^t({}^t f)$.)

LEMMA 2.1. *For a unitary involution $f \mapsto {}^t f$ in a C^*-tensor category $\mathcal{C}$, we have $\|{}^t f\| = \|f\|$ for any $f : X \to Y$.*

Proof. By the relations $\|{}^t f\|^2 = \|{}^t f({}^t f)^*\| = \|{}^t(f^* f)\|$ and $\|f\|^2 = \|f^* f\|$, we may assume that $f \in \mathrm{End}(X)$. Then $\mathrm{End}(X) \ni f \mapsto {}^t f \in \mathrm{End}(X)$ defines a $*$-homomorphism from the C^*-algebra $\mathrm{End}(X)$ into the opposite C^*-algebra $\mathrm{End}(X)^\circ$. Thus we have $\|{}^t f\| \leqslant \|f\|$ and hence $\|f\| = \|{}^{tt} f\| \leqslant \|{}^t f\|$. ∎

In the remaining of this section, we shall present the coherence theorem on involutive monoidal categories, which are reduced to the ordinary ones if we drop off the information on involutions. For proofs, we refer to [2].

THEOREM 2.2. (Coherence Theorem with Involution) *Let $(\mathcal{C}, \otimes, a, I, l, r, *, c, d)$ be an involutive C^*-tensor category and $X = (X_1, \dots, X_k)$, $Y = (Y_1, \dots, Y_l)$, k, $l \geqslant 1$, be finite sequences of objects in $\mathcal{C}$. Let $\widetilde{X}$ be an object in $\mathcal{C}$ obtained by repetition of $*$-operations and tensor products from $X_1, \dots, X_k$ and similarly for $\widetilde{Y}$. (For example, $(X_1 \otimes (X_2 \otimes X_3^{**}))^* \otimes X_4^*$.)*

If there are any unitaries $\widetilde{X} \to \widetilde{Y}$ which are the form of products of a, l, r, c, d with combinations of taking transposed morphisms or inverses allowed, then they all coincide.

DEFINITION 2.3. Let $\mathcal{C}$, $\mathcal{C}'$ be involutive C^*-tensor categories. A monoidal functor $F : \mathcal{C} \to \mathcal{C}'$ is *involutive* if there is a natural family $\{s_X : F(X^*) \to F(X)^*\}$ of unitaries in $\mathcal{C}'$ satisfying

$$\begin{array}{ccccc} F(Y^*) \otimes F(X^*) & \xrightarrow{s\otimes s} & F(Y)^* \otimes F(X)^* & \xrightarrow{c} & (F(X) \otimes F(Y))^* \\ {\scriptstyle m}\downarrow & & & & \uparrow{\scriptstyle {}^t m} \\ F(Y^* \otimes X^*) & \xrightarrow[F(c)]{} & F((X \otimes Y)^*) & \xrightarrow[s]{} & F(X \otimes Y)^* \end{array}$$

$$\begin{array}{ccc} F(X^{**}) & \xrightarrow{s} & F(X^*)^* \\ {\scriptstyle F(d^{-1})}\downarrow & & \uparrow{\scriptstyle {}^t s} \\ F(X) & \xrightarrow[d]{} & F(X)^{**} \end{array} .$$

PROPOSITION 2.4. *Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ be involutive C^*-tensor functors. Then the monoidal functor $GF : \mathcal{C} \to \mathcal{C}''$ is involutive as well with respect to $m^{GF} = G(m^F)m^G$ and $s^{GF} = s^G G(s^F)$.*

DEFINITION 2.5. Let $F, G : \mathcal{C} \to \mathcal{C}'$ be involutive monoidal functors. A natural unitary transformation $\{\varphi_X : F(X) \to G(X)\}$ is called an *involutive monoidal equivalence* if it is monoidal (multiplicative) and satisfies

$$\begin{array}{ccc} F(X^*) & \xrightarrow{s^F} & F(X)^* \\ {\scriptstyle \varphi_{X^*}} \downarrow & & \uparrow {\scriptstyle {}^t\varphi_X} \\ G(X^*) & \xrightarrow[s^G]{} & G(X)^* \end{array} .$$

DEFINITION 2.6. An involutive monoidal C^*-functor $F : \mathcal{C} \to \mathcal{C}'$ is called an *involutive monoidal C^*-isomorphism* if there are an involutive monoidal C^*-functor $G : \mathcal{C}' \to \mathcal{C}$ and two involutive monoidal equivalences

$$\varphi : FG \cong \mathrm{id}_{\mathcal{C}'}, \quad \psi : GF \cong \mathrm{id}_{\mathcal{C}}.$$

Two involutive C^*-tensor categories $\mathcal{C}$, $\mathcal{C}'$ are said to be *isomorphic* if there is an involutive monoidal isomorphism $F : \mathcal{C} \to \mathcal{C}'$.

PROPOSITION 2.7. (Involutive Categorical Equivalence) *For an involutive C^*-tensor functor $F : \mathcal{C} \to \mathcal{C}'$, the following are equivalent:*

(i) *F is an involutive monoidal isomorphism;*

(ii) *for each object X' in $\mathcal{C}'$, there is an object X in $\mathcal{C}$ such that $F(X) \cong X'$ and we have*

$$F : \mathrm{Hom}(X, Y) \cong \mathrm{Hom}(F(X), F(Y))$$

for any X, Y.

THEOREM 2.8. *Any involutive C^*-tensor category is C^*-isomorphic to a strictly involutive C^*-tensor category.*

3. RIGIDITY IN C^*-TENSOR CATEGORIES

By coherence theorem, C^*-tensor categories is unitarily isomorphic to strict ones, whence we may assume strictness in C^*-tensor categories without loss of generality.

An object X in a (strict) C^*-tensor category $\mathcal{C}$ with the unit object I is said to be *rigid* if we can find an object X^* and a pair of morphisms $\varepsilon : X \otimes Y \to I$, $\delta : I \to Y \otimes X$ satisfying the *hook identities*:

$$X \to X \otimes (Y \otimes X) \to (X \otimes Y) \otimes X \to X,$$

$$Y \to (Y \otimes X) \otimes Y \to Y \otimes (X \otimes Y) \to Y$$

are identity morphisms.

Given a rigid object X, the object X^* is unique up to an isomorphism and is called a dual object of X, whereas the pair (δ, ε) is referred to as a rigidity pair.

A C^*-tensor category $\mathcal{C}$ is *rigid* if every object in $\mathcal{C}$ is rigid and is isomorphic to a dual of another object.

Now the following is an easy exercise of manipulation of hook identities.

Lemma 3.1. (Transposed Morphism Formula) *Let X^*, Y^* be duals of X, Y with rigidity pairs $(\varepsilon_X, \delta_X)$, $(\varepsilon_Y, \delta_Y)$ respectively. Then, for morphisms $f : X \to Y$ and $g : Y^* \to X^*$, the following four conditions are equivalent:*

$$\begin{tikzcd} X \otimes Y^* \arrow[r, "f\otimes 1"] \arrow[d, "1\otimes g"'] & Y \otimes Y^* \arrow[d, "\varepsilon_Y"] \\ X \otimes X^* \arrow[r, "\varepsilon_X"'] & I \end{tikzcd}, \qquad \begin{tikzcd} Y^* \arrow[r, "\delta_X\otimes 1"] \arrow[d, "g"'] & X^* \otimes X \otimes Y^* \arrow[d, "1\otimes f\otimes 1"] \\ X^* \arrow[r, "1\otimes\varepsilon_Y"'] & X^* \otimes Y \otimes Y^* \end{tikzcd},$$

$$\begin{tikzcd} X \arrow[r, "1\otimes\delta_Y"] \arrow[d, "f"'] & X \otimes Y^* \otimes Y \arrow[d, "1\otimes g\otimes 1"] \\ Y \arrow[r, "\varepsilon_X\otimes 1"'] & X \otimes X^* \otimes Y \end{tikzcd}, \qquad \begin{tikzcd} I \arrow[r, "\delta_X"] \arrow[d, "\delta_Y"'] & X^* \otimes X \arrow[d, "1\otimes f"] \\ Y^* \otimes Y \arrow[r, "g\otimes 1"'] & X^* \otimes Y \end{tikzcd}.$$

Given a morphism $f : X \to Y$, the morphism g in the above lemma is often denoted by ${}^t f$ and called the transposed of f (with respect to $(\varepsilon_X, \varepsilon_Y)$).

An object X in a C^*-tensor category is *simple* (semisimple) if $\mathrm{End}(X) = \mathbb{C}1_X$ (it is isomorphic to a direct sum of simple objects).

Lemma 3.2. (Longo-Roberts) *Let $\mathcal{C}$ be a C^*-tensor category with simple unit object and X be a rigid object in $\mathcal{C}$. Then* $\mathrm{End}(X)$ *is finite-dimensional.*

Proof. Let $\varepsilon : X \otimes X^* \to I$, $\delta : I \to X^* \otimes X$ be a rigidity pair and $F : \mathrm{End}(X) \to \mathrm{Hom}(I, X^* \otimes X)$ be the associated Frobenius transform $F(f) = (1 \otimes f)\delta$ and $F^{-1}(g) = (\varepsilon \otimes 1)(1 \otimes g)$. Since

$$\|F(f)\|^2 = \delta^*(1 \otimes f^* f)\delta \leqslant \|f\|^2 \|\delta\|^2,$$
$$\|F^{-1}(g)\|^2 = (\varepsilon \otimes 1)(1 \otimes g g^*)(\varepsilon^* \otimes 1) \leqslant \|g\|^2 \|\varepsilon\|^2,$$

the C^*-algebra $\mathrm{End}(X)$ is continuously isomorphic to the Hilbert space $\mathrm{Hom}(I, X^* \otimes X)$ with the bounded inverse, whence $\mathrm{End}(X)$ is reflexive as a Banach space, proving $\dim(\mathrm{End}(X)) < +\infty$. ∎

Corollary 3.3. *In a rigid C^*-tensor category with simple unit object, every object is isomorphic to a direct sum of finitely many simple objects, i.e., a rigid C^*-tensor category is semisimple if the unit object is simple.*

Lemma 3.4. (cf. [11]) *For a simple object X in a rigid C^*-tensor category, X^* is simple and $X^{**} \cong X$.*

In what follows, the C^*-tensor category $\mathcal{C}$ is assumed to be rigid.

Definition 3.5. A morphism $\varphi : X \otimes Y \to I$ is called a *duality pairing* if there is a morphism $\psi : I \to Y \otimes X$ such that $\{\varphi, \psi\}$ forms a rigidity pair.

Lemma 3.6. (Characterization of Rigidity Pairings) *Let X be an object in $\mathcal{C}$ and Y be a dual object of X. For a morphism $\varphi : Y \otimes X \to I$, the following conditions are equivalent:*

(i) *the morphism φ is a duality pairing;*

(ii) *the positive linear functional*

$$\mathrm{End}(X) \ni a \mapsto \varphi(a \otimes 1)\varphi^*$$

is faithful;

(iii) *the positive linear functional*

$$\mathrm{End}(Y) \ni b \mapsto \varphi(1 \otimes b)\varphi^*$$

is faithful.

Moreover, if φ satisfies one of these conditions, then there is a unique copairing $\psi : I \to Y \otimes X$ such that (φ, ψ) is a rigidity pair.

Proof. Assume that φ is one half of a rigidity pair. Then $\varphi(aa^* \otimes 1)\varphi^* = 0$, i.e., $\varphi(a \otimes 1) = 0$ implies $a = 0$ by hook identities.

Conversely assume that the functional $\varphi(a \otimes 1)\varphi^*$ is faithful. As Y is a dual object of X, we can find a rigidity pair $\varepsilon : X \otimes Y \to I$, $\delta : I \to Y \otimes X$ and the positive linear functional $a \mapsto \varepsilon(a \otimes 1)\varepsilon^*$ is faithful as just observed. Let $f = (\varphi \otimes 1)(1 \otimes \delta) \in \mathrm{End}(X)$ be the Frobenius transform of φ with $\varphi = \varepsilon(f \otimes 1)$. By the faithfulness of the functional

$$a \mapsto \varphi(a \otimes 1)\varphi^* = \varepsilon(faf^* \otimes 1)\varepsilon^*,$$

f is invertible in $\mathrm{End}(X)$ ($\mathrm{End}(X)$ being finite-dimensional). If we set

$$\psi = (1 \otimes f^{-1})\delta,$$

the pair $\{\varphi, \psi\}$ satisfies hook identities (note that $\varepsilon(f \otimes 1) = \varepsilon(1 \otimes {}^t f)$ and $(1 \otimes f)\delta = ({}^t f \otimes 1)\delta$ with $({}^t f)^{-1} = {}^t(f^{-1})$).

If ψ' is another copairing, $g = (\varphi \otimes 1)(1 \otimes \psi') \in \mathrm{End}(X)$ satisfies $\psi' = (1 \otimes g)\psi$ by hook identities for (φ, ψ), whereas $g = 1$ by the hook identity for (φ, ψ'). ∎

LEMMA 3.7. (Tracial Pairing) *Let X^* be a dual object of X.*

(i) *There is a duality pairing $\varepsilon : X \otimes X^* \to I$ such that the associated linear functional is tracial.*

(ii) *A duality pairing $\varepsilon : X \otimes X^* \to I$ gives a tracial functional if and only if ${}^t(f^*) = ({}^t f)^*$ for $f \in \mathrm{End}(X)$, where ${}^t f \in \mathrm{End}(X^*)$ is defined by $\varepsilon(f \otimes 1) = \varepsilon(1 \otimes {}^t f)$.*

Proof. (i) Let $\varphi : X \otimes X^* \to I$ be a duality pairing. Let τ be a faithful tracial state on $\mathrm{End}(X)$ and $0 \leqslant h \in \mathrm{End}(X)$ be the Radon-Nikodym derivative of the functional $\varphi(a \otimes 1)\varphi^*$ with respect to τ. By the previous lemma, h is invertible and $\varepsilon = \varphi(h^{-1/2} \otimes 1)$ gives a rigidity pair, which satisfies

$$\varepsilon(a \otimes 1)\varepsilon^* = \tau(a), \quad a \in \mathrm{End}(X).$$

(ii) For $a, b \in \mathrm{End}(X)$,

$$\varepsilon(ab \otimes 1)\varepsilon^* = \varepsilon(1 \otimes {}^t b {}^t a)\varepsilon^* = \varepsilon(1 \otimes {}^t b)(1 \otimes {}^t a)\varepsilon^* = \varepsilon(b \otimes 1)(1 \otimes {}^t a)\varepsilon^*.$$

If we set $h = ({}^t a)^*$, then $(1 \otimes {}^t a)\varepsilon^* = (\varepsilon(1 \otimes h))^* = (\varepsilon({}^t h \otimes 1))^*$ shows that

$$\varepsilon(ab \otimes 1)\varepsilon^* = \varepsilon(b({}^t h)^* \otimes 1)\varepsilon^*.$$

Thus the functional is tracial if and only if $({}^t h)^* = a$, i.e., $({}^t a)^* = {}^t(a^*)$ for any $a \in \mathrm{End}(X)$. ∎

DEFINITION 3.8. A rigidity pair $\{\varepsilon : X\otimes X^* \to I, \delta : I \to X^*\otimes X\}$ (or just ε as δ is determined by ε) is *balanced* if

$$\varepsilon(a\otimes 1)\varepsilon^* = \delta^*(1\otimes a)\delta \quad \text{for any } a \in \mathrm{End}(X).$$

Note that, given a balanced pair $\{\varepsilon, \delta\}$, the pair $\{\delta^*, \varepsilon^*\}$ is also balanced because of

$$\delta^*(b\otimes 1)\delta = \delta^*(1\otimes {}^t b)\delta = \varepsilon({}^t b\otimes 1)\varepsilon^* = \varepsilon(1\otimes b)\varepsilon^*.$$

LEMMA 3.9. (Existence and Uniqueness)

(i) *For any X with a dual object X^*, there exists a balanced duality pairing.*

(ii) *For a balanced duality pairing $\varepsilon : X\otimes X^* \to I$, the associated functional $\varepsilon(a\otimes 1)\varepsilon^*$ is tracial.*

(iii) *Let $\varepsilon_X : X\otimes X^* \to I$ and $\varepsilon_Y : Y\otimes Y^* \to I$ be balanced duality pairings with X and Y isomorphic (whence $X^* \cong Y^*$). Then, given a unitary $v : X^* \to Y^*$, there is a unique unitary $u : X \to Y$ such that*

$$\varepsilon_X = \varepsilon_Y(u\otimes v).$$

Conversely, given a balanced duality pairing ε_Y and unitaries $u : X \to Y$, $v : X^* \to Y^*$, the duality pairing $\varepsilon_X = \varepsilon_Y(u\otimes v)$ is balanced.

Proof. For a simple object X, a rigidity pair $\{\varepsilon, \delta\}$ is balanced if and only if $\varepsilon\varepsilon^* = \delta^*\delta$. Given a rigidity $\{\varepsilon, \delta\}$ of X, $\{\mu\varepsilon, \mu^{-1}\delta\}$ is again a rigidity and hence we can choose $\mu \in \mathbb{C}$ so that $\{\mu\varepsilon, \mu^{-1}\delta\}$ is balanced.

Let $\mathcal{R}$ be a representative set of isomorphism classes of simple objects in $\mathcal{C}$ and for each $X \in \mathcal{R}$, take its dual X^* inside $\mathcal{R}$ (particularly $X^{**} = X$) and choose a balanced pair $\{\varepsilon_X, \delta_X\}$.

For an arbitrary object X with a specified dual object X^*, which can be decomposed as

$$X \cong \bigoplus_j m_j X_j, \quad X_j \in \mathcal{R},\ m_j \in \{1,2,3,\dots\},$$

choose mutually orthogonal coisometries $v_{j,k} : X \to X_j$, $w_{j,k} : X^* \to X_j^*$ for $k = 1,\dots,m_j$ and set

$$\varepsilon_X = \sum_j \sum_{k=1}^{m_j} \varepsilon_j(v_{j,k}\otimes w_{j,k}),$$

$$\delta_X = \sum_j \sum_{k=1}^{m_j} (w_{j,k}^*\otimes v_{j,k}^*)\delta_j,$$

where $\varepsilon_j = \varepsilon_{X_j}$ and $\delta_j = \delta_{X_j}$.

Then it is immediate to check the hook identities for the pair $\{\varepsilon_X, \delta_X\}$. Moreover, letting $e_{k,l} = v_{j,k}^* v_{j,l}$ with $1 \leqslant k,l \leqslant m_j$, we have

$$\begin{aligned}\varepsilon_X(e_{k,l}\otimes 1)\varepsilon_X^* &= \sum_{a,b} \varepsilon_j(v_{j,a}\otimes w_{j,a})(e_{k,l}\otimes 1)(v_{j,b}^*\otimes w_{j,b}^*)\varepsilon_j^* \\ &= \sum_a \varepsilon_j(v_{j,a}e_{k,l}v_{j,a}^*\otimes 1_{X_j^*})\varepsilon_j^* \\ &= \delta_{k,l}\varepsilon_j\varepsilon_j^*\end{aligned}$$

and similarly

$$\delta_X^*(1 \otimes e_{k,l})\delta_X = \delta_{k,l}\delta_j^*\delta_j,$$

which shows that the rigidity pair $\{\varepsilon_X, \delta_X\}$ is balanced and at the same time the associated functional is tracial.

Now let $u : X \to Y$, $v : X^* \to Y^*$ be unitaries. If we consider the pair $\varepsilon = \varepsilon_X(u^* \otimes 1) : Y \otimes X^* \to I$, $\delta = (1 \otimes u)\delta_X : I \to X^* \otimes Y$, then it satisfies one half of the hook identities

$$X^* \xrightarrow{\delta\otimes 1} X^* \otimes Y \otimes X^* \xrightarrow{1\otimes\varepsilon} X^*,$$

whereas the associated functional of ε is faithful and tracial. Then by the uniqueness of copairing (Lemma 3.6), δ is in fact the copairing of ε. In particular, it satisfies the other hook identity

$$Y \xrightarrow{1\otimes\delta} Y \otimes X^* \otimes Y \xrightarrow{\varepsilon\otimes 1} Y.$$

In this way we have a balanced tracial pair $\{\varepsilon_X(u^*\otimes 1), (1\otimes u)\delta_X\}$. Similarly, we know that $\{\varepsilon_X(1 \otimes v^*), (v \otimes 1)\delta_X\}$ is a balanced tracial pair.

Thus, for the proof of uniqueness, we may assume that we are given two balanced tracial pairings $\varepsilon : X \otimes X^* \to I$, $\varepsilon' : X \otimes X^* \to I$ and need to show the unique unitary $u \in \mathrm{End}(X)$ satisfying $\varepsilon' = \varepsilon(u \otimes 1)$.

Let $h \in \mathrm{End}(X)$ be the Frobenius transform of ε' with respect to the rigidity pair $\{\varepsilon, \delta\}$. Then $\varepsilon' = \varepsilon(h \otimes 1)$ and its copairing is given by $\delta' = (1 \otimes h^{-1})\delta$. The duality pairing ε' being assumed to be balanced, we have

$$\varepsilon(h \otimes 1)(a \otimes 1)(h^* \otimes 1)\varepsilon^* = \delta^*(1 \otimes (h^{-1})^*)(1 \otimes a)(1 \otimes h^{-1})\delta$$

for $a \in \mathrm{End}(X)$. Since $\tau(a) = \varepsilon(a\otimes 1)\varepsilon^* = \delta^*(1\otimes a)\delta$ is tracial by our assumption, the above condition is equivalent to $\tau(h^*ha) = \tau(h^{-1}(h^{-1})^*a)$, i.e., $h^*h = (h^*h)^{-1}$. Since $h^*h \geqslant 0$, we have $h^*h = 1$, which implies the unitarity of h because $\mathrm{End}(X)$ is finite-dimensional.

Finally, the functional

$$\varepsilon'(a \otimes 1)(\varepsilon')^* = \varepsilon(hah^* \otimes 1)\varepsilon^* = \varepsilon(a \otimes 1)\varepsilon^*$$

is tracial. ∎

COROLLARY 3.10. *For an object X, the trace τ_X on $\mathrm{End}(X)$ defined by*

$$\tau_X(a) = \varepsilon_X(a \otimes 1)\varepsilon_X^*$$

with $\varepsilon_X : X^* \otimes X \to I$ a balanced duality pairing is independent of the choice of ε_X.

LEMMA 3.11. (Multiplication and Addition) *Let $\varepsilon_X : X \otimes X^* \to I$ and $\varepsilon_Y : Y \otimes Y^* \to I$ be balanced duality pairings.*

(i) *The composite morphism*

$$X \otimes Y \otimes Y^* \otimes X^* \xrightarrow{1\otimes\varepsilon_Y\otimes 1} Y \otimes Y^* \xrightarrow{\varepsilon_X} I$$

gives a balanced duality pairing for the tensor product $X \otimes Y$.

(ii) *Let $p : X \oplus Y \to X$, $q : X \oplus Y \to Y$, $\overline{p} : X^* \oplus Y^* \to X^*$ and $\overline{q} : X^* \oplus Y^* \to Y^*$ be the obvious coisometries. Then the morphism*

$$\varepsilon_{X\oplus Y} = \varepsilon_X(p \otimes \overline{p}) + \varepsilon_Y(q \otimes \overline{q})$$

defines a balanced duality pairing for the direct sum $X \oplus Y$.

Proof. (i) Let $\delta_X : I \to X^* \otimes X$ and $\delta_Y : I \to Y^* \otimes Y$ be the accompanied copairings. Then $\varepsilon = \varepsilon_X(1 \otimes \varepsilon_Y \otimes 1)$ and $\delta = (1 \otimes \delta_X \otimes 1)\delta_Y$ form a rigidity pair. For $a \in \mathrm{End}(X \otimes Y)$, let $b \in \mathrm{End}(X)$ and $c \in \mathrm{End}(Y)$ be defined by

$$X \xrightarrow{1\otimes\varepsilon_Y^*} X \otimes Y \otimes Y^* \xrightarrow{a\otimes 1} X \otimes Y \otimes Y^* \xrightarrow{1\otimes\varepsilon_Y} X$$
$$Y \xrightarrow{\delta_X\otimes 1} X^* \otimes X \otimes Y \xrightarrow{1\otimes a} X^* \otimes X \otimes Y \xrightarrow{\delta_X^*\otimes 1} Y\,.$$

Then we have

$$\begin{aligned}
\varepsilon_X(1 \otimes \varepsilon_Y \otimes 1)(a \otimes 1)(1 \otimes \varepsilon_Y^* \otimes 1)\varepsilon_X^* &= \varepsilon_X(b \otimes 1)\varepsilon_X^* \\
&= \delta_X^*(1 \otimes b)\delta_X \\
&= (\delta_X^* \otimes \varepsilon_Y)(1 \otimes a \otimes 1)(\delta_X \otimes \varepsilon_Y^*) \\
&= \varepsilon_Y(\delta_X^* \otimes 1)(1 \otimes a \otimes 1)(d_X \otimes 1)\varepsilon_Y^* \\
&= \varepsilon_Y(c \otimes 1)\varepsilon_Y^* \\
&= \delta_Y^*(1 \otimes c)\delta_Y \\
&= \delta_Y^*(1 \otimes \delta_X^* \otimes 1)(1 \otimes a)(1 \otimes \delta_X \otimes 1)\delta_Y.
\end{aligned}$$

(ii) If we define a morphism $\delta_{X\oplus Y} : I \to (X^* \oplus Y^*) \otimes (X \oplus Y)$ by

$$\delta_{X\oplus Y} = (\overline{p} \otimes p)^*\delta_X + (\overline{q} \otimes q)^*\delta_Y,$$

it gives the accompanied copairing of $\varepsilon_{X\oplus Y}$ and, for $a \in \mathrm{End}(X \oplus Y)$,

$$\begin{aligned}
\varepsilon_{X\oplus Y}(a \otimes 1)\varepsilon_{X\oplus Y}^* &= \varepsilon_X(p \otimes \overline{p})(a \otimes 1)(p \otimes \overline{p})^*\varepsilon_X^* + \varepsilon_Y(q \otimes \overline{q})(a \otimes 1)(q \otimes \overline{q})^*\varepsilon_Y^* \\
&= \varepsilon_X(pap^* \otimes 1)\varepsilon_X^* + \varepsilon_Y(qaq^* \otimes 1)\varepsilon_Y^* \\
&= \delta_X^*(1 \otimes pap^*)\delta_X + \delta_Y^*(1 \otimes qaq^*)\delta_Y \\
&= \delta_{X\oplus Y}^*(1 \otimes a)\delta_{X\oplus Y}. \quad \blacksquare
\end{aligned}$$

4. FROBENIUS DUALITY

Let $\mathcal{C}$ be a rigid (strict) C^*-tensor category. By a Frobenius duality in $\mathcal{C}$, we shall mean a family of morphisms $\{\varepsilon_X : X \otimes X^* \to I\}$ together with a unitary involution $(*, t, c, d)$ in $\mathcal{C}$ satisfying:

(i) (multiplicativity)

$$\begin{array}{ccc}
X \otimes Y \otimes Y^* \otimes X^* & \xrightarrow{1\otimes c} & X \otimes Y \otimes (X \otimes Y)^* \\
{\scriptstyle 1\otimes\varepsilon_Y\otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_{XY}} \\
X \otimes X^* & \xrightarrow[\varepsilon_X]{} & I\,;
\end{array}$$

(ii) (naturality) For a morphism $f : X \to Y$,

$$\begin{array}{ccc}
X \otimes Y^* & \xrightarrow{f\otimes 1} & Y \otimes Y^* \\
{\scriptstyle 1\otimes {}^t f}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_Y} \\
X \otimes X^* & \xrightarrow[\varepsilon_X]{} & I\,;
\end{array}$$

(iii) (faithfulness) The map

$$\mathrm{Hom}(X,Y) \ni f \mapsto \varepsilon_Y \circ (f \otimes 1) \in \mathrm{Hom}(X \otimes Y^*, I)$$

is injective for X, $Y \in \mathrm{Object}(\mathcal{C})$;

(iv) (neutrality) For $f \in \mathrm{End}(X)$,

$$\varepsilon_X(fd_X^{-1} \otimes 1)c_{X,X^*}^{-1}\,{}^t\varepsilon_X = \varepsilon_{X^*}(1 \otimes d_X f d_X^{-1})^t(1 \otimes d_X)^t\varepsilon_{X^*}.$$

REMARK 4.1. The neutrality condition is simply written as

$$\varepsilon_X(f \otimes 1)^t\varepsilon_X = \varepsilon_{X^*}(1 \otimes f)^t\varepsilon_{X^*}$$

when the involution is strict.

DEFINITION 4.2. A Frobenius duality is *positive* if it satisfies

$$(d_X^{-1} \otimes 1)c_{X,X^*}^{-1}\,{}^t\varepsilon_X\varepsilon_I^{-1} = \varepsilon_X^*$$

for any object X in $\mathcal{C}$, where $\varepsilon_I : I \otimes I^* \to I$ is identified with the morphism $I^* \to I$ ($\mathcal{C}$ being assumed to be strict).

In what follows, we require the normalization $I^* = I$ and $d_I = 1_I$ for the involution to avoid inessential complications.

In a C^*-tensor category with positive Frobenius duality, hom-sets are naturally Hilbert spaces: For f, $g \in \mathrm{Hom}(X,Y)$, we define the inner product by

$$(f|g) = \tau_X(f^*g).$$

We record here the following fact to realize the role of neutrality in Frobenius dualities (see [21] for more information).

PROPOSITION 4.3. *The inner products in hom-sets are invariant under $f \mapsto f^*$, $f \mapsto {}^tf$ and Frobenius transforms.*

Proof. For f, $g \in \mathrm{Hom}(X,Y)$,

$$(g^*|f^*) = \tau_Y(gf^*) = \tau_X(f^*g) = (f|g)$$

and

$$({}^tf|{}^tg) = \tau_{Y^*}({}^tf^{*t}g) = \tau_{Y^*}({}^t(gf^*)) = \tau_Y(gf^*) = (f|g).$$

For f, $g \in \mathrm{Hom}(XY,Z)$, let F, $G \in \mathrm{Hom}(X,ZY^*)$ be their Frobenius transforms:

$$F = (f \otimes 1_{Y^*})(1_X \otimes \delta_Y), \quad G = (g \otimes 1_{Y^*})(1_X \otimes \delta_Y).$$

Then we have

$$\begin{aligned}\tau_X(F^*G) &= \tau_X((1_X \otimes \delta_Y^*)(f^*g \otimes 1_{Y^*})(1_X \otimes \delta_Y))\\ &= \delta_X^*(h \otimes 1_{X^*})\delta_X\\ &= \delta_X^*(1_X \otimes \delta_Y^* \otimes 1_{X^*})(f^*g \otimes_{Y^*X^*})(1_X \otimes \delta_Y \otimes 1_{X^*})\delta_X\\ &= \delta_{XY}^*(f^*g \otimes 1_{Y^*X^*})\delta_{XY}\\ &= \tau_{XY}(f^*g). \quad \blacksquare\end{aligned}$$

LEMMA 4.4. *The following diagram commutes:*

$$\begin{array}{ccc} X^* \otimes X^{**} & \xrightarrow{c} & (X^* \otimes X)^* \\ {\scriptstyle \varepsilon_{X^*}}\big\downarrow & & \big\downarrow{\scriptstyle {}^t\delta_X} \\ I & = & I^* \end{array}.$$

Proof. By the commutativity of the diagram

$$\begin{array}{ccccc} X^* \otimes X^{**} & \xrightarrow{\delta_X \otimes 1} & X^* \otimes X \otimes X^* \otimes X^{**} & \xrightarrow{1\otimes\varepsilon_X\otimes 1} & X^* \otimes X^{**} \\ {\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle 1\otimes c} & & \big\downarrow{\scriptstyle \varepsilon_{X^*}} \\ (X^* \otimes X)^* & \xrightarrow[\delta_X \otimes 1]{} & X^* \otimes X \otimes (X^* \otimes X)^* & \xrightarrow[\varepsilon_{X^*X}]{} & I \\ {\scriptstyle {}^t\delta_X}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_{X^*X}} & & \\ I^* & = & I & & \end{array}$$

we see that ${}^t\delta_X c$ is equal to $\varepsilon_{X^*}(1\otimes\varepsilon_X\otimes 1)(\delta_X\otimes 1) = \varepsilon_{X^*}$, which gives the claimed identity. ∎

LEMMA 4.5. *Given a positive Frobenius duality $\{*, t, c, d, \varepsilon\}$, the morphisms ε_X are balanced duality pairings.*

Proof. By the above lemma, ${}^t\delta_X = \varepsilon_{X^*}c^{-1}$. If we combine this with the commutative diagram

$$\begin{array}{ccccc} I^{**} & \xrightarrow{{}^{tt}\delta_X} & (X^*X)^{**} & \xrightarrow{{}^t c} & (X^*X^{**})^* \\ \| & & {\scriptstyle d_{X^*X}}\big\uparrow & & \big\uparrow{\scriptstyle c} \\ I & \xrightarrow[\delta_X]{} & X^*X & \xrightarrow[d_{X^*}\otimes d_X]{} & X^{***}X^{**} \end{array}$$

then we have

$$\delta_X = ({}^t d_X \otimes d_X^{-1})c^{-1}_{X^{**},X^*}\,{}^t\varepsilon_{X^*}.$$

Thus $\{\varepsilon_X, {}^t\varepsilon_{X^*}\}$ gives a rigidity pair, where ${}^t\varepsilon_{X^*}$ is used to denote the composite morphism

$$I \xrightarrow{{}^t\varepsilon_{X^*}} (X^*X^{**})^* \xrightarrow{c^{-1}} X^{***}X^{**} \xrightarrow{{}^t d\otimes d^{-1}} X^*X\,.$$

By using the positivity $\varepsilon_X^* = {}^t\varepsilon_X$, the neutrality property turns out to be nothing but the balancedness relation. ∎

In what follows, we shall show the existence and the uniqueness of the positive Frobenius duality in rigid C^*-tensor categories.

Let $\mathcal{C}$ be a rigid (strict) C^*-tensor category. We choose a representative set $\mathcal{R}_0$ of simple objects in $\mathcal{C}$ with $I \in \mathcal{R}_0$. For non self-dual object X in $\mathcal{R}_0$, there is a unique dual object X^* in $\mathcal{R}_0$. For a self-dual object X in $\mathcal{R}_0 \setminus \{I\}$, we choose its dual X^* so that $X^* \neq X$ (particularly, $X^* \notin \mathcal{R}_0$). We now set

$$\mathcal{R} = \mathcal{R}_0 \sqcup \{X^*; X \text{ is a self-dual object in } \mathcal{R}_0 \setminus \{I\}\}.$$

By letting $I^* = I$ and $(X^*)^* = X$ for $X \in \mathcal{R}_0$, we obtain an involution $*$ in $\mathcal{R}$ so that X^* is a dual object of $X \in \mathcal{R}$.

Let $\overline{\mathcal{R}}$ be the category with objects given by free (associative) products generated by the set $\mathcal{R}$ and hom-sets given by

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(\overline{X}, \overline{Y}),$$

where $\overline{X}$ and $\overline{Y}$ are objects in $\mathcal{C}$ defined inductively by

$$\overline{XY} = \overline{X} \otimes \overline{Y}.$$

(This is well-defined by the strictness assumption or the coherence theorem.)

The category $\overline{\mathcal{R}}$ is a C^*-tensor category in the obvious way and unitarily isomorphic to $\mathcal{C}$ by the obvious imbedding functor $\overline{\mathcal{R}} \to \mathcal{C}$. The operation $*$ is uniquely extended to objects in $\overline{\mathcal{R}}$ so that $(XY)^* = Y^*X^*$. Note that $X^{**} = X$ for any object X in $\overline{\mathcal{R}}$.

We choose balanced rigidity pairs $\{\varepsilon_X : X \otimes X^* \to I, \delta_X : I \to X^* \otimes X\}$ for objects in $\mathcal{R}$ so that

$$\varepsilon_X^* = \delta_{X^*} \quad \text{for } X \in \mathcal{R}.$$

(To ensure this relation, we have enlarged $\mathcal{R}_0$ to $\mathcal{R}$.)

For objects in $\overline{\mathcal{R}}$, rigidity pairs are defined inductively by the relation

$$\varepsilon_{XY} = \varepsilon_X(1_X \otimes \varepsilon_Y \otimes 1_{X^*}) : (XY)(Y^*X^*) \to I,$$
$$\delta_{XY} = (1_{Y^*} \otimes \delta_X \otimes 1_Y)\delta_Y : I \to (Y^*X^*)(XY),$$

which are balanced by Lemma 3.11.

Note that the relation $\varepsilon_X^* = \delta_{X^*}$ remains valid for objects in $\overline{\mathcal{R}}$.

LEMMA 4.6. *For morphisms $f : X \to Y$ and $g : Y^* \to X^*$ in $\overline{\mathcal{R}}$, the equality*

$$\varepsilon_Y(f \otimes 1_{Y^*}) = \varepsilon_X(1_X \otimes g)$$

holds if and only if

$$\varepsilon_{Y^*}(1 \otimes f) = \varepsilon_{X^*}(g \otimes 1).$$

Proof. The first equation is $g = {}^tf$. Taking the adjoint in the relation

$$\varepsilon_X(f^* \otimes 1) = \varepsilon_Y(1 \otimes {}^t(f^*)),$$

we have

$$(f \otimes 1)\delta_{X^*} = (f \otimes 1)\varepsilon_X^* = (1 \otimes {}^tf)\varepsilon_Y^* = (1 \otimes {}^tf)\delta_{Y^*}$$

(note that ${}^t(f^*) = ({}^tf)^*$ by Lemma 3.7, Lemma 3.9 and Lemma 3.11), i.e., ${}^tf$ is the transposed morphism of f with respect to the rigidity pairs $\{\varepsilon_{X^*}, \delta_{X^*}\}$, $\{\varepsilon_{Y^*}, \delta_{Y^*}\}$ and hence we have

$$\varepsilon_{Y^*}(1 \otimes f) = \varepsilon_{X^*}({}^tf \otimes 1) = \varepsilon_{X^*}(g \otimes 1)$$

by Lemma 3.1. ∎

THEOREM 4.7. *Every rigid C^*-tensor category admits a positive Frobenius duality, which is unique up to unitary isomorphisms, i.e., given two Frobenius dualities $\{\varepsilon_X, X^*, {}^tf, c, d\}$ and $\{\varepsilon'_X, X^{*'}, {}^{t'}f, c', d'\}$, we can find a family of unitaries $\{s_X : X^* \to X^{*'}\}$ satisfying:*

(i) (Equivariance)

$$\begin{array}{ccc} X^* \otimes X & \xrightarrow{\varepsilon_X} & I \\ {\scriptstyle s_X}\big\downarrow & & \big\| \\ X^{*'} \otimes X & \xrightarrow[\varepsilon'_X]{} & I \end{array} ;$$

(ii) (Naturality)

$$\begin{tikzcd} X^* \arrow[r, "s_X"] & X^{*'} \\ Y^* \arrow[u, "{}^t f"] \arrow[r, "s_Y"'] & Y^{*'} \arrow[u, "{}^{t'} f"'] \end{tikzcd}$$

for $f : X \to Y$*;*

(iii) (Multiplicativity)

$$\begin{tikzcd} Y^* \otimes X^* \arrow[r, "s \otimes s"] \arrow[d, "c"'] & Y^{*'} \otimes X^{*'} \arrow[d, "c'"] \\ (X \otimes Y)^* \arrow[r, "s"'] & (X \otimes Y)^{*'} \end{tikzcd} ;$$

(iv) (Duality)

$$\begin{tikzcd} X^{**} \arrow[r, "s_{X^*}"] & X^{**'} \\ X \arrow[u, "d"] \arrow[r, "d'"'] & X^{*'*'} \arrow[u, "{}^{t'} s_X"'] \end{tikzcd} .$$

Proof. Let $\mathcal{C}$ be a rigid (strict) C^*-tensor category. By the previous discussion, $\mathcal{C}$ is unitarily isomorphic to the C^*-tensor category $\overline{\mathcal{R}}$, which is furnished with a special family of duality pairings $\{\varepsilon_X : X^*X \to I\}$.

Given a morphism $f : X \to Y$ in $\overline{\mathcal{R}}$, we define ${}^t f : Y^* \to X^*$ by the relation

$$\varepsilon_Y(f \otimes 1_{Y^*}) = \varepsilon_X(1 \otimes {}^t f),$$

namely,

$${}^t f = (1 \otimes \varepsilon_Y)(1 \otimes f \otimes 1)(\delta_X \otimes 1)$$

by Lemma 3.1. Then we have ${}^t(f \otimes g) = {}^t g \otimes {}^t f$ from definition. Moreover, by the previous lemma and the fact that $X^{**} = X$ in $\overline{\mathcal{R}}$, we have ${}^{tt} f = f$. Since ε_X is balanced, Lemma 3.9 and Lemma 3.7 ensures the relation ${}^t(f^*) = ({}^t f)^*$. Thus we can define a unitary involution by letting $c_{X,Y} = 1_{Y^*X^*}$, $d = \{1_X\}$ and we see that the family $\{\varepsilon_X\}$ meets the requirements of Frobenius duality except for the neutrality condition.

We now claim ${}^t\varepsilon_X = \delta_{X^*}$. In fact, from Lemma 3.1 and our definitions, we have

$${}^t\varepsilon_X = (1 \otimes \varepsilon_X)(1 \otimes \delta_X \otimes 1)\delta_{X^*}$$

and then the hook identity is used to get

$$(\varepsilon_{X^*} \otimes 1_X)(1_{X^*} \otimes {}^t\varepsilon_X) = 1_{X^*},$$

which implies ${}^t\varepsilon_X = \delta_{X^*}$ by the uniqueness of copairings (Lemma 3.6). Since $\delta_{X^*} = \varepsilon_X^*$ by our choice, we have ${}^t\varepsilon_X = \varepsilon_X^*$, showing the neutrality condition as ε_X being balanced and, at the same time, the positivity of the Frobenius duality $\{\varepsilon_X\}$. So far we have checked the existence of a positive Frobenius duality in the tensor category $\overline{\mathcal{R}}$. Now the existence part is a consequence of Lemma 4.8 below.

Conversely, assume that two positive Frobenius dualities $\{\varepsilon, *, t, c, d\}$ and $\{\varepsilon', *', t', c', d'\}$ in a rigid (strict) C^*-tensor category $\mathcal{C}$ are given. Since $\{\varepsilon_X\}$ and $\{\varepsilon'_X\}$ are families of balanced duality pairings, the uniqueness result (Lemma 3.9) enables us to define a unitary $s_X : X^* \to X^{*'}$ so that $\varepsilon'_X(s_X \otimes 1_X) = \varepsilon_X$ for each X.

The family $\{s_X\}$ is obviously natural in X and multiplicative as can be easily seen, i.e., $s = \{s_X\}$ gives a monoidal equivalence between antimultiplicative functors t and t' ($t \overset{s}{\simeq} t'$). Then $t^2 \overset{s^2}{\simeq} t'^2$ and, if we compose this with the monoidal equivalence $t'^2 \overset{d'}{\simeq} \mathrm{id}$, we obtain another duality family d'' defined by

$$X \xrightarrow{d'_X} X^{*'*'} \xrightarrow{t' s_X} X^{**'} \xrightarrow{s^{-1}_{X^*}} X^{**}.$$

Thus, we may assume that

$$X^* = X^{*'}, \varepsilon_X = \varepsilon'_X, t = t', c = c'$$

for any X in $\mathcal{C}$ and the difference is stacked up to duality isomorphisms

$$d_X, d''_X : X \to X^{**}.$$

However, from positivity of Frobenius duality, we have

$$(1 \otimes d_X)\varepsilon^*_X = {}^t\varepsilon_X = (1 \otimes d''_X)\varepsilon^*_X$$

($X^* \otimes X^{**}$ and $(X^* \otimes X)^*$ being identified by the conjugate multiplicativity $c = c'$), where the faithfulness axiom can be used to conclude $d_X = d''_X$ for any X. ∎

LEMMA 4.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be rigid strict C^*-tensor categories with simple unit objects. Assume that $\mathcal{D}$ is furnished with a positive Frobenius duality with the strict involution and $G : \mathcal{D} \to \mathcal{C}$ be an isomorphic functor of C^*-tensor categories with the trivial multiplicativity ($G(XY) = G(X)G(Y)$ and $m^G_{X,Y} = \mathrm{id}$) such that $G(I_{\mathcal{D}}) = I_{\mathcal{C}}$.*

Then $\mathcal{C}$ admits a positive Frobenius duality.

Proof. Choose a C^*-tensor functor $F : \mathcal{C} \to \mathcal{D}$ with multiplicativity m^F so that $F(I_{\mathcal{C}}) = I_{\mathcal{D}}$ and the composite functors $G \circ F$, $F \circ G$ are unitarily and monoidally equivalent to the identity functors.

We can then define an involution in $\mathcal{C}$ by

$$X^* = G(F(X)^*), \quad {}^tf = G({}^tF(f)).$$

The conjugate multiplicativity $c_{X,Y}$ and the duality d_X are defined by

$$Y^*X^* = G((F(X)F(Y))^*) \xleftarrow{G({}^tm^F)} G(F(XY)^*) = (XY)^*$$

$$X \longleftarrow G(F(X)) = G(F(X)^{**}) \longrightarrow G((FG(F(X)^*))^*) = X^{**}.$$

Moreover, define a morphism $\varepsilon_X : XX^* \to I$ so that $F(\varepsilon_X)$ is given by

$$F(XX^*) \xleftarrow{m^F} F(X)F(X^*) = F(X)FG(F(X)^*) \longrightarrow F(X)F(X)^* \xrightarrow{\varepsilon_{F(X)}} I.$$

Now it is straightforward to check that these give a positive Frobenius duality in $\mathcal{C}$. ∎

5. FROBENIUS DUALITY IN TANNAKA DUALS

LEMMA 5.1. *Let V and W be finite-dimensional Hilbert spaces and $\varepsilon : W \otimes V \to \mathbb{C}$ be a non-degenerate bilinear form. Then there is a unique unitary map $u : V^* \to W$ such that*

$$\varepsilon(u \otimes 1) : V^* \otimes V \to \mathbb{C}$$

is positive.

Proof. This is just a reformulation of the existence and the uniqueness of polar decompositions of linear operators on Hilbert spaces. ∎

THEOREM 5.2. *Let $\mathcal{C}$ be a rigid C^*-tensor category and F be a C^*-tensor functor from $\mathcal{C}$ into the C^*-tensor category $\mathcal{H}$ of finite-dimensional Hilbert spaces. Let $\{\varepsilon_X\}$ be a positive Frobenius duality in $\mathcal{C}$. Then there is a unitarily equivalent C^*-tensor functor $G : \mathcal{C} \to \mathcal{H}$ satisfying:*

(i) *$G(X^*) = G(X)^*$, $G({}^t f) = {}^t G(f)$ and $G(d_X)$ is the canonical identification of $G(X)$ with $G(X)^{**}$ for any object X in $\mathcal{C}$;*

(ii) *$G(\varepsilon_X) : G(X)^* \otimes G(X) \to \mathbb{C}$ is positive definite.*

Such functors are unique in the sense that, if we have two such functors G_1 and G_2, then we can find a family of unitary maps $\{\varphi_X : G_1(X) \to G_2(X)\}$ such that $\varphi_{XY} = \varphi_X \otimes \varphi_Y$, $\varphi_{X^*} = \overline{\varphi}_X$ and

$$\begin{array}{ccc} G_1(X) & \xrightarrow{\varphi_X} & G_2(X) \\ {\scriptstyle G_1(f)}\downarrow & & \downarrow{\scriptstyle G_2(f)} \\ G_1(Y) & \xrightarrow[\varphi_Y]{} & G_2(Y) \end{array}$$

for $f : X \to Y$ in $\mathcal{C}$.

Proof. We use the strict model $\overline{\mathcal{R}}$ of $\mathcal{C}$ (see the description after the definition of positive Frobenius duality) and let $\{\varepsilon_X\}$ be the Frobenius duality constructed there.

For each $X \in \mathcal{R}_0$, the previous lemma gives a unitary map $u_{X^*} : F(X)^* \to F(X^*)$ such that

$$F(\varepsilon_X)(u_{X^*} \otimes 1) : F(X)^* \otimes F(X) \to \mathbb{C}$$

is positive definite.

We then define $G(X)$, $H(X)$ and $u_X : G(X) \to H(X)$ for X in $\overline{\mathcal{R}}$ inductively so that

$$G(XY) = G(X) \otimes G(Y),\ H(XY) = H(X) \otimes H(Y), \quad u_{XY} = u_X \otimes u_Y$$

with the initial conditions $H(X) = F(X)$ for $X \in \mathcal{R}$,

$$G(X) = \begin{cases} F(X) & \text{if } X \in \mathcal{R}_0, \\ F(X)^* & \text{if } X^* \in \mathcal{R}_0; \end{cases} \quad u_X = \begin{cases} 1_{F(X)} & \text{if } X \in \mathcal{R}_0, \\ u_X & \text{if } X^* \in \mathcal{R}_0. \end{cases}$$

Since both of $\mathcal{C}$ and $\mathcal{H}$ are strict, we have the ordinary associativity

$$\begin{array}{ccc} F(X) \otimes F(Y) \otimes F(Z) & \xrightarrow{m \otimes 1} & F(X \otimes Y) \otimes F(Z) \\ {\scriptstyle 1 \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ F(X) \otimes F(Y \otimes Z) & \xrightarrow[m]{} & F(X \otimes Y \otimes Z) \end{array}$$

for X, Y, Z in $\mathcal{C}$, whence

$$H(X) = F(X_1) \otimes \cdots \otimes F(X_n) \xrightarrow{m} F(X_1 \otimes \cdots \otimes X_n) = F(\overline{X})$$

is well-defined for $X = (X_1, \dots, X_n)$ in $\overline{\mathcal{R}}$.

Now we can extend G and H into functors so that

$$\begin{array}{ccccccc} H(X) & \xrightarrow{H(f)} & H(Y) & & G(X) & \xrightarrow{G(f)} & G(Y) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} & & {\scriptstyle u_X}\downarrow & & \downarrow{\scriptstyle u_Y} \\ F(\overline{X}) & \xrightarrow[F(\overline{f})]{} & F(Y) & & H(X) & \xrightarrow[H(f)]{} & H(Y) . \end{array}$$

From the multiplicativity of m and u, we see that these are strictly monoidal. Note here that G, H are unitarily equivalent to the tensor functor $F(\bar{\cdot})$ and $G(\varepsilon_X) : G(X)^* \otimes G(X) \to \mathbb{C}$ is positive by our construction.

Now we shall show that $G({}^t f)$ is the transposed map of $G(f)$. To this end, we shall work with the image tensor category $\mathcal{D}$ of G: $\mathcal{D}$ is a subtensor category of $\mathcal{H}$ satisfying:

(i) for V in $\mathcal{D}$, its dual Hilbert space V^* is again in $\mathcal{D}$;

(ii) there is a positive Frobenius duality $\{\varepsilon_V : V^* \otimes V \to \mathbb{C}\}$ in $\mathcal{D}$.

Let $\{S_i\}$ be a representative set of simple objects in $\mathcal{D}$ and $\mathcal{D}'$ be the enhanced tensor category of $\mathcal{D}$ by adding direct sums. Given an object V in $\mathcal{D}$, choose an irreducible decomposition

$$v : V \to V' \equiv \bigoplus_i m_i S_i.$$

We define a rigidity pairing $\varepsilon_{V'}$ in $\mathcal{D}'$ by

$$\varepsilon_{V'} = \bigoplus_i m_i \varepsilon_{S_i},$$

which is balanced as seen in Lemma 3.9.

SUBLEMMA 5.3. *The conjugation $\overline{v} : V^* \to (V')^*$ of v belongs to the hom-set in $\mathcal{D}'$ and we have*

$$\varepsilon_V = \varepsilon_{V'}(\overline{v} \otimes v).$$

Proof. By the uniqueness of balanced pairings, there is a unitary morphism $w : V^* \to (V')^*$ in $\mathcal{D}'$ such that $\varepsilon_V = \varepsilon_{V'}(w \otimes v)$. Since $\varepsilon_{V'}(\overline{v} \otimes v)$ is a positive bilinear form, the expression

$$\varepsilon_V = (\varepsilon_{V'}(\overline{v} \otimes v))\,(\overline{v}^{-1} w \otimes 1)$$

gives a polar decomposition in $\mathcal{H}$. Since ε_V is assumed to be positive, we conclude that $w = \overline{v}$ belongs to the hom-set in $\mathcal{D}'$. ∎

We now complete the proof of Theorem 5.2. Let W be an object in $\mathcal{D}$ and

$$w : W \to W' \equiv \bigoplus_j n_j S_j$$

be an irreducible decomposition in $\mathcal{D}'$. We define a balanced rigidity pairing $\varepsilon_{W'} : (W')^* \otimes W' \to \mathbb{C}$ by direct sum as before.

Given a morphism $f : V \to W$ in $\mathcal{D}$, $f' = wfv^* : V' \to W'$ is a morphism in $\mathcal{D}'$. From the definition of $\varepsilon_{V'}$ and $\varepsilon_{W'}$, it is easy to check that the rigidity transposed of f' is given by the transposed map ${}^t(f')$ of f'. In particular, ${}^t(f')$ is again a morphism in $\mathcal{D}'$ and we have

$$\varepsilon_{W'}(1 \otimes f') = \varepsilon_{V'}({}^t(f') \otimes 1).$$

Then ${}^tf = \overline{w}^{-1}\,{}^t(f')\overline{v}$ is a morphism in $\mathcal{D}$ because $\overline{v}$ and $\overline{w}$ are morphisms in $\mathcal{D}'$. Moreover, we see

$$\begin{aligned}\varepsilon_W(1 \otimes f) &= \varepsilon_{W'}(\overline{w} \otimes wf) = \varepsilon_{W'}(1 \otimes f')(\overline{w} \otimes v)\\ &= \varepsilon_{V'}({}^t(f') \otimes 1)(\overline{w} \otimes v) = \varepsilon_{V'}(\overline{v} \otimes v)({}^tf \otimes 1)\\ &= \varepsilon_V({}^tf \otimes 1). \quad \blacksquare\end{aligned}$$

REMARK 5.4. Related to Tannaka-Krein duality, it will be worth pointing out here that the Tannaka dual of a compact quantum group provides a rigid C^*-tensor category with simple unit object, whence it bears a unique positive Frobenius duality by the results discussed so far. On the other hand, it is immediate to see that a standard invariant of S. Popa (or equivalently a paragroup of A. Ocneanu) is associated to a positive Frobenius duality (cf. [8]). Thus we have a conceptual explanation of Theorem A in [1] which states that any finite-dimensional representation of a compact quantum group gives rise to a Popa's combinatorial structure.

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