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A DECOMPOSITION THEOREM FOR GENERATORS OF STRONGLY CONTINUOUS GROUPS ON HILBERT SPACES

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Abstract. For the generator A of a strongly continuous group on a Hilbert space, we modify Liapunov's method of changing the scalar product to obtain a decomposition $A = B + C$ with B skew-adjoint and C bounded and selfadjoint (with respect to the new scalar product). This yields a new proof of the fact that A has bounded H^∞–calculi on vertical strips. Furthermore we show that, with respect to the new scalar product, A^2 can be obtained by a closed sectorial form in the sense of Kato.

Keywords: *C_0-group, Liapunov's direct method, H^∞-calculus, cosine function, square root problem.*

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1. INTRODUCTION AND PRELIMINARIES

Every bounded linear operator A on a Hilbert space H has a canonical decomposition

$$A = \frac{A - A^*}{2} + \frac{A + A^*}{2} \tag{1.1}$$

as sum of a skew-adjoint and a selfadjoint operator. This decomposition reflects the canonical decomposition of the elements of the numerical range of A in real and imaginary parts. Furthermore, the commutator

$$\Big[\frac{A - A^*}{2}, \frac{A + A^*}{2}\Big] = \frac{1}{2}(AA^* - A^*A)$$

of both summands is selfadjoint. For unbounded operators such a decomposition fails, in general, for many reasons.

We show below that, if A generates a strongly continuous *group* on H, one can always find an equivalent scalar product such that the above decomposition remains valid (see Theorem 3.1 below).

Thus, a group generator on a Hilbert space always can be viewed as a bounded perturbation of a skew-adjoint operator (after changing the scalar product). This allows us to give a new proof of a theorem of Boyadzhiev and deLaubenfels (from [7]) which asserts that a group generator has a bounded H^∞-calculus on every vertical strip $\{z : |\operatorname{Re}(z)| < \omega\}$ where ω is greater than the group type (Theorem 4.2).

In the last section we examine squares of group generators on Hilbert spaces. We show that (after a suitable change of the scalar product) these operators always can be obtained by a closed sectorial form with Kato's Square Root Property (see Definition 5.7). Using the correspondence between squares of group generators and generators of cosine functions, we finally establish the theorem that generators of cosine functions always can be obtained by sectorial forms with the Square Root Property (Theorem 5.8).

In the following, H denotes a complex Hilbert space. By an operator A on H we always mean a *linear* operator whose domain $\mathrm{D}(A)$ is a linear subspace of H. If A is injective we will write A^{-1} for its inverse. If $A^{-1} \in \mathcal{L}(H)$, then A is called *invertible*. The (original) scalar product on H will be written as $(\,\cdot\mid\cdot\,)$, for new scalar products we use the same notation with additional subscripts. If $(\,\cdot\mid\cdot\,)_\circ$ is an equivalent scalar product, and A is a densely defined closed operator on H, we let A^* and A° denote the adjoint of A with respect to the original and new scalar product, respectively.

The well-known Lax-Milgram Theorem and some trivial computations imply the following proposition which is fundamental for our considerations.

PROPOSITION 1.1. *Let H be a Hilbert space with scalar product $(\,\cdot\mid\cdot\,)$. Then the mapping*

$$Q \mapsto (Q\cdot \mid \cdot\,)$$

is a bijection between the set of bounded, positive, invertible operators on H and the set of all scalar products on H which are equivalent to $(\,\cdot\mid\cdot\,)$.

If $(\,\cdot\mid\cdot\,)_\circ = (Q\cdot\mid\cdot\,)$ is such a new scalar product and A is a densely defined linear operator on H, then $\mathrm{D}(A^\circ) = Q^{-1}\mathrm{D}(A^*)$ and $A^\circ = Q^{-1}A^*Q$.

Note that all results using equivalent scalar products could be reformulated in terms of similarity. This is due to the fact that the operator $Q^{1/2}$ is a unitary equivalence between the Hilbert spaces $(H, (\,\cdot\mid\cdot\,))$ and $(H, (Q\cdot\mid\cdot\,))$. For example, the operator A is accretive with respect to the new scalar product $(Q\cdot\mid\cdot\,)$ if, and only if the operator $Q^{1/2}AQ^{-(1/2)}$ is accretive with respect to the original scalar product.

In the following, we will use standard results from semigroup theory without further reference. Proofs can be found in the books [1] and [11].

2. THE LIAPUNOV METHOD

Let A be the generator of a C_0-semigroup $T = (T(t))_{t\geqslant 0}$ on the Hilbert space H. If H is finite dimensional, the classical Liapunov Theorem establishes a connection between the spectral bound of A and the asymptotic behaviour of T. It also states that for an exponentially stable semigroup T there is an equivalent scalar product such that each $T(t)$ is contractive with respect to the new norm. This new scalar product is given by the formula

$$(x \mid y)_\circ = \int_0^\infty (T(t)x \mid T(t)y)\,\mathrm{d}t = (Qx \mid y) \tag{2.1}$$

with $Q = \int_0^\infty T(t)^*T(t)\,\mathrm{d}t$. Moreover, the *Liapunov equation*

$$A^*Qx + QAx = -x \tag{2.2}$$

holds for $x \in H$. This is still true if H is infinite dimensional and A is a bounded operator (see [9], Chapter I).

If the generator A is unbounded, definition (2.1) obviously still yields a positive definite, sesquilinear and continuous form on H, the operator Q now being defined by $Q :=$ strong-$\int_0^\infty T(t)^*T(t)\,\mathrm{d}t$. Moreover, the Liapunov equation remains valid, in the sense that $Q(\mathrm{D}(A)) \subset \mathrm{D}(A^*)$ and (2.2) holds for $x \in \mathrm{D}(A)$ (see [8], Theorem 5.1.3). In general, the scalar product $(\,\cdot \mid \cdot\,)_\circ = (Q\cdot \mid \cdot\,)$ is not equivalent to the original one. But it is if one supposes that T actually extends to a C_0-*group*. In fact, for each C_0-group $(T(t))_{t\in\mathbb{R}}$ on a Banach space X one can find $\omega_0 \geqslant 0, M \geqslant 1$ with

$$\|T(t)\| \leqslant M\mathrm{e}^{\omega_0|t|}, \quad t \in \mathbb{R}. \tag{2.3}$$

(The infimum of all such ω_0 is called the *group type* of the group $(T(t))_{t\in\mathbb{R}}$.) This implies $\|x\| \leqslant \|T(-t)\|\,\|T(t)x\| \leqslant M\mathrm{e}^{\omega_0 t}\|T(t)x\|$ for all $t \geqslant 0$. Hence,

$$\|x\|_\circ^2 = (x \mid x)_\circ = \int_0^\infty \|T(t)x\|^2\,\mathrm{d}t \geqslant \int_0^\infty M^{-2}\mathrm{e}^{-2\omega_0 t}\,\mathrm{d}t\|x\|^2 = \frac{1}{2\omega_0 M^2}\|x\|^2$$

holds for all $x \in H$. (Note that we have $\omega_0 > 0$, since the semigroup $(T(t))_{t\geqslant 0}$ is assumed to be exponentially stable.)

Proposition 2.1. *Let A be the generator of an exponentially stable C_0-semigroup T on H. If T extends to a group, then the operator Q defined by*

$$Q := \int_0^\infty T(t)^*T(t)\,\mathrm{d}t$$

is a bounded, positive, and invertible operator on H. Therefore,

$$(x \mid y)_\circ := (Qx \mid y) = \int_0^\infty (T(t)x \mid T(t)y)\,\mathrm{d}t$$

defines a scalar product which is equivalent to the original one. Moreover, one has

$$\mathrm{D}(A) = \mathrm{D}(A^\circ) \quad \text{with } A^\circ + A = -Q^{-1}. \tag{2.4}$$

Proof. Only the last statement remains to be shown. As mentioned above, the Liapunov equation $A^*Q + QA = -I$ holds on $\mathrm{D}(A)$. Since $A^\circ = Q^{-1}A^*Q$, we have $\mathrm{D}(A) \subset \mathrm{D}(A^\circ)$ and $A^\circ \supset -A - Q^{-1}$. But both operators $-A - Q^{-1}$ and A° generate C_0-groups, the first being a bounded perturbation and the second being the adjoint of a C_0-group generator. Hence $A^\circ = -A - Q^{-1}$. ∎

Before we examine the consequences of the above proposition, we look for a converse.

LEMMA 2.2. *Let A be the generator of an exponentially stable C_0-semigroup T on H. Assume that $Q := \int_0^\infty T(t)^*T(t)\,\mathrm{d}t$ is invertible. Then, for all $t > 0$, the operators $T(t)$ are injective with closed range. Furthermore, there is $r_0 > 0$ such that the operators $\lambda + A$ are injective with closed range for all λ with $\operatorname{Re}\lambda > r_0$.*

Proof. Since the set of invertible operators is an open subset of $\mathcal{L}(H)$, $T^*(t)\int_0^\infty T^*(s)T(s)\,\mathrm{d}s\,T(t) = \int_t^\infty T^*(s)T(s)\,\mathrm{d}s$ is invertible for small $t > 0$. Thus, $T^*(t)$ is surjective for small $t > 0$. From the Closed Range Theorem it is immediate that $T(t)$ is injective with closed range for small $t > 0$. To obtain the result for general $t > 0$, simply write $T(t) = T(t/n)^n$ with $n \in \mathbb{N}$ large enough. The Liapunov equation (2.2) yields $\mathrm{D}(A^\circ) \supset \mathrm{D}(A)$ and $-Ax = (A^\circ + Q^{-1})x$ for $x \in \mathrm{D}(A)$. From this it follows that $\mathrm{D}(A)$ is a closed subset of $\mathrm{D}(A^\circ)$, where the norm on $\mathrm{D}(A^\circ)$ is the usual graph norm. Now, $A^\circ + Q^{-1}$ generates a C_0-semigroup on H. This implies that there is $r_0 > 0$ such that $\lambda - (A^\circ + Q^{-1})$ is an isomorphism of $\mathrm{D}(A^\circ)$ onto H for each λ with $\operatorname{Re}\lambda > r_0$. Hence $\lambda + A$ is injective with closed range for each such λ. ∎

PROPOSITION 2.3. *Let T, A and Q be as in Lemma 2.2. The following assertions are equivalent:*

(i) *the semigroup T extends to a group;*
(ii) *the operator Q is invertible and $T(t)$ has dense range for some $t > 0$;*
(iii) *the operator Q is invertible and $T^*(t)$ is injective for some $t > 0$;*
(iv) *both operators Q and $\widetilde{Q} := \int_0^\infty T(t)T^*(t)\,\mathrm{d}t$ are invertible;*
(v) *The operator Q is invertible and no left halfplane is contained in the residual spectrum of A.*

Proof. Assume (i). Then $T^* = (T^*(t))_{t\geqslant 0}$ is also a group, and $-A$ is a C_0-semigroup generator. By Proposition 2.1, the assertions (ii), (iii), (iv) and (v) follow (one has to change the roles of T and T^* for the proof of (iv)).

From Lemma 2.2 and the first part of its proof it is clear that each one of the assertions (ii), (iii) and (iv) immediately implies (i). Suppose (v) holds and let r_0 be as in Lemma 2.2. By (v), there is λ with $\operatorname{Re}\lambda > 0$ such that $(\lambda + A) : \mathrm{D}(A) \to H$ is bijective. This implies $\mathrm{D}(A) = \mathrm{D}(A^\circ)$, hence $-A = A^\circ + Q^{-1}$ is a C_0-semigroup generator (cf. the proof of Lemma 2.2). This proves (i). ∎

COROLLARY 2.4. *Let T, A and Q be as in Lemma 2.2 and suppose that Q is invertible.*

(i) *If each $T(t)$ is a normal operator, then T extends to a group.*

(ii) *If T is a holomorphic semigroup, then A is bounded.*

Proof. If $T(t)$ is normal for each t, then $Q = \widetilde{Q}$. If $T(\cdot)$ is holomorphic, then also $T^*(\cdot)$ is holomorphic. From this follows that $T^*(t)$ is injective for each t. Apply now Proposition 2.3 to obtain that T extends to a group. In case T is a holomorphic semigroup, this implies that A is bounded. ∎

Let $(S(t))_{t\geqslant 0}$ be the right translation semigroup on the Hilbert space $L^2(0,\infty)$ (see [11], I.4.16) and let $\omega > 0$. Then $T(t) := \mathrm{e}^{-\omega t}S(t)$ defines an exponentially stable semigroup with $T^*(t)T(t) = \mathrm{e}^{-2\omega t}I$. Hence the associated operator Q is invertible. This shows that the invertibility of Q is not sufficient for having a group.

COMMENTS. In the case when A is bounded, the Liapunov method is used in Chapter I of the book ([9]) of Daleckiĭ and Kreĭn. There, the operator equation $QA + A^*Q = -I$ is directly linked to the problem of finding a Liapunov function for the semigroup (which is sometimes called "Liapunov's direct method"). For the unbounded case, the relevant facts are included in Theorem 5.1.3 of Curtain and Zwart's book ([8]), where a characterization of exponential stability of the semigroup is given in terms of the existence of an operator Q satisfying the Liapunov equation. It is shown in [24] that this method in fact gives an *equivalent* scalar product if the semigroup extends to a group.

3. THE MAIN THEOREM

Let A be the generator of a C_0-group T on the Hilbert space H, satisfying (2.3), and let $\omega > \omega_0$. We now define

$$(3.1)\qquad \begin{aligned}(x \mid y)_\circ &:= \int_{\mathbb{R}} (T(t)x \mid T(t)y)\mathrm{e}^{-2\omega|t|}\,\mathrm{d}t \\ &= \int_0^\infty (T(t)x \mid T(t)y)\mathrm{e}^{-2\omega t}\,\mathrm{d}t + \int_0^\infty (T(-t)x \mid T(-t)y)\mathrm{e}^{-2\omega t}\,\mathrm{d}t\end{aligned}$$

for $x, y \in H$, i.e., we apply the Liapunov method simultaneously to the rescaled "forward" and "backward" semigroups obtained from the group T. From Proposition 2.1 it is immediate that $(\,\cdot \mid \cdot\,)_\circ$ is a scalar product on H which is equivalent to the original one. The following theorem summarizes its properties.

THEOREM 3.1. *Let A be the generator of a C_0-group T on a Hilbert space H with $\|T(t)\| \leqslant M\mathrm{e}^{\omega_0|t|}$ for $t \in \mathbb{R}$, and let $\omega > \omega_0$. With respect to the (equivalent) scalar product $(\cdot \mid \cdot)_\circ$ defined by (3.1) the following assertions hold:*

(i) *The operators $A-\omega$ and $-A-\omega$ are both* m*-dissipative; i.e., $\|T(t)\|_\circ \leqslant \mathrm{e}^{\omega|t|}$ for all $t \in \mathbb{R}$.*

(ii) $\mathrm{D}(A) = \mathrm{D}(A^\circ)$ *and* $A = B + C$ *with*

$$B := \frac{1}{2}(A - A^\circ) \quad \textit{and} \quad C := \frac{1}{2}(A + A^\circ).$$

(iii) *B is skewadjoint with $\mathrm{D}(A) = \mathrm{D}(B)$.*

(iv) *C has an extension to a bounded and selfadjoint operator (also denoted by C) with $-\omega \leqslant C \leqslant \omega$.*

(v) *$\mathrm{D}(A)$ is C-invariant, i.e., $C(\mathrm{D}(A)) \subset \mathrm{D}(A)$, and $[B,C] = BC - CB$ has an extension to a bounded and selfadjoint operator on H.*

Proof. We first show (i). One has

$$\begin{aligned}
\|T(s)x\|_\circ^2 &= \int_{\mathbb{R}} \|T(t)T(s)x\|^2 \mathrm{e}^{-2\omega|t|}\,\mathrm{d}t = \int_{\mathbb{R}} \|T(t+s)x\|^2 \mathrm{e}^{-2\omega|t|}\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \|T(t)x\|^2 \mathrm{e}^{-2\omega|t-s|}\,\mathrm{d}t = \int_{\mathbb{R}} \|T(t)x\|^2 \mathrm{e}^{-2\omega|t|}\mathrm{e}^{2\omega(|t|-|t-s|)}\,\mathrm{d}t \\
&\leqslant \mathrm{e}^{2\omega|s|}\|x\|_0^2, \quad s \in \mathbb{R},\ x \in H
\end{aligned}$$

since $|t| - |t-s| \leqslant |s|$ for all $s, t \in \mathbb{R}$ by the triangle inequality.

To prove (ii), let

$$Q_\oplus := \int_0^\infty T(t)^* T(t) \mathrm{e}^{-2\omega t}\,\mathrm{d}t, \quad Q_\ominus := \int_0^\infty T(-t)^* T(-t) \mathrm{e}^{-2\omega t}\,\mathrm{d}t,$$

and

$$Q := Q_\oplus + Q_\ominus = \int_{\mathbb{R}} T(t)^* T(t) \mathrm{e}^{-2\omega|t|}\,\mathrm{d}t.$$

Then $(x \mid y)_\circ = (Qx \mid y)$ for all $x, y \in H$. The Liapunov equations for $Q_\oplus$ and $Q_\ominus$ read

$$Q_\oplus(A-\omega)x + (A-\omega)^* Q_\oplus x = -x \tag{3.2}$$

$$Q_\ominus(-A-\omega)x + (-A-\omega)^* Q_\ominus x = -x \tag{3.3}$$

for $x \in \mathrm{D}(A)$ (see (2.2)). (In particular this means that $Q_\oplus \mathrm{D}(A) \subset \mathrm{D}(A^*)$ and $Q_\ominus \mathrm{D}(A) \subset \mathrm{D}(A^*)$.) Adding equations (3.2) and (3.3) one obtains

$$QAx + A^*Qx = 2\omega(Q_\oplus - Q_\ominus)x$$

for $x \in \mathrm{D}(A)$. Note that $\mathrm{D}(A^\circ) = Q^{-1}\mathrm{D}(A^*)$ and $A^\circ = Q^{-1}A^*Q$ by Proposition 1.1. Hence we have $\mathrm{D}(A) \subset \mathrm{D}(A^\circ)$ and $A + A^\circ \subset 2\omega Q^{-1}(Q_\oplus - Q_\ominus)$. As in the proof of Proposition 2.1 it follows that $\mathrm{D}(A^\circ) = \mathrm{D}(A)$. This proves (ii), and a short computation also yields (iv).

For the proof of (iii) we note first that, by (iv), $B = -A + C$ is a bounded perturbation of the generator of a C_0-group. Therefore, B° and $-B$ are C_0-group generators as well. But it is easily seen that $B^\circ \supset -B$, whence it follows that $B^\circ = -B$.

The C-invariance of $\mathrm{D}(A)$ is clear from the formula $C = \omega Q^{-1}(Q_\oplus - Q_\ominus)$ and $\mathrm{D}(A) = \mathrm{D}(A^\circ) = Q^{-1}\mathrm{D}(A^*)$. Furthermore, using (3.2), (3.3) and the fact that $A^\circ = 2C - A$, we compute

$$\begin{aligned} CA &= \omega Q^{-1}(Q_\oplus A - Q_\ominus A) = \omega Q^{-1}(-I + 2\omega Q_\oplus - A^* Q_\oplus - I + 2\omega Q_\ominus + A^* Q_\ominus) \\ &= \omega Q^{-1}(-2I + 2\omega Q - A^*(Q_\oplus - Q_\ominus)) \\ &= -2\omega Q^{-1} + 2\omega^2 I - \omega A^\circ Q^{-1}(Q_\oplus - Q_\ominus) \\ &= -2\omega Q^{-1} + 2\omega^2 I - A^\circ C = -2\omega Q^{-1} + 2\omega^2 I - (2C - A)C \\ &= -2\omega Q^{-1} + 2\omega^2 I - 2C^2 + AC. \end{aligned}$$

This shows that $[B, C] = [A, C]$ has an extension to a bounded operator which is selfadjoint with respect to $(\,\cdot \mid \cdot\,)_\circ$. ∎

COROLLARY 3.2. *Let A generate a C_0-group T on a Hilbert space H. Then there exists a bounded operator C such that $B := A - C$ generates a bounded C_0-group. Moreover, the operator C can be chosen in such a way that* $\mathrm{D}(A)$ *is C-invariant and the commutator $[A, C] = AC - CA$ has an extension to a bounded operator on H.*

REMARK 3.3. Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group on H with generator A and such that $\|T(t)\| \leqslant \mathrm{e}^{\omega|t|}$ for all $t \in \mathbb{R}$ and some constant $\omega \geqslant 0$. Then it is easy to show that there is a sum decomposition $A = B + C$ such that B is skewadjoint and C is bounded and selfadjoint with $-\omega \leqslant C \leqslant \omega$. (Consider the symmetric, sesquilinear form $c(u, v) := (Au \mid v) + (u \mid Av)$ on $\mathrm{D}(A)$. Then apply the generalized Cauchy-Schwarz inequality (see [19], Chapter XII, Lemma 3.1) to obtain that c is continuous with respect to the norm on H. Define $C \in \mathcal{L}(H)$ by $c(u, v) = (Cu \mid v)$ and B by $B := A - C$.)

In general however, $\mathrm{D}(A)$ is not C-invariant. In fact, let $H := L^2(\mathbb{R})$ and $B = \mathrm{d}/\mathrm{d}t$ the generator of the shift group. Furthermore, let $C := (f \mapsto \omega m f)$ where $m(x) = \operatorname{sgn} x$ is the sign function. Then C is bounded and selfadjoint and $A := B + C$ generates a C_0-group T with $\|T(t)\| \leqslant \mathrm{e}^{\omega|t|}$. Obviously, $\mathrm{D}(A) = \mathrm{D}(B) = W^{1,2}(\mathbb{R})$ is not invariant with respect to multiplication by m.

This shows that part (v) of Theorem 3.1 is not trivial and is due to the particular way of renorming.

COMMENTS. The following well-known theorem by Sz.-Nagy (see [22]) can be regarded as the "limit case" in Theorem 3.1: Every generator of a bounded C_0-group on a Hilbert space is similar to a skew-adjoint operator. This result cannot be deduced directly from Theorem 3.1. However, Zwart in [24] gives a proof using the Liapunov renorming.

In [9] it is proved that, given a C_0-group T on a Hilbert space, one has $\|T(t)\|_\circ \leqslant \mathrm{e}^{\omega|t|}$ for some equivalent scalar product $(\,\cdot \mid \cdot\,)_\circ$ and some ω strictly larger than the group type ω_0. (This is covered by part (i) of our Theorem 3.1.) While the proof in [9] is based on the boundedness of the H^∞-calculus (see next section) and on a deep result of Paulsen, our approach is more direct and considerably shorter. In [21], Simard shows that in general it is not possible to take $\omega = \omega_0$ in Theorem 3.1.

4. THE H^∞-CALCULUS

In this section we show that the generator A of a C_0-group T on a Hilbert space H has a bounded $H^\infty(S_\alpha)$-functional calculus for each $\alpha > \omega_0$ where ω_0 is as in (2.3). Here S_α denotes the open strip

$$S_\alpha := \{z \in \mathbb{C} : |\operatorname{Re}(z)| < \alpha\}$$

and $H^\infty(S_\alpha)$ is the Banach algebra of all bounded analytic functions on S_α.

Choose $\omega_0 < \omega_1 < \alpha < \omega$, and let $\Gamma = \gamma_1 \oplus \gamma_2$ with $\gamma_1(r) = -\omega_1 - \mathrm{i}r$ and $\gamma_2(r) = \omega_1 + \mathrm{i}r$, $r \in \mathbb{R}$. For $f \in H^\infty(S_\alpha)$ and $x \in \mathrm{D}(A^2)$ we define

$$f(A)x := \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(z)}{z^2 - \omega^2} R(z, A)\, \mathrm{d}\lambda (A^2 - \omega^2)x. \tag{4.1}$$

(Note that the integral defines a bounded operator on H since the resolvent is bounded on vertical lines. Hence, each $f(A)$ defines a bounded operator from $\mathrm{D}(A^2)$ (with the graph norm) to H.) Cauchy's theorem implies that this definition is independent of the choice of ω_1 and ω. Standard arguments also show that

$$\left(\frac{1}{\lambda - \cdot}\right)(A)x = R(\lambda, A)x \quad \text{and} \quad \mathbb{1}(A)x = x$$

for $\lambda \notin \overline{S}_\alpha$ and $x \in \mathrm{D}(A^2)$, where $\mathbb{1}$ denotes the constant one function. Furthermore, one has $(fg)(A)x = f(A)g(A)x$ for all $x \in \mathrm{D}(A^4)$ and $f, g \in H^\infty(S_\alpha)$. If $f(A)$ extends to a bounded operator on H (note that $\mathrm{D}(A^2)$ is dense), we simply write $f(A) \in \mathcal{L}(H)$.

DEFINITION 4.1. We call the mapping

$$(f \mapsto f(A)) : H^\infty(S_\alpha) \to \mathcal{L}(\mathrm{D}(A^2), H),$$

defined by (4.1), the *natural $H^\infty(S_\alpha)$-calculus* for A. We say that this calculus is *bounded* if there is a constant c such that

$$\|f(A)x\|_H \leqslant c\|f\|_\infty \|x\|_H$$

for all $f \in H^\infty(S_\alpha)$ and $x \in \mathrm{D}(A^2)$.

By the Closed Graph Theorem the natural $H^\infty(S_\alpha)$-calculus for A is bounded if and only if $f(A) \in \mathcal{L}(H)$ for each $f \in H^\infty(S_\alpha)$. In this case, the mapping $(f \mapsto f(A)) : H^\infty(S_\alpha) \to \mathcal{L}(H)$ is a bounded algebra homomorphism.

We now come to the main result of this section.

THEOREM 4.2. *Let A be the generator of a strongly continuous group T on a Hilbert space H satisfying $\|T(t)\| \leqslant M\mathrm{e}^{\omega_0|t|}$, $t \in \mathbb{R}$, for some constants $M, \omega_0 \geqslant 0$, and let $\alpha > \omega_0$. Then the natural $H^\infty(S_\alpha)$-calculus for A is bounded.*

Before proving the theorem, let us examine a special case.

Proposition 4.3. *Let B be skew-adjoint, i.e., B generates a unitary group on H. Let $\Phi : C^{\mathrm{b}}(\mathrm{i}\mathbb{R}) \longrightarrow \mathcal{L}(H)$ be the usual functional calculus for B given by the spectral theorem. (Here, $C^{\mathrm{b}}(\mathrm{i}\mathbb{R})$ denotes the algebra of bounded continuous functions on $\mathrm{i}\mathbb{R}$.) Let $\alpha > 0$. Then, $f(B)x = \Phi(f|\mathrm{i}\mathbb{R})x$ for all $f \in H^\infty(S_\alpha)$ and $x \in \mathrm{D}(B^2)$. In particular, the natural $H^\infty(S_\alpha)$-calculus for B is bounded.*

Proof. By the spectral theorem we can assume $H = L^2(\Omega, \mu)$, where Ω is a locally compact space, μ is a positive regular Borel measure on Ω, and B is multiplication by a continuous function $m \in C(\Omega)$ such that $m(\Omega) \subset \mathrm{i}\mathbb{R}$ (see Remark 4.4 below). Then $f(B)$ is the multiplication by the function

$$g(\cdot) := (m(\cdot)^2 - \omega^2)\,\frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(\lambda)}{\lambda^2 - \omega^2}\,\frac{1}{\lambda - m(\cdot)}\,\mathrm{d}\lambda\,.$$

Since the integral converges in $C^{\mathrm{b}}(\Omega)$, one can evaluate pointwise, and by Cauchy's theorem we arrive at $g(p) = f(m(p))$ for all $p \in \Omega$. But this is exactly the way how the usual functional calculus is constructed. ∎

Remark 4.4. In the previous proof we used a version of the spectral theorem which is not standard, thus we give a sketch of its proof. Let A be a (possibly unbounded) selfadjoint operator on a Hilbert space H and let $T := (\mathrm{i} - A)^{-1} = R(\mathrm{i}, A)$. Then T is a bounded normal operator on H. By Scholium 9.4 in [20], there is a locally compact space Ω', a positive regular Borel measure μ' on Ω' and a bounded continuous function g' on Ω' such that $\mu'(U) > 0$ for every nonempty open subset of Ω', and (H, T) is unitarily equivalent to $(L^2(\Omega', \mu'), g')$. Since T is injective, the closed set $Z := (g' = 0)$ is locally μ-null. Then, with $\Omega := \Omega' \setminus Z$, $\mu := \mu'|\Omega$ and $g := g'|\Omega$, $(L^2(\Omega', \mu'), g')$ is unitarily equivalent to $(L^2(\Omega, \mu), g)$ and (H, A) is unitarily equivalent to $(L^2(\Omega, \mu), m)$, where $m := \mathrm{i} - g^{-1}$.

Proof of Theorem 4.2. It suffices to show that for fixed $f \in H^\infty(S_\alpha)$ there is a $c = c_f$ such that $\|f(A)x\|_H \leqslant c\|x\|_H$ for $x \in \mathrm{D}(A^2)$. By choosing a suitable equivalent scalar product and employing Theorem 3.1 we can assume that there is a skew-adjoint operator B with $\mathrm{D}(B) = \mathrm{D}(A)$ and a bounded selfadjoint operator C such that $A = B + C$.

Next, note that by the Plancherel Theorem $(r \mapsto R(\pm\omega_1 \pm \mathrm{i}r, A)x) \in L^2(\mathbb{R}, H)$ with a constant $c = c(A) \geqslant 0$, so

$$\int_{\mathbb{R}} \|R(\pm\omega_1 \pm \mathrm{i}r, A)x\|^2\,\mathrm{d}r \leqslant c(A)^2\|x\|^2 \tag{4.2}$$

for all $x \in H$. Therefore, by Hölder's inequality,

$$\int_\Gamma \frac{f(\lambda)}{\lambda - \omega} R(\lambda, A)\,\mathrm{d}\lambda \in \mathcal{L}(H), \tag{4.3}$$

where the integral is understood in the strong sense. It follows that

$$f(A)x = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(\lambda)}{\lambda - \omega} R(\lambda, A)\,\mathrm{d}\lambda\,(A - \omega)x \tag{4.4}$$

for $x \in \mathrm{D}(A^2)$. (Use Cauchy's theorem together with $\frac{1}{\lambda^2-\omega^2}R(\lambda,A)(A+\omega)$ $\cdot(A-\omega) = \frac{-1}{\lambda^2-\omega^2}(A-\omega) + \frac{1}{\lambda-\omega}R(\lambda,A)(A-\omega)$.)

Of course, the same considerations apply to B and A^* instead of A. Finally, we have

$$\int_\Gamma f(\lambda)\, R(\lambda,A)CR(\lambda,B)\,\mathrm{d}\lambda \in \mathcal{L}(H) \tag{4.5}$$

(again a strong integral) since

$$\begin{aligned}\Big|\Big(\int_\Gamma f(\lambda)R(\lambda,A)CR(\lambda,B)x\,\mathrm{d}\lambda \mid y\Big)\Big| &\leqslant \int_\Gamma |f(\lambda)|\,|(CR(\lambda,B)x \mid R(\lambda,A)^*y)|\,\mathrm{d}|\lambda| \\ \leqslant \|f\|_\infty\|C\| \int_\Gamma \|R(\lambda,B)x\|\,\|R(\overline{\lambda},A^*)y\|\,\mathrm{d}|\lambda| &\leqslant 2c(B)c(A^*)\|f\|_\infty\|C\|\,\|x\|\,\|y\|\end{aligned}$$

for $x,y \in H$. (Here we have used Hölder's inequality and (4.2) with A replaced by B and A^*.)

We can now complete the proof of Theorem 4.2. To simplify notation, we write "$F \approx G$" as an abbreviation for "F is bounded if and only if G is bounded", where a linear mapping $F : \mathrm{D}(A^2) \to H$ is called bounded if it extends to a bounded operator on H. Then

$$\begin{aligned}\int_\Gamma \frac{f(\lambda)}{\lambda-\omega}R(\lambda,A)\,\mathrm{d}\lambda\,(A-\omega) &\approx \int_\Gamma \frac{f(\lambda)}{\lambda-\omega}R(\lambda,A)\,\mathrm{d}\lambda\,(B-\omega) \\ &\approx \int_\Gamma \frac{f(\lambda)}{\lambda-\omega}(R(\lambda,A)-R(\lambda,B))\,\mathrm{d}\lambda\,(B-\omega) \\ &= \int_\Gamma \frac{f(\lambda)}{\lambda-\omega}R(\lambda,A)\,C\,R(\lambda,B)\,\mathrm{d}\lambda\,(B-\omega) \\ &= \int_\Gamma \frac{f(\lambda)}{\lambda-\omega}R(\lambda,A)\,C\,[-1+R(\lambda,B)(\lambda-\omega)]\,\mathrm{d}\lambda \\ &\approx \int_\Gamma \frac{f(\lambda)}{\lambda-\omega}R(\lambda,A)\,C\,R(\lambda,B)(\lambda-\omega)\,\mathrm{d}\lambda \\ &= \int_\Gamma f(\lambda)R(\lambda,A)\,C\,R(\lambda,B)\,\mathrm{d}\lambda.\end{aligned}$$

By (4.5), this last term is bounded. (We have applied (4.3) in the first line, (4.4) for B instead of A together with Proposition 4.3 in the second line, and again (4.3) in the fifth line.) This finishes the proof. ∎

REMARK 4.5. Let X be a Banach space, A the generator of a C_0-group T on X with M, ω_0 satisfying (2.3), and let $\alpha > \omega_0$. Then, the natural $H^\infty(S_\alpha)$-calculus can be constructed in the same way as in the Hilbert space setting.

Let $\mathcal{F}$ be a subalgebra of $H^\infty(S_\alpha)$ that contains all elementary rationals $r_\lambda := (\lambda - \cdot)^{-1}$, $\lambda \notin \overline{S}_\alpha$. An algebra homomorphism $\Phi : \mathcal{F} \to \mathcal{L}(X)$ such that $\Phi(r_\lambda) = R(\lambda, A)$ is called a *bounded $\mathcal{F}$-calculus* for A if there is c such that $\|\Phi(f)\|_{\mathcal{L}(X)} \leqslant c\|f\|_\infty$ for all $f \in \mathcal{F}$.

We let $\mathcal{R}_0(S_\alpha)$ denote the algebra which is generated by all the r_λ, $\lambda \notin \overline{S}_\alpha$ and set $\mathcal{R}(S_\alpha) := \mathcal{R}_0(S_\alpha) \oplus \mathbb{C}\mathbb{1}$. Obviously, there is one and only one algebra homomorphism $\Phi : \mathcal{R}_0(S_\alpha) \to \mathcal{L}(X)$ with $\Phi(r_\lambda) = R(\lambda, A)$ for each $\lambda \notin \overline{S}_\alpha$. From Chapter II, Theorem 10.4 in [12] one can deduce with the help of a Möbius transformation that $\mathcal{R}(S_\alpha)$ is uniformly dense in $H^\infty(S_\alpha) \cap C(K)$ where K is the closure of S_α in the Riemann sphere. A fortiori, $\mathcal{R}_0(S_\alpha)$ is uniformly dense in $H^\infty(S_\alpha) \cap C_0(\overline{S}_\alpha)$. Hence, there is at most one bounded $H^\infty(S_\alpha) \cap C_0(\overline{S}_\alpha)$-calculus for A and at most one bounded $H^\infty(S_\alpha) \cap C(K)$-calculus for A that maps $\mathbb{1}$ to the identity operator.

We say that a sequence $(f_n)_n \subset \mathcal{F}$ converges *boundedly and pointwise* on S_α to a function f, if $f_n \to f$ pointwise on S_α and $\sup_n \|f_n\|_\infty < \infty$.

A bounded $\mathcal{F}$-calculus $\Phi : \mathcal{F} \to \mathcal{L}(X)$ for A is said to be continuous with respect to bounded and pointwise convergence (in short: *b.p.-continuous*), if it has the following property: If $f_n, f \in \mathcal{F}$ such that $f_n \to f$ boundedly and pointwise then $\Phi(f_n) \to \Phi(f)$ strongly on X.

If the natural $H^\infty(S_\alpha)$-calculus is bounded, then it is also b.p.-continuous. This is due to the following McIntosh-type Convergence Lemma for the natural $H^\infty(S_\alpha)$-calculus which we state without proof (see Section 5 in [17] for the sectorial version).

LEMMA 4.6. *If $(f_n)_n \subset H^\infty(S_\alpha)$ is uniformly bounded and pointwise convergent to $f \in H^\infty(S_\alpha)$ with $\sup_n \|f_n(A)\| < \infty$, then $f(A) \in \mathcal{L}(X)$ and $f(A_n)$ converges strongly to $f(A)$.*

Now, for any $f \in H^\infty(S_\alpha)$ there exists a sequence of rational functions $r_n \in \mathcal{R}(S_\alpha)$ such that $\|r_n\|_\infty \leqslant \|f\|_\infty$ and $r_n \to f$ pointwise on S_α. This follows, again after applying a suitable Möbius transformation, from Chapter VI in Theorem 5.3 in [12]. Moreover, the constant function $\mathbb{1}$ is approximated pointwise by the rational functions $(n(n + \alpha - \cdot)^{-1})_{n\in\mathbb{N}}$. Therefore, there is at most one bounded and b.p.-continuous $H^\infty(S_\alpha)$-calculus for A. Furthermore, due to the Convergence Lemma, the natural $H^\infty(S_\alpha)$-calculus is bounded if and only if the (unique) $\mathcal{R}_0(S_\alpha)$-calculus is bounded.

COMMENTS. An extension of the natural calculus appears first in a paper by Bade (see [6]). For sectorial operators, an analogous construction was done by McIntosh and his co-workers (see [17]) who first put attention to the boundedness of the natural H^∞-calculus. Theorem 4.2 is originally due to Boyadzhiev and deLaubenfels (Theorem 3.2, [7]). Actually, they construct a bounded $H^\infty(S_\alpha) \cap C_0(\overline{S}_\alpha)$-functional calculus for the group generator A. The natural $H^\infty(S_\alpha)$-calculus is an extension of theirs, as can be seen from Remark 4.5 or directly from the construction (cf. [7], Lemma 2.6). It follows from Remark 4.5 that Theorem 4.2 can be viewed as a corollary of the Boyadzhiev-deLaubenfels Theorem. However, while in [7] the theory of regularized semigroups and the functional calculus for sectorial operators is used, our proof is much shorter and more transparent. In [13] we give another proof which imitates McIntosh's method for sectorial operators, thereby obtaining even a characterization of C_0-group generators on Hilbert spaces.

5. COSINE FUNCTIONS AND VARIATIONAL METHODS

A *cosine function* on a Banach space X is a strongly continuous mapping $\mathrm{Cos} : \mathbb{R} \to \mathcal{L}(X)$ such that $\mathrm{Cos}(0) = I$ and

$$2\,\mathrm{Cos}(t)\,\mathrm{Cos}(s) = \mathrm{Cos}(t+s) + \mathrm{Cos}(t-s), \quad t, s \in \mathbb{R}. \tag{5.1}$$

Inserting $t = 0$ in (5.1) yields $\mathrm{Cos}(-s) = \mathrm{Cos}(s)$ for all $s \in \mathbb{R}$, and interchanging s and t in (5.1) shows $\mathrm{Cos}(t)\,\mathrm{Cos}(s) = \mathrm{Cos}(s)\,\mathrm{Cos}(t)$ for all $s, t \in \mathbb{R}$. In the following, we cite some basic results of the theory of cosine functions from Sections 3.14–3.16 in [1].

Given a cosine function, one can take its Laplace transform and define its *generator* B by

$$\lambda R(\lambda^2, B)x = \int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{Cos}(t)x\,\mathrm{d}t$$

for $x \in X$ and $\mathrm{Re}\,\lambda$ sufficiently large. Then, for each pair $(x, y) \in X^2$, the function

$$u(t) := \mathrm{Cos}(t)x + \int_0^t \mathrm{Cos}(s)y\,\mathrm{d}s$$

is the unique mild solution of the second order abstract Cauchy problem

$$\begin{cases} u''(t) = B\,u(t), & t \geqslant 0, \\ u(0) = x, \\ u'(0) = y, \end{cases}$$

(cf. [1], Corollary 3.14.8). If B generates a cosine function, then it also generates a holomorphic semigroup of angle $\pi/2$ (cf. [1], Theorem 3.14.17).

PROPOSITION 5.1. (Theorem 3.14.11 in [1]) *Let A generate a cosine function on the Banach space X. Let the operator $\mathcal{A}$ on $X \times X$ be defined by*

$$\mathrm{D}(\mathcal{A}) := \mathrm{D}(A) \times X, \quad \mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ Ax \end{pmatrix}.$$

Then there exists a unique Banach space V such that $\mathrm{D}(A) \hookrightarrow V \hookrightarrow X$ and the part $\mathcal{B}$ of $\mathcal{A}$ in $V \times X$ generates a C_0-semigroup.

The space $V \times H$ is called the *phase space* associated with A. If A generates a cosine function and $\lambda \in \mathbb{C}$, then $A + \lambda$ generates a cosine function with the same phase space (cf. Corollary 3.14.13 in [1]).

The connection to the theory of C_0-groups is given by the following: If an operator A generates a C_0-group $(U(t))_{t\in\mathbb{R}}$ on the Banach space X, then A^2 generates a cosine function Cos with phase space $\mathrm{D}(A) \times X$, where $\mathrm{Cos}(t) = (U(t) + U(-t))/2$, $t \geqslant 0$ (cf. Example 3.14.15 in [1]). Moreover, a remarkable theorem of Fattorini states the partial converse (cf. 3.16.7 in [1]).

Theorem 5.2. (Fattorini) *Let B be the generator of a cosine function on an* UMD*-space X. If $-B$ is sectorial, then $A := \mathrm{i}(-B)^{1/2}$ generates a strongly continuous group and $A^2 = B$.*

This suggests to consider squares of group generators.

Theorem 5.3. *Let A generate a strongly continuous group T on the Hilbert space H. Assume that there is $\omega \geqslant 0$ such that*

$$\|T(t)\| \leqslant \mathrm{e}^{\omega|t|}, \quad t \in \mathbb{R},$$

i.e., both $A - \omega$ and $-A - \omega$ are m*-dissipative. Then, for every $0 \leqslant |\phi| < \pi/2$ the operator*

$$\mathrm{e}^{\mathrm{i}\phi}\Big(A^2 - \Big(\frac{\omega}{\cos\phi}\Big)^2\Big) \tag{5.2}$$

is m*-dissipative. Thus, A^2 generates a holomorphic semigroup $(S(z))_{\operatorname{Re} z>0}$ of angle $\pi/2$ such that*

$$\|S(z)\| \leqslant \mathrm{e}^{(\omega/\cos\phi)^2 \operatorname{Re} z}, \quad |\arg z| \leqslant \phi < \frac{\pi}{2}. \tag{5.3}$$

For the proof we will need the following lemma, whose proof is an easy combination of elementary facts on dissipativity (which can be found, e.g., in Section 2.1.1 of [23]).

Lemma 5.4. *Let A be an operator on the Hilbert space H. For $\lambda > 0$ and $\alpha \in \mathbb{R}$ the following assertions are equivalent:*

(i) *A is* m*-dissipative;*
(ii) *$A + \mathrm{i}\alpha$ is* m*-dissipative;*
(iii) *$\lambda \in \rho(A)$ and $\|(A+\lambda)(A-\lambda)^{-1}\| \leqslant 1$;*
(iv) *$A - \varepsilon$ is* m*-dissipative for all $\varepsilon > 0$.*

Proof of Theorem 5.3. The case $\omega = 0$ is trivial since then the group is unitary and A is skew-adjoint. This implies that A^2 is selfadjoint with $A^2 \leqslant 0$, and the assertions of the theorem are immediate.

Assume $\omega > 0$, let $0 \leqslant |\phi| < \pi/2$ and fix $\varepsilon > 0$. Define $\alpha = \omega\tan\phi$, i.e., $z := \omega - \mathrm{i}\alpha = (\omega/\cos\phi)\mathrm{e}^{-\mathrm{i}\phi}$. By assumption and Lemma 5.4, the operators $A - (\omega - \mathrm{i}\alpha)$ and $-A - (\omega - \mathrm{i}\alpha)$ are m-dissipative. It follows from Lemma 5.4 that

$$\Big\|\frac{A-(z-\varepsilon)}{A-(z+\varepsilon)}\Big\|, \Big\|\frac{A+(z-\varepsilon)}{A+(z+\varepsilon)}\Big\| \leqslant 1.$$

(Here and in the following we write $\frac{A+\lambda}{A+\mu}$ instead of $(A+\lambda)(A+\mu)^{-1}$ to make the computations more perspicuous.) Hence, it follows that

$$\Big\|\frac{A^2-(z-\varepsilon)^2}{A^2-(z+\varepsilon)^2}\Big\| = \Big\|\Big(\frac{A-(z-\varepsilon)}{A-(z+\varepsilon)}\Big)\Big(\frac{A+(z-\varepsilon)}{A+(z+\varepsilon)}\Big)\Big\| \leqslant 1.$$

One computes

$$\frac{A^2-(z-\varepsilon)^2}{A^2-(z+\varepsilon)^2} = \frac{A^2-(z^2+\varepsilon^2)+2z\varepsilon}{A^2-(z^2+\varepsilon^2)-2z\varepsilon} = \frac{\mathrm{e}^{\mathrm{i}\phi}[A^2-(z^2+\varepsilon^2)]+2\varepsilon|z|}{\mathrm{e}^{\mathrm{i}\phi}[A^2-(z^2+\varepsilon^2)]-2\varepsilon|z|}.$$

We can now apply Lemma 5.4 again (note that $2\varepsilon|z| > 0$) to conclude that

$$\mathrm{e}^{\mathrm{i}\phi}[A^2 - (z^2 + \varepsilon^2)] = \mathrm{e}^{\mathrm{i}\phi}A^2 - \Big(\frac{\omega}{\cos\phi}\Big)^2 \mathrm{e}^{-\mathrm{i}\phi} - \varepsilon^2 \mathrm{e}^{\mathrm{i}\phi}$$

is m-dissipative. Letting $\varepsilon \searrow 0$ it follows from Lemma 5.4 that $\mathrm{e}^{\mathrm{i}\phi}A^2 - (\frac{\omega}{\cos\phi})^2\mathrm{e}^{-\mathrm{i}\phi}$, and finally that

$$\mathrm{e}^{\mathrm{i}\phi}\Big(A^2 - \Big(\frac{\omega}{\cos\phi}\Big)^2\Big) = \mathrm{e}^{\mathrm{i}\phi}A^2 - \Big(\frac{\omega}{\cos\phi}\Big)^2 \mathrm{e}^{\mathrm{i}\phi}$$

is m-dissipative. This finishes the proof of the first part of the theorem. The second part follows from standard semigroup theory (see [1], Chapter 3.4 and Chapter 3.9). ∎

The next proposition is needed for the proof of Theorem 5.8.

PROPOSITION 5.5. *Let A be as in Theorem 5.3. Then*

$$\mathrm{D}(\mathrm{i}(\omega^2 - A^2)^{1/2}) = \mathrm{D}(A).$$

Proof. First note that the operator $\omega^2 - A^2$ is sectorial (since $A^2 - \omega^2$ is m-dissipative). Thus the square root is well defined. Since A generates a group, A^2 generates a cosine function with phase space $\mathrm{D}(A) \times H$. By general cosine function theory (see the remarks at the beginning of this section), $A^2 - \omega^2$ also generates a cosine function with the same phase space. Fattorini's Theorem 5.2 implies that $B := \mathrm{i}(\omega^2 - A^2)^{1/2}$ generates a group and $B^2 = A^2 - \omega$. Now $\mathrm{D}(B) = \mathrm{D}(A)$ follows from the uniqueness of the phase space. ∎

Suppose that we have the following situation: V is another Hilbert space, densely embedded into H, and $a : V \times V \to \mathbb{C}$ is a continuous, sesquilinear form on V which is *H-elliptic*, i.e., there is $\mu > 0$ such that $\operatorname{Re} a + \mu(\,\cdot \mid \cdot\,)_H$ is an equivalent scalar product on V. We briefly say that (a, V) is a *closed form.*

Given a closed form (a, V), an operator A on H is defined by

$$(u, v) \in \operatorname{graph}(A) \quad \text{if and only if} \quad u \in V,\ v \in H \text{ and } a(u, \cdot) = (v, \cdot\,)_H \text{ on } V.$$

If an operator A arises in this manner, we say that A is *variational.* (This is a tribute to the origin of the theory of closed forms in the Dirichlet principle.) The negative of a variational operator generates a holomorphic semigroup which is — after shifting — contractive on a whole sector in the complex plane. This is in fact a characterization due to the following result (see, e.g., Theorem 1.2 in [2]).

PROPOSITION 5.6. *Let A be an operator on a Hilbert space. The following assertions are equivalent:*

(i) *A is variational;*

(ii) *there are $w \in \mathbb{R}$ and $0 \leqslant \phi < \pi/2$ such that both operators $\mathrm{e}^{\mathrm{i}\phi}(A + w)$ and $\mathrm{e}^{-\mathrm{i}\phi}(A + w)$ are m-accretive.*

(Recall that an operator A is called *m-accretive* if $-A$ is m-dissipative.)

If A is variational, then $A + \lambda$ is m-accretive for all large $\lambda > 0$. Kato proved that in this case $\mathrm{D}((A + \lambda)^{\alpha}) = \mathrm{D}((A + \lambda)^{*\alpha})$ holds for all $0 \leqslant \alpha < 1/2$ (Theorem 1.1 in [14]). In the case $\alpha = 1/2$ this is no longer true in general (see [16]).

Definition 5.7. A variational operator A is called *square root regular* if

$$\mathrm{D}((A+\lambda)^{\frac{1}{2}}) = \mathrm{D}((A+\lambda)^{*\frac{1}{2}}) \tag{5.4}$$

for some $\lambda > 0$. A closed form (a, V) is said to have the *Square Root Property* if $V = \mathrm{D}((A+\lambda)^{1/2})$, where A is the operator associated with a and $\lambda \in \mathbb{R}$ is such that $(A+\lambda)$ is sectorial.

Note that if A is square root regular, then (5.4) holds for all λ such that $A+\lambda$ is sectorial since the domain of the square root is invariant with respect to shifting (see Chapter 3.8 in [1]). It is known (see Theorems 1 and 2, [15]) that (a, V) has the Square Root Property if and only if its associated operator A is square root regular.

Now we are prepared for the final theorem.

Theorem 5.8. *Let B be the generator of a cosine function on a Hilbert space H. Then $-B$ is variational and square root regular with respect to some equivalent scalar product.*

Proof. First, one can find β such that $-B+\beta$ is sectorial. Since $B-\beta$ generates a cosine function as well, we can apply Fattorini's Theorem. Thus, the operator $A := \mathrm{i}(\beta - B)^{1/2}$ generates a strongly continuous group T on H. Choose ω_0, M such that (2.3) holds and let $\omega > \omega_0$. By Theorem 3.1 we obtain a new scalar product $(\cdot \mid \cdot)_\circ$ making $A-\omega$ and $-A-\omega$ m-dissipative and such that $\mathrm{D}(A) = \mathrm{D}(A^\circ)$ holds. Apply now Theorem 5.3 together with Proposition 5.6 to conclude that $-A^2 = -B+\beta$ is variational. This implies that $-B$ is variational. Finally, we apply Proposition 5.5 to the operators A and A° and obtain

$$\begin{aligned}\mathrm{D}((\beta+\omega^2-B)^{1/2}) &= \mathrm{D}((\omega^2-A^2)^{1/2}) = \mathrm{D}(A) = \mathrm{D}(A^\circ)\\ &= \mathrm{D}((\omega^2-A^{\circ 2})^{1/2}) = \mathrm{D}((\beta+\omega^2-B)^{\circ\frac{1}{2}}).\end{aligned}$$

This completes the proof. ∎

Comments. The construction of a variational operator by means of a form depends on the particular scalar product of H. (In Example 3.2 in [2] a variational operator is constructed which loses this property after changing the scalar product.) Thus it is natural to ask if a given operator A is variational with respect to *some* equivalent scalar product. A general characterization was obtained in [2], and one can in fact derive Theorem 5.8 from it. However, its proof uses a deep result of Le Merdy, whereas our proof in the special case of generators of cosine functions is more direct.

Kato's original question, whether every variational operator is square root regular (see Section 5, Remark 1 in [14]) was subsequently answered by McIntosh in the negative (see [16]). Confining the question to elliptic operators, the problem is known as *Kato's Square Root Problem*, and it has been solved only recently by Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian (see [3] and [4]). More information on the Square Root Problem can be found in [5] and [18].

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