J. OPERATOR THEORY
52(2004), 61–87

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PARTIAL-ISOMETRIC CROSSED PRODUCTS BY SEMIGROUPS OF ENDOMORPHISMS

JANNY LINDIARNI and IAIN RAEBURN

Communicated by Kenneth R. Davidson

ABSTRACT. Let Γ^+ be the positive cone in a totally ordered abelian group Γ, and let α be an action of Γ^+ by endomorphisms of a C^*-algebra A. We consider a new kind of crossed-product C^*-algebra $A \times_\alpha \Gamma^+$, which is generated by a faithful copy of A and a representation of Γ^+ as partial isometries. We claim that these crossed products provide a rich and tractable family of Toeplitz algebras for product systems of Hilbert bimodules, as recently studied by Fowler, and we illustrate this by proving detailed structure theorems for actions by forward and backward shifts.

KEYWORDS: *C^*-algebra, endomorphism, semigroup, partial isometry, crossed product.*

MSC (2000): 46L55.

1. INTRODUCTION

Let Γ be a totally ordered abelian group with positive cone Γ^+, and consider an action $\alpha : \Gamma^+ \to \operatorname{End} A$ of Γ^+ by endomorphisms of a C^*-algebra A. We study covariant representations (π, V) of the system (A, Γ^+, α) in which the endomorphisms α_s are implemented by partial isometries V_s, and the corresponding crossed-product C^*-algebra $A \times_\alpha \Gamma^+$ which is generated by a universal covariant representation. We think these *partial-isometric crossed products* are likely to be of interest for several reasons.

Our first motivation comes from the analogous covariant representation theory in which the elements of Γ^+ are implemented by isometries. To avoid confusion, we shall refer to these as *covariant isometric representations* and the corresponding crossed products $A \times_\alpha^{\mathrm{iso}} \Gamma^+$ as *isometric crossed products.* Isometric crossed products by the semigroup $\mathbb{N} = \mathbb{Z}^+$ were first used to give a model for the Cuntz algebra $\mathcal{O}_n$ ([6], [21], [24], [4]). Subsequently, various authors considered the action τ of Γ^+ by right translation on a distinguished subalgebra B_{Γ^+} of $\ell^\infty(\Gamma^+)$, and used $B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+$ as a model for the Toeplitz algebra $\mathcal{T}(\Gamma)$. This program has also been effective for the more general quasi-lattice ordered groups, such as $\mathbb{N}^k$ ([15]). More recently, isometric crossed products by actions of $\mathbb{N}^k$ have proved to be useful models for Hecke algebras arising in number theory (see [16], [14], [17], for example).

The theory of isometric crossed products, however, yields no information about some systems (A, Γ^+, α). For example, consider the action σ of $\mathbb{N}$ by left translations on

$$\mathbf{c}_0 := \{f : \mathbb{N} \to \mathbb{C} : f(n) \to 0 \text{ as } n \to \infty\}.$$

Every covariant isometric representation (π, V) of $(\mathbf{c}_0, \mathbb{N}, \sigma)$ satisfies

$$\pi(f) = (V^*)^n \pi(\sigma_n(f)) V^n,$$

and since $\sigma_n(f) \to 0$ as $n \to \infty$ for every $f \in \mathbf{c}_0$, this is only possible if π is identically zero. Thus $\mathbf{c}_0 \times_\sigma^{\mathrm{iso}} \mathbb{N} = \{0\}$. Every system (A, Γ^+, α), on the other hand, admits covariant partial-isometric representations (π, V) in which π is faithful, and hence the partial-isometric crossed product contains full information about the system.

Our second motivation concerns the Toeplitz algebras of Hilbert bimodules ([10]). In Pimsner's original investigations of Hilbert bimodules ([22]), a key example was provided by an endomorphism α of a C^*-algebra A, and the Cuntz-Pimsner algebra of this bimodule is the isometric crossed product $A \times_\alpha^{\mathrm{iso}} \mathbb{N}$. The Toeplitz algebra of this bimodule, on the other hand, is our partial-isometric crossed product $A \times_\alpha \mathbb{N}$. Fowler has recently considered product systems of Hilbert bimodules over more general semigroups, and studied the Toeplitz algebras of these product systems. In particular, he has identified conditions under which the results of [10] carry over to his new family of Toeplitz algebras ([8]). Important examples of product systems are provided by endomorphic actions of semigroups, and our partial-isometric crossed products are Toeplitz algebras to which Fowler's results apply. Because of their concrete nature, partial-isometric crossed products form a particularly tractable family of Toeplitz algebras, and even for the systems $(B_\mathbb{N}, \mathbb{N}, \tau)$ and $(B_\mathbb{N}, \mathbb{N}, \sigma)$ the partial-isometric crossed products have rich structure. Thus our results confirm that there is a lot of interesting information lying between a Cuntz-Pimsner algebra and its Toeplitz algebra extension. In particular, for the Hilbert bimodules associated to the endomorphisms τ_1 and σ_1 of $B_\mathbb{N}$, there are many distinct relative Cuntz-Pimsner algebras as in [18] and [9].

A third point of interest lies in the form of our structure theorems for crossed products. Associated to any pair of ideals I, J in a C^*-algebra B there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I \cap J & \to & J & \to & J/(I\cap J) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I & \to & B & \to & B/I & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I/(I\cap J) & \to & B/J & \to & B/(I+J) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which all the rows and columns are exact. From this, it follows easily that B has a composition series $0 \leqslant I \cap J \leqslant I + J \leqslant B$ of ideals with subquotients

$$I \cap J, \quad (I+J)/(I \cap J) \cong (I/(I \cap J)) \oplus (J/(I \cap J)) \quad \text{and} \quad B/(I+J),$$

and it is often helpful to have a structure theorem which identifies these subquotients in familiar terms. Here, though, we can do more: we can identify in familiar terms the four extensions which make up the outside square. Thus, for example, Theorem 6.1 describes $B = B_{\mathbb{N}} \times_{\tau} \mathbb{N}$, and identifies both the right-hand and bottom exact sequences as the extension of $C(\mathbb{T})$ by $\mathcal{K}$ provided by the Toeplitz algebra $\mathcal{T}(\mathbb{Z})$.

We begin with a preliminary section containing background material about power partial isometries, Toeplitz algebras and isometric crossed products, and Hilbert bimodules. In Section 3, we discuss representations of totally ordered semigroups by partial isometries, and analyse the C^*-algebras generated by two semigroups of truncated shifts.

In Section 4, we discuss covariant partial-isometric representations and the partial-isometric crossed product. We show that every action $\alpha : \Gamma^+ \to \operatorname{End} A$ admits covariant partial-isometric representations in which A acts faithfully (Example 4.6), and show how the results of Fowler ([8]) allow us to identify the covariant partial-isometric representations (π, V) of (A, Γ^+, α) for which the associated representation $\pi \times V$ of $A \times_\alpha \Gamma^+$ is faithful (Theorem 4.8).

In Section 5, we give a structure theorem for the crossed product of the system $(B_{\Gamma^+}, \Gamma^+, \tau)$ arising in the analysis of Toeplitz algebras. While many of our observations work for arbitrary totally ordered abelian groups, the main Theorem 5.6 concerns subsemigroups of $\mathbb{R}^+$. In Theorem 6.1, we obtain more detailed information for the semigroup $\mathbb{N}$. In the last section, we consider the action σ of $\mathbb{N}$ by right translation on $B_{\mathbb{N}}$. Although the structure of $B_{\mathbb{N}} \times_\sigma \mathbb{N}$ is quite a bit simpler than that of $B_{\mathbb{N}} \times_\tau \mathbb{N}$, it is still a good deal more complicated than that of $B_{\mathbb{N}} \times_\sigma^{\mathrm{iso}} \mathbb{N} = \{0\}$ (see Theorem 7.4).

CONVENTIONS. Throughout this paper, Γ will be a totally ordered abelian group with positive cone Γ^+; sometimes Γ will be a subgroup of $\mathbb{R}$ but if so, we shall say so. Our main examples are the additive semigroup $\mathbb{N}$ (which for us always contains 0) and other subsemigroups of the additive group $\mathbb{R}$; we therefore use additive notation in Γ, so that the identity is 0 and $\Gamma^+ = \{x \in \Gamma : x \geqslant 0\}$. A subgroup I of Γ is an *order ideal* if $x \in I$ and $0 \leqslant y \leqslant x$ imply $y \in I$. We say that Γ is *simple* if it has no nontrivial order ideals; standard theorems say that Γ is simple if and only if it is *archimedean* in the sense that $\{y \in \Gamma^+ : y \leqslant nx \text{ for some } n \in \mathbb{N}\} = \Gamma^+$ for every nonzero $x \in \Gamma^+$, and hence if and only if Γ is order isomorphic to a subgroup of $\mathbb{R}$ ([11]).

We denote by $\mathcal{K}(H)$ the C^*-algebra of compact operators on a Hilbert space H. We write $\bar{\pi}$ for the extension of a non-degenerate representation $\pi : A \to B(H)$ to the multiplier algebra $M(A)$; similarly, if α is an extendible endomorphism of a C^*-algebra A, in the sense that there is an approximate identity $\{a_i\}$ such that $\alpha(a_i)$ converges strictly to a projection in $M(A)$ ([1], Section 2), we write $\bar{\alpha}$ for the extension of α to an endomorphism of $M(A)$.

2. PRELIMINARIES

2.1. POWER PARTIAL ISOMETRIES. A *partial isometry* V on a Hilbert space H is an operator on H such that $\|Vh\| = \|h\|$ for $h \in (\ker V)^{\perp}$. A bounded operator V is a partial isometry if and only if $VV^*V = V$, and then the operators V^*V and VV^* are the orthogonal projections on the initial space $(\ker V)^{\perp}$ and range VH respectively. An element v of a C^*-algebra A is called a partial isometry if $vv^*v = v$; it then becomes a partial isometry in the usual sense whenever we represent A on Hilbert space.

The product of two partial isometries is not in general a partial isometry: for example, if $\{e_1, e_2\}$ is an orthonormal set and P, Q are the orthogonal projections on $\operatorname{span}\{e_1\}$, $\operatorname{span}\{e_1 + e_2\}$, then e_1 is a unit vector in $(\ker QP)^{\perp}$ but $QPe_1 = (e_1 + e_2)/2$ does not have norm 1. Since we are interested in semigroups of partial isometries, we need to know when a product of partial isometries is a partial isometry. The answer is well-known, and is proved, for example, in Lemma 2 of [12].

PROPOSITION 2.1. *Let S and T be partial isometries. Then ST is a partial isometry if and only if S^*S commutes with TT^*.*

A partial isometry v is a *power partial isometry* if v^n is a partial isometry for $n \in \mathbb{N}$. Every isometry and coisometry is a power partial isometry, and the other key examples are the truncated shifts described in Example 2.2 — indeed, every power partial isometry is a direct sum of an isometry, a coisometry and some truncated shifts ([12]).

EXAMPLE 2.2. Let $\{e_i : 1 \leqslant i \leqslant k+1\}$ be the usual orthonormal basis for $\mathbb{C}^{k+1}$, and consider the *truncated shift* $J_k = \sum_{j=1}^{k} e_{j+1} \otimes \overline{e}_j$ on $\mathbb{C}^{k+1}$. Since the $e_i \otimes \overline{e}_j$ are matrix units, we have $J_k^* = \sum_{j=2}^{k+1} e_{j-1} \otimes \overline{e}_j$, and can check that J_k is a power partial isometry satisfying $J_k^{k+1} = 0$. The C^*-subalgebra $C^*(J_k)$ of $B(\mathbb{C}^{k+1})$ generated by J_k contains $e_i \otimes \overline{e}_j = (J_k^*)^{j-1} J_k^k (J_k^*)^k J_k^{i-1}$, and hence is all of $B(\mathbb{C}^{k+1})$.

The C^*-algebra $C^*\Big(\bigoplus_{k=1}^{\infty} J_k\Big)$ is universal among C^*-algebras generated by a power partial isometry ([13], Theorem 1.3). We shall need to know this about $C^*\Big(\bigoplus_{k=1}^{n} J_k\Big)$.

LEMMA 2.3. *The C^*-algebra $C^*\Big(\bigoplus_{k\leqslant n} J_k\Big)$ is isomorphic to $\bigoplus_{k\leqslant n} M_{k+1}(\mathbb{C})$.*

Proof. We prove this by induction on n. For $n = 1$, we have $C^*(J_1) \cong B(\mathbb{C}^2)$ by Example 2.2. Suppose $n > 1$ and we know the result for $n - 1$. Then since $J_k^n = 0$ for $k < n$, we have $\Big(\bigoplus_{k\leqslant n} J_k\Big)^n = 0 \oplus J_n^n = 0 \oplus (e_{n+1} \otimes \overline{e}_1)$. Since

$C^*(e_{n+1}\otimes\overline{e}_1) = B(\mathbb{C}^{n+1}) = C^*(J_n)$, this implies that $\Big(\bigoplus_{k<n} J_k\Big)\oplus 0 \in C^*\Big(\bigoplus_{k\leqslant n} J_k\Big)$, and that

$$C^*\Big(\bigoplus_{k\leqslant n} J_k\Big) = C^*\Big(\bigoplus_{k<n} J_k\Big) \oplus B(\mathbb{C}^{n+1}),$$

which is isomorphic to $\bigoplus_{k\leqslant n} B(\mathbb{C}^{k+1})$ by the inductive hypothesis. ∎

2.2. TOEPLITZ ALGEBRAS. We let $\{\varepsilon_r : r \in \Gamma^+\}$ denote the usual orthonormal basis of $\ell^2(\Gamma^+)$. There is a representation $T = T^\Gamma$ of Γ^+ by isometries on $\ell^2(\Gamma^+)$ such that $T_s(\varepsilon_r) = \varepsilon_{r+s}$ for $r, s \in \Gamma^+$. The *Toeplitz algebra* $\mathcal{T}(\Gamma)$ is the C^*-subalgebra of $B(\ell^2(\Gamma^+))$ generated by $\{T_s : s \in \Gamma^+\}$.

One way to analyse $\mathcal{T}(\Gamma)$ is by realising it as a semigroup crossed product. If α is an action of Γ^+ by endomorphisms of a C^*-algebra A, then a *covariant isometric representation* of (A, Γ^+, α) consists of a non-degenerate representation $\pi : A \to B(H)$ and an isometric representation $V$of Γ^+ on H such that

$$\pi(\alpha_s(a)) = V_s\pi(a)V_s^* \quad \text{for } s \in \Gamma^+ \text{ and } a \in A;$$

the semigroup crossed product of A by α is by definition universal for such representations. Here we call it the *isometric crossed product* and denote it by $A\times_\alpha^{\mathrm{iso}}\Gamma^+$; we write $\pi \times^{\mathrm{iso}} V$ for the representation of $A \times_\alpha^{\mathrm{iso}} \Gamma^+$ associated to a covariant isometric representation (π, V).

For $s \in \Gamma^+$, we denote by 1_s the characteristic function of $\{t \in \Gamma^+ : t \geqslant s\}$. Since $1_s1_t = 1_{\max\{s,t\}}$ and $1_s^* = 1_s$, $B_{\Gamma^+} := \overline{\operatorname{span}}\{1_t : t \in \Gamma^+\}$ is a C^*-subalgebra of $\ell^\infty(\Gamma^+)$. The action τ of Γ^+ by right translation on $\ell^\infty(\Gamma^+)$ satisfies $\tau_t(1_s) = 1_{s+t}$, and hence restricts to an action of Γ^+ by endomorphisms of B_{Γ^+}. For every isometric representation V of Γ^+ on H, there is a representation π_V of B_{Γ^+} such that (π_V, V) is a covariant isometric representation of $(B_{\Gamma^+}, \Gamma^+, \tau)$; if each V_s is non-unitary, then $\pi_V \times^{\mathrm{iso}} V$ is a faithful representation of $B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+$ ([3], Theorem 2.4). In particular, the representation $T = T^\Gamma : \Gamma^+ \to B(\ell^2(\Gamma^+))$ induces an isomorphism $\pi_T \times^{\mathrm{iso}} T$ of $B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+$ onto $\mathcal{T}(\Gamma)$.

Since every unitary representation is in particular an isometric representation, there is a canonical quotient map ψ_T of $\mathcal{T}(\Gamma) \cong B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+$ onto $C^*(\Gamma) \cong C(\widehat{\Gamma})$. Murphy proved in [19] that the kernel of ψ_T is the commutator ideal $\mathcal{C}_\Gamma$ of $\mathcal{T}(\Gamma)$. In Remark 3.3 of [2], this is deduced from properties of isometric crossed products; in particular, it is shown that $\mathcal{C}_\Gamma$ itself is an isometric crossed product $B_{\Gamma^+,\infty} \times_\tau^{\mathrm{iso}} \Gamma^+$. From this description it follows easily that $\mathcal{C}_\Gamma$ is generated by the elements $T_sT_s^* - T_tT_t^*$ for $s < t$.

Douglas proved that if Γ is a subgroup of $\mathbb{R}$ then $\mathcal{C}_\Gamma$ is a simple C^*-algebra, and Murphy proved the converse ([19], Theorem 4.3; see also p. 1141, [3]). The following description of $\mathcal{C}_\Gamma$ will be useful.

LEMMA 2.4. *For every totally ordered abelian group* Γ,

$$\mathcal{C}_\Gamma = \overline{\operatorname{span}}\{T_r(1 - T_uT_u^*)T_t^* : r, u, t \in \Gamma^+\}. \tag{2.1}$$

Proof. Let $\mathcal{I}$ denote the right-hand side of (2.1). Since each $1 - T_uT_u^* = [T_u^*, T_u]$ is a commutator, $\mathcal{I} \subset \mathcal{C}_\Gamma$. On the other hand, $\mathcal{C}_\Gamma$ is generated by the elements $T_sT_s^* - T_tT_t^* = T_s(1 - T_{t-s}T_{t-s}^*)T_s^*$ for $s < t$, so it suffices to prove that $\mathcal{I}$ is an ideal.

Since $\mathcal{T}(\Gamma)$ is generated by $\{T_s : s \in \Gamma^+\}$, we must show $T_s\mathcal{I} \subset \mathcal{I}$ and $T_s^*\mathcal{I} \subset \mathcal{I}$. We trivially have $T_s\mathcal{I} \subset \mathcal{I}$. Let $r, u, t \in \Gamma^+$. Then

$$\begin{aligned}
&T_s^*T_r(1 - T_uT_u^*)T_t^* \\
&\qquad = \begin{cases} T_{r-s}(1 - T_uT_u^*)T_t^* & \text{if } r \geqslant s, \\ T_{s-r}^*(1 - T_uT_u^*)T_t^* & \text{if } r < s, \end{cases} \\
&\qquad = \begin{cases} T_{r-s}(1 - T_uT_u^*)T_t^* & \text{if } r \geqslant s, \\ (T_{s-r}^* - T_{s-r}^*T_{s-r}T_{u-(s-r)}T_{u-(s-r)}^*T_{s-r}^*)T_t^* & \text{if } r < s \text{ and } s - r < u, \\ 0 & \text{otherwise,} \end{cases} \\
&\qquad = \begin{cases} T_{r-s}(1 - T_uT_u^*)T_t^* & \text{if } r \geqslant s, \\ (1 - T_{u-(s-r)}T_{u-(s-r)}^*)T_{s-r+t}^* & \text{if } r < s \text{ and } s - r < u, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and hence $T_s^*T_r(1 - T_uT_u^*)T_t^*$ belongs to $\mathcal{I}$. ∎

2.3. HILBERT BIMODULES. A *Hilbert bimodule* over a C^*-algebra A is a right Hilbert A-module X together with a homomorphism $\varphi : A \to \mathcal{L}(X)$, which gives a left action of A on X: $a \cdot x := \varphi(a)x$ for $a \in A$ and $x \in X$. A *Toeplitz representation* (ψ, π) of X in a C^*-algebra B consists of a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that

$$\psi(x \cdot a) = \psi(x)\pi(a), \quad \psi(x)^*\psi(y) = \pi(\langle x, y\rangle_A), \quad \text{and} \quad \psi(a \cdot x) = \pi(a)\psi(x)$$

for every $x \in X$ and $a \in A$.

The *Toeplitz algebra* of X is the C^*-algebra $\mathcal{T}_X$ generated by the range of a universal Toeplitz representation (i_X, i_A) of X; then for every Toeplitz representation (ψ, π) of X in a C^*-algebra B, there is a homomorphism $\psi \times \pi$ of $\mathcal{T}_X$ into B such that $(\psi \times \pi) \circ i_X = \psi$ and $(\psi \times \pi) \circ i_A = \pi$. For every Hilbert bimodule X, there is such a C^*-algebra and it is unique up to isomorphism ([10], Proposition 1.3).

DEFINITION 2.5. Suppose $X = \{X_s : s \in \Gamma^+\}$ is a family of Hilbert bimodules over a C^*-algebra A. We write $\varphi_s : A \to \mathcal{L}(X_s)$ for the left action of A on X_s. Fowler says that X is a *product system* over Γ^+ if there is an associative multiplication on X (strictly speaking, on the disjoint union of the X_s) such that for $s, t \in \Gamma^+$, the map $(x, y) \mapsto xy$ extends to an isomorphism of the Hilbert bimodule $X_s \otimes X_t$ onto X_{st} ([8], Definition 2.1). He also requires that $X_0 = A$ (with left and right actions defined by multiplication in A), and that the multiplications in X involving elements in $A = X_0$ satisfy $ax = a \cdot x$ and $xa = x \cdot a$ for $a \in A$ and $x \in X_s$.

DEFINITION 2.6. A *Toeplitz representation* ψ of a product system X in a C^*-algebra B is a map $\psi : X \to B$ such that for each $s \in \Gamma^+$, (ψ_s, ψ_0) is a Toeplitz representation of X_s, and $\psi(xy) = \psi(x)\psi(y)$ for $x, y \in X$.

Every product system X over Γ^+ has a *Toeplitz algebra* $\mathcal{T}_X$ which is generated by a universal Toeplitz representation i_X of X, and it is unique up to isomorphism ([8], Proposition 2.8).

3. PARTIAL-ISOMETRIC REPRESENTATIONS

Let Γ be a totally ordered abelian group with positive cone Γ^+. A *partial-isometric representation* of Γ^+ on a Hilbert space H is a map V of Γ^+ into $B(H)$ such that V_s is a partial isometry and $V_sV_t = V_{s+t}$ for every s,t in Γ^+. We denote by $C^*(V(\Gamma^+))$ the C^*-algebra generated by the operators V_s.

Example 3.1. Since $V_n = (V_1)^n$, a partial-isometric representation V of $\mathbb{N}$ is determined by the single partial isometry V_1; a single partial isometry W generates a partial isometric representation $V : n \mapsto W^n$ if and only if W is a power partial isometry. We often implicitly acknowledge this by writing V^n for V_n when the semigroup is $\mathbb{N}$.

The following property of partial-isometric representations will be used repeatedly.

Proposition 3.2. *Suppose V is a partial-isometric representation of Γ^+ on H. Then each V_s is a power partial isometry and $\{V_s^*V_s, V_tV_t^* : s,t \in \Gamma^+\}$ is a commuting family of projections.*

Proof. For each $s \in \Gamma^+$ and $n \in \mathbb{N}$, $V_s^n = V_{ns}$ is a partial isometry, so V_s is a power partial isometry. Because $V_sV_t = V_{s+t}$ is a partial isometry, each $V_s^*V_s$ commutes with each $V_tV_t^*$ by Proposition 2.1. To see that the range projections commute, we let $s,t \in \Gamma^+$ and compute:

$$\begin{aligned} V_sV_s^*V_tV_t^* &= \begin{cases} V_sV_{s-t}^*V_t^*V_tV_t^* & \text{if } t \leqslant s, \\ V_sV_s^*V_sV_{t-s}V_t^* & \text{if } s \leqslant t, \end{cases} \\ &= \begin{cases} V_sV_{s-t}^*V_t^* & \text{if } t \leqslant s, \\ V_sV_{t-s}V_t^* & \text{if } s \leqslant t, \end{cases} \\ &= \begin{cases} V_sV_s^* & \text{if } t \leqslant s, \\ V_tV_t^* & \text{if } s \leqslant t, \end{cases} \\ &= V_{s\vee t}V_{s\vee t}^*, \end{aligned}$$

where $s \vee t$ denotes $\max\{s,t\}$. Since $s \vee t = t \vee s$, this and the same calculation with s and t swapped show that $V_sV_s^*$ commutes with $V_tV_t^*$. A similar argument shows that $V_s^*V_sV_t^*V_t = V_{s\vee t}^*V_{s\vee t} = V_t^*V_tV_s^*V_s$. ∎

Every isometric representation $V : \Gamma^+ \to B(H)$ is also a partial-isometric representation, and so is the associated coisometric representation $V^* : s \mapsto V_s^*$. Thus there are natural representations T and T^* of Γ^+ by forward and backward shifts on $\ell^2(\Gamma^+)$. In the remainder of this section we discuss two partial-isometric representations by truncated shifts and the C^*-algebras they generate.

For $s \in \Gamma^+$, we consider two intervals

$$[0,s) := \{t \in \Gamma^+ : 0 \leqslant t < s\} \quad \text{and} \quad [0,s] := \{t \in \Gamma^+ : 0 \leqslant t \leqslant s\}.$$

For $t \in \Gamma^+$, there is a partial isometry K_t^s on $\ell^2([0,s))$ such that

$$K_t^s(\varepsilon_r) = \begin{cases} \varepsilon_{r+t} & \text{if } r+t \in [0,s), \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

and $K^s : \Gamma^+ \to B(\ell^2([0,s)))$ is a partial-isometric representation of Γ^+ satisfying $K^s_t = 0$ for $t \geqslant s$. Similarly, there are partial isometries J^s_t on $\ell^2([0,s])$ such that

$$J^s_t(\varepsilon_r) = \begin{cases} \varepsilon_{r+t} & \text{if } r+t \in [0,s], \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

and then $J^s : \Gamma^+ \to B(\ell^2([0,s]))$ is a partial-isometric representation of Γ^+ satisfying $J^s_t = 0$ for $t > s$.

We analyse $C^*(K^s(\Gamma^+))$ first.

PROPOSITION 3.3. *There is an order ideal I of Γ such that*

$$I^+ = \{t \in \Gamma^+ : 0 \leqslant t \leqslant ns \text{ for some } n \in \mathbb{N}\}, \tag{3.3}$$

and then $C^*(K^s(\Gamma^+))$ is Morita equivalent to the commutator ideal $\mathcal{C}_I$ in $\mathcal{T}(I)$.

Before we prove this proposition we identify $C^*(K^s(\Gamma^+))$ as a corner of the commutator ideal $\mathcal{C}_\Gamma$.

LEMMA 3.4. *For $s \in \Gamma^+$, $C^*(K^s(\Gamma^+))$ is isomorphic to $(1 - T_sT_s^*)\mathcal{C}_\Gamma(1 - T_sT_s^*)$.*

To prove Lemma 3.4, we need the following standard fact.

LEMMA 3.5. *Suppose K is a closed subspace of a Hilbert space H and P is the projection of H onto K. Then $PTP \mapsto PT|_K$ is an isomorphism of $PB(H)P$ onto $B(K)$.*

Proof of Lemma 3.4. We view $\ell^2([0,s))$ as the closed subspace of $\ell^2(\Gamma^+)$ spanned by $\{\varepsilon_t : t \in [0,s)\}$. Then $1 - T_sT_s^*$ is the projection of $\ell^2(\Gamma^+)$ onto $\ell^2([0,s))$. We have

$$(1 - T_sT_s^*)T_t(1 - T_sT_s^*)(\varepsilon_r) = \begin{cases} K^s_t(\varepsilon_r) & \text{if } r < s, \\ 0 & \text{if } r \geqslant s. \end{cases}$$

Thus the isomorphism of Lemma 3.5 identifies $C^*(K^s(\Gamma^+))$ with the C^*-subalgebra

$$D := C^*\big(\{(1 - T_sT_s^*)T_t(1 - T_sT_s^*) : t \in \Gamma^+\}\big)$$

of $\mathcal{T}(\Gamma)$. It therefore suffices to prove that $D = (1 - T_sT_s^*)\mathcal{C}_\Gamma(1 - T_sT_s^*)$.

Since $1 - T_sT_s^* = [T_s^*, T_s]$ belongs to $\mathcal{C}_\Gamma$, and $\mathcal{C}_\Gamma$ is an ideal, each generator $(1 - T_sT_s^*)T_t(1 - T_sT_s^*)$ belongs to $(1 - T_sT_s^*)\mathcal{C}_\Gamma(1 - T_sT_s^*)$. Thus

$$D \subset (1 - T_sT_s^*)\mathcal{C}_\Gamma(1 - T_sT_s^*).$$

Before proving the reverse inclusion, we recall from Lemma 2.4 that $\mathcal{C}_\Gamma$ is spanned by the elements of the form $T_r(1 - T_uT_u^*)T_t^*$. Since

$$\begin{aligned}
&(1 - T_sT_s^*)T_r(1 - T_uT_u^*)T_t^*(1 - T_sT_s^*) \\
&\quad = \begin{cases} (1 - T_sT_s^*)T_{r-t}(T_tT_t^* - T_{u+t}T_{u+t}^*)(1 - T_sT_s^*) & \text{if } 0 \leqslant r-t < s, \\ (1 - T_sT_s^*)(T_rT_r^* - T_{u+r}T_{u+r}^*)T_{t-r}^*(1 - T_sT_s^*) & \text{if } 0 \leqslant t-r < s, \\ 0 & \text{otherwise,} \end{cases} \\
&\quad = \begin{cases} (1 - T_sT_s^*)T_{r-t}(1 - T_sT_s^*)(T_tT_t^* - T_{u+t}T_{u+t}^*)(1 - T_sT_s^*) & \text{if } 0 \leqslant r-t < s, \\ (1 - T_sT_s^*)(T_rT_r^* - T_{u+r}T_{u+r}^*)(1 - T_sT_s^*)T_{t-r}^*(1 - T_sT_s^*) & \text{if } 0 \leqslant t-r < s, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

it suffices to prove that $(1-T_sT_s^*)T_tT_t^*(1-T_sT_s^*) \in D$ for every t. But since $T_t^*T_t = 1$ and $1 - T_sT_s^* \leqslant 1 - T_{t+s}T_{t+s}^*$, we calculate

$$\begin{aligned}(1-T_sT_s^*)T_t(1-T_sT_s^*) &= (1-T_sT_s^*)(T_t - T_tT_sT_s^*T_t^*T_t)\\ &= (1-T_sT_s^*)(1-T_{t+s}T_{t+s}^*)T_t = (1-T_sT_s^*)T_t,\end{aligned}$$

and deduce that

$$(1-T_sT_s^*)T_tT_t^*(1-T_sT_s^*) = (1-T_sT_s^*)T_t((1-T_sT_s^*)T_t)^*$$

is in D. This proves the reverse inclusion, and hence Lemma 3.4. ∎

Proof of Proposition 3.3. Since I^+ is a subsemigroup of Γ^+ and $0 \leqslant t \leqslant r \in I^+$ implies $t \in I^+$, $I := I^+ \cup (-I^+)$ is an order ideal. For $t \in \Gamma^+ \setminus I^+$, we have $K_t^s = 0$. Thus

$$C^*(K^s(\Gamma^+)) = C^*(\{K_t^s : t \in \Gamma^+\}) = C^*(\{K_t^s : t \in I^+\}) = C^*(K^s(I^+)).$$

By Lemma 3.4, $C^*(K^s(I^+))$ is isomorphic to the C^*-subalgebra $(1-V_sV_s^*)\mathcal{C}_I(1-V_sV_s^*)$ of the commutator ideal $\mathcal{C}_I$, where, to avoid eyestrain, we have written V for T^I. But $(1-V_sV_s^*)\mathcal{C}_I(1-V_sV_s^*)$ is Morita equivalent to the ideal $\overline{\mathcal{C}_I(1-V_sV_s^*)\mathcal{C}_I}$ ([23], Example 3.6), so it suffices to prove that $\mathcal{C}_I(1-V_sV_s^*)\mathcal{C}_I$ is dense in $\mathcal{C}_I$.

Lemma 2.4 implies that

$$\mathcal{C}_I = \overline{\operatorname{span}}\{V_r(1-V_tV_t^*)V_u^* : r,t,u \in I^+\}. \tag{3.4}$$

Since $1-V_sV_s^*$ is a projection in $\mathcal{C}_I$, $1-V_sV_s^*$ belongs to $\mathcal{C}_I(1-V_sV_s^*)\mathcal{C}_I$. Now suppose $t \in I^+$, say $t \leqslant Ns$. An induction argument shows that

$$\begin{aligned}1 - V_{ns}V_{ns}^* &= (1-V_{(n-1)s}V_{(n-1)s}^*) + (V_{(n-1)s}V_{(n-1)s}^* - V_{ns}V_{ns}^*)\\ &= (1-V_{(n-1)s}V_{(n-1)s}^*) + V_{(n-1)s}(1-V_sV_s^*)V_{(n-1)s}^*\end{aligned}$$

belongs to $\overline{\mathcal{C}_I(1-V_sV_s^*)\mathcal{C}_I}$ for all n, and hence so does $1-V_tV_t^* \leqslant 1-V_{Ns}V_{Ns}^*$. We deduce that $V_r(1-V_tV_t^*)V_u^*$ is in $\mathcal{C}_I(1-V_sV_s^*)\mathcal{C}_I$ for every $r,t,u \in I^+$, and (3.4) implies that $\mathcal{C}_I(1-V_sV_s^*)\mathcal{C}_I$ is dense in $\mathcal{C}_I$. ∎

We now consider $C^*(J^s(\Gamma^+))$.

PROPOSITION 3.6. *By identifying $\ell^2([0,s))$ with a closed subspace of $\ell^2([0,s])$, we can view $C^*(K^s(\Gamma^+))$ as a C^*-subalgebra of $C^*(J^s(\Gamma^+))$, and then*

$$C^*(J^s(\Gamma^+)) = C^*(K^s(\Gamma^+)) + \mathcal{K}(\ell^2([0,s])). \tag{3.5}$$

Proof. When we view $\ell^2([0,s))$ as $\overline{\operatorname{span}}\{\varepsilon_r : r \in [0,s)\} \subset \ell^2([0,s])$, $1 - J_s^s(J_s^s)^*$ is the projection of $\ell^2([0,s])$ onto $\ell^2([0,s))$. By Lemma 3.5, there is an isomorphism of $B(\ell^2([0,s)))$ onto $(1-J_s^s(J_s^s)^*)B(\ell^2([0,s]))(1-J_s^s(J_s^s)^*)$. Moreover, for $r < s$,

$$\begin{aligned}(1-J_s^s(J_s^s)^*)J_t^s(1-J_s^s(J_s^s)^*)(\varepsilon_r) &= (1-J_s^s(J_s^s)^*)J_t^s(\varepsilon_r)\\ &= \begin{cases}(1-J_s^s(J_s^s)^*)(\varepsilon_{r+t}) & \text{if } r+t \in [0,s],\\ 0 & \text{otherwise,}\end{cases}\\ &= \begin{cases}\varepsilon_{r+t} & \text{if } r+t \in [0,s),\\ 0 & \text{otherwise,}\end{cases}\\ &= K_t^s(\varepsilon_r),\end{aligned}$$

and $(1-J_s^s(J_s^s)^*)J_t^s(1-J_s^s(J_s^s)^*)(\varepsilon_s)=0$; thus we can identify $C^*(K^s(\Gamma^+))$ with the C^*-subalgebra of $C^*(J^s(\Gamma^+))$ generated by

$$\{(1-J_s^s(J_s^s)^*)J_t^s(1-J_s^s(J_s^s)^*) : t\in\Gamma^+\}.$$

We notice that the rank-one operator $\varepsilon_r\otimes\overline{\varepsilon}_t=(J_{s-r}^s)^*J_s^s(J_s^s)^*J_{s-t}^s$, and hence

$$\mathcal{K}(\ell^2([0,s]))=\overline{\operatorname{span}}\{(J_{s-r}^s)^*J_s^s(J_s^s)^*J_{s-t}^s : r,t\in[0,s]\}\subset C^*(J^s(\Gamma^+)). \tag{3.6}$$

Thus $C^*(K^s(\Gamma^+))+\mathcal{K}(\ell^2([0,s]))\subset C^*(J^s(\Gamma^+))$. Note that $C^*(K^s(\Gamma^+))+\mathcal{K}(\ell^2([0,s]))$ is a C^*-subalgebra of $C^*(J^s(\Gamma^+))$ because $\mathcal{K}(\ell^2([0,s]))$ is an ideal. On the other hand, we compute using the definitions of J^s and K^s that

$$(J_t^s-K_t^s)(\varepsilon_r)=\begin{cases}(\varepsilon_s\otimes\overline{\varepsilon}_{s-t})(\varepsilon_r) & \text{if } t\leqslant s,\\ 0 & \text{otherwise,}\end{cases}$$

for every $r\in[0,s]$, and hence

$$J_t^s=\begin{cases}K_t^s+(\varepsilon_s\otimes\overline{\varepsilon}_{s-t}) & \text{if } t\leqslant s,\\ 0 & \text{otherwise.}\end{cases}$$

Thus J_t^s belongs to $C^*(K^s(\Gamma^+))+\mathcal{K}(\ell^2([0,s]))$ for every $t\in\Gamma^+$, and $C^*(J^s(\Gamma^+))\subset C^*(K^s(\Gamma^+))+\mathcal{K}(\ell^2([0,s]))$. ∎

COROLLARY 3.7. *Let $\mathcal{J}=C^*(K^s(\Gamma^+))\cap\mathcal{K}(\ell^2([0,s]))$. Then $\mathcal{J}$ is an ideal of $C^*(K^s(\Gamma^+))$ and there is an exact sequence*

$$0\to\mathcal{K}(\ell^2([0,s]))\to C^*(J^s(\Gamma^+))\xrightarrow{R_s}C^*(K^s(\Gamma^+))/\mathcal{J}\to 0 \tag{3.7}$$

in which $R_s(J_t^s)=K_t^s+\mathcal{J}$.

Proof. We trivially have that $\mathcal{J}$ is an ideal of $C^*(K^s(\Gamma^+))$. By Lemma 3.6, $C^*(J^s(\Gamma^+))=C^*(K^s(\Gamma^+))+\mathcal{K}(\ell^2([0,s]))$. Then $C^*(J^s(\Gamma^+))/\mathcal{K}(\ell^2([0,s]))=C^*(K^s(\Gamma^+))/\mathcal{J}$, and we have (3.7). ∎

When Γ is archimedean, and hence isomorphic to a subgroup of the additive group of real numbers, we can say more.

PROPSITION 3.8. *Suppose Γ is archimedean. Then either Γ is singly generated, in which case $C^*(J^s(\Gamma^+))=\mathcal{K}(\ell^2([0,s]))$, or Γ is isomorphic to a dense subgroup of $\mathbb{R}$, in which case we have an exact sequence*

$$0\to\mathcal{K}(\ell^2([0,s]))\to C^*(J^s(\Gamma^+))\xrightarrow{R_s}C^*(K^s(\Gamma^+))\to 0 \tag{3.8}$$

such that $R_s(J_t^s)=K_t^s$.

Proof. If Γ is singly generated, then $\Gamma=\mathbb{Z}t$ for some t, $s=nt$ for some $n\geqslant 0$, J_t^s is the truncated shift on $\mathbb{C}^{n+1}\cong\ell^2([0,s])$, and $C^*(J^s(\Gamma^+))=C^*(J_t^s)$ is all of $\mathcal{K}(\ell^2([0,s]))=B(\ell^2([0,s]))$ by Example 2.2. Now suppose Γ is dense in $\mathbb{R}$. Then Γ is simple, and the order ideal I of Proposition 3.3 is all of Γ; since $\mathcal{C}_\Gamma$ is simple, Proposition 3.3 implies that $C^*(K^s)$ is simple too. Thus $\mathcal{J}:=C^*(K^s)\cap\mathcal{K}(\ell^2([0,s]))$ is either 0 or $C^*(K^s)$. But Γ is dense, $\ell^2([t,s])=K_t^s(\ell^2([0,s]))$ is infinite-dimensional for all $t<s$, and hence K_t^s is not compact whenever $t<s$. Thus $\mathcal{J}=0$, and Corollary 3.7 gives (3.8). ∎

4. PARTIAL-ISOMETRIC CROSSED PRODUCTS

4.1. COVARIANT PARTIAL-ISOMETRIC REPRESENTATIONS. We consider a dynamical system (A, Γ^+, α) consisting of an action α of Γ^+ by endomorphisms of a C^*-algebra A such that $\alpha_0 = \mathrm{id}_A$. We assume that each α_s is extendible, and hence extends to a strictly continuous endomorphism $\overline{\alpha}_s$ of the multiplier algebra $M(A)$.

DEFINITION 4.1. A *covariant partial-isometric representation* of (A, Γ^+, α) is a pair (π, V) consisting of a non-degenerate representation π of A on a Hilbert space H and a partial-isometric representation V of Γ^+ on H such that

$$\pi(\alpha_s(a)) = V_s\pi(a)V_s^* \quad \text{and} \quad V_s^*V_s\pi(a) = \pi(a)V_s^*V_s \quad \text{for } s \in \Gamma^+,\ a \in A. \tag{4.1}$$

We can also talk about covariant partial-isometric representations of (A, Γ^+, α) in a C^*-algebra B or a multiplier algebra $M(B)$.

LEMMA 4.2. *If (π, V) is a covariant partial-isometric representation of (A, Γ^+, α), then $(\overline{\pi}, V)$ is a covariant partial-isometric representation of $(M(A), \Gamma^+, \overline{\alpha})$.*

Proof. Let $\{a_i\}$ be an approximate identity for A. Then

$$\pi(\alpha_s(ma_i)) = V_s\pi(ma_i)V_s^* = V_s\overline{\pi}(m)\pi(a_i)V_s^*$$

converges strongly to $V_s\overline{\pi}(m)V_s^*$, because π is non-degenerate. On the other hand, since $\alpha_s(a_i)$ converges strictly to $\overline{\alpha}_s(1)$,

$$\pi(\alpha_s(ma_i)) = \overline{\pi}(\overline{\alpha}_s(m))\pi(\alpha_s(a_i))$$

converges strongly to $\overline{\pi}(\overline{\alpha}_s(m))\pi(\alpha_s(1)) = \overline{\pi}(\overline{\alpha}_s(m))$. Thus $\overline{\pi}(\overline{\alpha}_s(m)) = V_s\overline{\pi}(m)V_s^*$. A similar argument shows that $V_s^*V_s\overline{\pi}(m) = \overline{\pi}(m)V_s^*V_s$. ∎

Next, we give an alternative version of the covariance relation (4.1).

LEMMA 4.3. *Let A be a C^*-algebra and α be an extendible endomorphism of A. Suppose π is a non-degenerate homomorphism of A into a multiplier algebra $M(B)$ and v is a partial isometry in $M(B)$. Then*

$$\pi(\alpha(a))v = v\pi(a) \textit{ for every } a \in A \quad \textit{and} \quad vv^* = \overline{\pi}(\overline{\alpha}(1)) \tag{4.2}$$

if and only if

$$\pi(\alpha(a)) = v\pi(a)v^* \quad \textit{and} \quad v^*v\pi(a) = \pi(a)v^*v \quad \textit{for every } a \in A. \tag{4.3}$$

Proof. Suppose (4.2) holds. Then

$$\pi(\alpha(a)) = \pi(\alpha(a))\overline{\pi}(\overline{\alpha}(1)) = \pi(\alpha(a))vv^* = v\pi(a)v^*,$$

and

$$v^*v\pi(a) = v^*\pi(\alpha(a))v = (\pi(\alpha(a^*))v)^*v = (v\pi(a^*))^*v = \pi(a)v^*v.$$

Now suppose (4.3) holds, and $a \in A$. Then

$$\pi(\alpha(a))v = v\pi(a)v^*v = vv^*v\pi(a) = v\pi(a),$$

and Lemma 4.2 gives $\overline{\pi}(\overline{\alpha}(1)) = v\overline{\pi}(1)v^* = vv^*$. ∎

COROLLARY 4.4. *Let π be a non-degenerate representation of A on a Hilbert space H and V be a partial-isometric representation of Γ^+ on H. Then (π, V) is a covariant partial-isometric representation of (A, Γ^+, α) if and only if*

$$\pi(\alpha_s(a))V_s = V_s\pi(a) \quad \textit{and} \quad \overline{\pi}(\overline{\alpha}_s(1)) = V_sV_s^* \quad \textit{for } a \in A,\ s \in \Gamma^+.$$

COROLLARY 4.5. *If (π, V) is a covariant partial-isometric representation of (A, Γ^+, α) on H, then $V_0 = 1$.*

Proof. Using Proposition 3.2, we calculate

$$\begin{aligned} V_0 &= V_{0+0}V_{0+0}^*V_0 = V_0(V_0V_0^*)(V_0^*V_0) = V_0(V_0^*V_0)(V_0V_0^*) \\ &= (V_0V_0^*)V_{0+0}V_0^* = V_0V_0^*. \end{aligned}$$

Since we assume that α_0 is the identity endomorphism, and π is non-degenerate, we get $V_0V_0^* = \overline{\pi}(\overline{\alpha}_0(1)) = 1$. Thus $V_0 = 1$. ∎

EXAMPLE 4.6. Suppose $\alpha : \Gamma^+ \to \operatorname{End} A$, and π_0 is a non-degenerate representation of A on H. Define $\pi : A \to B(\ell^2(\Gamma^+, H))$ by

$$(\pi(a)\zeta)(r) = \pi_0(\alpha_r(a))(\zeta(r)).$$

Then π is a representation which is non-degenerate on

$$\mathcal{H} = \overline{\operatorname{span}}\{\zeta \in \ell^2(\Gamma^+, H) : \zeta(r) \in \overline{\pi}_0(\overline{\alpha}_r(1))H \text{ for all } r \in \Gamma^+\}.$$

For $s \in \Gamma^+$, define V_s on $\mathcal{H}$ by $V_s\zeta(r) = \zeta(r+s)$: for $\zeta \in \mathcal{H}$, $V_s\zeta(r)$ is in $\overline{\pi}_0(\overline{\alpha}_{r+s}(1))H \subset \overline{\pi}_0(\overline{\alpha}_r(1))H$, and hence $V_s\zeta \in \mathcal{H}$. Let $\zeta \in (\ker V_s)^\perp$. Then $\zeta(r) = 0$ for every $r < s$, and hence

$$\|V_s\zeta\|^2 = \sum_{r\in\Gamma^+} \|\zeta(r+s)\|^2 = \sum_{r\in\Gamma^+} \|\zeta(r)\|^2 = \|\zeta\|^2.$$

Thus V_s is a partial isometry. Indeed, $V : s \mapsto V_s$ is a partial-isometric representation of Γ^+ on $\mathcal{H}$, because

$$V_{s+t}\zeta(r) = \zeta(r+s+t) = V_t\zeta(r+s) = V_sV_t\zeta(r) \quad \text{for } \zeta \in \mathcal{H} \text{ and } s, t, r \in \Gamma^+.$$

We claim that $(\pi|_{\mathcal{H}}, V)$ is a covariant partial-isometric representation of (A, Γ^+, α). For $a \in A$, $\zeta \in \mathcal{H}$ and $r, s \in \Gamma^+$, we have

$$\begin{aligned} \big(\pi(\alpha_s(a))(V_s\zeta)\big)(r) &= \pi_0(\alpha_r(\alpha_s(a)))(V_s\zeta(r)) = \pi_0(\alpha_{r+s}(a))(\zeta(r+s)) \\ &= (\pi(a)\zeta)(r+s) = (V_s\pi(a)\zeta)(r), \end{aligned}$$

and hence $\pi(\alpha_s(a))V_s = V_s\pi(a)$. Since $V_sV_s^*$ is the projection on

$$\operatorname{range} V_s = \{\zeta \in \mathcal{H} : \zeta(r) \in \overline{\pi}_0(\overline{\alpha}_{r+s}(1))H \text{ for all } r \in \Gamma^+\},$$

we get

$$(\overline{\pi}(\overline{\alpha}_s(1))\zeta)(r) = \overline{\pi}_0(\overline{\alpha}_r(\overline{\alpha}_s(1)))(\zeta(r)) = \overline{\pi}_0(\overline{\alpha}_{r+s}(1))(\zeta(r)) = (V_sV_s^*\zeta)(r)$$

for every $\zeta \in \mathcal{H}$ and $r \in \Gamma^+$. Corollary 4.4 implies that (π, V) is covariant, as claimed.

Notice that if π_0 is faithful, so is π. Thus every system (A, Γ^+, α) admits covariant partial-isometric representations (π, V) with π faithful.

4.2. Crossed products and Toeplitz algebras of Hilbert bimodules. Suppose (A,Γ^+,α) is a dynamical system as in Subsection 4.1. Following Fowler ([8], Section 3) we give each $X_s := \{s\}\times\overline{\alpha}_s(1)A$ the structure of a Hilbert bimodule over A via

$$(s,x)\cdot a := (s,xa),\quad \langle (s,x),(s,y)\rangle_A := x^*y \quad\text{and}\quad a\cdot(s,x) = (s,\alpha_s(a)x),$$

and define a multiplication on $X = \bigsqcup\{X_s : s\in\Gamma^+\}$ by

$$(s,x)(t,y) = (s+t,\alpha_t(x)y) \quad \text{for } x\in\overline{\alpha}_s(1)A \text{ and } y\in\overline{\alpha}_t(1)A.$$

Then X is a product system of Hilbert bimodules over Γ^+ ([8], Lemma 3.2). The Toeplitz algebra $(\mathcal{T}_X, i_X)$ described in Proposition 2.8 of [8] is universal for covariant partial-isometric representations:

Propsition 4.7. *Let $\alpha : \Gamma^+ \to \operatorname{End} A$ be an action by extendible endomorphisms. Then there is a covariant partial-isometric representation (i_A, i_{Γ^+}) of (A,Γ^+,α) in $\mathcal{T}_X$ such that i_A is injective and*

(i) *for every covariant partial-isometric representation (π,V) of (A,Γ^+,α) on H there is a non-degenerate representation $\pi\times V$ of $\mathcal{T}_X$ on H such that $(\pi\times V)\circ i_A = \pi$ and $(\overline{\pi\times V})\circ i_{\Gamma^+} = V$; and*

(ii) *$\mathcal{T}_X$ is generated by $i_A(A)\cup i_{\Gamma^+}(\Gamma^+)$; indeed, we have*

$$\mathcal{T}_X = \overline{\operatorname{span}}\{i_{\Gamma^+}(s)^*i_A(a)i_{\Gamma^+}(t) : a\in A,\ s,t\in\Gamma^+\}. \tag{4.4}$$

If (j_A, j_{Γ^+}) is a covariant partial-isometric representation of (A,Γ^+,α) in a C^*-algebra B with properties (i) *and* (ii)*, then there is an isomorphism of B onto $\mathcal{T}_X$ which carries (j_A,j_{Γ^+}) into (i_A,i_{Γ^+}).*

Proof. By Proposition 3.4 of [8], there is a partial-isometric representation (i_A,i_{Γ^+}) in $\mathcal{T}_X$ such that $i_A(a) = i_X(0,a)$ and $i_{\Gamma^+}(s) = \lim i_X(s,\alpha_s(a_i))^*$, where $\{a_i\}$ is an approximate identity for A. Since Proposition 3.4 of [8] also says that we can recover i_X via $i_X(s,a) = i_{\Gamma^+}(s)^*i_A(a)$, the elements $i_A(a)$ and $i_{\Gamma^+}(s)$ generate. To verify (4.4), just check that the right-hand side is closed under multiplication, and hence is a C^*-algebra containing the generators.

Let (π,V) be a partial-isometric representation of (A,Γ^+,α) on H. Then Proposition 3.4 of [8] gives a Toeplitz representation ψ of X such that $\psi(s,x) = V_s^*\pi(x)$ and, by Proposition 2.8(a) of [8], there is a representation ψ_* of $\mathcal{T}_X$ with $\psi_*\circ i_X = \psi$. We take $\pi\times V = \psi_*$. Then $\pi\times V(i_A(a)) = \pi\times V(i_X(0,a)) = \psi(0,a) = \pi(a)$, and

$$\begin{aligned}\pi\times V(i_{\Gamma^+}(s)) &= \pi\times V(\lim i_X(s,\alpha_s(a_i))^*) = \lim \pi\times V(i_X(s,\alpha_s(a_i))^*)\\ &= \lim\psi_*(i_X(s,\alpha_s(a_i))^*) = \lim\psi(s,\alpha_s(a_i))^* = V_s.\end{aligned}$$

Thus (i_A,i_{Γ^+}) satisfies (i). Example 4.6 shows that there are covariant representations (π,V) with π faithful, and then the equation $\pi = (\pi\times V)\circ i_A$ shows that i_A is injective. The uniqueness follows by a standard argument using the two universal properties. ∎

We call $\mathcal{T}_X$ the *partial-isometric crossed product* of (A,Γ^+,α), and denote it $A\times_\alpha\Gamma^+$. Because it is the Toeplitz algebra of a product system, we can apply Fowler's results, and in particular Corollary 9.4 of [8]. This gives:

Theorem 4.8. *Suppose (π,V) is a covariant partial-isometric representation of the system (A,Γ^+,α) on H. Then $\pi\times V$ is faithful on $A\times_\alpha\Gamma^+$ if and only if π acts faithfully on $(V_s^*H)^\perp$ for every s in $\Gamma^+\setminus\{0\}$.*

5. THE CROSSED PRODUCT OF $(B_{\Gamma^+}, \Gamma^+, \tau)$

In this section we analyse the crossed product $B_{\Gamma^+} \times_\tau \Gamma^+$ by the action τ of Γ^+ by right translation on $B_{\Gamma^+} \subset \ell^\infty(\Gamma^+)$ which, by Proposition 9.6 of [8], is universal for partial-isometric representations of Γ^+. Theorem 5.6 concerns subgroups of $\mathbb{R}$, but until then Γ can be any totally ordered abelian group.

We begin by analysing some related crossed products associated to intervals in Γ^+. Let $s \in \Gamma^+$, and let I stand for one of $[0, s)$ or $[0, s]$. For $t \in \Gamma^+$, let 1_t^I be the characteristic function of $I \cap [t, \infty)$. Since $(1_t^I)^* = 1_t^I$ and $1_t^I 1_u^I = 1_{\max\{t,u\}}^I$, $B_I = \overline{\operatorname{span}}\{1_t^I : t \in I\}$ is a C^*-subalgebra of $\ell^\infty(I)$. We denote by τ^I the action of the semigroup Γ^+ by translation on $\ell^\infty(I)$:

$$\tau_r^I(f)(t) = \begin{cases} f(t-r) & \text{if } t - r \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tau_r^I(1_t^I)$ is 1_{r+t}^I if $r + t \in I$, and 0 otherwise, so B_I is invariant under τ^I. We thus obtain a dynamical system (B_I, Γ^+, τ^I). Since $\tau_r^I = 0$ when r is not in I, $B_I \times_{\tau^I} \Gamma^+$ for $I = [0, s]$ or $[0, s)$ is quite different in nature from $B_{\Gamma^+} \times_\tau \Gamma^+$. Nevertheless, the crossed products $B_I \times_{\tau^I} \Gamma^+$ play an important role in our structure theorem for $B_{\Gamma^+} \times_\tau \Gamma^+$, and hence we analyse $B_I \times_{\tau^I} \Gamma^+$ first. We begin by describing their universal property.

PROPOSITION 5.1. *The crossed product $B_I \times_{\tau^I} \Gamma^+$ is the universal C^*-algebra generated by a partial-isometric representation V of Γ^+ such that $V_r = 0$ for $r \notin I$.*

We let $(i_{B_I}, i_{\Gamma^+}^I)$ denote the universal covariant partial-isometric representation which generates $B_I \times_{\tau^I} \Gamma^+$. The covariance relation $i_{B_I}(1_r^I) = i_{\Gamma^+}^I(r) i_{\Gamma^+}^I(r)^*$ implies that the partial-isometric representation $i_{\Gamma^+}^I$ itself generates $B_I \times_{\tau^I} \Gamma^+$, and also that $i_{\Gamma^+}^I(r) = 0$ for $r \notin I$. So for any covariant partial-isometric representation (π, V) of (B_I, Γ^+, τ^I) we have $V_r = 0$ when r is not in I. It remains to prove that if V is a partial-isometric representation of Γ^+ such that $V_r = 0$ for $r \notin I$, then there is a representation π_V^I of B_I such that (π_V^I, V) is covariant, because then $V = (\pi_V^I \times V) \circ i_{\Gamma^+}^I$ factors through $i_{\Gamma^+}^I$. Thus the proposition will follow from Lemma 5.3 below.

For the proof of Lemma 5.3, we need the following variant of Proposition 2.2 of [3], and Proposition 1.3 of [15], which can be proved by making minor modifications to the proof of Proposition 2.2 of [3].

LEMMA 5.2. *Suppose $\{P_r : r \in I\}$ is a family of projections on H such that $P_r \geqslant P_t$ when $r \leqslant t$. Then there is a representation π_P of B_I on H such that $\pi_P(1_r^I) = P_r$ for $r \in I$, and π_P is faithful if and only if $P_r \neq P_t$ when $r \neq t$.*

LEMMA 5.3. *Let V be a partial-isometric representation of Γ^+ on H such that $V_r = 0$ when $r \notin I$. Then there is a representation π_V^I of B_I on H such that (π_V^I, V) is a covariant partial-isometric representation of the dynamical system (B_I, Γ^+, τ^I).*

Proof. First we prove that there is a representation π_V^I of B_I such that $\pi_V^I(1_r^I) = V_r V_r^*$. For $t > r$, we have

$$V_r V_r^* - V_t V_t^* = V_r(1 - V_{t-r}V_{t-r}^*)V_s^* = ((1 - V_{t-r}V_{t-r}^*)V_r^*)^*(1 - V_{t-r}V_{t-r}^*)V_r^*,$$

so $V_rV_r^* \geqslant V_tV_t^*$. Lemma 5.2 now gives the required representation.

To see that (π_V^I, V) is covariant, it suffices by Corollary 4.4 to show that $\pi_V^I(\tau_r^I(a))V_r = V_r\pi_V^I(a)$ for $r \in \Gamma^+$ and $a \in B_I$. By continuity, we need only do this for $a = 1_t^I$. For $r \notin I$ both sides of the equation are zero. For $r \in I$, we calculate using Proposition 3.2: if $t + r \in I$, we have

$$\pi_V^I(\tau_r^I(1_t^I))V_r = \pi_V^I(1_{t+r}^I)V_r = V_r(V_tV_t^*)(V_r^*V_r) = (V_rV_r^*V_r)(V_tV_t^*) = V_r\pi_V^I(1_t^I),$$

and if $t + r \notin I$, we have

$$\pi_V^I(\tau_r^I(1_t^I))V_r = \pi_V^I(0)V_r = 0 = V_{t+r}V_t^* = V_rV_tV_t^* = V_r\pi_V^I(1_t^I).$$

Thus (π_V^I, V) is covariant. ∎

This completes the proof of Proposition 5.1. ∎

Proposition 5.4. *Let V be a partial-isometric representation of Γ^+ such that $V_r = 0$ for $r \notin I$. Then the representation $\pi_V^I \times V$ of $B_I \times_{\tau^I} \Gamma^+$ is faithful if and only if*

$$(1 - V_r^*V_r)(V_uV_u^* - V_tV_t^*) \neq 0 \quad \textit{for every } r > 0,\ u, t \in I \textit{ and } u < t.$$

Proof. By Theorem 4.8, $\pi_V^I \times V$ is faithful if and only if $\pi_V^I|_{\mathrm{range}(1-V_r^*V_r)}$ is faithful for every $r > 0$. Let $r > 0$ and set $P_u := (1 - V_r^*V_r)V_uV_u^*$ for $u \in I$, which is a projection by Proposition 3.2. The same proposition implies that for $u \leqslant t$,

$$\begin{aligned} P_u - P_t &= (1 - V_r^*V_r)V_uV_u^*(1 - V_r^*V_r) - (1 - V_r^*V_r)V_tV_t^*(1 - V_r^*V_r) \\ &= (1 - V_r^*V_r)(V_uV_u^* - V_uV_{t-u}V_{t-u}^*V_u^*)(1 - V_r^*V_r) \\ &= (1 - V_r^*V_r)V_u(1 - V_{t-u}V_{t-u}^*)V_u^*(1 - V_r^*V_r) \geqslant 0. \end{aligned}$$

Thus by Lemma 5.2, there is a representation π_P of B_I on $(1 - V_rV_r^*)H$ such that $\pi_P(1_u^I) = P_u$. Let $h \in (1 - V_r^*V_r)H$. Then

$$\pi_P(1_u^I)h = (1 - V_r^*V_r)V_uV_u^*h = V_uV_u^*(1 - V_r^*V_r)h = V_uV_u^*h = \pi_V^I(1_u^I)h,$$

and since the 1_u^I generate B_I, we deduce that $\pi_P = \pi_V^I|_{(1-V_r^*V_r)H}$. The proposition therefore follows from the second part of Lemma 5.2. ∎

Corollary 5.5. *Let J^r be the partial-isometric representation of Γ^+ satisfying* (3.2). *Then the representation $\bigoplus_{r\in I}(\pi_{J^r}^I \times J^r)$ of $B_I \times_{\tau^I} \Gamma^+$ on $\bigoplus_{r\in I} \ell^2([0, r])$ is faithful.*

Proof. By Proposition 5.4, it is faithful if and only if

$$\bigoplus_{r\in I}\big(1 - (J_v^r)^*J_v^r\big)\big(J_u^r(J_u^r)^* - J_t^r(J_t^r)^*\big) \neq 0 \tag{5.1}$$

for every $v > 0$ and $0 \leqslant u < t \in I$. But the summand $\big(1 - (J_v^u)^*J_v^u\big)\big(J_u^u(J_u^u)^* - J_t^u(J_t^u)^*\big)$ is nonzero, so (5.1) holds. ∎

The crossed products $B_I \times_{\tau^I} \Gamma^+$ are important because they arise as quotients of $B_{\Gamma^+} \times_\tau \Gamma^+$. Each generating semigroup $i^I_{\Gamma^+} : \Gamma^+ \to B_I \times_{\tau^I} \Gamma^+$ is a partial-isometric representation, and the universal property of $B_{\Gamma^+} \times_\tau \Gamma^+$ gives surjective homomorphisms $q_I : B_{\Gamma^+} \times_\tau \Gamma^+ \to B_I \times_{\tau^I} \Gamma^+$ such that $q_I(i_{\Gamma^+}(r)) = i^I_{\Gamma^+}(r)$.

For $s > 0$, we write $q_s := q_{[0,s]}$ and $q_s^- := q_{[0,s)}$; for $s = 0$, we have only $q_0 := q_{[0,0]}$. Note that

$$\ker q_r^- \subset \ker q_s \subset \ker q_s^- \subset \ker q_t \subset \ker q_0 \quad \text{for } t < s < r.$$

Our structure theorem for $B_{\Gamma^+} \times_\tau \Gamma^+$ involves these ideals and the natural homomorphisms $\varphi_T := \pi_T^{\Gamma^+} \times T : B_{\Gamma^+} \times_\tau \Gamma^+ \to \mathcal{T}(\Gamma)$ associated to the Toeplitz representation $T : \Gamma^+ \to \mathcal{T}(\Gamma)$ and $\varphi_{T^*} := \pi_{T^*}^{\Gamma^+} \times T^* : B_{\Gamma^+} \times_\tau \Gamma^+ \to \mathcal{T}(\Gamma)$ associated to its adjoint $T^* : r \mapsto T_r^*$; both φ_T and φ_{T^*} are surjective because the T_r generate $\mathcal{T}(\Gamma)$. The theorem also involves the homomorphisms ψ_T and ψ_{T^*} of $\mathcal{T}(\Gamma)$ onto $C(\widehat{\Gamma})$ which carry T_r to the evaluation maps $\varepsilon_r : \gamma \mapsto \gamma(r)$ and, respectively, $\varepsilon_{-r} : \gamma \mapsto \gamma(-r)$.

THEOREM 5.6. *Suppose Γ is a subgroup of $\mathbb{R}$. Let*

$$\mathcal{I} = \big(\ker(\pi_T^{\Gamma^+} \times T)\big) \cap \big(\ker(\pi_{T^*}^{\Gamma^+} \times T^*)\big).$$

Then we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I} & \to & \ker(\pi_{T^*}^{\Gamma^+} \times T^*) & \to & \mathcal{C}_\Gamma & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \ker(\pi_T^{\Gamma^+} \times T) & \to & B_{\Gamma^+} \times_\tau \Gamma^+ & \overset{\varphi_T}{\to} & \mathcal{T}(\Gamma) & \to & 0 \\
 & & \downarrow & & {\scriptstyle\varphi_{T^*}}\downarrow & & {\scriptstyle\psi_{T^*}}\downarrow & & \\
0 & \to & \mathcal{C}_\Gamma & \to & \mathcal{T}(\Gamma) & \overset{\psi_T}{\to} & C(\widehat{\Gamma}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which all the rows and columns are exact. For $s \in \Gamma^+$, let $\mathcal{I}_s := \mathcal{I} \cap (\ker q_s)$, and for $s > 0$, let $\mathcal{I}_s^- := \mathcal{I} \cap (\ker q_s^-)$. Then:

(i) *$\mathcal{I}/\mathcal{I}_0 \cong \mathbb{C}$;*
(ii) *$\mathcal{I}_s^-/\mathcal{I}_s \cong \mathcal{K}(\ell^2([0,s]))$ for every $s > 0$;*
(iii) *$\mathcal{I}_s^- = \bigcap_{t<s} \mathcal{I}_t$ for every $s > 0$; and*
(iv) *$\mathcal{I}_s = \overline{\bigcup_{r>s} \mathcal{I}_r^-}$ for every $s \in \Gamma^+$.*

The proof of this theorem will occupy the rest of the section.

The right-hand exact sequence is due to Douglas ([7], Proposition 3). If Ψ denotes the automorphism of $C(\widehat{\Gamma})$ induced by the homeomorphism $\gamma \mapsto \gamma^{-1}$, then $\psi_{T^*} = \Psi \circ \psi_T$, and hence the bottom sequence is also exact. Since the middle sequences are exact by definition of φ_T and φ_{T^*}, we have the following diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I} & \to & \ker\varphi_{T^*} & & \mathcal{C}_\Gamma & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \ker\varphi_T & \to & B_{\Gamma^+}\times_\tau\Gamma^+ & \overset{\varphi_T}{\to} & \mathcal{T}(\Gamma) & \to & 0 \\
 & & & & {\scriptstyle\varphi_{T^*}}\downarrow & & {\scriptstyle\psi_{T^*}}\downarrow & & \\
0 & \to & \mathcal{C}_\Gamma & \to & \mathcal{T}(\Gamma) & \overset{\psi_T}{\to} & C(\widehat{\Gamma}) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The top left-hand square commutes because all the maps are inclusions, and the bottom right-hand square commutes because

$$\psi_{T^*}\circ\varphi_T(i_{\Gamma^+}(t)) = \psi_{T^*}(T_t) = \varepsilon_{-t} = \varepsilon_t^* = \psi_T(T_t^*) = \psi_T\circ\varphi_{T^*}(i_{\Gamma^+}(t))$$

for every $t\in\Gamma^+$. The equation $\psi_{T^*}\circ\varphi_T = \psi_T\circ\varphi_{T^*}$ also implies that φ_T maps $\ker\varphi_{T^*}$ into $\ker\psi_{T^*} = \mathcal{C}_\Gamma$, and it maps $\ker\varphi_{T^*}$ onto $\mathcal{C}_\Gamma$ because each of the spanning elements $T_r(1-T_vT_v^*)T_t^*$ in Lemma 2.4 has the form $\varphi_T(b)$ for $b = i_{\Gamma^+}(r)(1-i_{\Gamma^+}(v)i_{\Gamma^+}(v)^*)i_{\Gamma^+}(t)^*$ in $\ker\varphi_{T^*}$. Since $\ker(\varphi_T|_{\ker\varphi_{T^*}})$ is by definition $\mathcal{I}$, this gives exactness of the top row, and exactness of the left-hand column follows similarly.

It remains to prove the assertions about the structure of $\mathcal{I}$. Of these, (a) is easy: the homomorphism q_0 is nonzero on the elements $(1-i_{\Gamma^+}(u)^*i_{\Gamma^+}(u))(1-i_{\Gamma^+}(v)i_{\Gamma^+}(v)^*)$ of $\mathcal{I}$, and has one-dimensional range. For the next two parts, we need a lemma.

Lemma 5.7. *For each interval I,* $\ker q_I = \bigcap_{r\in I}\ker(\pi_{J^r}^{\Gamma^+}\times J^r)$.

Proof. For $t\in\Gamma^+$, we have

$$\bigoplus_{r\in I}(\pi_{J^r}^{\Gamma^+}\times J^r)(i_{\Gamma^+}(t)) = \bigoplus_{r\in I} J_t^r = \bigoplus_{r\in I}(\pi_{J^r}^{I}\times J^r)\circ q_I(i_{\Gamma^+}(t)).$$

Since $\bigoplus_{r\in I}(\pi_{J^r}^I\times J^r)$ is faithful on $B_I\times_{\tau^I}\Gamma^+$ by Corollary 5.5, it follows that $\ker q_I = \ker\bigoplus_{r\in I}(\pi_{J^r}^{\Gamma^+}\times J^r)$. ∎

We now prove the remaining parts of Theorem 5.6. It is convenient to do (iii) first.

Proof of (iii). From Lemma 5.7, we have

$$\bigcap_{t<s}\ker q_t = \bigcap_{t<s}\Big(\bigcap_{r\leqslant t}\ker(\pi_{J^r}^{\Gamma^+}\times J^r)\Big) = \bigcap_{r<s}\ker(\pi_{J^r}^{\Gamma^+}\times J^r) = \ker q_s^-,$$

and intersecting with $\mathcal{I}$ gives (iii). ∎

Proof of (ii). We shall prove that $\pi_{J^s}^{\Gamma^+} \times J^s$ is a surjection of $\ker q_s^-$ onto $\mathcal{K}(\ell^2([0,s]))$ with kernel $\ker q_s$. From (3.6) we see that $\mathcal{K}(\ell^2([0,s]))$ is spanned by the elements

$$(J_{s-t}^s)^* J_s^s (J_s^s)^* J_{s-r}^s = \pi_{J^s}^{\Gamma^+} \times J^s(i_{\Gamma^+}(s-t)^* i_{\Gamma^+}(s) i_{\Gamma^+}(s)^* i_{\Gamma^+}(s-r))$$

of $\pi_{J^s}^{\Gamma^+} \times J^s(\ker q_s^-)$, so $\pi_{J^s}^{\Gamma^+} \times J^s(\ker q_s^-) \supset \mathcal{K}(\ell^2([0,s]))$. We next show the reverse inequality.

If Γ is singly generated, then $C^*(J^s(\Gamma^+)) = \mathcal{K}(\ell^2([0,s]))$ by Example 2.2, so suppose Γ is not singly generated. Then by Proposition 3.8 there is a homomorphism R_s of $C^*(J^s(\Gamma^+))$ onto $C^*(K^s(\Gamma^+))$ such that $R_s(J_t^s) = K_t^s$ and $\ker R_s = \mathcal{K}(\ell^2([0,s]))$. But then we can verify on generators that

$$(\pi_{K^s}^{[0,s)} \times K^s) \circ q_s^- = R_s \circ (\pi_{J^s}^{\Gamma^+} \times J^s),$$

and this implies that $\pi_{J^s}^{\Gamma^+} \times J^s(\ker q_s^-) \subset \ker R_s = \mathcal{K}(\ell^2([0,s]))$, as claimed.

From two applications of Lemma 5.7 and (iii), we obtain

$$\begin{aligned}\ker(\pi_{J^s}^{\Gamma^+} \times J^s) \cap (\ker q_s^-) &= \ker(\pi_{J^s}^{\Gamma^+} \times J^s) \cap \Big(\bigcap_{t<s} \ker(\pi_{J^t}^{\Gamma^+} \times J^t)\Big) \\ &= \bigcap_{t\leqslant s} \ker(\pi_{J^t}^{\Gamma^+} \times J^t) = \ker q_s,\end{aligned}$$

and hence $\ker(\pi_{J^s}^{\Gamma^+} \times J^s|_{\ker q_s^-}) = \ker q_s$.

We have now proved that $\pi_{J^s}^{\Gamma^+} \times J^s$ induces an isomorphism of $(\ker q_s^-)/(\ker q_s)$ onto $\mathcal{K}(\ell^2([0,s]))$. However, for any $r > 0$ and $t > s$ the element

$$\big(i_{\Gamma^+}(s) i_{\Gamma^+}(s)^* - i_{\Gamma^+}(t) i_{\Gamma^+}(t)^*\big)\big(1 - i_{\Gamma^+}(r)^* i_{\Gamma^+}(r)\big)$$

belongs to $\mathcal{I}_s^- = (\ker q_s^-) \cap \mathcal{I}$ but not to $\ker q_s$, and hence has nonzero image in $\mathcal{K}(\ell^2([0,s]))$. Since the image of $\mathcal{I}_s^-$ is an ideal in $\mathcal{K}(\ell^2([0,s]))$ and $\mathcal{K}(\ell^2([0,s]))$ is simple, this image must be all of $\mathcal{K}(\ell^2([0,s]))$. So $\pi_{J^s}^{\Gamma^+} \times J^s$ also induces an isomorphism of $\mathcal{I}_s^-/\mathcal{I}_s$ onto $\mathcal{K}(\ell^2([0,s]))$. ∎

Proof of (iv). We trivially have $\overline{\bigcup_{r>s} \ker q_r^-} \subset \ker q_s$. Since $\overline{\bigcup_{r>s} \ker q_r^-}$ is an ideal in $B_{\Gamma^+} \times_\tau \Gamma^+$, there is a representation π of $B_{\Gamma^+} \times_\tau \Gamma^+$ on a Hilbert space H such that $\ker \pi = \overline{\bigcup_{r>s} \ker q_r^-}$, and then $\pi(i_{\Gamma^+}(t)) = 0$ for $t \geqslant r > s$. By Proposition 5.1, there is a representation π_s of $B_{[0,s]} \times_\tau \Gamma^+$ on H such that $\pi_s(i_{\Gamma^+}^{[0,s]}(t)) = \pi(i_{\Gamma^+}(t))$ for $t \in [0,s]$, and then $\pi_s \circ q_s = \pi$. Thus $\ker q_s \subset \ker \pi = \overline{\bigcup_{r>s} \ker q_r^-}$. Now intersecting with $\mathcal{I}$ gives $\mathcal{I}_s = \overline{\bigcup_{r>s} (\ker q_r^-) \cap \mathcal{I}} = \overline{\bigcup_{r>s} \mathcal{I}_r^-}$, by, for example, Lemma 1.3 of [3]. ∎

This completes the proof of Theorem 5.6. ∎

6. THE CROSSED PRODUCT BY THE FORWARD SHIFT

We now show that when Γ^+ is the additive semigroup $\mathbb{N}$, we can obtain more detailed information about the left-hand and top exact sequences in Theorem 5.6.

By viewing functions on $\mathbb{N}$ as sequences, we can identify $B_{\mathbb{N}}$ with the C^*-algebra of convergent sequences $\mathbf{c}$. Under this identification, the action τ of $\mathbb{N}$ is generated by the usual shift $\tau_1 : (x_0, x_1, x_2, \ldots) \mapsto (0, x_0, x_1, x_2, \ldots)$. The partial-isometric crossed product $\mathbf{c} \times_\tau \mathbb{N}$ is generated by the power partial isometry $i_{\mathbb{N}}(1)$, and by Proposition 9.6 of [8], is the universal C^*-algebra generated by a power partial isometry.

Since $[0, n) = [0, n-1]$ for this semigroup, the maps q_n^- and q_{n-1} coincide, and $\mathcal{I}_n^- = \mathcal{I}_{n-1}$ for $n > 0$. Thus Theorem 5.6 says that $\mathcal{I}/\mathcal{I}_0 \cong \mathbb{C}$ and $\mathcal{I}_{n-1}/\mathcal{I}_n \cong \mathcal{K}(\ell^2([0, n])) = M_{n+1}(\mathbb{C})$ for $n > 0$. We will prove that $\mathcal{I}$ is isomorphic to $\bigoplus_{n\in\mathbb{N}} M_{n+1}(\mathbb{C})$. To describe the extensions in the top and left-hand sequences, we let $P_n := 1 - T^{n+1}(T^*)^{n+1}$ be the projection onto $\operatorname{span}\{e_i : 0 \leqslant i \leqslant n\}$, and define

$$\mathcal{A} = \big\{f : \mathbb{N} \to \mathcal{K}(\ell^2(\mathbb{N})) : f(n) \in P_n\mathcal{K}(\ell^2(\mathbb{N}))P_n \text{ and } \varepsilon_\infty(f) := \lim_{n\to\infty} f(n) \text{ exists}\big\};$$

note that $\mathcal{A}_0 := \{f \in \mathcal{A} : \varepsilon_\infty(f) = 0\}$ is isomorphic to $\bigoplus_{n\in\mathbb{N}} M_{n+1}(\mathbb{C})$. Our refinement of Theorem 5.6 is:

Theorem 6.1. *There are isomorphisms $\pi : \ker\varphi_{T^*} \to \mathcal{A}$ and $\pi^* : \ker\varphi_T \to \mathcal{A}$ and an automorphism α of $\mathcal{A}_0$ such that the following diagram commutes and has all rows and columns exact:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{A}_0 & \xrightarrow{\mathrm{id}} & \mathcal{A} & \xrightarrow{\varepsilon_\infty} & \mathcal{K}(\ell^2(\mathbb{N})) & \to & 0 \\
 & & \alpha\downarrow & & \pi^{-1}\downarrow & & \downarrow & & \\
0 & \to & \mathcal{A} & \xrightarrow{(\pi^*)^{-1}} & \mathbf{c}\times_\tau\mathbb{N} & \xrightarrow{\varphi_T} & \mathcal{T}(\mathbb{Z}) & \to & 0 \\
 & & \varepsilon_\infty\downarrow & & \varphi_{T^*}\downarrow & & \psi_{T^*}\downarrow & & \\
0 & \to & \mathcal{K}(\ell^2(\mathbb{N})) & \to & \mathcal{T}(\mathbb{Z}) & \xrightarrow{\psi_T} & C(\mathbb{T}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{6.1}$$

Applying the universal property of $\mathbf{c} \times_\tau \mathbb{N}$ to the power partial isometries P_nTP_n and $P_nT^*P_n$ gives representations π_n and π_n^* of $\mathbf{c} \times_\tau \mathbb{N}$ on $\ell^2(\mathbb{N})$ such that $\pi_n(i_{\mathbb{N}}(1)) = P_nTP_n$ and $\pi_n^*(i_{\mathbb{N}}(1)) = P_nT^*P_n$. We will prove that $\lim_{n\to\infty} \pi_n(a)$ exists for all $a \in \ker\varphi_{T^*}$, so that $\pi(a) := \{\pi_n(a)\}$ belongs to $\mathcal{A}$, and similarly for $\pi^*(b) := \{\pi_n^*(b)\}$ when $b \in \ker\varphi_T$. To do this, we need to identify spanning families for $\ker\varphi_{T^*}$ and $\ker\varphi_T$.

LEMMA 6.2. *For $i, j, m \in \mathbb{N}$, let*

$$f_{i,j}^m := i_{\mathbb{N}}(i) i_{\mathbb{N}}(m)^* i_{\mathbb{N}}(m)(1 - i_{\mathbb{N}}(1) i_{\mathbb{N}}(1)^*) i_{\mathbb{N}}(j)^*,$$

and

$$g_{i,j}^m := i_{\mathbb{N}}(i)^* i_{\mathbb{N}}(m) i_{\mathbb{N}}(m)^*(1 - i_{\mathbb{N}}(1)^* i_{\mathbb{N}}(1)) i_{\mathbb{N}}(j).$$

Then $\ker \varphi_{T^*} = \overline{\operatorname{span}}\{f_{i,j}^m : i, j, m \in \mathbb{N}\}$ *and* $\ker \varphi_T = \overline{\operatorname{span}}\{g_{i,j}^m : i, j, m \in \mathbb{N}\}$.

Proof. First we claim that $\mathcal{E} := \overline{\operatorname{span}}\{f_{i,j}^m : i, j, m \in \mathbb{N}\}$ is an ideal. To see this, it suffices to show that $i_{\mathbb{N}}(1)\mathcal{E} \subset \mathcal{E}$ and $i_{\mathbb{N}}(1)^*\mathcal{E} \subset \mathcal{E}$. The first is trivial. To show $i_{\mathbb{N}}(1)^*\mathcal{E} \subset \mathcal{E}$, we first let $i > 0$ and compute using Proposition 3.2:

$$\begin{aligned} i_{\mathbb{N}}(1)^* i_{\mathbb{N}}(i) &= i_{\mathbb{N}}(1)^* \big(i_{\mathbb{N}}(1) i_{\mathbb{N}}(i-1)\big) \\ &= i_{\mathbb{N}}(1)^* i_{\mathbb{N}}(1) \big(i_{\mathbb{N}}(i-1) i_{\mathbb{N}}(i-1)^* i_{\mathbb{N}}(i-1)\big) \\ &= \big(i_{\mathbb{N}}(i-1) i_{\mathbb{N}}(i-1)^*\big) \big(i_{\mathbb{N}}(1)^* i_{\mathbb{N}}(1)\big) i_{\mathbb{N}}(i-1) \\ &= i_{\mathbb{N}}(i-1) i_{\mathbb{N}}(i)^* i_{\mathbb{N}}(i). \end{aligned}$$

Thus for $i, j, m \in \mathbb{N}$, we have

$$\begin{aligned} i_{\mathbb{N}}(1)^* f_{i,j}^m &= \begin{cases} i_{\mathbb{N}}(1)^* i_{\mathbb{N}}(i) i_{\mathbb{N}}(m)^* i_{\mathbb{N}}(m)\big(1 - i_{\mathbb{N}}(1) i_{\mathbb{N}}(1)^*\big) i_{\mathbb{N}}(j)^* & \text{if } i > 0, \\ i_{\mathbb{N}}(1)^* \big(1 - i_{\mathbb{N}}(1) i_{\mathbb{N}}(1)^*\big) i_{\mathbb{N}}(m)^* i_{\mathbb{N}}(m) i_{\mathbb{N}}(j)^* & \text{if } i = 0, \end{cases} \\ &= \begin{cases} i_{\mathbb{N}}(i-1) i_{\mathbb{N}}(i)^* i_{\mathbb{N}}(i) i_{\mathbb{N}}(m)^* i_{\mathbb{N}}(m)\big(1 - i_{\mathbb{N}}(1) i_{\mathbb{N}}(1)^*\big) i_{\mathbb{N}}(j)^* & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \\ &= \begin{cases} i_{\mathbb{N}}(i-1) i_{\mathbb{N}}(i \vee m)^* i_{\mathbb{N}}(i \vee m)\big(1 - i_{\mathbb{N}}(1) i_{\mathbb{N}}(1)^*\big) i_{\mathbb{N}}(j)^* & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \end{aligned}$$

which belongs to $\mathcal{E}$. This proves the claim.

Since $T^*(T^*)^* = 1$, each $f_{i,j}^m$ belongs to $\ker \varphi_{T^*}$, and $\mathcal{E} \subset \ker \varphi_{T^*}$. Suppose φ is a non-degenerate representation of $\mathbf{c} \times_\tau \mathbb{N}$ on a Hilbert space H with $\ker \varphi = \mathcal{E}$. Then

$$1 - \varphi(i_{\mathbb{N}}(1))\varphi(i_{\mathbb{N}}(1))^* = \varphi(1 - i_{\mathbb{N}}(1) i_{\mathbb{N}}(1)^*) = 0,$$

so $\varphi(i_{\mathbb{N}}(1))$ is a coisometry, and $\varphi(i_{\mathbb{N}}(1))^*$ is an isometry. By Coburn's Theorem, there is a representation ψ of $\mathcal{T}(\mathbb{Z})$ on H such that $\psi(T) = \varphi(i_{\mathbb{N}}(1))^*$. Then, since $\psi \circ \varphi_{T^*}(i_{\mathbb{N}}(1)) = \varphi(i_{\mathbb{N}}(1))$, we have $\psi \circ \varphi_{T^*} = \varphi$, and $\ker \varphi_{T^*} \subset \ker \varphi = \mathcal{E}$.

Similar arguments give the description of $\ker \varphi_T$. ∎

COROLLARY 6.3. *We have $\pi_n(a) \to \varphi_T(a)$ for every $a \in \ker \varphi_{T^*}$, and $\pi_n^*(b) \to \varphi_{T^*}(b)$ for every $b \in \ker \varphi_T$.*

Proof. Since $P_n T^i = 0$ unless $i \leqslant n$, we have

$$\pi_n(f_{i,j}^m) = \pi_n^*(g_{i,j}^m) = \begin{cases} T^i(1 - TT^*)(T^*)^j & \text{if } i, j, m \leqslant n, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

Since all the homomorphisms have norm 1 and $\pi_n(f_{i,j}^m) = \varphi_T(f_{i,j}^m)$ for $n \geqslant m$, an $\varepsilon/3$-argument shows that $\pi_n(a) \to \varphi_T(a)$ for all $a \in \ker \varphi_{T^*}$. Similar arguments give the second assertion. ∎

Before proving that π and π^* map onto $\mathcal{A}$, we show that they restrict to isomorphisms of $\mathcal{I} := (\ker \varphi_T) \cap (\ker \varphi_{T^*})$ onto $\mathcal{A}_0$. For this we need a spanning family for $\mathcal{I}$.

LEMMA 6.4. *For* $0 \leqslant i, j \leqslant m$ *we have*

$$f^m_{i,j} - f^{m+1}_{i,j} = g^m_{m-i,m-j} - g^{m+1}_{m-i,m-j}, \tag{6.3}$$

and the elements (6.3) *span* $\mathcal{I}$.

Proof. For $i \leqslant m$, we compute using Proposition 3.2:

$$\begin{aligned} i_{\mathbb{N}}(i)i_{\mathbb{N}}(m)^* &= i_{\mathbb{N}}(i)\big(i_{\mathbb{N}}(i)^*i_{\mathbb{N}}(m-i)^*\big) \\ &= i_{\mathbb{N}}(i)i_{\mathbb{N}}(i)^*\big(i_{\mathbb{N}}(m-i)^*i_{\mathbb{N}}(m-i)i_{\mathbb{N}}(m-i)^*\big) \\ &= \big(i_{\mathbb{N}}(m-i)^*i_{\mathbb{N}}(m-i)\big)\big(i_{\mathbb{N}}(i)i_{\mathbb{N}}(i)^*\big)i_{\mathbb{N}}(m-i)^* \\ &= i_{\mathbb{N}}(m-i)^*i_{\mathbb{N}}(m)i_{\mathbb{N}}(m)^*. \end{aligned}$$

Thus for $0 \leqslant i, j \leqslant m$, we have

$$\begin{aligned} f^m_{i,j} - f^{m+1}_{i,j} &= i_{\mathbb{N}}(i)\big(i_{\mathbb{N}}(m)^*i_{\mathbb{N}}(m) - i_{\mathbb{N}}(m+1)^*i_{\mathbb{N}}(m+1)\big)\big(1 - i_{\mathbb{N}}(1)i_{\mathbb{N}}(1)^*\big)i_{\mathbb{N}}(j)^* \\ &= i_{\mathbb{N}}(i)i_{\mathbb{N}}(m)^*\big(1 - i_{\mathbb{N}}(1)^*i_{\mathbb{N}}(1)\big)i_{\mathbb{N}}(m)\big(1 - i_{\mathbb{N}}(1)i_{\mathbb{N}}(1)^*\big)i_{\mathbb{N}}(j)^* \\ &= i_{\mathbb{N}}(m-i)^*i_{\mathbb{N}}(m)i_{\mathbb{N}}(m)^*\big(1 - i_{\mathbb{N}}(1)^*i_{\mathbb{N}}(1)\big) \\ &\qquad \cdot i_{\mathbb{N}}(m-j)\big(i_{\mathbb{N}}(j)i_{\mathbb{N}}(j)^* - i_{\mathbb{N}}(j+1)i_{\mathbb{N}}(j+1)^*\big) \\ &= i_{\mathbb{N}}(m-i)^*i_{\mathbb{N}}(m)i_{\mathbb{N}}(m)^*\big(1 - i_{\mathbb{N}}(1)^*i_{\mathbb{N}}(1)\big) \\ &\qquad \cdot \big(i_{\mathbb{N}}(m)i_{\mathbb{N}}(m)^* - i_{\mathbb{N}}(m+1)i_{\mathbb{N}}(m+1)^*\big)i_{\mathbb{N}}(m-j) \\ &= i_{\mathbb{N}}(m-i)^*\big(i_{\mathbb{N}}(m)i_{\mathbb{N}}(m)^* - i_{\mathbb{N}}(m+1)i_{\mathbb{N}}(m+1)^*\big) \\ &\qquad \cdot \big(1 - i_{\mathbb{N}}(1)^*i_{\mathbb{N}}(1)\big)i_{\mathbb{N}}(m-j) \\ &= g^m_{m-i,m-j} - g^{m+1}_{m-i,m-j}. \end{aligned}$$

Let $\mathcal{E}_0 = \overline{\operatorname{span}}\{f^m_{i,j} - f^{m+1}_{i,j} : m \in \mathbb{N}, 0 \leqslant i, j \leqslant m\}$. Equation (6.3) implies that $\mathcal{E}_0 \subset \mathcal{I}$. Since $\ker \varphi_T$ and $\ker \varphi_{T^*}$ are ideals, $\ker \varphi_T \cap \ker \varphi_{T^*} = \ker \varphi_T \ker \varphi_{T^*}$. A routine calculation using Proposition 3.2 shows that for $i, j, m, p, r, n \in \mathbb{N}$

$$g^m_{i,j} f^n_{p,r} = \begin{cases} f^{j+p}_{j+p-i,r} - f^{j+p+1}_{j+p-i,r} & \text{if } r, i, m, n \leqslant j + p, \\ 0 & \text{otherwise,} \end{cases}$$

which is in $\mathcal{E}_0$. Since $g^m_{i,j}$ and $f^n_{p,r}$ span $\ker \varphi_T$ and $\ker \varphi_{T^*}$, it follows that $\ker \varphi_T \ker \varphi_{T^*}$ is contained in $\mathcal{E}_0$. Thus $\mathcal{I} = \ker \varphi_T \ker \varphi_{T^*} = \mathcal{E}_0$. ∎

PROPSITION 6.5. *The homomorphisms* $\pi : a \mapsto \{\pi_n(a)\}$ *and* $\pi^* : b \mapsto \{\pi^*_n(b)\}$ *restrict to isomorphisms of* $\mathcal{I}$ *onto* $\mathcal{A}_0$.

Proof. The ideal $\mathcal{A}_0$ is spanned by the functions $\{e^m_{ij} : 0 \leqslant i, j \leqslant m\}$ given by

$$e^m_{ij}(n) = \begin{cases} T^i(1 - TT^*)(T^*)^j & \text{if } m = n, \\ 0 & \text{otherwise;} \end{cases} \tag{6.4}$$

indeed, for each fixed m they span $P_m\mathcal{K}(\ell^2(\mathbb{N}))P_m$. Equation (6.2) implies that

$$\pi_n(f^m_{i,j} - f^{m+1}_{i,j}) = \begin{cases} T^i(1 - TT^*)(T^*)^j & \text{if } m = n \text{ and } i, j \leqslant m, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

This proves that $\pi(\mathcal{I}) = \mathcal{A}_0$, and similarly $\pi^*(\mathcal{I}) = \mathcal{A}_0$.

The relations (6.5) and (6.4) also show how to construct an inverse for π: since

$$\big\{\{f_{i,j}^m - f_{i,j}^{m+1} : i,j \leqslant m\} : m \in \mathbb{N}\big\}$$

consists of mutually orthogonal families of matrix units, there is a homomorphism of $\bigoplus M_{m+1}(\mathbb{C}) \cong \mathcal{A}_0$ onto $\mathcal{I}$ which takes e_{ij}^m to $f_{i,j}^m - f_{i,j}^{m+1}$. Similar arguments using $\mathcal{I} = \overline{\operatorname{span}}\{g_{i,j}^m - g_{i,j}^{m+1}\}$ give the corresponding property of π^*. ∎

COROLLARY 6.6. *Both* $\pi : \ker \varphi_{T^*} \to \mathcal{A}$ *and* $\pi^* : \ker \varphi_T \to \mathcal{A}$ *are surjective.*

Proof. Since we know that $\pi(\ker \varphi_{T^*}) \supset \mathcal{A}_0$, it suffices to show that for each $K \in \mathcal{K}(\ell^2(\mathbb{N}))$, there exists $g \in \ker \varphi_{T^*}$ with $\varepsilon_\infty(\pi(g)) = K$. Indeed, because the range of the homomorphism $\varepsilon_\infty \circ \pi$ is closed, it suffices to do this for $K = T^i(1 - TT^*)(T^*)^j$. But a computation shows that $\pi(f_{i,j}^{i\vee j})(n) = T^i(1-TT^*)(T^*)^j$ for $n \geqslant i \vee j$. We similarly have $\pi^*(g_{i,j}^{i\vee j})(n) = T^i(1 - TT^*)(T^*)^j$, and the result follows. ∎

To see that π and π^* are injective, we need to know that $\mathcal{I}$ is an essential ideal in $\mathbf{c} \times_\tau \mathbb{N}$. To do this, we need the following example of a faithful representation of $\mathbf{c} \times_\tau \mathbb{N}$.

EXAMPLE 6.7. Let V be the partial isometry on $\ell^2(\mathbb{N} \times \mathbb{N})$ such that

$$V(\varepsilon_{k,l}) = \begin{cases} \varepsilon_{k+1,l-1} & \text{if } l \geqslant 1, \\ 0 & \text{otherwise,} \end{cases}$$

and note that V is a power partial-isometry, so that we have a non-degenerate representation $\pi_V \times V$ of $\mathbf{c} \times_\tau \mathbb{N}$ on $\ell^2(\mathbb{N} \times \mathbb{N})$ such that $\pi_V \times V(i_{\mathbb{N}}(1)) = V$. If $m > 0$ and $i < j$, then

$$\big(1 - (V^*)^m V^m\big)\big(V^i(V^*)^i - V^j(V^*)^j\big)(\varepsilon_{i,0}) = \varepsilon_{i,0}$$

so Proposition 5.4 implies that $\pi_V \times V$ is faithful on $\mathbf{c} \times_\tau \mathbb{N}$.

LEMMA 6.8. *The ideal* $\mathcal{I}$ *is essential in* $\mathbf{c} \times_\tau \mathbb{N}$.

Proof. Let V be the power partial-isometric representation in Example 6.7. Then $\pi_V S \times V$ is a faithful representation of $\mathbf{c} \times_\tau \mathbb{N}$ on $\ell^2(\mathbb{N} \times \mathbb{N})$. Since $(\pi_V \times V)(f_{i,i}^{i+j} - f_{i,i}^{i+j+1})\varepsilon_{i,j} = \varepsilon_{i,j}$, $\pi_V \times V$ is non-degenerate on $\mathcal{I}$, and it follows that $\mathcal{I}$ is essential. ∎

PROPSITION 6.9. *Both* $\pi : \ker \varphi_{T^*} \to \mathcal{A}$ *and* $\pi^* : \ker \varphi_T \to \mathcal{A}$ *are isomorphisms.*

Proof. Corollary 6.6 says they are surjective. To see that π is injective, suppose $a \in \ker \varphi_{T^*}$ and $\pi(a) = 0$. Then for every $c \in \mathcal{I}$, we have $\pi(ac) = 0$, $ac = 0$, and $a = 0$ by Lemma 6.8. Thus π is injective. A similar argument shows that π^* is injective. ∎

Proof of Theorem 6.1. We have now proved that we can identify the top and left-hand sequences with

$$0 \to \mathcal{A}_0 \to \mathcal{A} \overset{\varepsilon_\infty}{\to} \mathcal{K}(\ell^2(\mathbb{N})) \to 0. \tag{6.6}$$

However, since the isomorphisms $\pi|_{\mathcal{I}}$ and $\pi^*|_{\mathcal{I}}$ are not the same, to make the top left-hand square commute, we have to introduce an automorphism α of $\mathcal{A}_0$. The required automorphism is defined on the spanning elements of (6.4) by $\alpha(e_{i,j}^n) = e_{n-i,n-j}^n$. That the diagram commutes then follows from Lemma 6.4. ∎

7. THE CROSSED PRODUCT BY THE BACKWARD SHIFT

The backward shifts σ_k on $\ell^\infty(\mathbb{N})$ satisfy

$$\sigma_k(1_n) = \begin{cases} 1_{n-k} & \text{if } n \geqslant 1, \\ 1 & \text{otherwise,} \end{cases}$$

and hence give an action $\sigma : \mathbb{N} \to \operatorname{End} \mathbf{c}$. In this section we prove a structure theorem for the crossed product $(\mathbf{c} \times_\sigma \mathbb{N}, k_{\mathbf{c}}, k_{\mathbb{N}})$.

Our first task is to determine the universal property of $\mathbf{c} \times_\sigma \mathbb{N}$, which is quite different from that of $\mathbf{c} \times_\tau \mathbb{N}$. The first difference is that the partial isometries V^n in a covariant partial-isometric representation (π, V) of $(\mathbf{c}, \mathbb{N}, \sigma)$ are coisometries:

$$V_n V_n^* = V_n \pi(1) V_n^* = \pi(\sigma_n(1)) = \pi(1) = 1 \quad \text{for every } n \in \mathbb{N}.$$

The second difference is that the partial isometries $k_{\mathbb{N}}(n)$ no longer generate $\mathbf{c} \times_\sigma \mathbb{N}$: we cannot recover $k_{\mathbf{c}}(1_n)$ from $k_{\mathbb{N}}(n) k_{\mathbb{N}}(n)^*$ alone. More precisely:

PROPSITION 7.1. *Let (π, V) be a covariant partial-isometric representation of $(\mathbf{c}, \mathbb{N}, \sigma)$ on H, and write V for the generator V_1. Define*

$$Q_0 = 1 - V^*V \quad \textit{and} \quad Q_n := \pi(1_n) - V^* \pi(1_{n-1}) V \quad \textit{for } n > 0. \tag{7.1}$$

Then $\{Q_n\}$ is a sequence of projections satisfying

$$\cdots \leqslant Q_{n+1} \leqslant Q_n \leqslant Q_{n-1} \leqslant \cdots \leqslant Q_0, \tag{7.2}$$

and we can recover π via

$$\pi(1_n) = (V^*)^n V^n + \sum_{k=0}^{n-1} (V^*)^k Q_{n-k} V^k \quad \textit{for } n > 0. \tag{7.3}$$

Conversely, for any coisometry V on H and any sequence of projections Q_n satisfying (7.2), there is a covariant partial-isometric representation $(\pi_{V,Q}, V)$ of $(\mathbf{c}, \mathbb{N}, \sigma)$ on H such that $\pi_{V,Q}$ satisfies (7.3).

Proof. Suppose (π, V) is a covariant partial-isometric representation of $(\mathbf{c}, \mathbb{N}, \sigma)$, and define $\{Q_n\}$ using (7.1). Then for $n > 0$

$$Q_n = \pi(1_n) - V^* \pi(\sigma_1(1_n)) V = \pi(1_n) - V^* \big(V \pi(1_n) V^* \big) V = (1 - V^*V) \pi(1_n) \tag{7.4}$$

is the product of commuting projections, and hence is a projection. We have $Q_1 = \pi(1_1)(1 - V^*V) \leqslant 1 - V^*V = Q_0$, and for $n > 0$, (7.4) gives

$$Q_n - Q_{n+1} = (1 - V^*V)(\pi(1_n) - \pi(1_{n+1})) \geqslant 0.$$

This gives (7.2). When we plug the formulas for Q_{n-k} into the right-hand side of (7.3), the sum telescopes, and we are left with $\pi(1_n)$.

For the converse, let

$$P_0 := 1 \quad \text{and} \quad P_n := (V^*)^n V^n + \sum_{k=0}^{n-1} (V^*)^k Q_{n-k} V^k \quad \text{for } n > 0.$$

Since $Q_n \leqslant 1 - V^*V$, then $VQ_n = 0$. So, for fixed n, $\{(V^*)^k Q_{n-k} V^k : k < n\}$ are mutually orthogonal and orthogonal to $(V^*)^n V^n$, and hence P_n is a projection. We have $P_0 - P_1 = (1 - V^*V) - Q_1 \geqslant 0$, and for $n > 0$,

$$\begin{aligned}
&P_n - P_{n+1} \\
&\quad = (V^*)^n V^n + \sum_{k=0}^{n-1} (V^*)^k Q_{n-k} V^k - (V^*)^{n+1} V^{n+1} - \sum_{k=0}^{n} (V^*)^k Q_{n+1-k} V^k \\
&\quad = (V^*)^n V^n - (V^*)^{n+1} V^{n+1} - (V^*)^n Q_1 V^n + \sum_{k=0}^{n-1} (V^*)^k (Q_{n-k} - Q_{n-k+1}) V^k \\
&\quad = (V^*)^n (1 - V^*V - Q_1) V^n + \sum_{k=0}^{n-1} (V^*)^k (Q_{n-k} - Q_{n-k+1}) V^k,
\end{aligned}$$

and hence $P_n \geqslant P_{n+1}$. By Proposition 1.3 of [15], there is a representation $\pi_{V,Q}$ of $\mathbf{c}$ such that $\pi_{V,Q}(1_n) = P_n$.

We now prove that $(\pi_{V,Q}, V)$ is covariant. Since each V^p is a coisometry, we have $\pi_{V,Q}(\sigma_p(1)) = \pi_{V,Q}(1) = 1 = V^p (V^*)^p$. We also have

$$V^p \pi_{V,Q}(1_0) = \pi_{V,Q}(1_0) V^p = \pi_{V,Q}(\sigma_p(1_0)) V^p.$$

For $n > 0$, we compute using $VQ_n = 0$ and Proposition 3.2:

$$\begin{aligned}
V^p \pi_{V,Q}(1_n) &= V^p \Big((V^*)^n V^n + \sum_{k=0}^{n-1} (V^*)^k Q_{n-k} V^k \Big) \\
&= \begin{cases} V^p (V^*)^p (V^*)^{n-p} V^{n-p} V^p + \sum_{k=p}^{n-1} (V^*)^{k-p} Q_{n-k} V^k & \text{if } n > p, \\ V^{p-n} V^n + \sum_{k=0}^{n-1} V^{p-k} Q_{n-k} V^k & \text{if } n \leqslant p, \end{cases} \\
&= \begin{cases} (V^*)^{n-p} V^{n-p} V^p + \sum_{k=0}^{n-p-1} (V^*)^k Q_{n-p-k} V^{k+p} & \text{if } n > p, \\ V^p & \text{if } n \leqslant p, \end{cases} \\
&= \begin{cases} \big((V^*)^{n-p} V^{n-p} + \sum_{k=0}^{n-p-1} (V^*)^k Q_{n-p-k} V^k \big) V^p & \text{if } n > p, \\ V^p & \text{if } n \leqslant p, \end{cases} \\
&= \begin{cases} \pi_{V,Q}(1_{n-p}) V^p & \text{if } n > p, \\ \pi_{V,Q}(1) V^p & \text{if } n \leqslant p, \end{cases} \\
&= \pi_{V,Q}(\sigma_p(1_n)) V^p.
\end{aligned}$$

It therefore follows from Corollary 4.4 that $(\pi_{V,Q}, V)$ is covariant. ∎

We write q_n for the element $k_{\mathbf{c}}(1_n) - k_{\mathbb{N}}(1)^* k_{\mathbf{c}}(\sigma_n(1)) k_{\mathbb{N}}(1)$ of $\mathbf{c} \times_\sigma \mathbb{N}$. Proposition 7.1 implies that $\{q_n\}$ is a decreasing sequence of projections, which together with $k_{\mathbb{N}}(1)$ generates $\mathbf{c} \times_\sigma \mathbb{N}$. The next example shows that the q_n are distinct.

EXAMPLE 7.2. Let $V : \mathbb{N} \to B(\ell^2(\mathbb{N}\times\mathbb{N}))$ be the coisometric representation such that

$$V^n(\varepsilon_{k,l}) = \begin{cases} \varepsilon_{k,l-n} & \text{if } l \geqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

so that $(V^*)^n(\varepsilon_{k,l}) = \varepsilon_{k,l+n}$, and let Q_n be the projection on $\overline{\operatorname{span}}\{\varepsilon_{k,0} : k \geqslant n\}$. Then $\cdots < Q_{n+1} < Q_n < \cdots < Q_1 < 1 - V^*V$. In the representation $\pi_{V,Q}$ of Proposition 7.1, $\pi_{V,Q}(1_n)$ is the projection on $\overline{\operatorname{span}}\{\varepsilon_{k,l} : k + l \geqslant n\}$.

We can now characterise the faithful representations of $\mathbf{c} \times_\sigma \mathbb{N}$:

PROPSITION 7.3. *Suppose $(\pi_{V,Q}, V)$ is a covariant partial-isometric representation of $(\mathbf{c}, \mathbb{N}, \sigma)$. Then the representation $\pi_{V,Q} \times V$ of $\mathbf{c} \times_\sigma \mathbb{N}$ is faithful if and only if $Q_n \neq Q_{n+1}$ for all $n \geqslant 0$.*

Proof. By Proposition 1.3 of [15], there is a representation π_Q of $\mathbf{c}$ such that $\pi_Q(1_n) = Q_n$ for $n \in \mathbb{N}$. For $h \in \operatorname{range}(1 - V^*V) = (V^*H)^\perp$, we have $\pi_{V,Q}(1)h = \pi_Q(1)h$ and

$$\pi_{V,Q}(1_n)h = \Big((V^*)^nV^n + \sum_{k=0}^{n-1}(V^*)^kQ_{n-k}V^k\Big)(1 - V^*V)h = Q_nh = \pi_Q(1_n)h;$$

thus $\pi_Q = \pi_{V,Q}|_{(V^*H)^\perp}$. By Lemma 4.8, $\pi_{V,Q} \times V$ is faithful if and only if $\pi_{V,Q}$ is faithful on $(V^*H)^\perp$, and by Proposition 1.3 in [15], $\pi_{V,Q} = \pi_Q$ is faithful on $(V^*H)^\perp$ if and only if $Q_n \neq Q_{n+1}$ for all $n \geqslant 0$. ∎

We are now ready to describe $\mathbf{c} \times_\sigma \mathbb{N}$. Recall that $T = T_1$ is the unilateral shift on $\ell^2(\mathbb{N})$, and denote by F the constant function $F : n \mapsto T^*$. Then F is a coisometry in the C^*-algebra $C_{\mathrm{b}}(\mathbb{N}, B(\ell^2(\mathbb{N})))$. For $m \in \mathbb{N}$, define $Q_m \in C_{\mathrm{b}}(\mathbb{N}, B(\ell^2(\mathbb{N})))$ by

$$Q_m(n) = \begin{cases} 1 - TT^* & \text{if } n \geqslant m, \\ 0 & \text{if } n < m. \end{cases}$$

Then $\{Q_m\}$ is a decreasing sequence of projections with $Q_0 = 1 - F^*F$. By Proposition 7.1, there is a homomorphism $\pi_{F,Q} \times F : \mathbf{c} \times_\sigma \mathbb{N} \to C_{\mathrm{b}}(\mathbb{N}, B(\ell^2(\mathbb{N})))$ such that $\pi_{F,Q} \times F(k_{\mathbb{N}}(1)) = F$, $\pi_{F,Q} \times F(q_m) = Q_m$, and

$$\begin{aligned}
\pi_{F,Q} \times F(k_{\mathbf{c}}(1_m))(n) &= \pi_{F,Q} \times F\Big(k_{\mathbb{N}}(m)^*k_{\mathbb{N}}(m) + \sum_{k=0}^{m-1} k_{\mathbb{N}}(k)^*q_{m-k}k_{\mathbb{N}}(k)\Big)(n) \\
&= \begin{cases} T^m(T^*)^m + \sum\limits_{k=m-n}^{m-1} T^k(1 - TT^*)(T^*)^k & \text{if } n \leqslant m, \\ T^m(T^*)^m + \sum\limits_{k=0}^{m-1} T^k(1 - TT^*)(T^*)^k & \text{if } n > m. \end{cases} \\
&= \begin{cases} T^{m-n}(T^*)^{m-n} & \text{if } n \leqslant m, \\ 1 & \text{if } n > m. \end{cases}
\end{aligned}$$

THEOREM 7.4. *The homomorphism $\pi_{F,Q} \times F$ is an isomorphism of $\mathbf{c} \times_\sigma \mathbb{N}$ onto*

$$\mathcal{B} := \{f \in C(\mathbb{N} \cup \{\infty\}, \mathcal{T}(\mathbb{Z})) : \psi_T(f(n)) \text{ is constant}\}.$$

Proof. Since $Q_m \neq Q_{m+1}$ for every $m \geqslant 0$, Proposition 7.3 implies that $\pi_{F,Q} \times F$ is faithful. Recall from Proposition 4.7 that

$$\mathbf{c} \times_\sigma \mathbb{N} = \overline{\operatorname{span}}\{k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_m) k_{\mathbb{N}}(j) : i, j, m \in \mathbb{N}\}.$$

For $i, j, m \in \mathbb{N}$, we have

$$\pi_{F,Q} \times F\big(k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_m) k_{\mathbb{N}}(j)\big)(n) = \begin{cases} T^{i+(m-n)}(T^*)^{j+(m-n)} & \text{if } n \leqslant m, \\ T^i(T^*)^j & \text{if } n > m. \end{cases} \tag{7.5}$$

Thus $\lim\limits_{n\to\infty} \pi_{F,Q} \times F(a)(n)$ exists for every $a \in \operatorname{span}\{k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_m) k_{\mathbb{N}}(j)\}$, and by an $\varepsilon/3$ argument we can extend this to $a \in \mathbf{c} \times_\sigma \mathbb{N}$. Equation (7.5) also implies that

$$\psi_T\big(\pi_{F,Q} \times F(k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_m) k_{\mathbb{N}}(j))(n)\big) = \varepsilon_{i-j}$$

for every $n \in \mathbb{N}$, and hence $\pi_{F,Q} \times F(\mathbf{c} \times_\sigma \mathbb{N}) \subset \mathcal{B}$.

Let $f \in \mathcal{B}$, and define $g \in \mathcal{B}$ by $g(n) = f(\infty)$ for every $n \in \mathbb{N}$. Then

$$\psi_T\big((f-g)(n)\big) = \psi_T(f(n)) - \psi_T(g(n)) = \psi_T(f(n)) - \psi_T(f(\infty)) = 0,$$

so $(f-g)(n) \in \ker \psi_T = \mathcal{K}(\ell^2(\mathbb{N}))$ for all $n \in \mathbb{N}$; since $\lim\limits_{n\to\infty}(f-g)(n) = 0$, $f-g$ belongs to $C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N})))$. But for $i, j, m \in \mathbb{N}$,

$$\pi_{F,Q} \times F\big(k_{\mathbb{N}}(i)^* k_{\mathbf{c}}(1_m - 1_{m+1}) k_{\mathbb{N}}(j) - k_{\mathbb{N}}(i+1)^* k_{\mathbf{c}}(1_{m-1} - 1_m) k_{\mathbb{N}}(j+1)\big)$$

is the matrix unit e_{ij}^m of (6.4), so $C_0(\mathbb{N}, \mathcal{K}(\ell^2(\mathbb{N}))) = \overline{\operatorname{span}}\{e_{ij}^m : i, j, m \in \mathbb{N}\}$ is contained in $\pi_{F,Q} \times F(\mathbf{c} \times_\sigma \mathbb{N})$. The function g is constant, so it belongs to

$$C^*(F) = \pi_{F,Q} \times F(C^*(k_{\mathbb{N}}(1))) \subset \pi_{F,Q} \times F(\mathbf{c} \times_\sigma \mathbb{N}),$$

and hence so does $f = (f-g) + g$. ∎

COROLLARY 7.5. *There is an exact sequence*

$$0 \to C(\mathbb{N} \cup \{\infty\}, \mathcal{K}(\ell^2(\mathbb{N}))) \to \mathbf{c} \times_\sigma \mathbb{N} \to C(\mathbb{T}) \to 0.$$

Acknowledgements. This research was supported by the Australian Research Council.

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JANNY LINDIARNI
School of Mathematical and Physical Sciences
University of Newcastle
Newcastle, NSW 2308
AUSTRALIA
E-mail: janny@frey.newcastle.edu.au

IAIN RAEBURN
School of Mathematical and Physical Sciences
University of Newcastle
Newcastle, NSW 2308
AUSTRALIA
E-mail: iain@frey.newcastle.edu.au

Received June 6, 2002.