J. OPERATOR THEORY
52(2004), 89–101

PERIPHERAL POINT SPECTRUM AND GROWTH OF POWERS OF OPERATORS

OMAR EL-FALLAH and THOMAS RANSFORD

Communicated by Nikolai K. Nikolski

ABSTRACT. Let E be a closed subset of the unit circle. A result of Nikolski shows that, if T is an operator on a separable Hilbert space whose point spectrum contains E, and if $0 < \alpha < \dim_{\mathrm{H}} E$ (the Hausdorff dimension of E), then $\sum\limits_n n^{\alpha-1}\|T^n\|^{-2} < \infty$. We complement this result by showing that, for each $\beta > \overline{\dim}_{\mathrm{B}} E$ (the upper box dimension of E), there exists an operator T on a separable Hilbert space, whose point spectrum contains E, and such that $\sum\limits_n n^{\beta-1}\|T^n\|^{-2} = \infty$. We also prove some more refined results along the same lines.

KEYWORDS: *Point spectrum, powers of operators, resolvent.*

MSC (2000): 47A10.

1. INTRODUCTION

Let T be a bounded linear operator on a complex Banach space X. We write $\sigma(T)$ for the spectrum of T and $\sigma_{\mathrm{p}}(T)$ for the point spectrum of T. We also denote by Γ the unit circle. In this article we investigate the connection between the size of the set $\sigma_{\mathrm{p}}(T) \cap \Gamma$ and the growth of $\|T^n\|$ as $n \to \infty$. The results we consider are of interest only if the spectral radius of T equals one. In that case, $\sigma_{\mathrm{p}}(T) \cap \Gamma$ is the peripheral point spectrum of T.

Jamison ([3]) has shown that if X is separable and T is power-bounded, then $\sigma_{\mathrm{p}}(T) \cap \Gamma$ is at most countable (see also Theorem 4.1 below). For a general Banach space, this is about all that can be said, because it is possible to have $\sigma_{\mathrm{p}}(T) \supset \Gamma$ and yet have $\|T^n\| \to \infty$ arbitrarily slowly. Indeed, given a sequence $\omega(n) \to \infty$ (without loss of generality $\omega(0) = 1$ and $\omega(m+n) \leqslant \omega(m)\omega(n)$), let $X = c_0(1/\omega) = \{(x_n) : x_n/\omega(n) \to 0\}$ and let T be the left shift on X. Then for each $\lambda \in \Gamma$, the vector $x_\lambda = (1, \lambda, \lambda^2, \ldots)$ is an eigenvector of T with eigenvalue λ, and $\|T^n\| = \omega(n)$ for all $n \geqslant 1$.

The situation is different if X is a separable *Hilbert* space. In this case, Nikolski has proved the following theorem (see p. 239, Theorem 11 in [7]). (For the definition of γ-capacity and further information on the subject, we refer to Chapter III in [4].)

THEOREM 1.1. *Let T be an operator on a separable Hilbert space.*

(i) *If $\sigma_{\mathrm{p}}(T)\cap\Gamma$ has positive Lebesgue measure, then $\sum_n \|T^n\|^{-2}<\infty$.*

(ii) *If $\sigma_{\mathrm{p}}(T)\cap\Gamma$ has positive γ-capacity, where $\gamma:\Gamma\to(0,\infty)$ is an integrable function with positive Fourier coefficients, then there exists an integer $N\geqslant 0$ such that*

$$\sum_n \widehat{\gamma}(n+N)\|T^n\|^{-2}<\infty.$$

Our aim in this paper is to complement this result by constructing operators T on a separable Hilbert space, with $\sigma_{\mathrm{p}}(T)\cap\Gamma$ prescribed in advance, and such that $\|T^n\|\to\infty$ relatively slowly. Given a closed subset E of Γ, we write E_δ for the set of points of Γ whose arc-length distance from E is at most δ, and $|E_\delta|$ for the Lebesgue measure of this set.

THEOREM 1.2. *Let E be a closed subset of Γ, and let (η_n) be a positive sequence such that $\eta_n\to\infty$. Then there exists an operator T on a separable Hilbert space such that $\sigma_{\mathrm{p}}(T)=\sigma(T)=E$ and*

$$\sum_n \eta_n|E_{1/n}|\,\|T^n\|^{-2}=\infty. \tag{1.1}$$

For example, if $E=\Gamma$, then the theorem yields an operator T with $\sigma_{\mathrm{p}}(T)=\Gamma$ and $\sum_n \eta_n\|T^n\|^{-2}=\infty$. Though η_n may tend to infinity arbitrarily slowly, Theorem 1.1(i) shows that we cannot remove it entirely.

To help compare Theorems 1.1 and 1.2, we prove a corollary which contains weaker but simpler forms of both results. We write $\dim_{\mathrm{H}} E$ and $\overline{\dim}_{\mathrm{B}}E$ for the Hausdorff dimension and upper box dimension of E respectively. (For the definition of these dimensions and further information on the subject, see for example Chapters 2 and 3 in [2].)

COROLLARY 1.3. *Let E be a closed subset of Γ.*

(i) *If T is an operator on a separable Hilbert space such that $\sigma_{\mathrm{p}}(T)$ contains E, and if $0<\alpha<\dim_{\mathrm{H}} E$, then*

$$\sum_n n^{\alpha-1}\|T^n\|^{-2}<\infty.$$

(ii) *For each $\beta>\overline{\dim}_{\mathrm{B}}E$, there exists an operator T on a separable Hilbert space such that $\sigma_{\mathrm{p}}(T)$ contains E and*

$$\sum_n n^{\beta-1}\|T^n\|^{-2}=\infty.$$

Proof. (i) Let $0<\alpha<\dim_{\mathrm{H}} E$ and define $\gamma(t)=|\sin(t/2)|^{-\alpha}$. By p. 34, Théorème I in [4], the set E is of positive γ-capacity. Also, $\widehat{\gamma}(n)=(A+\mathrm{o}(1))n^\alpha$ as $n\to\infty$, where $A\neq 0$ (see p. 40 in [4]). The result now follows directly from Theorem 1.1(ii).

(ii) Let $\beta > \beta' > \overline{\dim}_{\mathrm{B}} E$. By Proposition 3.2, [2], we have $|E_{1/n}| = \mathrm{O}(n^{\beta'-1})$ as $n \to \infty$. Apply Theorem 1.2 with $\eta_n = n^{\beta-\beta'}$. ∎

For many reasonably regular sets, the Hausdorff and upper box dimensions coincide. Thus, for these sets at least, the corollary shows that Theorems 1.1 and 1.2 are not too far apart.

The basic construction of the operators T will be carried out in Section 2, where we will prove a more general technical result. Theorem 1.2 will be deduced from this in Section 3. Finally, in Section 4, we shall make some comments and pose a few questions.

2. CONSTRUCTION OF THE OPERATORS

To carry out the construction of the operators T, we shall make use of a weight function $\omega : \mathbb{Z} \to (0, \infty)$ satisfying the following conditions:

$$\begin{cases} \omega(-n) = \omega(n) \geqslant \omega(0), \\ \frac{\omega(n)}{\sqrt{n}} \text{ is increasing for } n \geqslant 1, \\ \omega(2n) \leqslant c\,\omega(n), \\ \sum_n \frac{1}{\omega(n)^2} < \infty. \end{cases} \tag{2.1}$$

For example, $\omega(n) = \sqrt{|n|}(1 + \log |n|)$, $n \neq 0$, would do, though, for the proof of Theorem 1.2, we shall eventually need to make a more careful choice. Our aim in this section is to prove the following result.

THEOREM 2.1. *Let $\omega : \mathbb{Z} \to (0, \infty)$ be a function satisfying* (2.1). *Let E be a closed subset of Γ. Then there exists an operator T on a separable Hilbert space such that $\sigma_{\mathrm{p}}(T) = \sigma(T) = E$ and*

$$\frac{1}{n} \sum_{k=0}^{n-1} \|T^k\|^2 \leqslant \text{const}\ \omega(n)^2 |E_{1/n}|, \quad n \geqslant 1. \tag{2.2}$$

The basic idea for the construction is contained in the following simple lemma. Given a Banach space X, we write X^* for the dual of X.

LEMMA 2.2. *Let A be a commutative Banach algebra with identity and let $a \in A$. Let $T : A^* \to A^*$ be defined by $T\varphi(x) = \varphi(ax)$, $\varphi \in A^*$, $x \in A$. Then $\sigma_{\mathrm{p}}(T) = \sigma(T) = \sigma(a)$.*

Proof. The operator T is just the adjoint of the multiplication operator $S : x \mapsto ax$. Hence $\sigma_{\mathrm{p}}(T) \subset \sigma(T) = \sigma(S) \subset \sigma(a)$. To prove the reverse inclusion, take $\lambda \in \sigma(a)$. Then there is a character χ on A such that $\chi(a) = \lambda$. For $x \in A$, we have $T\chi(x) = \chi(ax) = \chi(a)\chi(x) = \lambda\chi(x)$, so $T\chi = \lambda\chi$, and $\lambda \in \sigma_{\mathrm{p}}(T)$. ∎

Thus our strategy will be to construct a commutative Banach algebra A which is isomorphic to a separable Hilbert space, and an element $a \in A$ with $\sigma(a) = E$. After that, we shall need to estimate the norms of the powers of the operator T thereby obtained.

The first step is to take a separable Hilbert space and turn it into a Banach algebra. Here we follow an idea of Nikolski ([6]). Denote by $L^2(\omega)$ the Hilbert space of functions $f \in L^2(\Gamma)$ such that

$$\|f\|_\omega^2 := \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2\omega(n)^2 < \infty.$$

Lemma 2.3. *Let $\omega : \mathbb{Z} \to (0,\infty)$ be a function satisfying* (2.1). *Then $L^2(\omega)$ is an algebra with respect to pointwise multiplication, and there exists a constant C_ω such that $\|fg\|_\omega \leqslant C_\omega\|f\|_\omega\|g\|_\omega$ for $f, g \in L^2(\omega)$. Also $C^\infty(\Gamma) \subset L^2(\omega) \subset C(\Gamma)$.*

Proof. Let $f, g \in L^2(\omega)$. Then

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\widehat{fg}(n)|^2\omega(n)^2 &= \sum_{n\in\mathbb{Z}}\Big|\sum_{k\in\mathbb{Z}} \widehat{f}(k)\widehat{g}(n-k)\Big|^2\omega(n)^2\\ &\leqslant \sum_{n\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} |\widehat{f}(k)\widehat{g}(n-k)|^2\omega(k)^2\omega(n-k)^2\Big)\Big(\sum_{k\in\mathbb{Z}}\frac{\omega(n)^2}{\omega(k)^2\omega(n-k)^2}\Big).\end{aligned}$$

Now if $|k| > |n|/2$, then $\omega(n) \leqslant \omega(2k) \leqslant c\,\omega(k)$. Likewise, if $|k| \leqslant |n|/2$, then $\omega(n) \leqslant \omega(2n-2k) \leqslant c\,\omega(n-k)$. Hence, for each $n \in \mathbb{Z}$,

$$\sum_{k\in\mathbb{Z}}\frac{\omega(n)^2}{\omega(k)^2\omega(n-k)^2} \leqslant \sum_{|k|\leqslant|n|/2}\frac{c^2}{\omega(k)^2} + \sum_{|k|>|n|/2}\frac{c^2}{\omega(n-k)^2} \leqslant 2\sum_{k\in\mathbb{Z}}\frac{c^2}{\omega(k)^2} < \infty.$$

Writing $C_\omega^2 = 2\sum_k c^2/\omega(k)^2$, we thus have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\widehat{fg}(n)|^2\omega(n)^2 &\leqslant C_\omega^2\sum_{n\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} |\widehat{f}(k)\widehat{g}(n-k)|^2\omega(k)^2\omega(n-k)^2\Big)\\ &= C_\omega^2\sum_{k\in\mathbb{Z}}|\widehat{f}(k)|^2\omega(k)^2\sum_{l\in\mathbb{Z}}|\widehat{g}(l)|^2\omega(l)^2 < \infty.\end{aligned}$$

Hence $fg \in L^2(\omega)$ and $\|fg\|_\omega \leqslant C_\omega\|f\|_\omega\|g\|_\omega$.

The conditions (2.1) imply that $\omega(n) = \mathrm{O}(n^\alpha)$ as $n \to \pm\infty$, where $\alpha = \log_2 c$, and the inclusion $C^\infty(\Gamma) \subset L^2(\omega)$ follows easily from this. Also, if $f \in L^2(\omega)$, then

$$\sum_{n\in\mathbb{Z}}|\widehat{f}(n)| \leqslant \Big(\sum_{n\in\mathbb{Z}}|\widehat{f}(n)|^2\omega(n)^2\Big)^{1/2}\Big(\sum_{n\in\mathbb{Z}}\frac{1}{\omega(n)^2}\Big)^{1/2} < \infty,$$

and this yields the other inclusion $L^2(\omega) \subset C(\Gamma)$. ∎

We are now ready to define the operator T. Let $\omega : \mathbb{Z} \to (0,\infty)$ be a function satisfying (2.1). By the preceding lemma, $L^2(\omega)$ can be given an equivalent norm making it a regular Banach function algebra on Γ. Given a closed subset E of Γ, the ideal

$$I(E) := \{f \in L^2(\omega) : f|_E = 0\}$$

is closed in $L^2(\omega)$, and the quotient $L^2(\omega)/I(E)$ is a commutative Banach algebra whose character space can be identified with E. In particular, if we write u for the function $u(z) = z$, and $\pi : L^2(\omega) \to L^2(\omega)/I(E)$ for the quotient map, then the spectrum of $\pi(u)$ in $L^2(\omega)/I(E)$ is precisely E. Note that $L^2(\omega)/I(E)$ is also isomorphic to a separable Hilbert space. We define $A = L^2(\omega)/I(E)$ and $a = \pi(u)$, and then T as in Lemma 2.2. By that lemma, T is an operator on a separable Hilbert space with $\sigma_{\mathrm{p}}(T) = \sigma(T) = E$.

What remains is to obtain bounds for the norms of powers of T. We shall do this indirectly, by first establishing an estimate for the norm of the resolvent $(T - \lambda I)^{-1}$.

Suppose once again that $\omega : \mathbb{Z} \to (0,\infty)$ satisfies (2.1). Define $\psi : [1,\infty) \to (0,\infty)$ by setting $\psi(n) = \omega(n)\sqrt{n}$, $n \geqslant 1$, and then interpolating so as to make $\psi(t)/t$ linear on each interval $[n, n+1]$. Notice that from (2.1) it follows that the function $\psi(t)/t$ is increasing, and that $\psi(2t) \leqslant 2c^2\psi(t)$ for all $t \geqslant 1$.

LEMMA 2.4. *With this notation,*

$$\|(T - \lambda I)^{-1}\| \leqslant \text{const}\ \psi\Big(\frac{3}{\operatorname{dist}(\lambda, E)}\Big), \quad \frac{1}{2} < |\lambda| < 2,\ \lambda \notin E. \tag{2.3}$$

This result in turn depends on two further lemmas.

LEMMA 2.5. *Suppose that ω satisfies* (2.1). *Let $u \in L^2(\omega)$ be the function $u(z) = z$. Then*

$$\|(u - \lambda)^{-1}\|_\omega \leqslant \text{const}\ \psi\Big(\frac{1}{\operatorname{dist}(\lambda, \Gamma)}\Big), \quad \frac{1}{2} < |\lambda| < 2,\ \ \lambda \notin \Gamma.$$

Proof. Suppose that $\frac{1}{2} < |\lambda| < 1$. Then

$$\|(u-\lambda)^{-1}\|_\omega^2 = \Big\| \sum_{k \geqslant 1} \lambda^{k-1} u^{-k} \Big\|_\omega^2 \leqslant 4 \sum_{k\geqslant 1} |\lambda|^{2k} \omega(k)^2 \leqslant \frac{4}{1 - |\lambda|} \sup_{k \geqslant 1} |\lambda|^k \omega(k)^2.$$

If $1 \leqslant k \leqslant 1/(1 - |\lambda|)$, then, since $\psi(t)^2/t$ is increasing,

$$|\lambda|^k \omega(k)^2 \leqslant \omega(k)^2 = \frac{\psi(k)^2}{k} \leqslant \psi\Big(\frac{1}{1 - |\lambda|}\Big)^2 (1 - |\lambda|).$$

On the other hand, if $k > 1/(1 - |\lambda|)$, then we may choose an integer $j \geqslant 1$ such that $2^{j-1}/(1 - |\lambda|) < k \leqslant 2^j/(1 - |\lambda|)$, and then

$$\begin{aligned} |\lambda|^k \omega(k)^2 &\leqslant \Big(1 - \frac{2^{j-1}}{k}\Big)^k \psi\Big(\frac{2^j}{1 - |\lambda|}\Big)^2 \frac{(1 - |\lambda|)}{2^j} \\ &\leqslant \mathrm{e}^{-2^{j-1}} (2c^2)^{2j} \psi\Big(\frac{1}{1 - |\lambda|}\Big)^2 \frac{(1 - |\lambda|)}{2^j}. \end{aligned}$$

Combining these estimates, we obtain

$$\|(u - \lambda)^{-1}\|_\omega^2 \leqslant C\psi\Big(\frac{1}{1 - |\lambda|}\Big)^2, \quad \frac{1}{2} < |\lambda| < 1,$$

where $C = 4 \sup_{j \geqslant 0} \mathrm{e}^{-2^{j-1}} (2c^4)^j$. The case $1 < |\lambda| < 2$ is similar. ∎

Lemma 2.6. *Let E be a closed subset of Γ and let $v : \mathbb{C} \setminus E \to [0, \infty)$ be a subharmonic function. Suppose that*

$$v(\lambda) \leqslant \varphi\Big(\frac{1}{\operatorname{dist}(\lambda, \Gamma)}\Big), \quad \lambda \in \mathbb{C} \setminus \Gamma,$$

where $\varphi : (0, \infty) \to (0, \infty)$ is a continuous increasing function such that $\varphi(2t) \leqslant$ const $\varphi(t)$. Then

$$v(\lambda) \leqslant \text{const}\ \varphi\Big(\frac{1}{\operatorname{dist}(\lambda, E)}\Big), \quad \lambda \in \mathbb{C} \setminus E.$$

Proof. See pp. 366–367 in [9]. ∎

Proof of Lemma 2.4. As remarked earlier, the spectrum of T is equal to E. Thus the resolvent $\lambda \mapsto (T - \lambda I)^{-1}$ is holomorphic on $\mathbb{C} \setminus E$. The result now follows by applying Lemma 2.6 with $v(\lambda) = \|(T - \lambda I)^{-1}\|$ and $\varphi(t) = \text{const}\ \psi(t)$ (extended so as to be constant on $(0, 1]$). ∎

The final step is to use the resolvent estimate (2.3) to obtain bounds on the norms of powers of T. This type of problem was studied in [1], where it was shown that if S is an operator on an arbitrary Banach space, whose resolvent satisfies

$$\|(S - \lambda I)^{-1}\| \leqslant \frac{C}{\operatorname{dist}(\lambda, E)}, \quad |\lambda| > 1,$$

then $\|S^{n-1}\| \leqslant \frac{1}{2}\mathrm{e}C^2\Phi_E(n)$, $n \geqslant 1$, where $\Phi_E(n) = (n/\pi)|E_{\pi/2n}|$. We shall need a refinement of that result for operators on a Hilbert space.

Lemma 2.7. *Let S be an operator on a Hilbert space H. Suppose that*

$$\|(S - \lambda I)^{-1}\| \leqslant \varphi\Big(\frac{1}{\operatorname{dist}(\lambda, E)}\Big), \quad 1 < |\lambda| < 2,$$

where E is a closed subset of Γ, and where $\varphi : (1/3, \infty) \to (0, \infty)$ is a function such that $\varphi(t)/t$ is increasing. Then

$$\frac{1}{n}\sum_{k=0}^{n-1} \|S^k x\|^2 \leqslant \frac{\mathrm{e}^2}{2}\frac{\varphi(n)^2}{n^2}\Phi_E(n)\|x\|^2, \quad x \in H,\ n \geqslant 1. \tag{2.4}$$

Proof. For $n = 1$ the result is elementary. Fix $n \geqslant 2$ and $x \in H$. Let $1 < r < 2$, and consider

$$J := \frac{1}{2\pi r}\int_{|\lambda|=r} \|(S - \lambda I)^{-1}x\|^2\,|\mathrm{d}\lambda|.$$

On the one hand, because H is a Hilbert space, we can apply Parseval's theorem to get

$$J = \sum_{k=0}^{\infty} \frac{\|S^k x\|^2}{r^{2k+2}} \geqslant \frac{1}{r^{2n}}\sum_{k=0}^{n-1} \|S^k x\|^2.$$

On the other hand, from the resolvent estimate for S and the fact that $\varphi(t)/t$ is increasing,

$$J \leqslant \frac{1}{2\pi r} \int\limits_{|\lambda|=r} \varphi\Big(\frac{1}{\operatorname{dist}(\lambda, E)}\Big)^2 \|x\|^2 \,|\mathrm{d}\lambda| \leqslant \frac{\varphi\big(\frac{1}{r-1}\big)^2}{\frac{1}{(r-1)^2}} \frac{\|x\|^2}{2\pi r} \int\limits_{|\lambda|=r} \frac{|\mathrm{d}\lambda|}{\operatorname{dist}(\lambda, E)^2}.$$

Combining these inequalities and rearranging gives

$$\frac{1}{n}\sum_{k=0}^{n-1} \|S^k x\|^2 \leqslant \frac{\varphi\big(\frac{1}{r-1}\big)^2}{\frac{1}{(r-1)^2}} \frac{\|x\|^2 r^{2n}}{2\pi r n} \int\limits_{|\lambda|=r} \frac{|\mathrm{d}\lambda|}{\operatorname{dist}(\lambda, E)^2}. \tag{2.5}$$

We now take $r = (n+1)^{1/2}/(n-1)^{1/2}$. As in the proof of Theorem 1.1 of [1],

$$\frac{r^n}{2\pi n} \int\limits_{|\lambda|=r} \frac{|\mathrm{d}\lambda|}{\operatorname{dist}(\lambda, E)^2} \leqslant \frac{\mathrm{e}}{2} \Phi_E(n).$$

Also $1/(r-1) \leqslant n$ and $r^{n-1} = \big(1 + 2/(n-1)\big)^{(n-1)/2} \leqslant \mathrm{e}$. Substituting these estimates into (2.5) and once again exploiting the fact that $\varphi(t)/t$ is increasing yields (2.4). ∎

Proof of Theorem 2.1. All that remains is to prove (2.2). Let $H = L^2(\omega)/I(E)$, and let S be the operator on H of multiplication by $\pi(u)$. Since T is defined as the Banach-space adjoint of S, the resolvent estimate (2.3) holds equally for S. Recall also that $\psi(t)/t$ is increasing. Therefore we can apply the preceding lemma with $\varphi(t) = \text{const}\ \psi(3t)$, to obtain

$$\frac{1}{n}\sum_{k=0}^{n-1} \|S^k x\|^2 \leqslant \text{const}\ \frac{\psi(3n)^2}{n^2} \Phi_E(n) \|x\|^2, \quad x \in H,\ n \geqslant 1. \tag{2.6}$$

Now because S is a multiplication operator, it has the additional property that there exists a vector $x_0 \in H$ such that $\|S^k x_0\| \geqslant \|S^k\|$ for all $k \geqslant 0$. Indeed,

$$\|S^k x\| = \|\pi(u^k) x\| = \|\pi(u)^k \pi(1) x\| \leqslant \text{const}\ \|S^k \pi(1)\|\, \|x\|, \quad x \in H,\ k \geqslant 0,$$

so taking x_0 to be a large enough multiple of $\pi(1)$ will do. Putting $x = x_0$ in (2.6), and recalling the definitions of ψ and Φ_E, we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1} \|S^k\|^2 \leqslant \text{const}\ \frac{\omega(3n)^2}{n} \frac{n}{\pi} |E_{\pi/2n}| \leqslant \text{const}\ \omega(n)^2 |E_{1/n}|, \quad n \geqslant 1.$$

Finally, since T is the adjoint of S, its powers satisfy the same inequalities. ∎

3. COMPLETION OF THE PROOF OF THEOREM 1.1

To deduce Theorem 1.2 from Theorem 2.1, we need to make a careful choice of the weight function ω. We require two elementary lemmas.

Lemma 3.1. *Let $(a_n)_{n\geqslant 0}$ be a positive sequence, and let $s_n = (1/n)\sum_{k=0}^{n-1} a_k$, $n \geqslant 1$. Let $(w_n)_{n\geqslant 1}$ be a positive sequence such that $\sum_n w_n/s_n$ diverges. Suppose, in addition, that there exists $c > 0$ such that $w_m \leqslant cw_n$ whenever $n \leqslant m \leqslant 2n$. Then $\sum_n w_n/a_n$ diverges.*

Proof. Let $(v_n)_{n\geqslant 1}$ be a strictly positive, increasing sequence with $v_n \to \infty$ such that $\sum_n w_n/(v_n s_n)$ diverges. Given $p \geqslant 1$, define

$$N_p = \{n \in \mathbb{Z} \cap [2^{p-1}, 2^p) : a_n \leqslant v_{2^p} s_{2^p}\}.$$

Then, for each $p \geqslant 1$,

$$2^p s_{2^p} = \sum_0^{2^p-1} a_n \geqslant \sum_{n\in[2^{p-1},2^p)\setminus N_p} a_n \geqslant (2^{p-1} - |N_p|)v_{2^p} s_{2^p}.$$

It follows that $|N_p| \geqslant 2^{p-1} - 2^p/v_{2^p}$. In particular, since $v_n \to \infty$, there exists p_0 such that $|N_p| \geqslant 2^{p-2}$ for all $p \geqslant p_0$.

Now, if $n \leqslant m \leqslant 2n$, then $w_m \leqslant cw_n$ and $s_m \geqslant s_n/2$. Hence

$$\begin{aligned}\sum_{n\geqslant 1} \frac{w_n}{a_n} &\geqslant \sum_{p\geqslant p_0} \sum_{n\in N_p} \frac{w_n}{a_n} \geqslant \sum_{p\geqslant p_0} |N_p| \frac{c^{-1} w_{2^p}}{v_{2^p} s_{2^p}} \\ &\geqslant \sum_{p\geqslant p_0} \frac{|N_p|}{2^p} \sum_{n=2^p}^{2^{p+1}-1} \frac{c^{-2} w_n}{v_n 2 s_n} = \frac{1}{8c^2} \sum_{n\geqslant p_0} \frac{w_n}{v_n s_n},\end{aligned}$$

and this last sum diverges. ∎

Lemma 3.2. *Given a positive sequence $(\eta_n)_{n\geqslant 1}$ such that $\eta_n \to \infty$, there exist positive sequences $(\alpha_n)_{n\geqslant 1}$ and $(\beta_n)_{n\geqslant 1}$ such that:*

(i) *$\alpha_n \leqslant \eta_n$ for all sufficiently large n;*
(ii) *$\sum_n (\beta_n/n)$ converges and $\sum_n (\alpha_n\beta_n/n)$ diverges;*
(iii) *(α_n) is increasing and $\alpha_{2n} \leqslant 2\alpha_n$ for all n;*
(iv) *(β_n) is decreasing and $\beta_{2n} \geqslant \beta_n/2$ for all n.*

Proof. Choose an increasing sequence of positive integers $(m_k)_{k\geqslant 1}$ such that

$$\eta_n \geqslant 2^k \quad \text{for } n \in (2^{m_k}, 2^{m_{k+1}}].$$

We can further suppose that $m_{k+1} \geqslant 2m_k$ and $m_{k+1}2^{-m_{k+1}} < (1/2)m_k 2^{-m_k}$ for all k. Our first attempt is to set

$$\alpha_n = 2^k \quad \text{and} \quad \beta_n = \frac{2^{-k}}{m_{k+1}} \quad \text{for } n \in (2^{m_k}, 2^{m_{k+1}}].$$

Then (i) is clearly satisfied, and (ii) follows from the observation that, for each $k \geqslant 1$,

$$\sum_{n\in(2^{m_k},2^{m_{k+1}}]} \frac{\beta_n}{n} = \frac{2^{-k}}{m_{k+1}} \sum_{n\in(2^{m_k},2^{m_{k+1}}]} \frac{1}{n}$$
$$= \frac{2^{-k}}{m_{k+1}}\Big(\log\Big(\frac{2^{m_{k+1}}}{2^{m_k}}\Big) + \mathrm{O}\Big(\frac{1}{2^{m_k}}\Big)\Big) \asymp 2^{-k}.$$

Also (iii) is clear, as is the first part of (iv). However, the second part of (iv) fails, because if $n = m_k$, then $\beta_{2n}/\beta_n = m_k/2m_{k+1}$, which may be very small.

So we modify the sequence (β_n) as follows. For $k \geqslant 1$ and $1 \leqslant j \leqslant m_{k+1}-m_k$, set

$$\widetilde{\beta}_n = \max(\beta_n,\ 2^{-j}\beta_{2^{m_k}}) \quad \text{for } n \in (2^{m_k+j-1}, 2^{m_k+j}].$$

Note that $\widetilde{\beta}_{2^{m_k}} = \beta_{2^{m_k}}$ for all k (this amounts to checking that $m_{k+1}2^{-m_{k+1}} < (1/2)m_k 2^{-m_k}$, which we were careful to assume at the outset). It follows easily that the modified sequence $(\widetilde{\beta}_n)$ now satisfies (iv). It remains to check (ii). The divergence of $\sum_n \alpha_n\widetilde{\beta}_n/n$ is clear, because $\widetilde{\beta}_n \geqslant \beta_n$. Also, the convergence of $\sum_n \widetilde{\beta}_n/n$ will follow if we can show that $\sum_n (\widetilde{\beta}_n - \beta_n)/n$ converges. But this latter sum is majorized by

$$\sum_{k\geqslant 1}\sum_{j\geqslant 0} \frac{\beta_{2^{m_k}} 2^{-j}(2^{m_k+j} - 2^{m_k+j-1})}{2^{m_k+j-1}} = 2\sum_{k\geqslant 1}\beta_{2^{m_k}} \leqslant 2\sum_{k\geqslant 1} 2^{-k} < \infty,$$

so $\sum_n \widetilde{\beta}_n/n$ does indeed converge, and the proof of the lemma is complete. ∎

Proof of Theorem 1.2. Given $\eta_n \to \infty$, choose sequences (α_n) and (β_n) satisfying the conclusions of the preceding lemma. Set $\omega(n) = \sqrt{n/\beta_n}$ for $n \geqslant 1$, and extend ω to $\mathbb{Z}$ by defining $\omega(-n) = \omega(n)$ and $\omega(0) = \omega(1)$. The properties (β_n) ensure that ω satisfies (2.1).

By Theorem 2.1, given a closed subset E of Γ, there exists an operator T on a separable Hilbert space with $\sigma_{\mathrm{p}}(T) = \sigma(T) = E$ such that (2.2) holds. We are going to apply Lemma 3.1 with

$$a_n = \|T^n\|^2, \quad s_n = \frac{1}{n}\sum_{k=0}^{n-1}\|T^k\|^2 \quad \text{and} \quad w_n = \alpha_n|E_{1/n}|.$$

From (2.2), we have

$$\sum_n \frac{w_n}{s_n} \geqslant \text{const}\sum_n \frac{\alpha_n|E_{1/n}|}{\omega(n)^2|E_{1/n}|} = \text{const}\sum_n \frac{\alpha_n\beta_n}{n} = \infty.$$

Also, if $n \leqslant m \leqslant 2n$, then $w_m = \alpha_m|E_{1/m}| \leqslant \alpha_{2n}|E_{1/n}| \leqslant 2\alpha_n|E_{1/n}| = 2w_n$. Hence, Lemma 3.1 applies, and we deduce that $\sum_n w_n/a_n$ diverges, i.e.

$$\sum_n \alpha_n|E_{1/n}|\,\|T^n\|^{-2} = \infty.$$

Finally, since $\alpha_n \leqslant \eta_n$ for all large enough n, we deduce that (1.1) holds. This completes the proof. ∎

4. COMMENTS AND QUESTIONS

(i). For countable subsets E of Γ there is a much simpler construction. We can just take T to be a diagonal operator with entries from E: this is unitary and satisfies $\sigma_{\mathrm{p}}(T) = E$. In fact this observation can be exploited to improve slightly upon Theorem 2.1. Given a closed subset E of Γ, we can decompose it as $E = E^{\mathrm{p}} \cup E^{\mathrm{c}}$, where E^{p} is perfect and E^{c} is countable. Let T_1 be the operator associated to E^{p} by Theorem 2.1, and let T_2 be the diagonal operator with entries from E^{c}. Writing T as the direct sum of T_1 and T_2, we then have $\sigma_{\mathrm{p}}(T) = E$, and (2.2) now holds with $|E_{1/n}|$ replaced by $|E^{\mathrm{p}}_{1/n}|$, which in principle grows less rapidly.

(ii). The same circle of ideas leads to a partial extension of Theorem 2.1 to general F_σ sets. It is natural to seek such an extension, in view of a theorem of Nikolskaja ([5]) to the effect that the point spectrum of an operator on a separable Hilbert space is always an F_σ set. Given an F_σ subset F of Γ, write $F = \bigcup_{j\geqslant 1} E^j$, where the E^j are closed in Γ. For each j, Theorem 2.1 provides us with an operator T_j on a separable Hilbert space such that $\sigma_{\mathrm{p}}(T_j) = E^j$ and (2.2) holds with E replaced by E^j. A close inspection of the proof of Theorem 2.1 reveals that the constant in (2.2) depends only on ω, not on the sets E^j, so the norms of the T_j are uniformly bounded, and their direct sum T is thus still a bounded linear operator on a separable Hilbert space. This T then satisfies $\sigma_{\mathrm{p}}(T) = F$ and

$$\Big\|\sum_{k=0}^{n-1} T^{*k}T^k\Big\| \leqslant \text{const}\ \omega(n)^2 \sup_{j\geqslant 1} |E^j_{1/n}|, \quad n \geqslant 1.$$

We do not know if it is possible to replace the left-hand side by $\sum_{k=0}^{n-1} \|T^k\|^2$.

(iii). The method of proof of Theorem 2.1 breaks down if $\omega(n) = \mathrm{O}(\sqrt{n})$, because this would lead to an operator satisfying (2.3) with $\psi(t) = \text{const}\ t$, and it then follows from a result of Nikolski ([7], p. 209, Corollary 2) that $\sigma_{\mathrm{p}}(T) \cap \Gamma$ is at most countable. In fact Nikolski's result applies to any separable, reflexive Banach space. We here give another proof which shows that the hypothesis of reflexivity is unnecessary.

Theorem 4.1. *Let X be a separable Banach space and let T be an operator on X satisfying*

$$\|(T - \lambda I)^{-1}\| \leqslant \frac{C}{\operatorname{dist}(\lambda, \Gamma)}, \quad |\lambda| > 1. \tag{4.1}$$

Then $\sigma_{\mathrm{p}}(T) \cap \Gamma$ is at most countable.

Proof. Let μ, ν be distinct elements of $\sigma_{\mathrm{p}}(T) \cap \Gamma$, and let e_μ, e_ν be corresponding eigenvectors of norm 1. For $r > 1$, set $Q_r = (T - \nu I)(T - r\nu I)^{-1}$. Then

$$Q_r(e_\mu - e_\nu) = (\mu - \nu)(\mu - r\nu)^{-1} e_\mu, \quad r > 1.$$

On the other hand, since $Q_r = I + (r-1)(T - r\nu I)^{-1}$, the resolvent condition (4.1) implies that $\|Q_r\| \leqslant 1 + C$, $r > 1$. Hence

$$(1+C)\,\|e_\mu - e_\nu\| \geqslant |(\mu-\nu)(\mu - r\nu)^{-1}|, \quad r > 1.$$

Letting $r \to 1$, we deduce that $\|e_\mu - e_\nu\| \geqslant 1/(1+C)$. As X is separable, this implies that $\sigma_{\mathrm{p}}(T) \cap \Gamma$ is at most countable. ∎

Note that every power-bounded operator satisfies (4.1), as well as some that are not power-bounded (see e.g. [1], [8], [10]). This raises the question as to whether Theorem 1.1 can also be generalized in terms of resolvents.

(iv). Let T be an operator on a Hilbert space satisfying the resolvent condition (4.1). Then Lemma 2.7, applied with $S = T$, $E = \Gamma$ and $\varphi(t) = Ct$, shows that

$$\sum_{k=0}^{n-1} \|T^k x\|^2 \leqslant \mathrm{e}^2 C^2 n^2 \|x\|^2, \quad x \in H,\ n \geqslant 1.$$

In particular, if there exists vector $x_0 \in H$ such that

$$\|T^n x_0\| \geqslant \|T^n\|, \quad n \geqslant 0, \tag{4.2}$$

then it follows that $\min_{n \leqslant k < 2n} \|T^k\| = \mathrm{O}(\sqrt{n})$ as $n \to \infty$. Is it true that $\|T^n\| = \mathrm{O}(\sqrt{n})$? This question was posed by Shields in p. 373 in [8], without the extra condition (4.2), but in this generality the answer is now known to be negative (see [10]).

(v). The operator T constructed in the proof of Theorem 2.1 satisfies (4.2) for some x_0 because it is the adjoint of a multiplication operator. As such, it even enjoys the stronger property that

$$\|f(T)x_0\| \geqslant \|f(T)\| \tag{4.3}$$

for all functions f holomorphic on a neighbourhood of $\sigma(T)$. For such operators, the technique of Lemma 2.7 can also be used to prove a lower bound for the left-hand side of (2.4). As before, we write $\Phi_E(n) = (n/\pi)|E_{\pi/2n}|$.

Theorem 4.2. *Let T be an operator on a Hilbert space H whose spectral radius equals one, and set $E = \sigma(T) \cap \Gamma$. Suppose that there exists $x_0 \in H$ such that* (4.3) *holds for all functions f holomorphic on a neighbourhood of $\sigma(T)$. Then*

$$\frac{1}{n}\sum_{k=0}^{n-1} \|T^k x_0\|^2 \geqslant \frac{(1-\pi/4)^3}{8}\Phi_E(n), \quad n \geqslant 1.$$

Proof. For $n = 1$ the result is elementary. Fix $n \geqslant 2$. Let $0 < r < 1$, and consider

$$J := \frac{1}{2\pi}\int_{|z|=r} \|(I - z^n T^n)(I - zT)^{-1}x_0\|^2\,|\mathrm{d}z|.$$

On the one hand, by Parseval,

$$J = \frac{1}{2\pi}\int_{|z|=r} \Big\|\sum_{k=0}^{n-1} z^k T^k x_0\Big\|^2 |\mathrm{d}z| = \sum_{k=0}^{n-1} r^{2k}\|T^k x_0\|^2 \leqslant \sum_{k=0}^{n-1}\|T^k x_0\|^2.$$

On the other hand, using (4.3) with $f(T) = (I - z^n T^n)(I - zT)^{-1}$, together with the fact that the norm of $f(T)$ is at least as large as its spectral radius, we have

$$J \geqslant \frac{1}{2\pi} \int_{|z|=r} \frac{(1-|z|^n)^2}{\operatorname{dist}(z,E)^2} \,|\mathrm{d}z| \geqslant \frac{1}{2\pi} \int_{z \in rE_{1-r}} \frac{(1-r^n)^2}{(2-2r)^2} \,|\mathrm{d}z| \geqslant \frac{1}{8\pi} n^2 r^{2n-1} |E_{1-r}|.$$

Combining these inequalities, and setting $r = 1 - \pi/2n$, we obtain

$$\frac{1}{n} \sum_{k=0}^{n-1} \|T^k x_0\|^2 \geqslant \frac{1}{8}\Big(1 - \frac{\pi}{2n}\Big)^{2n-1} \Phi_E(n) \geqslant \frac{1}{8}\Big(1 - \frac{\pi}{4}\Big)^3 \Phi_E(n),$$

as desired. ∎

COROLLARY 4.3. *If, in addition, T is power-bounded, then $\sigma(T) \cap \Gamma$ is a finite set.*

Proof. If T is power-bounded, then the theorem shows that $\Phi_E(n)$ remains bounded as $n \to \infty$. This implies that E is a finite set. ∎

COROLLARY 4.4. *Let A be a Banach algebra which is isomorphic to a Hilbert space. If a is a power-bounded element of A, then $\sigma(a) \cap \Gamma$ is finite.*

Proof. Let $T : A \to A$ be the multiplication operator $x \mapsto ax$. Then (4.3) holds with x_0 a multiple of the identity. The result therefore follows from Corollary 4.3. ∎

Acknowledgements. This work started when the first author was visiting Université Laval, and he thanks Département de mathématiques et de statistique for its hospitality. The second named author's research was partially supported by grants from NSERC (Canada) and FQRNT (Québec).

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OMAR EL-FALLAH
Département de Mathématiques et Informatique
Université Mohammed V
Avenue Ibn Battouta
BP 1014, Rabat
MOROCCO

E-mail: elfallah@fsr.ac.ma

THOMAS RANSFORD
Département de mathématiques et de statistique
Université Laval
Québec (QC), G1K 7P4
CANADA

E-mail: ransford@mat.ulaval.ca

Received July 2, 2002.