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LOGARITHMIC GROWTH FOR WEIGHTED HILBERT TRANSFORMS AND VECTOR HANKEL OPERATORS

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ABSTRACT. We give an example of an operator weight W satisfying the operator Hunt-Muckenhoupt-Wheeden $\mathbb{A}_2$ condition, but for which the Hilbert transform on $L^2(W)$ is unbounded. The construction relates weighted boundedness with the boundedness of vector Hankel operators. We establish a relationship between the norm of a vector Hankel operator and a certain natural (but not Nehari-Page) BMO norm of its symbol, which is logarithmic in dimension.

KEYWORDS: *Weighted Hilbert transform, vector Hankel operators.*

MSC (2000): 42B20, 42A50, 47B35.

1. INTRODUCTION

Let W be an operator weight on the unit circle $\mathbb{T}$, i.e. a function $W : \mathbb{T} \to \mathcal{B}(\mathcal{H})$ such that $W(t)$ is positive and invertible almost everywhere. Here, $\mathcal{B}(\mathcal{H})$ denotes the bounded linear operators on a separable finite or infinite-dimensional Hilbert space $\mathcal{H}$. The weight W gives rise to the operator-weighted L^2-space

$$L^2(W) = \left\{ f : \mathbb{T} \to \mathcal{H} : \int_{\mathbb{T}} \langle W(t)f(t), f(t)\rangle \, \mathrm{d}t < \infty \right\}. \tag{1.1}$$

We consider whether the matrix Hunt-Muckenhoupt-Wheeden Theorem from [27] holds for infinite-dimensional $\mathcal{H}$, i.e. whether the operator $\mathbb{A}_2$ condition

$$\|W\|_{\mathbb{A}_2} := \sup_{I \subseteq \mathbb{T},\, I \text{ interval}} \|\langle W^{1/2}\rangle_I \langle W^{-1/2}\rangle_I\| < \infty \tag{1.2}$$

is equivalent to the boundedness of the Hilbert transform H on $L^2(W)$. We show that this is not the case in the infinite-dimensional situation, and we give a non-trivial lower bound for the dimensional growth. Here is the main result of this paper:

THEOREM 1.1. *There exists an operator-valued $\mathbb{A}_2$-weight such that the Hilbert transform in unbounded on $L^2(W)$. More precisely, there exist constants $a, A > 0$ such that for each positive integer n, there exists an $n \times n$ matrix weight W such that $\|W\|_{\mathbb{A}_2} \leqslant A$ but*

$$\|H\|_{L^2(W)\to L^2(W)} \geqslant a \cdot \log n. \tag{2.3}$$

The boundedness of H in $L^2(W)$ is of great interest in the theory of multivariate stationary stochastic processes. An operator weight W is the spectral measure of a regular multivariate stationary stochastic process if and only if the "angle between past and future" of the process is nonzero, i.e. the Hilbert transform is bounded on $L^2(W)$. We show that for processes taking values in an infinite-dimensional Hilbert space, regularity can no longer be characterized in terms of the $\mathbb{A}_2$ condition of the spectral measure.

Our main tool is a linearization of the problem coming from [10] and [9]. This linearization reduces the question about the validity of the Hunt-Muckenhoupt-Wheeden Theorem in infinite dimensions for a special class of weights to the comparison of two operator-valued BMO-spaces. More precisely, let B be an $\mathcal{B}(\mathcal{H})$-valued function on $\mathbb{T}$, and let the operator weight $W : \mathbb{T} \to \mathcal{B}(\mathcal{H}^2)$ be defined by

$$W = V_B^* V_B = \begin{pmatrix} \mathbb{1} & B \\ 0 & \mathbb{1} \end{pmatrix}^* \begin{pmatrix} \mathbb{1} & B \\ 0 & \mathbb{1} \end{pmatrix}. \tag{2.4}$$

Let $\mathrm{BMO}_{\mathrm{so}}$ denote the space of matrix functions B on $\mathbb{T}$ such that

$$\sup_J \langle \|(B - \langle B\rangle_J)e\|^2\rangle_J \leqslant C\|e\|^2 \quad \forall e \in \mathcal{H}, \tag{2.5}$$

where J denotes an arbitrary interval on $\mathbb{T}$. We will call the best $\sqrt{C}$ in (2.5) the norm of B, denoting it by $\|B\|_{\mathrm{BMO}_{\mathrm{so}}}$. Furthermore, let $\|B\|_{\mathrm{BMO}_{\mathrm{so}*}} = \|B^*\|_{\mathrm{BMO}_{\mathrm{so}}}$.

THEOREM 1.2. ([10], [9]) *Let W be an operator weight of the form* (2.4). *Then*

$$\|W\|_{\mathbb{A}_2} = 1 + \max\{\|B\|_{\mathrm{BMO}_{\mathrm{so}}}, \|B^*\|_{\mathrm{BMO}_{\mathrm{so}}}\}, \tag{2.6}$$

and

$$\|[H, B]\| \leqslant \|H\|_{L^2(W)\to L^2(W)} \leqslant \|[H, B]\| + 1. \tag{2.7}$$

We shall prove the following result:

THEOREM 1.3. *There is a constant $a > 0$ such that for each positive integer n, there exists an $n \times n$ matrix function B on $\mathbb{T}$ such that*

$$\|B\|_{\mathrm{BMO}_{\mathrm{so}}} \leqslant 1, \quad \|B^*\|_{\mathrm{BMO}_{\mathrm{so}}} \leqslant 1,$$

but

$$\|HB - BH\|_{L^2(\mathbb{C}^n,\mathrm{d}t)\to L^2(\mathbb{C}^n,\mathrm{d}t)} \geqslant a \cdot \log n. \tag{2.8}$$

G. Pisier has pointed out to us that the proof of this result can be extracted from the articles [1] and [11], if one considers the extension of Fourier multipliers on $H^1(\mathbb{T})$ to the analytic parts of both BMO spaces. This argument is presented in

Section 2 of this paper. For an elegant direct proof of the theorem which combines the ideas of [1] and [11], see the preprint [19].

The growth in Theorem 1.3 is sharp (see [23]), and this result is of independent interest. It describes the dimensional growth of the norm of Hankel operators in terms of a natural BMO norm of the symbol, and it has been used to obtain sharp dimensional growth in the Matrix Carleson Embedding Theorem (see [19] together with the earlier results [15], [20]). It can also be seen in the context of a "Fefferman-type duality" for operator-valued functions (see Section 2).

Notice that $\mathcal{H}$ here is just the n-dimensional Hilbert space, and usually the dimension will be clear from the context. Using this convention, we say that $\mathrm{BMO}_{\mathrm{so}}$ consists of operator functions such that Be lies in $\mathrm{BMO}(\mathcal{H})$ for every $e \in \mathcal{H}$. We use the same definition for operator functions whose values are operators in the infinite dimensional $\mathcal{H}$.

The goal of the paper is three-fold. We prove a certain non-trivial estimate from below for the Hankel operator H_B in terms of n and $\|B\|_{\mathrm{BMO}_{\mathrm{so}}}$, as in Theorem 1.3. It is convenient to use the commutator with the Hilbert transform $HB-BH$ instead of H_B. For self-adjoint B, there is no difference. The second goal is to construct positive definite $n \times n$ matrix functions W, the $\mathbb{A}_2$ norms of which (see the definition in [27] and below) are bounded by an absolute constant, but for which the norm of H on $L^2(W)$ grows with n, as in Theorem 1.1. This shows that the Hunt-Muckenhoupt-Wheeden result does not hold for operator valued weights. Finally, we indicate some interesting connections with a factorization problem for $H^1(\mathbb{T}, S^1)$ and with extensions of Bonsall's theorem to the case of operator-valued functions.

Let us recall that the boundedness of H on $L^2(W)$ is an important ingredient in characterizing regularity properties for stationary stochastic processes in terms of their spectral measure (see [22], [14], [21], and [27]). The theory of vector valued, i.e. multivariate, stationary processes taking values in a Hilbert space was started by Kolmogoroff and Wiener in the early 50's (see e.g. [28] and [18]).

The characterization of such n-dimensional processes for which the angle between past and future is positive has been given in [27]. It turns out that an $n \times n$ positive definite matrix function W is the spectral density of such a process if and only if W satisfies the $\mathbb{A}_2$ condition:

$$\|W\|_{\mathbb{A}_2} := \sup_J \|\langle W\rangle^{1/2}\langle W^{-1}\rangle^{1/2}\| < \infty.$$

The present article shows that for infinite dimensional processes (processes taking values in an infinite dimensional Hilbert space), this characterization is no longer true.

In [9], similar results were proved for "dyadic Hankel operators" and for martingale transforms respectively. The "continuous case" of the present article ("continuous" in the sense that we replace dyadic Hankel operators by usual Hankel operators, i.e. commutators of multiplication with H, and martingale transforms by the Hilbert transform) is based on [9], but it uses different methods.

2. MULTIPLIERS ON $H^1(\mathbb{T}, S^1)$

Let $\mathcal{H}$ be an infinite or finite dimensional Hilbert space. We fix an orthonormal basis (e_i) of $\mathcal{H}$. Consider the space

$$\mathrm{BMO_c} = \{B : \mathbb{T} \to \mathcal{B}(\mathcal{H}) : \|[H,B]\|_{L^2(\mathcal{H},\mathrm{d}t)\to L^2(\mathcal{H},\mathrm{d}t)} + \|\widehat{B}(0)\|_{\mathcal{B}(\mathcal{H})} < \infty\}. \tag{2.1}$$

Let $\mathrm{BMOA_c} = \{B \in \mathrm{BMO_c},\ B \text{ analytic}\}$. (By analytic, we mean that all negative Fourier coefficients of B vanish.) Let S^1 denote the space of trace-class operators on $\mathcal{H}$ with the trace norm.

Lemma 2.1. *Let $H^1(\mathbb{T}, S^1)$ denote the complex Hardy space of S^1-valued functions on $\mathbb{T}$. Then*

$$\mathrm{BMOA_c} \subseteq H^1(\mathbb{T}, S^1)^*, \tag{2.2}$$

where $B \in \mathrm{BMOA_c}$ acts on $H^1(\mathbb{T}, S^1)$ by $\langle F, B\rangle = \int_{\mathbb{T}} \mathrm{tr}(B^*(t)F(t))\mathrm{d}t$ for $F \in H^1(\mathbb{T}, S^1)$. The $\mathrm{BMOA_c}$-norm is equivalent to the norm in $H^1(\mathbb{T}, S^1)^*$. If $\dim \mathcal{H} < \infty$, then the inclusion (2.2) is an equality.

Proof. The proof is very similar to the scalar situation (see e.g. [25]). The role of the factorization of scalar-valued H^1 into $H^2 \cdot H^2$ is taken by the factorization of matrix H^1 functions [28] respectively by Sarason's factorization Theorem ([26]), which can be stated in the following way. The canonical product map

$$H^2(\mathbb{T}, \mathcal{H}) \widehat{\otimes} H^2(\mathbb{T}, \mathcal{H}) \to H^1(\mathbb{T}, S^1) \tag{2.3}$$

is surjective. Moreover, each element of $H^1(\mathbb{T}, S^1)$ has an inverse image of the same norm (see also [11], Theorem 1.5).

We outline the argument for the convenience of the reader. Let $B : \mathbb{T} \to \mathcal{B}(\mathcal{H})$ be analytic. Define the Hankel operator $H^2_0(\mathbb{T}, \mathcal{H})^\perp \to H^2(\mathbb{T}, \mathcal{H})$, $f \mapsto PBP_0^\perp f$, where P denotes the Riesz projection $L^2(\mathbb{T}, \mathcal{H}) \to H^2(\mathbb{T}, \mathcal{H})$, P_0 the orthogonal projection $L^2(\mathbb{T}, \mathcal{H}) \to H^2_0(\mathbb{T}, \mathcal{H})$ and orthogonal complements are taken with respect to the Hilbert space $L^2(\mathbb{T}, \mathcal{H})$. We will further need the orthogonal projection $P_\mathrm{c} : L^2(\mathbb{T}, \mathcal{H}) \to \mathcal{H}$ onto the constants. We first show that $PBP_0^\perp$ is bounded if and only if $B \in \mathrm{BMOA_c}$.

A standard calculation yields

$$\begin{aligned} \langle [H,B]f, g\rangle &= \langle [-\mathrm{i}(P + P_0 - \mathbb{1}), B]f, g\rangle \\ &= -\mathrm{i}(2\langle PBP_0^\perp f, g\rangle - \langle PBP_\mathrm{c} f, g\rangle - \langle P_\mathrm{c} BP_0^\perp f, g\rangle. \end{aligned} \tag{2.4}$$

If $B \in \mathrm{BMOA_c}$, then in particular $B \in \mathrm{BMO_{so}} \cap \mathrm{BMO_{so*}}$ with a corresponding estimate of norms. To see this, one can either use the dimension-independent reverse direction of the matrix Hunt-Muckenhoupt-Wheeden Theorem (see [27] together with the technique from [9]), or, more simply, use the fact that the $\mathrm{BMO}(\mathcal{H})$-norm of Be is equivalent to the norm of the operator $[H, Be] : L^2(\mathbb{T}) \to L^2(\mathbb{T}, \mathcal{H})$, and that $[H, Be]f = [H, B]ef$ for $e \in \mathcal{H}$, $f \in L^2(\mathbb{T})$. This version of the Coifman-Rochberg-Weiss Theorem follows from the version of Fefferman's Duality Theorem for $\mathcal{H}$-valued functions, see e.g. [4]. Thus by (2.4), $\|PBP_0^\perp\| \leqslant C(\|[H,B]\| + 2\|\widehat{B}(0)\| + \|B\|_{\mathrm{BMO_{so}}} + \|B^*\|_{\mathrm{BMO_{so}}})$ for a suitable constant $C > 0$. Conversely, just consider $\langle PBP_0^\perp f, g\rangle$, $\langle PBP_0^\perp f(0), g\rangle$ and $\langle PBP_0^\perp f, g(0)\rangle$.

Now we can complete the proof of the lemma as follows. For $F \in H^1(\mathbb{T}, S^1)$, write $F = \sum_{i\geqslant 1} f_i \otimes g_i$ with $f_i, g_i \in H^2(\mathbb{T}, \mathcal{H})$ for all i and $\|F\|_{H^1(\mathbb{T},S^1)} = \sum_{i\geqslant 1} \|f_i\|_2 \|g_i\|_2$. Then

$$
\begin{aligned}
|\langle F, B\rangle| &= \Big| \int_{\mathbb{T}} \operatorname{tr}(B^*(t)F(t))\mathrm{d}t \Big| = \Big| \sum_{i\geqslant 1} \langle B^* f_i, \bar{g}_i\rangle \Big| \\
&\leqslant \|P_0^\perp B^* P\| \sum_{i\geqslant 1} \|f_i\|_2 \|g_i\|_2 = \|PBP_0^\perp\| \|F\|_{H^1(\mathbb{T},S^1)}.
\end{aligned} \tag{2.5}
$$

Conversely, apply B to elementary tensors $f \otimes g \in H^1(\mathbb{T}, S^1)$.

To prove that the inclusion (2.2) is an equality for finite-dimensional $\mathcal{H}$, one just has to note that by a trivial extension of Fefferman's Duality Theorem, in this case each linear functional on $H^1(\mathbb{T}, S^1)$ is given by a matrix-valued function. ∎

Using results in [9], we can now formulate the question of the equivalence of the $\mathbb{A}_2$ condition and boundedness of the Hilbert transform for operator weights of the form

$$
W = \begin{pmatrix} \mathbb{1} & B \\ 0 & \mathbb{1} \end{pmatrix}^* \begin{pmatrix} \mathbb{1} & B \\ 0 & \mathbb{1} \end{pmatrix} \quad \text{with } B \text{ analytic} \tag{2.6}
$$

in the following way: Is $\mathrm{BMOA}_{\mathrm{so}} \cap \mathrm{BMOA}_{\mathrm{so}*}$ topologically embedded in $H^1(\mathbb{T}, S^1)^*$? We know that for finite-dimensional $\mathcal{H}$, $H^1(\mathbb{T}, S^1)^* \subseteq \mathrm{BMOA}_{\mathrm{so}} \cap \mathrm{BMOA}_{\mathrm{so}*}$ with a corresponding dimension-independent estimate of norms, as already discussed above.

Concerning the question as to whether the reverse inclusion is also true, G. Pisier has pointed out to us that a negative answer can be extracted from the literature by considering the Fourier multipliers of the spaces $H^1(\mathbb{T}, S^1)^*$ and $\mathrm{BMOA}_{\mathrm{so}} \cap \mathrm{BMOA}_{\mathrm{so}*}$ and using a result of F. Lust-Piquard. This can be seen as follows:

Lemma 2.2. *Let $m = (m(n))_{n\geqslant 0}$ be a bounded Fourier multiplier on $H^1(\mathbb{T})$. Then m acts boundedly on $\mathrm{BMOA}_{\mathrm{so}} \cap \mathrm{BMOA}_{\mathrm{so}*}$ and, for an absolute constant C, we have $\|m\|_{\mathcal{M}(\mathrm{BMOA})} \leqslant C\|m\|_{\mathcal{M}(H^1(\mathbb{T}))}$.*

Proof. Let $m = (m(n))_{n\geqslant 0}$ be a bounded Fourier multiplier on $H^1(\mathbb{T})$ with $\|m\|_{\mathcal{M}(H^1(\mathbb{T}))} = 1$. By the Marcinkiewicz-Zygmund principle (see e.g. [6], p. 203), m is also a bounded Fourier multiplier on $H^1(\mathbb{T}, \mathcal{H})$ with the same multiplier norm. We write $m * h$ for the action of m on $h \in H^1(\mathbb{T}, \mathcal{H})$.

We know that $B \in \mathrm{BMOA}_{\mathrm{so}}$ if and only if $Be \in \mathrm{BMOA}(\mathcal{H})$ uniformly for all e in the unit ball of $\mathcal{H}$. Using the duality $\mathrm{BMOA}(\mathbb{T}, \mathcal{H}) = H^1(\mathbb{T}, \mathcal{H})^*$ (see, e.g., [4]), we see that $\|B\|_{\mathrm{BMOA}_{\mathrm{so}}}$ is equivalent to the norm of the bilinear map $H^1(\mathbb{T}, \mathcal{H}) \times \mathcal{H} \to \mathbb{C}$, $(h, e) \mapsto \langle h, B\bar{e}\rangle$. Here the symbol $\bar{\ }$ denotes the natural complex conjugation on $\mathcal{H}$ associated with the fixed basis (e_i). Similarly, $\|B\|_{\mathrm{BMOA}_{\mathrm{so}*}}$ is equivalent to the norm of the bilinear map $\mathcal{H} \times H^1(\mathbb{T}, \mathcal{H}) \to \mathbb{C}$, $(e, h) \mapsto \langle e, B^*\bar{h}\rangle$. Now

$$
|\langle m*Be, h\rangle| = |\langle Be, \overline{m}*h\rangle| \leqslant C\|B\|_{\mathrm{BMOA}_{\mathrm{so}}} \|\overline{m}*h\|_1 \leqslant C\|B\|_{\mathrm{BMOA}_{\mathrm{so}}} \|h\|_1 \|e\| \tag{2.7}
$$

for an absolute constant $C > 0$, and we also obtain a corresponding estimate for $|\langle e, (m * B)^* \bar{h}\rangle|$. ∎

On the other hand, by [17], Théorème 1 together with Theorems 6.1, 6.2 in [24], not all Fourier multipliers on $H^1(\mathbb{T})$ extend to uniformly bounded multipliers on $H^1(\mathbb{T}, S_n^1)$ or, equivalently, to $H^1(\mathbb{T}, S_n^1)^*$, for all $n \in \mathbb{N}$. Here, S_n^1 denotes the $n \times n$ matrices with the trace norm. In particular, this implies by Lemma 2.1 that not all Fourier multipliers on $H^1(\mathbb{T})$ extend to bounded multipliers on $\mathrm{BMOA_c}$. Consequently, $\mathrm{BMOA_c}$ and $\mathrm{BMOA_{so}} \cap \mathrm{BMOA_{so*}}$ are not equal, and the infinite-dimensional matrix Hunt-Muckenhoupt-Wheeden Theorem fails.

Using results by Haagerup and Pisier ([11]) and by Blower ([1]), we can now obtain the desired dimensional estimate. Let X be a Banach space. Blower proves that all multipliers in $\mathcal{M}(H^1(\mathbb{T}))$ satisfying the strong Hörmander-Mikhlin conditions extend to bounded multipliers on $H^1(\mathbb{T}, X)$ if and only if X has the so-called AUMD property, i.e. all X-valued analytic martingales are bounded on $L^1(\mathbb{T}^{\mathbb{N}}, X)$. More precisely, Blower proves the following by a simple transference argument (see [1], Theorem 3, Equation 6). Let $\mathcal{C}(X)_{\mathrm{AUMD}}$ denote the supremum of the norms of the analytic martingale transforms on $L^1(\mathbb{T}^{\mathbb{N}}, X)$, and let $\mathcal{C}(X)_{\mathcal{M}}$ denote the supremum of the norms on $H^1(\mathbb{T}, X)$ of Hörmander-Mikhlin multipliers m satisfying $\|m\|_{\mathcal{M}(H^1(\mathbb{T}))} = 1$. Then $\mathcal{C}(X)_{\mathrm{AUMD}} \leqslant c\,\mathcal{C}(X)_{\mathcal{M}}$ for some absolute constant c.

On the other hand, Haagerup and Pisier prove that $\mathcal{C}(S_n^1)_{\mathrm{AUMD}} \geqslant a' \log n$ by a clever comparison with the triangular projection on S_n^1, the norm of which grows with $\log n$ by [16].

Putting this together, we obtain the following. Assume $B \in H^1(S_n^1)^*$ with $\|B\|_{H^1(S_n^1)^*} = 1$ and let $m \in \mathcal{M}(H^1(\mathbb{T}))$ with $\|m\|_{\mathcal{M}(H^1(\mathbb{T}))} = 1$ such that $\|m * B\|_{H^1(S_n^1)^*} \geqslant \frac{1}{2}\mathcal{C}(S_n^1)_{\mathcal{M}}$. Then

$$\begin{gathered}\|m * B\|_{H^1(S_n^1)^*} \geqslant \frac{1}{2c}\,\mathcal{C}(S_n^1)_{\mathrm{AUMD}} \geqslant \frac{a'}{2c}\log n, \\ \text{but } \|m * B\|_{\mathrm{BMOA_{so}} \cap \mathrm{BMOA_{so*}}} \leqslant C'\end{gathered} \tag{2.8}$$

for absolute constants a', c, C'. This finishes the proof of Theorem 1.3. ∎

It is now easy to prove Theorem 1.1. Consider a $2n \times 2n$ matrix function

$$V_B = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}.$$

Consider the matrix weight $W := V_B^* V_B$. It follows from Theorem 1.3 and [10], [9] that W satisfies Theorem 1.1.

This finishes the proof of the main result. ∎

Remark 2.3. We can also formulate our comparison of operator BMO spaces in terms of a special type of weak factorization of $H^1(\mathbb{T}, S^1)$.

Corollary 2.4. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then the canonical product map*

$$(H^1(\mathbb{T}, \mathcal{H}) \widehat{\otimes} \mathcal{H}) \oplus (\mathcal{H} \widehat{\otimes} H^1(\mathbb{T}, \mathcal{H})) \to H^1(\mathbb{T}, S^1), \quad (h \otimes e, e' \otimes h') \mapsto h \otimes e + e' \otimes h'$$

is not surjective.

Proof. First, recall that the $\mathrm{BMOA_{so}}$ (respectively $\mathrm{BMOA_{so*}}$) norm is equivalent to the norm of the associated bilinear form on $H^1(\mathbb{T}, \mathcal{H}) \times \mathcal{H}$ (respectively $\mathcal{H} \times$

$H^1(\mathbb{T},\mathcal{H})$). The space of bilinear forms on $H^1(\mathbb{T},\mathcal{H})\times\mathcal{H}$ is isometrically isomorphic to the dual of the projective tensor product $H^1(\mathbb{T},\mathcal{H})\widehat{\otimes}\mathcal{H}$, and the corresponding statement holds for the space of bilinear forms on $\mathcal{H}\times H^1(\mathbb{T},\mathcal{H})$. Thus we have a natural embedding

$$\mathrm{BMOA}_{\mathrm{so}}\cap\mathrm{BMOA}_{\mathrm{so}*}\subseteq ((H^1(\mathbb{T},\mathcal{H})\widehat{\otimes}\mathcal{H})\oplus(\mathcal{H}\widehat{\otimes}H^1(\mathbb{T},\mathcal{H})))^*, \tag{2.9}$$

and $\mathrm{BMOA}_{\mathrm{so}}\cap\mathrm{BMOA}_{\mathrm{so}*}$ is closed in the space on the right hand side.

So the fact that the $\mathrm{BMOA}_{\mathrm{so}}\cap\mathrm{BMOA}_{\mathrm{so}*}$-norm and the $\mathrm{BMOA}_{\mathrm{c}}$-norm are not equivalent implies that $\mathrm{BMOA}_{\mathrm{c}}$ is not a closed subspace of $((H^1(\mathbb{T},\mathcal{H})\widehat{\otimes}\overline{\mathcal{H}})\oplus(\overline{\mathcal{H}}\widehat{\otimes}H^1(\mathbb{T},\mathcal{H})))^*$. In particular, the space $H^1(\mathbb{T},S^1)^*$, which can be naturally embedded into $((H^1(\mathbb{T},\mathcal{H})\widehat{\otimes}\overline{\mathcal{H}})\oplus(\overline{\mathcal{H}}\widehat{\otimes}H^1(\mathbb{T},\mathcal{H})))^*$, is not a closed subspace of the latter space by Lemma 2.1. Passing to the preduals, we see that the canonical map

$$\begin{aligned}&(H^1(\mathbb{T},\mathcal{H})\widehat{\otimes}\mathcal{H})\oplus(\mathcal{H}\widehat{\otimes}H^1(\mathbb{T},\mathcal{H}))\to H^1(\mathbb{T},S^1),\\&(h\otimes e,e'\otimes h')\mapsto h\otimes e+e'\otimes h'\end{aligned}\tag{2.10}$$

is not surjective. ∎

3. EXTENSIONS OF BONSALL'S THEOREM TO THE VECTOR VALUED CASE

Another way of comparing the spaces $\mathrm{BMOA}_{\mathrm{so}}\cap\mathrm{BMOA}_{\mathrm{so}*}$ and $\mathrm{BMOA}_{\mathrm{c}}$ is to consider vector-valued versions of Bonsall's Theorem (see [2], Theorem 1).

THEOREM 3.1. ([2]) *Let Γ be a Hankel operator on the unit circle, $\Gamma: H^2(\mathbb{T})\to H^2(\mathbb{T})^\perp$. Then Γ is bounded on $H^2(\mathbb{T})$ if and only if Γ is bounded on the set of normalized Szegö kernels $\{k_z : z\in\mathbb{D}\}$, where $k_z(\xi)=(1-|z|^2)^{1/2}\frac{1}{1-\overline{z}\xi}$.*

The proof consists of a clever computation and an application of Fefferman's Duality Theorem. Let $\Gamma=\Gamma_{\overline{b}}=P^\perp\overline{b}P$ with the antianalytic symbol $\overline{b}$. Then a computation gives $\|\Gamma k_z\|^2=|b|^2(z)-|b(z)|^2$ for $z\in\mathbb{D}$. It is well-known that boundedness of the set $\{|b|^2(z)-|b(z)|^2 : z\in\mathbb{D}\}$ is equivalent to the BMO condition, and Fefferman's Duality Theorem then implies the boundedness of the Hankel operator Γ.

If we consider an operator-valued symbol B^*, where B is analytic, and let the vector Hankel operator $\Gamma=\Gamma_{B^*}=P^\perp B^*P$ act on $H^2(\mathbb{T},\mathcal{H})$, then

$$\|\Gamma_{B^*}k_ze\|^2=\langle(BB^*)(z)e,e\rangle-\langle B(z)B^*(z)e,e\rangle\quad z\in\mathbb{D},\ e\in\mathcal{H}\tag{3.1}$$

by exactly the same computation. The BMO space corresponding to the uniform boundedness of these expressions on the unit ball of $\mathcal{H}$ is $\mathrm{BMOA}_{\mathrm{so}*}$ (see [7], p. 234, Corollary VI.2.4). Now let $\widetilde{B}(z)=B^*(\overline{z})$. By the same argument,

$$\begin{aligned}\|(\Gamma_{B^*})^*\overline{k_z}\overline{e}\|^2&=\|\Gamma_{\widetilde{B}^*}k_ze\|^2\\&=\langle(B^*B)(\overline{z})e,e\rangle-\langle B^*(\overline{z})B(\overline{z})e,e\rangle\quad z\in\mathbb{D},\ e\in\mathcal{H},\end{aligned}\tag{3.2}$$

and the space corresponding to the uniform boundedness of these expressions on the unit ball of $\mathcal{H}$ is $\mathrm{BMOA}_{\mathrm{so}}$. However, the BMO space corresponding to the boundedness of Γ_{B^*} is $\mathrm{BMOA}_{\mathrm{c}}$. So we have shown the following theorem.

Theorem 3.2. *There exists an antianalytic operator symbol B^* and a constant $C > 0$ such that $|\langle \Gamma_{B^*} k_z e, f\rangle| \leqslant C\|e\|\|f\|_2$ and $|\langle \Gamma_{B^*} f, \overline{k_z} e\rangle| \leqslant C\|e\|\|f\|_2$ for all $z \in D$, $f \in H^2(\mathbb{T}, \mathcal{H})$, $e \in \mathcal{H}$, but such that Γ_{B^*} does not extend to a bounded linear operator on $H^2(\mathbb{T}, \mathcal{H})$. In this sense, Bonsall's Theorem does not extend to the infinite-dimensional Hilbert space valued case.*

4. REMARKS

It is also possible to obtain a counterexample to the operator Hunt-Muckenhoupt-Wheeden Theorem by comparing the norm of vector Hankel operators $[H, B]$ to the norm of vector paraproducts π_B and then use dimensional growth in the dyadic matrix Carleson Embedding Theorem (see [20]). This was done in [8]. However, this only gives the weaker dimensional growth $\|[H, B]\| \geqslant (\log n)^{1/2}\|B\|_{\mathrm{BMO}_{\mathrm{so}}}$.

On the other hand, using methods from [23], the $\log n$ dimensional growth of the Hankel operators can also be used to obtain $\log n$ dimensional growth in the matrix Carleson Embedding Theorem (see the preprint [19]). By earlier results in [20] and [15], this is sharp.

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