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PROJECTIVE MODULES ON FOCK SPACES

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Abstract. A Hilbert module over the free algebra generated by n noncommutative variables is a Hilbert space $\mathcal{H}$ with n bounded linear operators. In this paper we use Hilbert module language to study the semi-invariant subspaces of a family of weighted Fock spaces and their quotients that includes the Full Fock space, the symmetric Fock space, the Dirichlet algebra, and the reproducing kernel Hilbert spaces with a Nevanlinna-Pick kernel. We prove a commutant lifting theorem, obtain explicit resolutions and characterize the strongly orthogonally projective subquotients of each algebra. We use the symbols associated with the commutant lifting theorem to prove that two minimal projective resolutions are unitarily equivalent.

Keywords: *Fock spaces, Hilbert modules, Commutant Lifting Theorem, projective modules, projective resolution.*

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1. INTRODUCTION AND PRELIMINARIES

In [13], Douglas and Paulsen reformulated a part of single variable operator theory, including aspects of the Sz.-Nagy–Foiaş dilation theory, into the Hilbert module language and proposed it as a guide to study multivariate function algebras. This approach was strengthened by Muhly and Solel in [18], who studied more general operator algebras. In this paper we use the Hilbert module language to study the semi-invariant subspaces of a family of weighted Fock spaces and their quotients. This family includes the Full Fock space, the symmetric Fock space, the Dirichlet algebra, and the reproducing kernel Hilbert spaces with a Nevanlinna-Pick kernel.

We first prove a commutant lifting theorem, based on the recent paper of Clancy and McCullough ([9]). Then we use the Poisson kernels of [29] and [4] to obtain explicit resolutions, in a way similar to Theorem 1.4 of [7]. We combine the Poisson kernels and the commutant lifting theorems to characterize the strongly orthogonally projective subquotients of each algebra, and we use the symbols associated with the commutant lifting theorem to prove that two minimal resolutions are unitarily equivalent.

We obtain some applications. The Hilbert module language we use makes the recent results of the paper by McCullough and Trent ([17]) on invariant subspace and Nevanlinna-Pick kernels very transparent. We also prove that their conjecture is true. We characterize the strongly orthogonally projective subquotients of the symmetric Fock space (the pure Hilbert modules in the notation of [7]) and show that they are exactly the free modules. And we find counter-examples to a question of Muhly and Solel ([18], page 20). We show that strongly orthogonally projective subquotients of a quotient algebra are, in general, not strongly orthogonally projective for the algebra. In the Full Fock space, we prove that a subquotient is strongly orthogonally projective if and only if it is a submodule. The "if part" follows also from the work of Muhly and Solel ([19]), who characterized the strongly orthogonally projective modules of a large family of C^*-correspondences that includes the Full Fock space as a particular case.

A Hilbert module $(\mathcal{H}; L_1, \ldots, L_n)$ over the free algebra generated by n-noncommutative variables consists of a Hilbert space $\mathcal{H}$ and n bounded linear operators $L_1, \ldots, L_n$. We will consider a fixed Hilbert module $(\mathcal{H}; L_1, \ldots, L_n)$, where $\mathcal{H}$ denotes either a weighted Fock space or a quotient of a weighted Fock space, but most results will be stated for the Hilbert module $(\mathcal{H}\otimes\ell_2; L_1\otimes I_{\ell_2}, \ldots, L_n\otimes I_{\ell_2})$. If $\mathcal{E} \subset \mathcal{H}\otimes\ell_2$ is invariant under $L_i\otimes I_{\ell_2}$ for $i \leqslant n$, we say that $(\mathcal{E}; L_1\otimes I_{\ell_2}, \ldots, L_n\otimes I_{\ell_2})$ is a *submodule* of $\mathcal{H}\otimes\ell_2$. If $\mathcal{E} \subset \mathcal{H}\otimes\ell_2$ is invariant under $(L_i\otimes I_{\ell_2})^*$ for $i \leqslant n$, and $V_i = P_{\mathcal{E}}(L_i\otimes I_{\ell_2})|_{\mathcal{E}}$ for $i \leqslant n$, we say that $(\mathcal{E}; V_1, \ldots, V_n)$ is a $*$-*submodule* of $\mathcal{H}\otimes\ell_2$. And if $\mathcal{E} \subset \mathcal{H}\otimes\ell_2$ is semi-invariant under $L_i\otimes I_{\ell_2}$ for $i \leqslant n$, and $W_i = P_{\mathcal{E}}(L_i\otimes I_{\ell_2})|_{\mathcal{E}}$ for $i \leqslant n$, we say that $(\mathcal{E}; W_1, \ldots, W_n)$ is a *subquotient* of $\mathcal{H}\otimes\ell_2$. Recall that $\mathcal{E} \subset \mathcal{H}\otimes\ell_2$ is semi-invariant under an algebra $A \subset B(\mathcal{H}\otimes\ell_2)$ if the compression to $\mathcal{E}$ is a multiplicative map. That is, if $a, b \in A$ then $P_{\mathcal{E}}aP_{\mathcal{E}}bP_{\mathcal{E}} = P_{\mathcal{E}}abP_{\mathcal{E}}$. Sarason proved that $\mathcal{E}$ is semi-invariant if and only if there exist two submodules $\mathcal{E}_0$ and $\mathcal{E}_1$ such that $\mathcal{E}_0 \oplus \mathcal{E} = \mathcal{E}_1$. Every submodule and every $*$-submodule is a subquotient, but the reverse is not always true.

A bounded linear operator $f : \mathcal{H} \to \mathcal{K}$ between the Hilbert modules $(\mathcal{H}; L_1, \ldots, L_n)$ and $(\mathcal{K}; V_1, \ldots, V_n)$ is a module map if $f(L_i x) = V_i f(x)$ for every $x \in \mathcal{H}$ and $i \leqslant n$. The set of all bounded module maps is denoted by $\mathrm{Hom}(\mathcal{H}, \mathcal{K})$. The Hilbert modules $\mathcal{H}$ and $\mathcal{K}$ are isomorphic if there exists an invertible map $f : \mathcal{H} \to K$ such that f and f^{-1} are isometric module maps. Notice that the orthogonal projection $P_{\mathcal{E}} : \mathcal{H}\otimes I_{\ell_2} \to \mathcal{E}$ is a module map if and only if $\mathcal{E} \subset \mathcal{H}\otimes\ell_2$ is a $*$-submodule of $\mathcal{H}\otimes\ell_2$; and the inclusion $\iota : \mathcal{E} \to \mathcal{H}\otimes\ell_2$ is a module map if and only if $\mathcal{E}$ is a submodule of $\mathcal{H}\otimes\ell_2$.

A subquotient $\mathcal{E}$ is *strongly orthogonally projective* if whenever $\mathcal{K}_0$ and $\mathcal{K}_1$ are subquotients, $\Phi : \mathcal{K}_1 \to \mathcal{K}_0$ is a surjective coisometric module map, and $f : \mathcal{E} \to \mathcal{K}_0$ is a module map, then there exists a module map $F : \mathcal{E} \to \mathcal{K}_1$ such that $\|F\| = \|f\|$ and $f = \Phi \circ F$,

$$\begin{array}{ccc} & & \mathcal{K}_1 \\ & {\scriptstyle F} \nearrow & \downarrow {\scriptstyle \Phi} \\ \mathcal{E} & \xrightarrow{f} & \mathcal{K}_0 \end{array} .$$

This property was introduced in [13] under the name *hypoprojective*. Muhly and Solel renamed it *strongly orthogonally projective*, and introduced the weaker notion of *orthogonally projective*. The two notions coincide in our setting.

In Section 2 we describe a family of weighted Fock spaces, $\mathcal{F}^2(\omega_\alpha)$. In Section 3 we adapt a technique of Clancy and McCullough ([9]) to prove a Commutant Lifting Theorem for the weighted Fock spaces of Section 2. The proof of the Commutant Lifting Theorem is quite simple and short, and we use it to simplify the proof of the Nevanlinna-Pick Interpolation Theorem of [4]. We mention here that this noncommutative interpolation result implies commutative Nevanlinna-Pick theorems of Agler, Quiggen, and McCullough, and the noncommutative theorems of [4], [5], and [12]. We refer to [4] for details.

In Section 4 we use the Poisson kernels to obtain explicit resolutions. We show that if $\mathcal{E}$ is a subquotient of $\mathcal{F}^2(\omega_\alpha) \otimes \ell_2$, there exists a strongly orthogonally projective subquotient P_1 and a surjective coisometric module map Φ_0 such that

$$P_1 \xrightarrow{\Phi_0} \mathcal{E} \longrightarrow 0.$$

If we repeat this process for $\text{Ker}\Phi_0$, which is a submodule of $\mathcal{F}^2(\omega_\alpha) \otimes \ell_2$, and continue indefinitely, we obtain a projective resolution for $\mathcal{E}$

$$\cdots \xrightarrow{\Phi_4} P_4 \xrightarrow{\Phi_3} P_3 \xrightarrow{\Phi_2} P_2 \xrightarrow{\Phi_1} P_1 \xrightarrow{\Phi_0} \mathcal{E} \longrightarrow 0.$$

We can take the projective resolution of $\mathcal{E}$ to be minimal, and then we prove in Section 6 that any two minimal projective resolutions of $\mathcal{E}$ are unitarily equivalent.

If $\mathcal{E}$ and $\mathcal{F}$ are subquotients of $\mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ with projective resolutions (P_i) and (Q_i) respectively, and $f : \mathcal{E} \to \mathcal{F}$ is a module map

$$\begin{array}{ccccccccccccc} \cdots & \xrightarrow{\Phi_4} & P_4 & \xrightarrow{\Phi_3} & P_3 & \xrightarrow{\Phi_2} & P_2 & \xrightarrow{\Phi_1} & P_1 & \xrightarrow{\Phi_0} & \mathcal{E} & \longrightarrow & 0 \\ & & & & & & & & & & \downarrow f & & \\ \cdots & \xrightarrow{\Psi_4} & Q_4 & \xrightarrow{\Psi_3} & Q_3 & \xrightarrow{\Psi_2} & Q_2 & \xrightarrow{\Psi_1} & Q_1 & \xrightarrow{\Psi_0} & \mathcal{F} & \longrightarrow & 0 \end{array} ,$$

then there exists a family of module maps $f_i : P_i \to Q_i$ that commute with the maps of the above diagram.

In the Full Fock space we can take minimal projective resolutions of length two

$$0 \longrightarrow P_2 \xrightarrow{\Phi_1} P_1 \xrightarrow{\Phi_0} \mathcal{E} \longrightarrow 0.$$

This reformulates some aspects of noncommutative dilation theory for C_0 row contractions, which are always isomorphic to $*$-submodules of $\mathcal{F}^2 \otimes \ell_2$. The map Φ_0 is the adjoint of the minimal isometric dilation (see [14], [8], and [22]) and Φ_1

is Popescu's characteristic function of $\mathcal{E}$ ([21]). We use this as a guide to prove that the module map $\Phi_1 : P_2 \to P_1$ of the minimal projective resolution

$$\cdots \xrightarrow{\Phi_4} P_4 \xrightarrow{\Phi_3} P_3 \xrightarrow{\Phi_2} P_2 \xrightarrow{\Phi_1} P_1 \xrightarrow{\Phi_0} \mathcal{E} \longrightarrow 0$$

of $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha)$ is a unitary invariant of $\mathcal{E}$. We call Φ_1 the "characteristic function" of $\mathcal{E}$, although it is not an isometry when $\mathcal{F}^2(\omega_\alpha)$ is not the Full Fock space.

A strongly orthogonally projective resolution of a subquotient $\mathcal{E}$ of $\mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ has the form

$$\cdots \xrightarrow{\Phi_3} \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_2 \xrightarrow{\Phi_2} \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1 \xrightarrow{\Phi_1} \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_0 \xrightarrow{\Phi_0} \mathcal{E} \longrightarrow 0$$

for some $\mathcal{H}_i \subset \ell_2, i \in \mathbb{N}$. This resolution induces a natural complex

$$\cdots \xrightarrow{\Psi_4} \mathcal{H}_3 \xrightarrow{\Psi_3} \mathcal{H}_2 \xrightarrow{\Psi_2} \mathcal{H}_1 \xrightarrow{\Psi_1} \mathcal{H}_0.$$

In Section 6 we follow ideas of Greene ([15]) to complete this complex into the following commutative diagram for which all columns, except perhaps the last one, and the first two rows are exact:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\widehat{\Phi}_3} & Q_2 & \xrightarrow{\widehat{\Phi}_2} & Q_1 & \xrightarrow{\widehat{\Phi}_1} & Q_0 & \xrightarrow{\widehat{\Phi}_0} & \overline{\mathcal{E}} & \longrightarrow & 0 \\
 & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\
\cdots & \xrightarrow{\Phi_3} & \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_2 & \xrightarrow{\Phi_2} & \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1 & \xrightarrow{\Phi_1} & \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_0 & \xrightarrow{\Phi_0} & \mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & & & \\
\cdots & \xrightarrow{\Psi_3} & \mathcal{H}_2 & \xrightarrow{\Psi_2} & \mathcal{H}_1 & \xrightarrow{\Psi_1} & \mathcal{H}_0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array} \cdot$$

A standard argument identifies the homology of the last row with the homology of the last column, and in particular we obtain that $\text{Im}\Psi_{i+1} = \text{Ker}\Psi_i$ for $i = 2, 3, \ldots$.

In Section 5 we use the Poisson kernels to study quotients of Fock spaces. Our main applications are for quotients of the Full Fock space $\mathcal{F}^2 = \mathcal{F}^2(\ell_2^n)$. In 1995, Popescu [29] found a very simple proof of his noncommutative von Neumann inequality [24], which can be reformulated in the following way: If $\mathcal{E}$ is a subquotient of $\mathcal{F}^2 \otimes \ell_2$, there exists an isometry

$$K : \mathcal{E} \to \mathcal{F}^2 \otimes \mathcal{E} \quad \text{such that} \quad K^* : \mathcal{F}^2 \otimes \mathcal{E} \to \mathcal{E} \quad \text{is a module map.}$$

Since the formula of K resembles the formula of the classical Poisson kernel, Popescu named it the Poisson kernel of $\mathcal{E}$. This map was rediscovered by Arveson in his study of the d-shifts ([6]). If $\mathcal{E}$ is a subquotient of the symmetric Fock space $\mathcal{F}_+^2 = \mathcal{F}_+^2(\ell_2^d)$ (Arveson called this space $\mathcal{H}_d^2$), then $\mathcal{E}$ is also a subquotient of $\mathcal{F}^2(\ell_2^d)$, and hence it has a Poisson kernel $K : \mathcal{E} \to \mathcal{F}^2 \otimes \mathcal{E}$. Arveson [6] made the key observation that this map satisfies

$$K(\mathcal{E}) \subset \mathcal{F}_+^2 \otimes \mathcal{E} \quad \text{and hence} \quad K^*|_{\mathcal{F}_+^2 \otimes \mathcal{E}} : \mathcal{F}_+^2 \otimes \mathcal{E} \to \mathcal{E} \quad \text{is a module map.}$$

In [5] we noticed that this works for arbitrary quotients of F^∞ (the WOT-closed algebra generated by the left creation operators on the Full Fock space) and not just for the symmetric Fock space (which corresponds to the quotient of F^∞ by the commutator ideal). In [5] we used this observation to find simple representations of quotients of F^∞ (see Theorem 1.2 below).

This applies to reproducing kernel Hilbert spaces (RKHS) with Nevanlinna-Pick property. Using the work of Quiggin ([32]), Agler and McCarthy ([2]) proved that an irreducible RKHS with the Nevanlinna-Pick property H_K is isomorphic to a $*$-submodule of $\mathcal{H}^2_d = \mathcal{F}^2_+(\ell^d_2)$ for some $d = 1, 2, \dots, \infty$. Since $\mathcal{F}^2_+$ is a $*$-submodule of $\mathcal{F}^2$, it follows that a subquotient $\mathcal{E}$ of $\mathcal{H}_K \otimes \ell_2$ is also a subquotient of $\mathcal{F}^2 = \mathcal{F}^2(\ell^d_2)$ and hence it has a Poisson kernel $K : \mathcal{E} \to \mathcal{F}^2 \otimes \mathcal{E}$. Since H_K corresponds to a quotient of F^∞, we have that

$$K(\mathcal{E}) \subset H_K \otimes \mathcal{E} \quad \text{and hence} \quad K^*|_{H_K\otimes\mathcal{E}} : H_K \otimes \mathcal{E} \to \mathcal{E} \quad \text{is a module map.}$$

In Section 6 we use the map $K^*|_{H_K\otimes\mathcal{E}}$ to give a very transparent proof of a theorem of McCullough and Trent ([17]) and to prove that their conjecture is true. In Section 5 we prove that if a subquotient of $\mathcal{H}^2_d$ is not free (in the language of [7]), then all of their free resolutions have infinite length.

The Poisson kernels are related to the following problem of representation of quotient algebras. Let $(\mathcal{H}; L_1, \dots, L_n)$ be a Hilbert module. Assume that A, the w^*-closure of the algebra generated by $L_1, \dots, L_n$ and the identity $I_\mathcal{H}$, is an algebra, that $J \subset A$ is a w^*-closed 2-sided ideal, and that $\mathcal{N}_J \subset \mathcal{H}$ is the orthogonal complement of the subspace generated by the image of J in $\mathcal{H}$.

Problem 1.1. *When is the completely contractive representation $\Phi : A/J \to B(\mathcal{N}_J)$, defined by $\Phi(a + J) = P_{\mathcal{N}_J} a|_{\mathcal{N}_J}$, completely isometric?*

The motivation for this problem comes from [32] (see also [4], [5], and [13]). The solution of this problem follows easily from the existence of projective resolutions. The proof below is not new. It appeared in [4] and [5]. However, the proof is more transparent in this setting, and it includes one of the two steps we use to study quotients of weighted Fock spaces.

Theorem 1.2. *Let $(\mathcal{H}; L_1, \dots, L_n)$ be a Hilbert module with the property that for every $\mathcal{E} \subset \mathcal{H}\otimes\ell_2$ semi-invariant under $L_i \otimes I_{\ell_2}$, $i \leqslant n$, there exists a surjective coisometric module map $F : \mathcal{H} \otimes \ell_2 \to \mathcal{E}$. Then the map $\Phi : A/J \to B(\mathcal{N}_J)$ of Problem* 1.1 *is a complete isometry.*

Proof. Let $Q : A \to A/J$ be the quotient map. By Proposition 5.1 of [5] there exists a semi-invariant subspace $\mathcal{E} \subset \mathcal{H} \otimes \ell_2$ such that $\widehat{\Psi} : A/J \to B(\mathcal{E})$ defined by $\widehat{\Psi}(a + J) = P_\mathcal{E}(a \otimes I_{\ell_2})|_\mathcal{E}$ is a completely isometric representation. Since $F(L_i \otimes I_{\ell_2}) = P_\mathcal{E}(L_i \otimes I_{\ell_2})|_\mathcal{E} F = [\widehat{\Psi} \circ Q(L_i)]F$ for $i \leqslant n$ and A is the w^*-closure of the span of the products of the L_i's and the identity, we get that for every $a \in A$,

$$F(a \otimes I_\mathcal{E}) = [\widehat{\Psi} \circ Q(a)]F \quad \text{and hence} \quad F(a \otimes I_\mathcal{E})F^* = \widehat{\Psi} \circ Q(a).$$

We now check that $F^*(\mathcal{E}) \subset \mathcal{N}_J \otimes \ell_2$. Let $x_1 \in \mathcal{E}$, $b \in J$, $h \in H$, and $x_2 \in \ell_2$. Then $\langle F^*x_1, bh \otimes x_2\rangle = \langle F^*x_1, (b \otimes I_{\ell_2})(h \otimes x_2)\rangle = \langle x_1, F(b \otimes I_{\ell_2})(h \otimes x_2)\rangle = \langle x_1, [\widehat{\Psi} \circ Q(b)]F(h \otimes x_2)\rangle = 0$, because $Qb = 0$. Then for every $a \in A$,

$$F\big(P_{\mathcal{N}_J} a|_{\mathcal{N}_J} \otimes I_\mathcal{E}\big)F^* = \widehat{\Psi}\circ Q(a), \quad \text{and therefore} \quad F(\Phi(a+J)\otimes I_\mathcal{E})F^* = \widehat{\Psi}(a+J).$$

Since Φ is completely contractive, $\widehat{\Psi}$ is completely isometric, and F^* is an isometry, we conclude that Φ is completely isometric. ∎

2. WEIGHTED FOCK SPACES

In this section we give a unitarily equivalent description of the weighted Fock spaces of [4]. The second and third conditions of the weights are based on Quiggin's paper [31] and are chosen to obtain the more transparent conditions (2.1) and (2.2). We only use condition (ω_2) to estimate the norm of some maps in Lemma 2.1. But besides this, all other computations use only (2.1) and (2.2). Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \dots, g_n$ and unit e. We choose weights $(\omega_\alpha)_{\alpha\in\mathbb{F}_n^+}$ satisfying conditions $(\omega_1), (\omega_2)$ and (ω_3) which are listed below. A reader who wants to avoid the technical conditions can assume first that all the weights satisfy $\omega_\alpha = 1$ for every $\alpha \in \mathbb{F}_n^+$ (this corresponds to the Full Fock space), or that $\omega_\alpha = |\alpha| + 1$, where $|\alpha| = k$ if α is a word in $\mathbb{F}_n^+$ of length k (this corresponds to the Dirichlet algebra). The first condition is

$$(\omega_1) \qquad \omega_\alpha > 0 \quad \text{for every } \alpha \in \mathbb{F}_n^+ \text{ and } \omega_0 = 1,$$

where ω_0 is the weight associated to the identity of $\mathbb{F}_n^+$. Let $\mathcal{F}^2(\omega_\alpha)$ be the Hilbert space with a complete orthogonal system $\{\delta_\alpha : \alpha \in \mathbb{F}_n^+\}$ and with $\langle \delta_\alpha, \delta_\alpha \rangle = \omega_\alpha$. For $i \leqslant n$, define the left creation operator L_i and the right creation operator R_i on $\mathcal{F}^2(\omega_\alpha)$ by $L_i\delta_\alpha = \delta_{g_i\alpha}$ and $R_i\delta_\alpha = \delta_{\alpha g_i}$. The second condition on the weights:

$$(\omega_2) \qquad \frac{\omega_{g_i\alpha g_j}}{\omega_{\alpha g_j}} \leqslant \frac{\omega_{g_i\alpha}}{\omega_\alpha} \qquad \text{for all } i, j \leqslant n \text{ and } \alpha \in \mathbb{F}_n^+,$$

implies that the maps L_i and R_i are bounded. Indeed, it is very easy to check that with this condition, $\|L_i\| = \|L_i\delta_0\| = \sqrt{\omega_i}$ and $\|R_i\| = \|R_i\delta_0\| = \sqrt{\omega_i}$ (see Lemma 2.1). We look at the algebra generated by the L_i's, at the algebra generated by the R_i's, and use $\mathbb{F}_n^+$ to index products in the usual way. That is, if $W_1, \dots, W_n$ are bounded linear operators on a Hilbert space $\mathcal{H}$ and $\lambda_1, \dots, \lambda_n$ are complex numbers, we set

$$W_\alpha := \begin{cases} W_{i_1}W_{i_2}\cdots W_{i_k} & \text{if } \alpha = g_{i_1}\cdots g_{i_k}, \\ I_{\mathcal{H}} & \text{if } \alpha = e; \end{cases}$$

$$\lambda_\alpha := \begin{cases} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k} & \text{if } \alpha = g_{i_1}\cdots g_{i_k}, \\ 1 & \text{if } \alpha = e. \end{cases}$$

Then we have that $L_\alpha L_\beta = L_{\alpha\beta}$, $R_\alpha R_\beta = R_{\alpha\beta}$, and $\lambda_\alpha\lambda_\beta = \lambda_{\alpha\beta}$. The last condition we impose on the weights is an invertibility condition. For every $(\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n$, the unit ball of $\mathbb{C}^n$, we want the operator $\sum_{\alpha\in\mathbb{F}_n^+} \frac{\lambda_\alpha}{\omega_\alpha} L_\alpha$ to be invertible. More precisely, we require that there exists $(a_\alpha)_{\alpha\in\mathbb{F}_n^+}$ such that for every $(\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n$,

$$(\omega_3) \qquad \Big(\sum_{\alpha\in\mathbb{F}_n^+} \frac{\lambda_\alpha}{\omega_\alpha} L_\alpha\Big)^{-1} = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha\lambda_\alpha L_\alpha.$$

From (ω_3) we have $\Big(\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha\lambda_\alpha L_\alpha\Big)\Big(\sum_{\beta\in\mathbb{F}_n^+} \frac{\lambda_\beta}{\omega_\beta} L_\beta\Big) = \sum_{\gamma\in\mathbb{F}_n^+}\Big(\sum_{\alpha\beta=\gamma} \frac{a_\alpha}{\omega_\beta}\Big)\lambda_\gamma L_\gamma = I$ and $\Big(\sum_{\beta\in\mathbb{F}_n^+} \frac{\lambda_\beta}{\omega_\beta} L_\beta\Big)\Big(\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha\lambda_\alpha L_\alpha\Big) = \sum_{\gamma\in\mathbb{F}_n^+}\Big(\sum_{\beta\alpha=\gamma} \frac{a_\alpha}{\omega_\beta}\Big)\lambda_\gamma L_\gamma = I$. Hence

$$(2.1) \qquad a_0 = 1 \quad \text{and for every } |\gamma| \geqslant 1, \quad \sum_{\alpha\beta=\gamma} \frac{a_\alpha}{\omega_\beta} = 0 \quad \text{and} \quad \sum_{\beta\alpha=\gamma} \frac{a_\alpha}{\omega_\beta} = 0.$$

Recall that $|\gamma|$ denotes the number of letters of the word $\gamma \in \mathbb{F}_n^+$. That is, $|\gamma| = k$ if $\gamma = g_{i_1} g_{i_2} \cdots g_{i_k}$ and $|\gamma| = 0$ if γ is the identity of $\mathbb{F}_n^+$.

An easy induction step, see [4], gives that

$$a_0 = 1 \quad \text{and} \quad a_\alpha \leqslant 0 \quad \text{for } |\alpha| \geqslant 1. \tag{2.2}$$

Since most computations use this condition, we will sketch its proof: It is clear that $a_0 = 1$. Since $\frac{a_0}{\omega_{g_i}} + \frac{a_{g_i}}{\omega_0} = 0$ for $i \leqslant n$, then $a_{g_i} \leqslant 0$. Assume that $a_\alpha \leqslant 0$ for $|\alpha| \leqslant k$. Let $\gamma = \beta g_i$ with $|\beta| = k$. Since $\sum_{\alpha\sigma=\beta} \frac{a_\alpha}{\omega_\sigma} = 0$, $a_0 = -\sum_{\substack{\alpha\sigma=\beta \\ |\alpha|\geqslant 1}} \frac{\omega_\beta}{\omega_\sigma} a_\alpha$. Condition (ω_2) implies that $\frac{\omega_\beta}{\omega_\sigma} \geqslant \frac{\omega_{\beta g_i}}{\omega_{\sigma g_i}}$. Hence, $a_0 \geqslant -\sum_{\substack{\alpha\varepsilon=\gamma \\ |\alpha|\geqslant 1 \\ |\varepsilon|\geqslant 1}} \frac{\omega_\gamma}{\omega_\varepsilon} a_\alpha$. And this implies that $\sum_{\substack{\alpha\varepsilon=\gamma \\ |\varepsilon|\geqslant 1}} \frac{a_\alpha}{\omega_\varepsilon} \geqslant 0$. Since $0 = \sum_{\alpha\varepsilon=\gamma} \frac{a_\alpha}{\omega_\varepsilon} = \left[\sum_{\substack{\alpha\varepsilon=\gamma \\ |\varepsilon|\geqslant 1}} \frac{a_\alpha}{\omega_\varepsilon}\right] + \frac{a_\gamma}{\omega_0}$, we conclude that $a_\gamma \leqslant 0$.

It is easy to check that $L_\alpha^* \delta_\beta = \frac{\omega_\beta}{\omega_\gamma} \delta_\gamma$ if $\beta = \alpha\gamma$ and is zero otherwise. Then

$$L_\alpha L_\alpha^* \delta_\beta = \begin{cases} \frac{\omega_\beta}{\omega_\gamma} \delta_\beta & \text{if } \beta = \alpha\gamma, \\ 0 & \text{otherwise}. \end{cases}$$

From here it follows that the sequence

$$\Big(\sum_{|\alpha|\leqslant N} a_\alpha L_\alpha L_\alpha^*\Big)_{N\in\mathbb{N}} \text{ is nonnegative, nonincreasing, and } \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha L_\alpha L_\alpha^* = P_0, \tag{2.3}$$

where P_0 is the orthogonal projection onto the span of δ_0. Indeed, for any $\beta \in \mathbb{F}_n^+$, $\sum_{|\alpha|\leqslant N} a_\alpha L_\alpha L_\alpha^* \delta_\beta = b_{\beta,N} \delta_\beta$ where $b_{\beta,N} = \sum_{\substack{\alpha\gamma=\beta \\ |\alpha|\leqslant N}} \frac{a_\alpha}{\omega_\gamma} \omega_\beta$. Since $\sum_{\alpha\gamma=\beta} \frac{a_\alpha}{\omega_\gamma} = 0$, and since $a_\alpha \leqslant 0$ for $|\alpha| \geqslant 1$, it follows that the sequence $(b_{\beta,N})_{N\in\mathbb{N}}$ is nonnegative, non increasing, and converges to zero. Actually its terms are zero when $N > |\beta|$.

We obtain a similar result for the maps $R_i, i \leqslant n$. Notice that $R_\alpha \delta_\beta = \delta_{\beta\tilde{\alpha}}$, where

$$\widetilde{\alpha} = g_{i_k} g_{i_{k-1}} \cdots g_{i_2} g_{i_1} \quad \text{if } \alpha = g_{i_1} g_{i_2} \cdots g_{i_k}.$$

We easily check that

$$R_\alpha^* \delta_\beta = \begin{cases} \frac{\omega_\beta}{\omega_\gamma} \delta_\gamma & \text{if } \beta = \gamma\widetilde{\alpha}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

And then we get that $R_\alpha R_\alpha^* \delta_\beta = \frac{\omega_\beta}{\omega_\gamma} \delta_\beta$ if $\beta = \gamma\widetilde{\alpha}$ and is zero otherwise. Arguing as in (2.3), we obtain

$$\Big(\sum_{|\alpha|\leqslant N} a_{\tilde{\alpha}} R_\alpha R_\alpha^*\Big)_{N\in\mathbb{N}} \text{is nonnegative, non increasing, and } \sum_{\alpha\in\mathbb{F}_n^+} a_{\tilde{\alpha}} R_\alpha R_\alpha^* = P_0. \tag{2.5}$$

We need the following in Section 4.

Lemma 2.1. *For every* $\alpha \in \mathbb{F}_n^+$, $\|L_\alpha\| = \|L_\alpha(\delta_0)\| = \sqrt{\omega_\alpha}$.

Proof. Since L_α maps the orthogonal basis $\{\delta_\beta : \beta \in \mathbb{F}_n^+\}$ to an orthogonal set, it is enough to check that for every $\alpha \in \mathbb{F}_n^+$, $\|L_\alpha \frac{\delta_\beta}{\sqrt{\omega_\beta}}\| \leqslant \sqrt{\omega_\alpha}$. That is, we need to prove that for every $\alpha, \beta \in \mathbb{F}_n^+$, $\frac{\omega_{\alpha\beta}}{\omega_\beta} \leqslant \omega_\alpha$. Suppose that $\alpha = g_{i_1} \cdots g_{i_k}$ and that $\beta = g_{j_1} \cdots g_{j_l}$. To apply condition (ω_2), we first write $\frac{\omega_{\alpha\beta}}{\omega_\beta}$ as a product of k terms, and then we apply (ω_2) to each term

$$\begin{aligned} \frac{\omega_{\alpha\beta}}{\omega_\beta} &= \frac{\omega_{\alpha\beta}}{\omega_{g_{i_2}\cdots g_{i_k}\beta}} \frac{\omega_{g_{i_2}\cdots g_{i_k}\beta}}{\omega_{g_{i_3}\cdots g_{i_k}\beta}} \cdots \frac{\omega_{g_{i_k}\beta}}{\omega_\beta} \\ &\leqslant \frac{\omega_{\alpha g_{j_1}\cdots g_{j_{l-1}}}}{\omega_{g_{i_2}\cdots g_{i_k} g_{j_1}\cdots g_{j_{l-1}}}} \frac{\omega_{g_{i_2}\cdots g_{i_k} g_{j_1}\cdots g_{j_{l-1}}}}{\omega_{g_{i_3}\cdots g_{i_k} g_{j_1}\cdots g_{j_{l-1}}}} \cdots \frac{\omega_{g_{i_k} g_{j_1}\cdots g_{j_{l-1}}}}{\omega_{g_{j_1}\cdots g_{j_{l-1}}}} = \frac{\omega_{\alpha g_{j_1}\cdots g_{j_{l-1}}}}{\omega_{g_{j_1}\cdots g_{j_{l-1}}}}. \end{aligned}$$

By iterating this inequality $l-1$ times, we conclude that $\frac{\omega_{\alpha\beta}}{\omega_\beta} \leqslant \frac{\omega_\alpha}{\omega_0} = \omega_\alpha$. ∎

3. Commutant lifting theorem and interpolation problems

The Hilbert module $(\mathcal{H}, V_1, \ldots, V_n)$ is *orthogonally projective* in the category of its $*$-submodules if whenever $\mathcal{K}$ is a $*$-submodule of $\mathcal{H}$ and $f : \mathcal{H} \to \mathcal{K}$ is a module map, there exists a module map $F : \mathcal{H} \to \mathcal{K}$ such that $\|F\| = \|f\|$ and $f = P_\mathcal{K} \circ F$.

$$\begin{array}{ccc} & & \mathcal{H} \\ & {\scriptstyle F} \nearrow & \downarrow {\scriptstyle P_\mathcal{K}} \\ \mathcal{H} & \xrightarrow{f} & \mathcal{K} \end{array}.$$

Theorem 3.1. *For every Hilbert space* $\mathcal{H}$, $(\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}; L_1\otimes I_\mathcal{H}, \ldots, L_n\otimes I_\mathcal{H})$ *is orthogonally projective in the category of* $*$*-submodules of* $(\mathcal{F}^2(\omega_\alpha) \otimes \ell_2; L_1 \otimes I_{\ell_2}, \ldots, L_n \otimes I_{\ell_2})$.

Theorem 3.2. *For every Hilbert space* $\mathcal{H}$, $(\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}; R_1 \otimes I_\mathcal{H}, \ldots, R_n \otimes I_\mathcal{H})$ *is orthogonally projective in the category of* $*$*-submodules of* $(\mathcal{F}^2(\omega_\alpha)\otimes\ell_2; R_1\otimes I_{\ell_2}, \ldots, R_n \otimes I_{\ell_2})$.

When $n = 1$ and L_1 is a pure isometry, these theorems are proved in [32] and [33], when $n > 1$ and $\omega_\alpha = 1$, they are proved in [25], and when $n = 1$, they are proved in [9]. We adapt the proof of [9] to our setting.

We will give a detailed proof of Theorem 3.1 and provide the modifications for the proof of Theorem 3.2. To simplify notation, we say that $\mathcal{M} \subset \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}$ is a $*$-L-submodule of $\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}$ if $(\mathcal{M}; V_1, \ldots, V_n)$ is a $*$-submodule of $(\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}; L_1 \otimes I_\mathcal{H}, \ldots, L_n \otimes I_\mathcal{H})$. We say that $f : \mathcal{M} \to \mathcal{N}$ is an L-module map if $\mathcal{N}, \mathcal{M}$ are $*$-L-submodules of $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$ and f is a module map in the category of $*$-submodules of $(\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}; L_1 \otimes I_\mathcal{H}, \ldots, L_n \otimes I_\mathcal{H})$. Similarly, we define $*$-R-submodules and R-module maps.

Let $(\mathcal{M}; V_1, \ldots, V_n)$ be a $*$-L-submodule of $\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}$ and let $f : \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H} \to \mathcal{M}$ be an L-module map. For each $\alpha \in \mathbb{F}_n^+$, define

$$f_\alpha : \mathcal{H} \to \mathcal{M} \quad \text{by} \quad f_\alpha(x) = f(\delta_\alpha \otimes x).$$

Then each f_α, and hence f, is determined by $f_0 : \mathcal{H} \to \mathcal{M}$. Indeed,

$$f_\alpha(x) = f(\delta_\alpha \otimes x) = f\big((L_\alpha \otimes I_{\mathcal{H}})(\delta_0 \otimes x)\big) = V_\alpha f(\delta_0 \otimes x) = V_\alpha f_0(x). \tag{3.1}$$

One easily checks that $f^* : \mathcal{M} \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$ has the form $f^*(x) = \sum_{\alpha \in \mathbb{F}_n^*} \delta_\alpha \otimes \frac{f_\alpha^*(x)}{\omega_\alpha}$. Then, since $f_\alpha = V_\alpha f_0$,

$$\|f\| \leqslant 1 \quad \Leftrightarrow \quad ff^* \leqslant I \quad \Leftrightarrow \quad \sum_{\alpha \in \mathbb{F}_n^*} \frac{1}{\omega_\alpha} V_\alpha f_0 f_0^* V_\alpha^* \leqslant I. \tag{3.2}$$

Summarizing, we have that $f : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} \to \mathcal{M}$ is a module map if and only if there exists an $f_0 : \mathcal{H} \to \mathcal{M}$ such that $f^* : \mathcal{M} \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$ is given by

$$f^*(x) = \sum_{\alpha \in \mathbb{F}_n^+} \frac{\delta_\alpha}{\omega_\alpha} \otimes f_0^* V_\alpha^*(x). \tag{3.3}$$

Moreover, $\|f\| \leqslant C$ if and only if $\sum_{\alpha \in \mathbb{F}_n^+} \frac{1}{\omega_\alpha} V_\alpha f_0 f_0^* V_\alpha^* \leqslant C^2$.

Proof of Theorem 3.1. Let $(\mathcal{M}; V_1, \ldots, V_n)$ be a $*$-L-submodule of $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$ and let $f : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} \to \mathcal{M}$ be an L-module map with $\|f\| = 1$. We will find a $*$-L-submodule $\mathcal{N}$ containing $\mathcal{M}$ and an L-module map $F : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} \to \mathcal{N}$ such that $\|F\| = \|f\|$ and $f = P_{\mathcal{M}} \circ F$. By iterating this process, or by applying a maximality argument, we finish the proof.

If $\delta_0 \otimes \mathcal{H} \not\subset \mathcal{M}$, let $\mathcal{N}$ be the closure of $\delta_0 \otimes \mathcal{H} + \mathcal{M}$. If $\delta_0 \otimes \mathcal{H} \subset \mathcal{M}$, find $\alpha \in \mathbb{F}_n^+$ such that $\delta_\alpha \otimes \mathcal{H} \not\subset \mathcal{M}$ but such that if $\alpha = \beta\gamma$ and $|\beta| \geqslant 1$ then $\delta_\gamma \otimes \mathcal{H} \subset \mathcal{M}$. Then let $\mathcal{N}$ be the closure of $\delta_\alpha \otimes \mathcal{H} + \mathcal{M}$. In either case

$$(L_\alpha \otimes I_{\mathcal{H}})^* \mathcal{N} \subset \mathcal{M} \quad \text{if } |\alpha| \geqslant 1. \tag{3.4}$$

Let $W_i = P_{\mathcal{N}}(L_i \otimes I_{\mathcal{H}})P_{\mathcal{N}}$ for $i \leqslant n$. Then $(\mathcal{N}; W_1, \ldots, W_n)$ is a $*$-L-submodule of $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$, and $\mathcal{M}$ is a $*$-L-submodule of $\mathcal{N}$. It follows from (3.4) that

$$W_\alpha P_{\mathcal{N} \ominus \mathcal{M}} = 0 \quad \text{if } |\alpha| \geqslant 1. \tag{3.5}$$

and that

$$W_{\alpha\beta} P_{\mathcal{M}} = W_\alpha V_\beta P_{\mathcal{M}}. \tag{3.6}$$

Indeed, $W_{\alpha\beta}P_{\mathcal{M}} - W_\alpha V_\beta P_{\mathcal{M}} = W_\alpha(W_\beta P_{\mathcal{M}} - V_\beta P_{\mathcal{M}}) = W_\alpha(P_{\mathcal{N}}(L_\beta \otimes I_{\mathcal{H}})P_{\mathcal{M}} - P_{\mathcal{M}}(L_\beta \otimes I_{\mathcal{H}})P_{\mathcal{M}}) = W_\alpha(P_{\mathcal{N}} - P_{\mathcal{M}})(L_\beta \otimes I_{\mathcal{H}})P_{\mathcal{M}} = W_\alpha P_{\mathcal{N}\ominus\mathcal{M}}(L_\beta \otimes I_{\mathcal{H}})P_{\mathcal{M}} = 0$.

The goal now is to find an L-module map $F : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} \to \mathcal{N}$ satisfying $\|F\| = \|f\|$ and $P_{\mathcal{M}}F = f$. Like all L-module maps, F will be determined by $F_0 : \mathcal{H} \to \mathcal{M}$. Recall from (3.1) that $F_\alpha = W_\alpha F_0$ for any $\alpha \in \mathbb{F}_n^+$. However, it follows from (3.5) that for $|\alpha| \geqslant 1$, F_α is already determined by f:

$$F_\alpha h = W_\alpha F_0 h = W_\alpha P_{\mathcal{N}\ominus\mathcal{M}} F_0 h + W_\alpha P_{\mathcal{M}} F_0 h = W_\alpha P_{\mathcal{M}} F_0 h = W_\alpha f_0 h.$$

Decompose $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} = [\delta_0 \otimes \mathcal{H}] \oplus [\delta_0 \otimes \mathcal{H}]^\perp$ and $\mathcal{N} = [\mathcal{N} \ominus \mathcal{M}] \oplus \mathcal{M}$ and write F as a block matrix with respect to this decomposition. That is,

$$F = \begin{bmatrix} g_0 & a \\ b & c \end{bmatrix}.$$

Since $f = P_{\mathcal{M}}F = [b\ c]$, the second row of F is already determined and $\|[b\ c]\| \leqslant 1$. The second column of F is also already determined. Indeed, it is easy to see that

$$\begin{bmatrix} a \\ c \end{bmatrix}^* = \big(F_{|[\delta_0 \otimes \mathcal{H}]|^\perp}\big)^* : \mathcal{N} \to \bigoplus_{|\alpha|\geqslant 1} \delta_\alpha \otimes \mathcal{H} \quad \text{is given by } \sum_{|\alpha|\geqslant 1} \delta_\alpha \otimes \frac{F_\alpha^*(x)}{\omega_\alpha}.$$

Claim 3.3. $\sum\limits_{|\alpha|\geqslant 1} \frac{1}{\omega_\alpha} W_\alpha f_0 f_0^* W_\alpha^* \leqslant I.$

Since $F_\alpha = W_\alpha f_0$ for $|\alpha| \geqslant 1$, it follows that the Claim 3.3 is equivalent to $\|[\begin{smallmatrix} a \\ c \end{smallmatrix}]\| \leqslant 1$. Once we prove the Claim 3.3, we use Parrott's Lemma (see [20]) to find g_0 such that $\|F\| = 1$. Then we can find a module map $F : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} \to \mathcal{N}$ such that $\|F\| = \|f\|$ and $f = P_{\mathcal{M}} \circ F$, which is what we need to iterate the process and finish the proof.

It remains to prove the Claim 3.3. It follows from (2.3) that $\sum\limits_{\alpha\in\mathbb{F}_n^+} a_\alpha (L_\alpha \otimes I_{\mathcal{H}})(L_\alpha \otimes I_{\mathcal{H}})^* = P_{\delta_0\otimes\mathcal{H}} \geqslant 0$, where $P_{\delta_0\otimes\mathcal{H}}$ is the orthogonal projection onto $\delta_0 \otimes \mathcal{H}$. Since $a_\alpha \leqslant 0$ for $|\alpha| \geqslant 1$ and $a_0 = 1$, then $I \geqslant \sum\limits_{|\alpha|\geqslant 1} -a_\alpha (L_\alpha \otimes I_{\mathcal{H}})(L_\alpha \otimes I_{\mathcal{H}})^* \geqslant 0$. Hence

$$I \geqslant \sum_{|\alpha|\geqslant 1} -a_\alpha P_{\mathcal{N}} (L_\alpha \otimes I_{\mathcal{H}})(L_\alpha \otimes I_{\mathcal{H}})^* P_{\mathcal{N}} = \sum_{|\alpha|\geqslant 1} -a_\alpha W_\alpha W_\alpha^* \geqslant 0.$$

Recall from (3.2) that $\sum\limits_{\beta\in\mathbb{F}_n^+} \frac{1}{\omega_\beta} V_\beta f_0 f_0^* V_\beta^* \leqslant I$ because $f : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} \to \mathcal{M}$ is contractive. Then from (3.6) and (2.1) we get,

$$\begin{aligned} I &\geqslant \sum_{|\alpha|\geqslant 1} -a_\alpha W_\alpha W_\alpha^* \geqslant \sum_{|\alpha|\geqslant 1} -a_\alpha W_\alpha \Big(\sum_{\beta\in\mathbb{F}_n^+} \frac{1}{\omega_\beta} V_\beta f_0 f_0^* V_\beta^* \Big) W_\alpha^* \\ &= \sum_{|\alpha|\geqslant 1} \sum_{\beta\in\mathbb{F}_n^+} \frac{-a_\alpha}{\omega_\beta} W_\alpha V_\beta f_0 f_0^* V_\beta^* W_\alpha^* = \sum_{|\alpha|\geqslant 1} \sum_{\beta\in\mathbb{F}_n^+} \frac{-a_\alpha}{\omega_\beta} W_{\alpha\beta} f_0 f_0^* W_{\alpha\beta}^* \\ &= \sum_{|\gamma|\geqslant 1} \Big[\sum_{\substack{\alpha\beta=\gamma \\ |\alpha|\geqslant 1}} \frac{-a_\alpha}{\omega_\beta} \Big] W_\gamma f_0 f_0^* W_\gamma^* = \sum_{|\gamma|\geqslant 1} \frac{1}{\omega_\gamma} W_\gamma f_0 f_0^* W_\gamma^*, \end{aligned}$$

which is what we wanted to prove. The last equality follows from (2.1). Since $\sum\limits_{\alpha\beta=\gamma} \frac{-a_\alpha}{\omega_\beta} = 0$, then $\sum\limits_{\substack{\alpha\beta=\gamma \\ |\alpha|\geqslant 1}} \frac{-a_\alpha}{\omega_\beta} = \frac{a_0}{\omega_\gamma} = \frac{1}{\omega_\gamma}$. ∎

Sketch of the proof of Theorem 3.2. Let $(\mathcal{M}; V_1, \ldots, V_n)$ be a $*$-submodule of $(\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}; R_1 \otimes I_{\mathcal{H}}, \ldots, R_n \otimes I_{\mathcal{H}})$, and let $f : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} \to \mathcal{M}$ be an R-module map with $\|f\| = 1$. For each $\alpha \in \mathbb{F}_n^+$, define $f_\alpha : \mathcal{H} \to \mathcal{M}$ by $f_\alpha(x) = f(\delta_\alpha \otimes x)$. As before, f_α, and hence f, is determined by f_0, but we get now that $f_\alpha = V_{\tilde{\alpha}} f_0$. And since $\|f\| = 1$, we get $\sum\limits_{\alpha\in\mathbb{F}_n^+} V_{\tilde{\alpha}} f_0 f_0^* V_{\tilde{\alpha}}^* \leqslant I$.

We now choose a $*$-R-submodule $\mathcal{N}$ that contains $\mathcal{M}$. If $\delta_0 \otimes \mathcal{H} \not\subset \mathcal{M}$, let $\mathcal{N}$ be the closure of $\delta_0 \otimes \mathcal{H} + \mathcal{M}$. If $\delta_0 \otimes \mathcal{H} \subset \mathcal{M}$, find $\alpha \in \mathbb{F}_n^+$ such that $\delta_\alpha \otimes \mathcal{H} \not\subset \mathcal{M}$ but such that if $\alpha = \beta\gamma$ and $|\gamma| \geqslant 1$ then $\delta_\beta \otimes \mathcal{H} \subset \mathcal{M}$. Let $\mathcal{N}$ be the closure of

$\delta_\alpha \otimes \mathcal{H} + \mathcal{M}$. Define $W_i = P_{\mathcal{N}}(R_i \otimes I_{\mathcal{H}})P_{\mathcal{N}}$, and check that (3.4), (3.5), and (3.6) of the proof of Theorem 3.1 are satisfied.

We now want to define an R-module map $F : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} \to \mathcal{N}$ such that $\|F\| = \|f\|$ and $P_{\mathcal{M}} \circ F = f$. As before, F_α is already determined for $|\alpha| \geqslant 1$ and it is equal to $F_\alpha = W_{\tilde{\alpha}} f_0$. The decomposition of F into a 2×2 block matrix is identical to the one in the proof of Theorem 3.1. And the proof of Theorem 3.2 follows from the proof of the following

CLAIM 3.4. $\sum_{|\alpha|\geqslant 1} \frac{1}{\omega_\alpha} W_{\tilde{\alpha}} f_0 f_0^* W_{\tilde{\alpha}}^* \leqslant I.$

We have from (2.5) that $I \geqslant \sum_{|\alpha|\geqslant 1} (-a_{\tilde{\alpha}}(R_\alpha \otimes I_{\mathcal{H}})(R_\alpha \otimes I_{\mathcal{H}})^* \geqslant \sum_{|\alpha|\geqslant 1} -a_{\tilde{\alpha}} W_\alpha W_\alpha^*$. Then

$$
\begin{aligned}
I &\geqslant \sum_{|\alpha|\geqslant 1} -a_{\tilde{\alpha}} W_\alpha W_\alpha^* \geqslant \sum_{|\alpha|\geqslant 1} -a_{\tilde{\alpha}} W_\alpha \Big(\sum_{\beta \in \mathbb{F}_n^+} \frac{1}{\omega_\beta} V_{\tilde{\beta}} f_0 f_0^* V_{\tilde{\beta}}^* \Big) W_\alpha^* \\
&= \sum_{|\gamma|\geqslant 1} \Big[\sum_{\substack{\widetilde{\beta\alpha}=\gamma \\ |\alpha|\geqslant 1}} \frac{-a_{\tilde{\alpha}}}{\omega_\beta} \Big] W_{\tilde{\gamma}} f_0 f_0^* W_{\widetilde{\gamma}}^* = \sum_{|\gamma|\geqslant 1} \frac{1}{\omega_\gamma} W_{\tilde{\gamma}} f_0 f_0^* W_{\widetilde{\gamma}}^* ,
\end{aligned}
$$

which proves the claim. ∎

From Theorems 3.1 and 3.2 we immediately get

COROLLARY 3.5. *If $\mathcal{M}$ and $\mathcal{N}$ are $*$-L-submodules of $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$ and $T : \mathcal{M} \to \mathcal{N}$ is an L-module map, then there exists an L-module map $\widehat{T} : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$ such that $\|T\| = \|\widehat{T}\|$ and $TP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{T}$. And if $\mathcal{M}$ and $\mathcal{N}$ are $*$-R-submodules of $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$ and $T : \mathcal{M} \to \mathcal{N}$ is an R-module map, then there exists an R-module map $\widehat{T} : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$ such that $\|T\| = \|\widehat{T}\|$ and $TP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{T}$.*

Proof. Apply the theorems to the module map $TP_{\mathcal{M}}$. ∎

We will now describe the L-module maps and the R-module maps on $\mathcal{F}^2(\omega_\alpha)$. Notice that $T : \mathcal{F}^2(\omega_\alpha) \to \mathcal{F}^2(\omega_\alpha)$ is an L-module map if and only if T commutes with $L_1, \ldots, L_n$ and that $T : \mathcal{F}^2(\omega_\alpha) \to \mathcal{F}^2(\omega_\alpha)$ is an R-module map if and only if T commutes with $R_1, \ldots, R_n$.

There is a natural product on the set of δ_α's given by $\delta_\alpha \otimes \delta_\beta := \delta_{\alpha\beta}$ (the tensor product notation is used to emphasize the noncommutative nature of the product). Using this formula, we can take formal products of elements of $\mathcal{F}^2(\omega_\alpha)$, although the formal product does not have to belong to $\mathcal{F}^2(\omega_\alpha)$. Define

$$F^\infty(\omega_\alpha) = \{f \in \mathcal{F}^2(\omega_\alpha) : \forall g \in \mathcal{F}^2(\omega_\alpha), f \otimes g \in \mathcal{F}^2(\omega_\alpha)\},$$

with the norm

$$\|f\|_\infty = \sup\left\{\|f \otimes g\|_2 : g \in \mathcal{F}^2(\omega_\alpha), \|g\|_2 \leqslant 1\right\},$$

which is the norm of the left multiplication operator $L_f : \mathcal{F}^2(\omega_\alpha) \to \mathcal{F}^2(\omega_\alpha)$. If $f, g \in F^\infty(\omega_\alpha)$ then $f \otimes g \in F^\infty(\omega_\alpha)$ corresponds to the operator $L_{f\otimes g} = L_f \circ L_g$. Therefore, we view $F^\infty(\omega_\alpha)$ as a subalgebra of $B(\mathcal{F}^2(\omega_\alpha))$.

Similarly, we define $R^\infty(\omega_\alpha)$ as the set of $f \in \mathcal{F}^2(\omega_\alpha)$ such that $g \otimes f \in \mathcal{F}^2(\omega_\alpha)$ for every $g \in \mathcal{F}^2(\omega_\alpha)$, and we give $f \in R^\infty(\omega_\alpha)$ the norm of the right multiplication operator $R_f : \mathcal{F}^2(\omega_\alpha) \to \mathcal{F}^2(\omega_\alpha)$.

Proposition 3.6. *$T : \mathcal{F}^2(\omega_\alpha) \to \mathcal{F}^2(\omega_\alpha)$ is an L-module map if and only if there exists $g \in R^\infty(\omega_\alpha)$ such that $T = R_g$. And $T : \mathcal{F}^2(\omega_\alpha) \to \mathcal{F}^2(\omega_\alpha)$ is an R-module map if and only if there exists $g \in F^\infty(\omega_\alpha)$ such that $T = L_g$. Hence $F^\infty(\omega_\alpha)$ and $R^\infty(\omega_\alpha)$ are equal to their double commutant.*

Proof. Let $T : \mathcal{F}^2(\omega_\alpha) \to \mathcal{F}^2(\omega_\alpha)$ be an L-module map and set $f = T\delta_0$. Since $T\delta_\alpha = TL_\alpha\delta_0 = L_\alpha T\delta_0 = L_\alpha f = \delta_\alpha \otimes f$ it follows that for every $g \in \mathcal{F}^2(\omega_\alpha)$, $Tg = g \otimes f$. Hence $f \in R^\infty(\omega_\alpha)$ and $T = R_f$. Conversely, if $f \in R^\infty(\omega_\alpha)$, $L_iR_f\delta_\beta = L_i(\delta_\beta \otimes f) = \delta_{g_i\beta} \otimes f = R_f(\delta_{g_i\beta}) = R_fL_i(\delta_\beta)$. Hence R_f is an L-module map. The proof for R-module maps is identical. ∎

The characterization of the commutant of the left creation operators of the Full Fock space (i.e., when $\omega_\alpha = 1$ for all $\alpha \in \mathbb{F}_n^+$) appears in [26]. We are following that approach here. Proposition 3.6 extends to $*$-submodules of $\mathcal{F}^2(\omega_\alpha) \otimes \ell_2$. Recall that the commutant of $F^\infty(\omega_\alpha) \otimes I_{\ell_2}$ is $R^\infty(\omega_\alpha)\overline{\otimes}B(\mathcal{H})$ and that the commutant of $R^\infty(\omega_\alpha) \otimes I_{\ell_2}$ is $F^\infty(\omega_\alpha)\overline{\otimes}B(\mathcal{H})$. If $\mathcal{H}$ is k-dimensional, these spaces are $M_k(R^\infty(\omega_\alpha))$ and $M_k(F^\infty(\omega_\alpha))$.

Corollary 3.7. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. $T : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1 \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_2$ is an L-module map if and only if there exist operators $A_\alpha \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $T = \sum_{\alpha \in \mathbb{F}_n^+} R_\alpha \otimes A_\alpha \in R^\infty(\omega_\alpha)\overline{\otimes}B(\mathcal{H}_1, \mathcal{H}_2)$. And $T : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1 \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_2$ is an R-module map if and only if there exist operators $B_\alpha \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $T = \sum_{\alpha \in \mathbb{F}_n^+} L_\alpha \otimes B_\alpha \in F^\infty(\omega_\alpha)\overline{\otimes}B(\mathcal{H}_1, \mathcal{H}_2)$.*

Sketch of the proof. An L-module map $T : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1 \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_2$ is determined by the operator $T_0 : \mathcal{H}_1 \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_2$ defined by $T_0(x) = T(\delta_0 \otimes x)$. For each $x \in \mathcal{H}_1$, $T_0(x) = \sum_{\alpha \in \mathbb{F}_n^+} \delta_\alpha \otimes x_\alpha$ for some $x_\alpha \in \mathcal{H}_2$. The map that sends x to x_α is a bounded linear map $A_\alpha \in B(\mathcal{H}_1, \mathcal{H}_2)$. We now check easily that $T = \sum_{\alpha \in \mathbb{F}_n^+} R_\alpha \otimes A_\alpha$. ∎

Combining Corollary 3.5 and 3.7, we get

Corollary 3.8. *If $\mathcal{M}$ is a $*$-L-submodule of $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1$, $\mathcal{N}$ is a $*$-L-submodules of $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_2$, and $f : \mathcal{M} \to \mathcal{N}$ is an L-module map, then there exists $T = \sum_{\alpha \in \mathbb{F}_n^+} R_\alpha \otimes A_\alpha \in R^\infty(\omega_\alpha)\overline{\otimes}B(\mathcal{H}_1, \mathcal{H}_2)$ such that $\|T\| = \|f\|$ and $P_{\mathcal{M}}T = fP_{\mathcal{N}}$. If $\mathcal{M}$ is a $*$-R-submodule of $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1$, $\mathcal{N}$ is a $*$-R-submodules of $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_2$, and $f : \mathcal{M} \to \mathcal{N}$ is an R-module map, then there exists $T = \sum_{\alpha \in \mathbb{F}_n^+} L_\alpha \otimes A_\alpha \in F^\infty(\omega_\alpha)\overline{\otimes}B(\ell_2)$ such that $\|T\| = \|f\|$ and $P_{\mathcal{M}}T = fP_{\mathcal{N}}$.*

Remark 3.9. The unitary flip operator $U : \mathcal{F}^2(\omega_\alpha) \to \mathcal{F}^2(\omega_\alpha)$ is defined by $U\delta_\alpha = \sqrt{\frac{\omega_\alpha}{\omega_{\tilde{\alpha}}}}\delta_{\tilde{\alpha}}$. If $\omega_\alpha = \omega_{\tilde{\alpha}}$, this map provides a nice description of $R^\infty(\omega_\alpha)$ in terms of $F^\infty(\omega_\alpha)$ given by $UF^\infty(\omega_\alpha)U = R^\infty(\omega_\alpha)$. In general, we cannot describe one in terms of the other.

We need some preliminaries to state the Nevanlinna-Pick interpolation problem. An element $f \in \mathcal{F}^2(\omega_\alpha)$ has the form $f = \sum_\alpha c_\alpha \delta_\alpha$ where $\|f\|_2 = \sqrt{\sum_\alpha |c_\alpha|^2 \omega_\alpha}$. Using the product $\delta_\alpha \otimes \delta_\beta = \delta_{\alpha\beta}$, we view $f = f(\delta_{g_1}, \dots, \delta_{g_n})$ as a formal power series in n noncommutative variables $\delta_{g_1}, \dots, \delta_{g_n}$. If $\lambda = (\lambda_1, \dots, \lambda_n)$ is an n-tuple of complex numbers, we define $f(\lambda) = f(\lambda_1, \dots, \lambda_n) = \sum_\alpha c_\alpha \lambda_\alpha$, if the expression makes sense. Notice that such map is multiplicative in the sense that if f, g, and $f \otimes g$ are in $\mathcal{F}^2(\omega_\alpha)$, then $(f \otimes g)(\lambda) = f(\lambda) g(\lambda)$, and is determined by the values $\delta_\alpha(\lambda) = \lambda_\alpha$. We will show that the evaluation by λ makes sense whenever $\lambda \in \mathbb{B}_n$, the unit ball of $\mathbb{C}^n$.

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{B}_n$. Use (ω_3) to define $z_\lambda = \Big(\sum_{\alpha \in \mathbb{F}_n^+} \frac{\bar{\lambda}_\alpha}{\omega_\alpha} L_\alpha \Big) \delta_0 = \sum_{\alpha \in \mathbb{F}_n^+} \frac{\bar{\lambda}_\alpha}{\omega_\alpha} \delta_\alpha \in \mathcal{F}^2(\omega_\alpha)$ and check that $\langle \delta_\beta, z_\lambda \rangle = \lambda_\beta$. Then for any $f \in \mathcal{F}^2(\omega_\alpha)$, $f(\lambda) = \langle f, z_\lambda \rangle$. Since $\|f\|_{\mathcal{F}^2(\omega_\alpha)} \leqslant \|f\|_\infty$, it follows that the map $\Phi_\lambda : F^\infty(\omega_\alpha) \to \mathbb{C}$ defined by $\Phi(f) = f(\lambda)$ is multiplicative and WOT-continuous.

The z_λ's have useful properties. From (2.3) and (2.4) we easily check that for $i \leqslant n$, $L_i^* z_\lambda = \bar{\lambda}_i z_\lambda$ and $R_i^* z_\lambda = \bar{\lambda}_i z_\lambda$. Hence if $f \in F^\infty(\omega_\alpha)$ we have that $(L_f)^* z_\lambda = \overline{f(\lambda)} z_\lambda$. Moreover we also check that if $f \in M_k(F^\infty(\omega_\alpha))$, $f(\lambda) = A \in M_k$, and $x \in \ell_2^k$, then

$$(L_f)^*(z_\lambda \otimes x) = f(\lambda)^* x = A^* x. \tag{3.7}$$

THEOREM 3.10. ([4]) *Let $\mu_1, \dots, \mu_N \in \mathbb{B}_n$ be N-distinct points, and let $A_1, \dots, A_N \in M_k$ be $k \times k$ matrices. Then there exists f in $M_k(F^\infty(\omega_\alpha))$ such that $\|f\|_\infty \leqslant 1$ and $f(\mu_i) = A_i$ for $i \leqslant N$ if and only if the matrix*

$$[\langle z_{\mu_i}, z_{\mu_j} \rangle (I_k - A_j A_i^*)]_{i,j \leqslant N} \tag{3.8}$$

is positive semidefinite.

Proof. Let $\mathcal{N} = \operatorname{span}\{z_{\mu_1}, \dots, z_{\mu_N}\}$. It follows from the properties of the z_λ's that $\mathcal{N}$ is a $*$-R-submodule and a $*$-L-submodule of $\mathcal{F}^2(\omega_\alpha)$. Let $f \in M_k(F^\infty(\omega_\alpha))$ be any element that satisfies $f(\mu_i) = A_i$ for $i \leqslant N$ (it is easy to see that such an f always exists, although it could have big norm). We claim that the matrix (3.8) is positive semidefinite if and only if the compression of L_f to $\mathcal{N} \otimes \ell_2^k$ is a contraction. Let $T = (P_{\mathcal{N}_J} \otimes I_{\ell_2^k}) L_f|_{\mathcal{N}_J \otimes \ell_2^k}$. Then T is a contraction if and only if $I - TT^* \geqslant 0$, and if we use (3.7), we see that this is equivalent to (3.8) being positive semidefinite.

Suppose now that there exists $f \in M_k(F^\infty(\omega_\alpha))$ such that $\|f\|_\infty \leqslant 1$ and $f(\mu_i) = A_i$ for $i \leqslant N$. Since the compression of L_f to $\mathcal{N} \otimes \ell_2^k$ is clearly contractive, (3.8) is positive semidefinite.

Conversely, suppose that $[\langle z_{\mu_i}, z_{\mu_j} \rangle (I_k - A_j^* A_i)]_{i,j \leqslant N}$ is positive semidefinite. Then, the operator T on $\mathcal{N}_J \otimes \ell_2^k$, defined by $T^*(z_{\mu_i} \otimes x) = z_{\mu_i} \otimes A_i^* x$ for $i \leqslant N$ and $x \in \ell_2^k$, commutes with the $P_{\mathcal{N} \otimes \ell_2^k}(R_i \otimes I_{\ell_2^k})|_{\mathcal{N} \otimes \ell_2^k}$'s and satisfies $\|T\| \leqslant 1$. By Corollary 3.8, there exists $f \in M_k(F^\infty(\omega_\alpha))$ such that $\|f\|_\infty = \|T\| \leqslant 1$ and $P_\mathcal{N} L_f P_\mathcal{N} = T$. Since $\mathcal{N}$ is a $*$-L-submodule, it follows from (3.7) that $f(\mu_i) = A_i$ for $i \leqslant N$. ∎

REMARK 3.11. (i) We can use Corollary 3.11 to deduce Caratheodory interpolation problems in $F^\infty(\omega_\alpha)$. And with more work, we can follow [28] to obtain tangential Nevanlinna-Pick interpolation problems in $F^\infty(\omega_\alpha)$.

(ii) In the Full Fock space case (when $\omega_\alpha = 1$) the elements z_λ, $\lambda \in \mathbb{B}_n$, were introduced by Popescu in [27]. They were studied in [3], [11], and particularly in [6], where Arveson proved that the closed span of these vectors is the Symmetric Fock space.

4. POISSON KERNELS AND PROJECTIVE RESOLUTIONS

In this section we recall the construction of the Poisson kernels of [30] and [5], and show that they lead to projective resolutions. We start with the following

LEMMA 4.1. ([4]) *Let $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ be semi-invariant under the maps $L_i \otimes I_{\ell_2}$, $i \leqslant n$, and let $W_i = P_{\mathcal{E}}(L_i \otimes I_{\ell_2})|_{\mathcal{E}}$ for $i \leqslant n$. Then:*

(i) *The sequence $\Delta_N = \sum\limits_{|\alpha|\leqslant N} a_\alpha W_\alpha W_\alpha^*$ is non-negative and nonincreasing, and*

(ii) $\lim\limits_{N\to\infty} \sum\limits_{|\gamma|>N} \Big[\sum\limits_{\substack{\alpha\beta=\gamma \\ |\alpha|\leqslant N}} \frac{a_\beta}{\omega_\alpha} \Big] W_\gamma W_\gamma^* = 0.$

Consequently we get

(iii) $\Delta = \lim\limits_{N\to\infty} \Delta_N$ *exists, $0 \leqslant \Delta \leqslant I$, and Δ is not equal to zero, and*

(iv) $\sum\limits_{\alpha\in\mathbb{F}_n^+} \frac{1}{\omega_\alpha} W_\alpha \Delta W_\alpha^* = I_{\mathcal{E}}$.

Proof. We first verify the lemma for $\mathcal{E} = \mathcal{F}^2(\omega_\alpha)$. In this case, Δ_N and its limit are computed in (2.3), where it is shown that $\Delta = P_0 \geqslant 0$, the orthogonal projection onto the span of δ_0. This implies that $\sum\limits_{|\alpha|\leqslant N} \frac{1}{\omega_\alpha} L_\alpha \Delta L_\alpha^*$ is the orthogonal projection onto the span of the δ_β with $|\delta| \leqslant N$. It follows from (2.1) that

$$\sum_{|\alpha|\leqslant N} \frac{1}{\omega_\alpha} L_\alpha \Delta L_\alpha^* = \sum_{|\alpha|\leqslant N} \frac{1}{\omega_\alpha} L_\alpha \Big(\sum_{\beta\in\mathbb{F}_n^+} a_\beta L_\beta L_\beta^* \Big) L_\alpha^* = I + \sum_{|\gamma|>N} \Big[\sum_{\substack{\alpha\beta=\gamma \\ |\alpha|\leqslant N}} \frac{a_\beta}{\omega_\alpha} \Big] L_\gamma L_\gamma^*,$$

and this implies that $-\sum\limits_{|\alpha|>N} \Big[\sum\limits_{\substack{\alpha\beta=\gamma \\ |\alpha|\leqslant N}} \frac{a_\beta}{\omega_\alpha} \Big] L_\gamma L_\gamma^*$ is the orthogonal projection onto the closed span of the δ_γ's with $|\gamma| > N$, and hence this sequence of operators converges to zero in the strong operator topology. Notice that $\sum\limits_{\substack{\alpha\beta=\gamma \\ |\alpha|\leqslant N}} \frac{a_\beta}{\omega_\alpha} \leqslant 0$ whenever $|\gamma| > N$.

Suppose now that $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ is semi-invariant and $W_i = P_{\mathcal{E}}(L_i \otimes I_{\ell_2})|_{\mathcal{E}}$ for $i \leqslant n$. The proof of the lemma will follows from $W_\alpha W_\alpha^* = P_{\mathcal{E}}(L_\alpha L_\alpha^* \otimes I_{\ell_2})P_{\mathcal{E}}$ if $|\alpha| = 0$ and $W_\alpha W_\alpha^* \leqslant P_{\mathcal{E}}(L_\alpha L_\alpha^* \otimes I_{\ell_2})P_{\mathcal{E}}$ if $|\alpha| \geqslant 1$. To see this, notice that for $\alpha \in \mathbb{F}_n^+$, $P_{\mathcal{E}} \leqslant I$. Then $(L_\alpha \otimes I_{\ell_2})P_{\mathcal{E}}(L_\alpha \otimes I_{\ell_2})^* \leqslant (L_\alpha \otimes I_{\ell_2})I(L_\alpha \otimes I_{\ell_2})^*$. Hence, $W_\alpha W_\alpha^* = P_{\mathcal{E}}(L_\alpha \otimes I_{\ell_2})P_{\mathcal{E}}(L_\alpha \otimes I_{\ell_2})^* \leqslant P_{\mathcal{E}}(L_\alpha \otimes I_{\ell_2})(L_\alpha \otimes I_{\ell_2})^* P_{\mathcal{E}} = P_{\mathcal{E}}(L_\alpha L_\alpha^* \otimes I_{\ell_2})P_{\mathcal{E}}$.

Since $a_\alpha \leqslant 0$ for $|\alpha| \geqslant 1$, $a_\alpha W_\alpha W_\alpha^* \geqslant P_{\mathcal{E}}(a_\alpha L_\alpha L_\alpha^* \otimes I_{\ell_2})P_{\mathcal{E}}$. Hence, $\sum\limits_{|\alpha|\leqslant N} a_\alpha W_\alpha W_\alpha^* \geqslant P_{\mathcal{E}}\Big(\sum\limits_{|\alpha|\leqslant N} a_\alpha L_\alpha L_\alpha^* \otimes I_{\ell_2}\Big)P_{\mathcal{E}} \geqslant 0$, and (i) follows. Similarly,

$$0 \leqslant -\sum_{|\gamma|>N}\Big[\sum_{\substack{\alpha\beta=\gamma\\|\alpha|\leqslant N}} \frac{a_\beta}{\omega_\alpha}\Big] W_\gamma W_\gamma^* \leqslant P_{\mathcal{E}}\Big(-\sum_{|\gamma|>N}\Big[\sum_{\substack{\alpha\beta=\gamma\\|\alpha|\leqslant N}} \frac{a_\beta}{\omega_\alpha}\Big] L_\gamma L_\gamma^* \otimes I_{\ell_2}\Big)P_{\mathcal{E}}.$$

Since condition (ii) is true for the L_α's, we get (ii) for the W_α's. Conditions (iii) and (iv) follows easily from (i), (ii), and from the identity

$$\sum_{|\alpha|\leqslant N} \frac{1}{\omega_\alpha} W_\alpha \Delta W_\alpha^* = I + \sum_{|\gamma|>N}\Big[\sum_{\substack{\alpha\beta=\gamma\\|\alpha|\leqslant N}} \frac{a_\beta}{\omega_\alpha}\Big] W_\gamma W_\gamma^*. \qquad ∎$$

THEOREM 4.2. ([4]; Poisson kernels) *Let $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha)\otimes \ell_2$ be semi-invariant under $L_i \otimes I_{\ell_2}$, $i \leqslant n$, with $W_i = P_{\mathcal{E}}(L_i \otimes I_{\ell_2})|_{\mathcal{E}}$ for $i \leqslant n$. Define $D = \Delta^{1/2}$, where Δ is the non-negative operator of the Lemma* 4.1. *Define*

$$K : \mathcal{E} \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{E} \quad \textit{by} \quad K(x) = \sum_{\alpha\in\mathbb{F}_n^+} \frac{\delta_\alpha}{\omega_\alpha} \otimes DW_\alpha^* x.$$

Then K is an isometry and K^* is a module map. We call K the Poisson kernel of $\mathcal{E}$.

Proof. It follows from (iv) of Lemma 4.1 and from (3.3) that K is an isometry and that K^* is a module map. ∎

We write the next lemma in a more general setting.

LEMMA 4.3. *Let $(\mathcal{E}_0; W_1, \dots, W_n)$ and $(\mathcal{E}_1; V_1, \dots, V_n)$ be Hilbert modules and let $\Phi : \mathcal{E}_1 \to \mathcal{E}_0$ be a surjective coisometric module map. Then:*

(i) *$\Phi^*(\mathcal{E}_0)$ is invariant under $V_1^*, \dots, V_n^*$. Hence, $(\Phi^*(\mathcal{E}_0); T_1, \dots, T_n)$ is a $*$-submodule of $\mathcal{E}_1$ with $T_i = P_{\Phi^*(\mathcal{E}_0)} V_i|_{\Phi^*(\mathcal{E}_0)}$ for $i \leqslant n$,*

(ii) *$\Phi|_{\Phi^*(\mathcal{E}_0)} : \Phi^*(\mathcal{E}_0) \to \mathcal{E}_0$ is a module map, and*

(iii) *$\Phi^* : \mathcal{E}_0 \to \Phi^*(\mathcal{E}_0)$ is a module map.*

Consequently, $\mathcal{E}_0$ is isomorphic to $\Phi^*(\mathcal{E}_0)$.

Proof. Since $\Phi V_i = W_i \Phi$ for $i \leqslant n$, we have that $V_i^*\Phi^* = \Phi^* W_i^*$ and (i) follows. Then $\Phi^* W_i^* = V_i^* \Phi^* = P_{\Phi^*(\mathcal{E}_0)} V_i^*|_{\Phi^*(\mathcal{E}_0)}\Phi^* = T_i^*\Phi^*$, and this implies $\Phi|_{\Phi^*(\mathcal{E}_0)} T_i = W_i \Phi|_{\Phi^*(\mathcal{E}_0)}$ and (ii) follows. Moreover, since Φ^* is an isometry, we have that for $i \leqslant n$, $W_i = \Phi T_i \Phi^*$. Finally, $\Phi^* W_i = \Phi^*(\Phi T_i \Phi^*) = (\Phi^*\Phi) T_i \Phi^* = T_i \Phi^*$ and (iii) follows. ∎

From Theorem 4.2 and Lemma 4.3 we get

COROLLARY 4.4. *Let $\mathcal{E}$ be a subquotient of $\mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ and let $K : \mathcal{E} \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{E}$ be the Poisson kernel of $\mathcal{E}$. Then $\mathcal{E}$ is isomorphic to $K(\mathcal{E})$.*

We are ready to prove one of the main results of this paper.

THEOREM 4.5. *For any Hilbert space $\mathcal{H} \subset \ell_2$, $(\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}; L_1\otimes I_{\mathcal{H}}, \ldots, L_n\otimes I_{\mathcal{H}})$ is strongly orthogonally projective in the category of subquotients of $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$.*

Proof. Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be two subquotient of $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ and let $\Phi : \mathcal{E}_1 \to \mathcal{E}_0$ be a surjective coisometric module map. Let $\mathcal{H} \subset \ell_2$ and let $f : \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H} \to \mathcal{E}_0$ be a module map. The goal is to find a module map $F : \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H} \to \mathcal{E}_1$ such that $\|F\| = \|f\|$ and $\Phi \circ F = f$.

The first step consists in replacing $\mathcal{E}_1$ with $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$. Let $K_1 : \mathcal{E}_1 \to \mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ be the Poisson kernel of $\mathcal{E}_1$ and define the coisometric module map $\Psi = \Phi \circ K_1^* : \mathcal{F}^2(\omega_\alpha)\otimes\ell_2 \to \mathcal{E}_0$. Let $g = \Psi^* \circ f : \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H} \to \Psi^*(\mathcal{E}_0)$, which is a module map by Lemma 4.3. Since $\Psi^*(\mathcal{E}_0)$ is a $*$-submodule of $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$, it follows from Theorem 3.1 that there exists a module map $G : \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H} \to \mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ such that $\|G\| = \|g\|$ and $g = P_{\Psi^*(\mathcal{E}_0)} \circ G$. Let $F = K_1^* \circ G : \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H} \to \mathcal{E}_1$. F is a module map, $\|F\| = \|G\| = \|g\| = \|f\|$, and since $\Psi = \Psi \circ P_{\Psi^*(\mathcal{E}_0)}$, then $\Phi \circ F = (\Phi \circ K_1^*) \circ G = \Psi \circ G = \Psi \circ (P_{\Psi^*(\mathcal{E}_0)} \circ G) = \Psi \circ g = \Psi \circ \Psi^* \circ f = f$. ∎

As an immediate corollary we get

COROLLARY 4.6. *For every subquotient $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ there exist a family of strongly orthogonally projective subquotients P_i and partial isometric module maps Φ_i such that the following sequence is exact:*

$$\cdots \xrightarrow{\Phi_4} P_4 \xrightarrow{\Phi_3} P_3 \xrightarrow{\Phi_2} P_2 \xrightarrow{\Phi_1} P_1 \xrightarrow{\Phi_0} \mathcal{E} \longrightarrow 0.$$

Proof. Let $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ be a subquotient with Poisson kernel $K_0 : \mathcal{E} \to \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{E}$. Let $P_1 = \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{E}$ and $\Phi_0 = K_0^* : P_1 \to \mathcal{E}$. To construct the next strongly orthogonally projective subquotient, define $\mathcal{H}_1 = \mathrm{Ker}(\Phi_0)$. $\mathcal{H}_1$ is a submodule of $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ with Poisson kernel $K_1 : \mathcal{H}_1 \to \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_1$. Let $P_2 = \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_1$ and $\Phi_1 = K_1^* : P_2 \to P_1$. Since $\mathcal{H}_1$ is a submodule of $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$, $K_1^* : P_2 \to P_1$ is a module map with image equal to the kernel of K_0^*. Proceeding this way, we finish the proof. ∎

The previous two theorems show that there are enough strongly orthogonally projective subquotients to obtain projective resolutions. The following two results will help us identify all strongly orthogonally projective subquotients in particular examples. We need a definition

DEFINITION 4.7. If $\mathcal{M} \subset \mathcal{F}^2(\omega_\alpha)\otimes\ell_2$, the *right slice* of $\mathcal{M}$ is the smallest Hilbert space $\mathcal{H} \subset \ell_2$ such that $\mathcal{M} \subset \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}$.

If $x = \sum_{\alpha\in\mathbb{F}_n^+} \delta_\alpha \otimes x_\alpha \in \mathcal{M}$, then all of the x_α's belong to the right slice of $\mathcal{M}$. In fact, it is easy to see that the right slice of $\mathcal{M}$ is the closure of the linear span of the x_α's for all $x \in \mathcal{M}$.

THEOREM 4.8. *A subquotient $\mathcal{E}$ of $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ is strongly orthogonally projective in the category of subquotients of $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ if and only if $K(\mathcal{E}) = \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}$ for some $\mathcal{H} \subset \mathcal{E}$. Moreover, if $\mathcal{E}$ is strongly orthogonally projective, then $\mathcal{E}$ is a submodule of $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$.*

Proof. Suppose that $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ is semi-invariant and $W_i = P_{\mathcal{E}}(L_i \otimes I_{\ell_2})|_{\mathcal{E}}$ for $i \leqslant n$ and that $\mathcal{E}$ is strongly orthogonally projective. Since $K(\mathcal{E})$ is isomorphic to $\mathcal{E}$, it is also strongly orthogonally projective, and since $K(\mathcal{E})$ is a $*$-submodule, the orthogonal projection $P : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{E} \to K(\mathcal{E})$ is a module map. Then there exists a contractive module map $\Phi : K(\mathcal{E}) \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{E}$ such that $P \circ \Phi = I_{\mathcal{E}}$. The norm condition forces Φ to be the inclusion map, and we conclude that $\mathcal{K}(E)$ is also a submodule. That is, $K(\mathcal{E})$ is reducing for the maps $L_i \otimes I_{\mathcal{E}}$. According to (2.3), the orthogonal projection onto the span of δ_0 is equal to $P_0 = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha L_\alpha L_\alpha^*$. Since $K(\mathcal{E})$ is invariant under $L_i \otimes I_{\mathcal{E}}$ and $(L_i \otimes I_{\mathcal{E}})^*$, it follows that $P_0 \otimes I_{\mathcal{E}}$ maps $K(\mathcal{E})$ into $K(\mathcal{E})$. If $x = \sum_{\beta \in \mathbb{F}_n^+} \delta_\beta \otimes x_\beta \in K(\mathcal{E})$, then we have that $\delta_0 \otimes x_0 = (P_0 \otimes I_{\mathcal{E}})x \in K(\mathcal{E})$. More generally, we have

$$\big[(L_\alpha \otimes I_{\mathcal{E}})(P_0 \otimes I_{\mathcal{E}})(L_\beta \otimes I_{\mathcal{E}})^*\big](x) = \delta_\alpha \otimes x_\beta \in K(\mathcal{E}).$$

Then $\mathcal{F}^2(\omega_\alpha) \otimes x_\beta \subset \mathcal{E}$. If $\mathcal{H}$ is the closed span of such x_β's for all $x = \sum_{\beta \in \mathbb{F}_n^+} \delta_\beta \otimes x_\beta \in K(\mathcal{E})$, we have that $K(\mathcal{E}) = \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$, (i.e., $\mathcal{H}$ is the "right slice" of $K(\mathcal{E})$.)

We will now check that $\mathcal{E}$ is invariant under $L_i \otimes I_{\ell_2}$. For each $z \in \mathcal{H}$, $KK^*\big(\delta_\beta \otimes z\big) = \delta_\beta \otimes z$. Since $K^*(\delta_\beta \otimes z) = W_\beta K^*(\delta_0 \otimes z)$ then

$$\delta_\beta \otimes z = \sum_{\alpha \in \mathbb{F}_n^+} \frac{\delta_\alpha}{\omega_\alpha} \otimes D W_\alpha^*(K^*(\delta_\beta \otimes z)) = \sum_{\alpha \in \mathbb{F}_n^+} \delta_\alpha \otimes D \frac{W_\alpha^* W_\beta}{\omega_\alpha} K^*(\delta_0 \otimes z).$$

Therefore,

$$z = D \frac{W_\beta^* W_\beta}{\omega_\beta} K^*(\delta_0 \otimes z) \quad \text{for every } \beta \in \mathbb{F}_n^+.$$

We now claim that $D \frac{W_\beta^* W_\beta}{\omega_\beta}$ is a contraction. Indeed D satisfies $0 \leqslant D \leqslant I$ because Δ satisfies $0 \leqslant \Delta \leqslant I$ (see Lemma 4.1), and since $\|L_\beta\| = \sqrt{\omega_\beta}$ (see Lemma 2.1), we also have that $\frac{W_\beta^* W_\beta}{\omega_\beta}$ is a contraction. Since $\|z\| = \|K^*(\delta_0 \otimes z)\|$, it follows that

$$\Big[D \frac{W_\beta^* W_\beta}{\omega_\beta}\Big](K^*(\delta_0 \otimes z)) = (K^*(\delta_0 \otimes z)) \quad \text{for every } \beta \in \mathbb{F}_n^+. \tag{4.1}$$

(An easy convexity argument gives that if T is a contraction, $Tx_1 = Tx_2$, and $\|x_1\| = \|x_2\| = \|Tx_1\|$, then $x_1 = x_2$.) Now, $0 \leqslant \frac{W_\beta^* W_\beta}{\omega_\beta} \leqslant I$ and $\frac{W_\beta^* W_\beta}{\omega_\beta} = \frac{1}{\omega_\beta} P_{\mathcal{E}}(L_\beta \otimes I_{\ell_2})^* P_{\mathcal{E}}(L_\beta \otimes I_{\ell_2})|_{\mathcal{E}}$. It follows from (4.1) that the norm of $P_{\mathcal{E}}(L_\beta \otimes I_{\ell_2})K^*(\delta_0 \otimes z)$ cannot be smaller that the norm of $(L_\beta \otimes I_{\ell_2})K^*(\delta_0 \otimes z)$. Then we get that

$$\big(L_\beta \otimes I_{\ell_2}\big)(K^*(\delta_0 \otimes z)) \in \mathcal{E} \quad \text{for every } \beta \in \mathbb{F}_n^+.$$

Since $(L_\beta \otimes I_{\ell_2})(K^*(\delta_\alpha \otimes z)) = (L_\beta \otimes I_{\ell_2})(L_\alpha \otimes I_{\ell_2})(K^*(\delta_0 \otimes z)) = (L_{\beta\alpha} \otimes I_{\ell_2})(K^*(\delta_0 \otimes z)) = K^*(\delta_{\beta\alpha} \otimes z)$, we see that

$$\big(L_\beta \otimes I_{\ell_2}\big)(K^*(\delta_\alpha \otimes z)) \in \mathcal{E} \quad \text{for every } \alpha, \beta \in \mathbb{F}_n^+.$$

Since the set of $K^*(\delta_\alpha \otimes z)$ where $\alpha \in \mathbb{F}_n^+$ and $z \in \mathcal{H}$ spans $\mathcal{E}$, we conclude that $\mathcal{E}$ is invariant under the maps $L_i \otimes I_{\ell_2}$, $i \leqslant n$. ∎

COROLLARY 4.9. *A subquotient $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ is strongly orthogonally projective if and only if there exist $\mathcal{H}_1 \subset \ell_2$ and a surjective coisometric module map $\Phi_0 : \mathcal{F}^2(\omega_\alpha) \otimes \ell_2 \to \mathcal{E}$ such that the following sequence is exact:*

$$0 \longrightarrow \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1 \longrightarrow \mathcal{F}^2(\omega_\alpha) \otimes \ell_2 \xrightarrow{\Phi_0} \mathcal{E} \longrightarrow 0.$$

Proof. If $\mathcal{E}$ is strongly orthogonally projective, then $K(\mathcal{E}) = \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$ for some $\mathcal{H} \subset \ell_2$, and $\mathrm{Ker}(K^*) = K(\mathcal{E})^\perp = \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}^\perp$. The short exact sequence is obtained if $\Phi = K^*$. Conversely, suppose that there exists a coisometric module map Φ satisfying the conditions of Corollary 4.9. Since $\Phi^*(\mathcal{E}) = \mathrm{Ker}(\Phi)^\perp$, then $\Phi^*(\mathcal{E}) = \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1^\perp$. By Theorem 4.5, $\Phi^*(\mathcal{E})$ is strongly orthogonally projective. Since $\mathcal{E}$ is isomorphic to $\Phi^*(\mathcal{E})$, we finish the proof. ∎

Theorem 4.8 is the best possible. In the next section we will show the invariant submodules of the Full Fock space are strongly orthogonally projective.

PROPOSITION 4.10. *Let $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ be a strongly orthogonally projective subquotient of $\mathcal{F}^2(\omega_\alpha) \otimes \ell_2$, and assume that the ω_α's have the property that if $\|L_\alpha g\| = \|L_\alpha\| \|g\|$ for all α, then $g \in \mathcal{F}^2(\omega_\alpha)$ is a multiple of δ_0. Then there exists $\mathcal{H} \subset \ell_2$ such that $\mathcal{E} = \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$.*

Proof. It follows from Theorem 4.8 that $\mathcal{E}$ is invariant under $L_i \otimes I_{\ell_2}, i \leqslant n$ and that $K(\mathcal{E}) = \mathcal{F}^2(\omega_\alpha) \otimes \widehat{\mathcal{H}}$ for some $\widehat{\mathcal{H}} \subset \ell_2$. For each $x \in \widehat{\mathcal{H}}$, $\|x\| = 1$, define $T_x : \mathcal{F}^2(\omega_\alpha) \to \mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ by $T_x(y) = K^*(y \otimes x)$. It is easy to check that T_x is an isometric module map. Let (e_n) be an orthonormal basis of ℓ_2 and write $T_x(\delta_0) = \sum_{n \geqslant 1} g_n \otimes e_n$ for some $g_n \in \mathcal{F}^2(\omega_\alpha)$. Then $T_x(\delta_\alpha) = \sum_{n \geqslant 1} L_\alpha g_n \otimes e_n$. Since $\|T_x \delta_\alpha\| = \|\delta_\alpha\| = \sqrt{\omega_\alpha}$ and since $\|L_\alpha g_n\| \leqslant \sqrt{\omega_\alpha} \|g_n\| = \|L_\alpha\| \|g_n\|$ (see Lemma 2.1), it follows that for each $n \in \mathbb{N}$ and each $\alpha \in \mathbb{F}_n^+$, $\|L_\alpha g_n\| = \|L_\alpha\| \|g_n\|$. By hypothesis, this implies that each g_n is a multiple of δ_0; that is, $g_n = c_n \delta_0$ for some $c_n \in \mathbb{C}$. Then

$$T_x(\delta_0) = K^*(\delta_0 \otimes x) = \sum_{n \geqslant 1} c_n \delta_0 \otimes e_n = \delta_0 \otimes \Big[\sum_{n \geqslant 1} c_n e_n \Big] = \delta_0 \otimes y \in \mathcal{E}.$$

Since $\mathcal{E}$ is invariant under $L_i \otimes I_{\ell_2}$, $i \leqslant n$, we see that $\mathcal{F}^2(\omega_\alpha) \otimes y \subset \mathcal{E}$. Since the set of $K^*(\delta_\alpha \otimes x)$'s where $\alpha \in \mathbb{F}_n^+$ and $x \in \mathcal{K}$ is dense in $\mathcal{E}$, there exists $\mathcal{H}$ such that $\mathcal{E} = \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$. ∎

Proposition 4.10 applies to the Dirichlet algebra. Let $\mathcal{H}^2(n+1)$ be the Hilbert space with orthogonal basis $\{e_n : n \in \mathbb{N}\}$ but with weights $\langle e_n, e_n \rangle = n + 1$. We consider the Hilbert module $(\mathcal{H}^2(n + 1); L)$ where L is the shift operator $Le_n = e_{n+1}$. The algebra generated by L is called the Dirichlet algebra.

COROLLARY 4.11. *A subquotient $\mathcal{E}$ of $\mathcal{H}^2(n+1) \otimes \ell_2$ is strongly orthogonally projective if and only if there exists $\mathcal{H} \subset \ell_2$ such that $\mathcal{E} = \mathcal{H}^2(n+1) \otimes \mathcal{H}$. Moreover, for every subquotient $\mathcal{E}$ of $\mathcal{H}^2(n+1) \otimes \ell_2$ there exist a family of strongly orthogonally projective subquotients P_i and partial isometric module maps Φ_i such that the following sequence is exact:*

$$\cdots \xrightarrow{\Phi_4} P_4 \xrightarrow{\Phi_3} P_3 \xrightarrow{\Phi_2} P_2 \xrightarrow{\Phi_1} P_1 \xrightarrow{\Phi_0} \mathcal{E} \longrightarrow 0.$$

5. EXAMPLES

5.1. THE FULL FOCK SPACE $\mathcal{F}^2$. The Full Fock space $\mathcal{F}^2$ is the Hilbert space $\mathcal{F}^2(\omega_\alpha)$ when $\omega_\alpha = 1$ for every $\alpha \in \mathbb{F}_n^+$, and $F^\infty(\omega_\alpha)$ is the Fock space F^∞ that was introduced by Popescu in 1991 in connection to a noncommutative von Neumann's inequality [24]. We easily check that in this case, $a_0 = 1$, $a_{g_1} = a_{g_2} = \cdots a_{g_n} = -1$ and that $a_\alpha = 0$ for $|\alpha| \geqslant 2$.

In this paper we have used the tensor product $\otimes$ in two different ways: as the formal product of elements in $\mathcal{F}^2(\omega_\alpha)$ and as the Hilbert tensor product of Hilbert spaces. Until now it has been easy to distinguish the different meanings, but this is more difficult here. Accordingly, for this subsection only, we use the symbol $\otimes$ to denote the product of elements of $\mathcal{F}^2(\omega_\alpha)$, and the symbol $\otimes_2$ to denote the Hilbert tensor product. For example, if $\phi, \psi \in \mathcal{F}^2$ and $\phi \in F^\infty$, $\phi \otimes \psi$ belongs to $\mathcal{F}^2$ and denotes the product of ϕ with ψ; but $\phi \otimes_2 \psi$ denotes the element of the Hilbert tensor product $\mathcal{F}^2 \otimes_2 \mathcal{F}^2$.

The invariant subspaces of $\mathcal{F}^2$ were characterized by Popescu in [21] in 1989. He proved that $\mathcal{M} \subset \mathcal{F}^2$ is invariant under $L_1, \ldots, L_n$ if and only if there exists a family φ_i of elements of $\mathcal{F}^2$ satisfying:

(i) for every $\psi \in \mathcal{F}^2$, $\|\psi\|_2 = \|\psi \otimes \varphi\|_2$,
(ii) for $i \neq j$, $\mathcal{F}^2 \otimes \varphi_i$ and $\mathcal{F}^2 \otimes \varphi_j$ are orthogonal, and
(iii) $\mathcal{M} = \bigoplus\limits_i \mathcal{F}^2 \otimes \varphi_i$.

The closed span of the φ_i's, which is denoted by $\mathcal{L}$, is the wandering subspace of $\mathcal{M}$.

We use the following simple lemma to describe the strongly orthogonally projective submodules of $\mathcal{F}^2$.

LEMMA 5.1. *Let $\mathcal{M} \subset \mathcal{F}^2$ be an invariant subspace with Poisson kernel K. Then $K(\mathcal{M}) = \mathcal{F}^2 \otimes_2 \mathcal{L}$.*

Proof. Let $\mathcal{M}$ be an invariant subspace of $\mathcal{F}^2$, with $V_i = P_\mathcal{M} L_i|_\mathcal{M}$ for $i \leqslant n$. Find a family of $\varphi_i \in \mathcal{F}^2$ such that $\mathcal{M} = \bigoplus\limits_{i \in I} \mathcal{F}^2 \otimes \varphi_i$. Then it follows that $\{\delta_\alpha \otimes \varphi_i : \alpha \in \mathbb{F}_n^+, i \in I\}$ is an orthonormal basis of $\mathcal{M}$. To find the Poisson kernel of $\mathcal{M}$, we need to compute Δ of Lemma 4.1. Since the a_α's are very simple, we have that $\Delta = \sum\limits_{\alpha \in \mathbb{F}_n^+} a_\alpha V_\alpha V_\alpha^* = I_\mathcal{M} - V_1 V_1^* - V_2 V_2^* - \cdots - V_n V_n^*$, and we easily check that
$$\Delta \delta_\alpha \otimes \varphi_i = \begin{cases} \varphi_i & \text{if } |\alpha| = 0, \\ 0 & \text{if } |\alpha| \geqslant 1. \end{cases}$$
It follows then that Δ, and hence D of Theorem 4.2, are equal to $P_\mathcal{L}$, the orthogonal projection onto the wandering subspace $\mathcal{L}$. Then $K : \mathcal{M} \to \mathcal{F}^2(\omega_\alpha) \otimes_2 \mathcal{F}^2(\omega_\alpha)$ is given by
$$K(\delta_\beta \otimes \varphi_i) = \sum_{\alpha \in \mathbb{F}_n^+} \delta_\alpha \otimes_2 P_\mathcal{L} V_\alpha^* (\delta_\beta \otimes \varphi_i) = \delta_\beta \otimes_2 \varphi_i.$$
And $K(\mathcal{M}) = \mathcal{F}^2 \otimes_2 \mathcal{L}$. ∎

Lemma 5.1 is true for submodules of $\mathcal{F}^2 \otimes_2 \ell_2$. We gave the proof in the simpler case $\mathcal{F}^2$ to minimize the confusion of the symbols $\otimes$ and $\otimes_2$, but the proof for the general case is identical. It follows from this lemma and Theorem 4.8 that the strongly orthogonally projective subquotients are precisely the submodules of $\mathcal{F}^2 \otimes \ell_2$. Part of this follows from the work of Muhly and Solel ([19]), who characterized the strongly orthogonally projective modules of a large class of C^*-correspondences that include the Full Fock space as a particular case. If $\mathcal{E}$ is a subquotient of $\mathcal{F}^2 \otimes \ell_2$, there exists a projective resolution $\Phi_0 : \mathcal{F}^2 \otimes_2 \mathcal{H}_1 \to \mathcal{E}$ where Φ_0 is the adjoint of the Poisson kernel K_0 of $\mathcal{E}$ and $\mathcal{H}_1$ is the right slice of $K_0(\mathcal{E})$ (i.e., the smallest subspace of ℓ_2 such that $K_0(\mathcal{E})$ is a subset of $\mathcal{F}^2$ tensored with this Hilbert space). We repeat this process for $\mathrm{Ker}\Phi_0$. But this is an invariant subspace and its Poisson kernel $K_1 : \mathrm{Ker}\Phi_0 \to \mathcal{F}^2 \otimes \ell_2$ is an isometry with range $\mathcal{F}^2 \otimes_2 \mathcal{H}_2$, where $\mathcal{H}_2$ is the wandering subspace of $\mathrm{Ker}\Phi_0$. Then $K_1^* : \mathcal{F}^2 \otimes_2 \mathcal{H}_2 \to \mathrm{Ker}\Phi_0$ is an isometric module map and $\Phi_1 = \iota \circ K_1^* : \mathcal{F}^2 \otimes_2 \mathcal{H}_1 \to \mathcal{F}^2 \otimes_2 \mathcal{H}_1$ is a partial isometry module map. Then we get

COROLLARY 5.2. *A subquotient $\mathcal{E}$ of $\mathcal{F}^2 \otimes \ell_2$ is strongly orthogonally projective if and only if $\mathcal{E}$ is invariant under $L_i \otimes I_{\ell_2}$, $i \leqslant n$. Moreover, every subquotient $\mathcal{E}$ of $\mathcal{F}^2 \otimes \ell_2$ admits a projective resolution*

$$0 \longrightarrow \mathcal{F}^2 \otimes \mathcal{H}_2 \xrightarrow{\Phi_1} \mathcal{F}^2 \otimes \mathcal{H}_1 \xrightarrow{\Phi_0} \mathcal{E} \longrightarrow 0,$$

where $\mathcal{H}_1, \mathcal{H}_2 \subset \ell_2$ and Φ_0, Φ_1 are partial isometric module maps.

We will see in the next section that the maps Φ_0 and Φ_1 reformulate aspects of noncommutative dilation theory for C_0-row contractions. Φ_0 is the adjoint of the minimal isometric dilation of $\mathcal{E}$ (see [14], [8] and [22]), and Φ_1 is Popescu's characteristic function of $\mathcal{E}$ (see [21]).

5.2. QUOTIENT SPACES. Let J be a w^*-closed 2-sided ideal of $F^\infty(\omega_\alpha)$, and let $\mathcal{N}_J$ be the orthogonal complement of the image of J. Since J is a left ideal, it follows that $\mathcal{N}_J$ is a $*$-submodule of $\mathcal{F}^2(\omega_\alpha)$ with $V_i = P_{\mathcal{N}_J} L_i|_{\mathcal{N}_J}$ for $i \leqslant n$.

The following lemma is implicit in the proof of Theorem 1.2 (see also Proposition 3.4 of [4]).

LEMMA 5.3. *Let $\mathcal{E} \subset \mathcal{N}_J \otimes \ell_2$ be semi-invariant under V_i, $i \leqslant n$ and let $K : \mathcal{E} \to \mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ be the Poisson kernel of $\mathcal{E}$ (notice that $\mathcal{E}$ is also a subquotient of $F^2(\omega_\alpha) \otimes \ell_2$). Then $K(\mathcal{E}) \subset \mathcal{N}_J \otimes \ell_2$. Consequently, $(K^*)|_{\mathcal{N}_J \otimes \ell_2} : \mathcal{N}_J \otimes \ell_2 \to \mathcal{E}$ is a surjective coisometric module map.*

Proof. Recall from Theorem 1.2 that $F^\infty(\omega_\alpha)/J$ is completely isometric to $P_{N_J} F^\infty(\omega_\alpha)|_{\mathcal{N}_J}$. Since $W_i = P_{\mathcal{E}}(L_i \otimes I_{\ell_2})|_{\mathcal{E}} = P_{\mathcal{E}}\big[P_{\mathcal{N}_J \otimes \ell_2}(L_i \otimes I_{\ell_2})|_{\mathcal{N}_J \otimes \ell_2}\big]|_{\mathcal{E}} = P_{\mathcal{E}} V_i|_{\mathcal{E}}$, the map that sends $\varphi \in F^\infty(\omega_\alpha)$ to $P_{\mathcal{E}}(L_\varphi \otimes I_{\ell_2})|_{\mathcal{E}} \in B(\mathcal{E})$ factors through $F^\infty(\omega_\alpha)/J$. That is, $P_{\mathcal{E}}(L_\varphi \otimes I_{\ell_2})|_{\mathcal{E}} = \Psi \circ Q(L_\varphi)$ where $Q : F^\infty(\omega_\alpha) \to F^\infty(\omega_\alpha)/J$ is the quotient map and $\Psi : F^\infty(\omega_\alpha)/J \to B(\mathcal{E})$ is defined by $\Psi(L_\varphi + J) = P_{\mathcal{E}}(L_\varphi \otimes I_{\ell_2})|_{\mathcal{E}}$. Let $K : \mathcal{E} \to \mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ be the Poisson kernel of $\mathcal{E}$ and notice that for every $\varphi \in F^\infty(\omega_\alpha)$, $K^*(L_\varphi \otimes I_{\ell_2}) = \Psi \circ Q(L_\varphi)K^*$.

Let $x_1 \in \mathcal{E}$, $b \in J$, $h \in \mathcal{F}^2(\omega_\alpha)$, and $x_2 \in \ell_2$. Then $\langle Kx_1, bh \otimes x_2\rangle = \langle Kx_1, (b \otimes I_{\ell_2})(h \otimes x_2)\rangle = \langle x_1, K^*(b \otimes I_{\ell_2})(h \otimes x_2)\rangle = \langle x_1, [\widehat{\Psi} \circ Q(b)]F(h \otimes x_2)\rangle = 0$, because $Qb = 0$. Therefore $K(\mathcal{E}) \subset \mathcal{N}_J \otimes \ell_2$ and $(K^*)|_{\mathcal{N}_J \otimes \ell_2} : \mathcal{N}_J \otimes \ell_2 \to \mathcal{E}$ is a surjective coisometry. Since $\mathcal{N}_J$ is $*$-invariant, we also get that $(K^*)|_{\mathcal{N}_J \otimes \ell_2}$ is a module map. ∎

THEOREM 5.4. *A subquotient $\mathcal{E} \subset \mathcal{N}_J \otimes \ell_2$ is strongly orthogonally projective if and only if it is isomorphic to $\mathcal{N}_J \otimes \mathcal{H}$ for some $\mathcal{H} \subset \ell_2$. Moreover, for every subquotient $\mathcal{E}$ of $\mathcal{N}_J \otimes \ell_2$ there exist a family of strongly orthogonally projective subquotients P_i and partial isometric module maps Φ_i such that the following sequence is exact:*

$$\cdots \xrightarrow{\Phi_4} P_4 \xrightarrow{\Phi_3} P_3 \xrightarrow{\Phi_2} P_2 \xrightarrow{\Phi_1} P_1 \xrightarrow{\Phi_0} \mathcal{E} \longrightarrow 0.$$

Proof. We show first for every $\mathcal{H} \subset \ell_2$, $\mathcal{N}_J \otimes \mathcal{H}$ is strongly orthogonally projective in the category of $*$-submodules (i.e., it satisfies Theorem 3.1). We observe that since J is also a right ideal, $\mathcal{N}_J$ is invariant under $R_i^*, i \leqslant n$, the adjoint of the right creation operators. This means that if $\varphi \in R^\infty(\omega_\alpha)$, the map $T = P_{\mathcal{N}_J} R_\varphi|_{\mathcal{N}_J} : \mathcal{N}_J \to \mathcal{N}_J$ is a module map. Indeed, $V_i^* T^* = P_{\mathcal{N}_J} L_i^* P_{\mathcal{N}_J} R_\varphi^* P_{\mathcal{N}_J} = P_{\mathcal{N}_J} L_i^* R_\varphi^* P_{\mathcal{N}_J} = P_{\mathcal{N}_J} R_\varphi^* L_i^* P_{\mathcal{N}_J} = P_{\mathcal{N}_J} R_\varphi^* P_{\mathcal{N}_J} L_i^* P_{\mathcal{N}_J} = T^* V_i^*$ for $i \leqslant n$.

Let $\mathcal{K}$ be a $*$-submodule of $\mathcal{N}_J \otimes \ell_2$ and $f : \mathcal{N}_J \otimes \mathcal{H} \to \mathcal{K}$ a module map. We claim that there exists a module map $F : \mathcal{N}_J \otimes \mathcal{H} \to \mathcal{N}_J \otimes \ell_2$ such that $\|F\| = \|f\|$ and $P_\mathcal{K} \circ F = f$. Since $\mathcal{N}_J \otimes \mathcal{H}$ and $\mathcal{K}$ are $*$-L-submodules of $\mathcal{F}^2(\omega_\alpha) \otimes \ell_2$, it follows from Corollary 3.5 that there exists an L-module map $\widehat{T} : \mathcal{F}^2(\omega_\alpha) \otimes \ell_2 \to \mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ such that $\|f\| = \|\widehat{T}\|$ and $P_\mathcal{K} \widehat{T}|_{\mathcal{N}_J \otimes \mathcal{H}} = f$. Since $\mathcal{N}_J$ is also a $*$-R-submodule, it follows that $T = P_{\mathcal{N}_J \otimes \ell_2} \widehat{T}|_{\mathcal{N}_J \otimes \ell_2} : \mathcal{N}_J \otimes \ell_2 \to \mathcal{N}_J \otimes \ell_2$ is a module map. Since $\mathcal{N}_J \otimes \mathcal{H}$ is a submodule of $\mathcal{N}_J \otimes \ell_2$, the function $F = T|_{\mathcal{N}_J \otimes \mathcal{H}} : \mathcal{N}_J \otimes \mathcal{H} \to \mathcal{N}_J \otimes \ell_2$ is a module map. It is now clear that $P_\mathcal{K} \circ F = f$ and that $\|F\| = \|f\|$. This means that $\mathcal{N}_J \otimes \mathcal{H}$ is strongly orthogonally projective in the category of the $*$-submodules. Using Lemma 5.3, we follow the proof of Theorem 4.5 and Corollary 4.6 to conclude that $\mathcal{N}_J \otimes \mathcal{H}$ is strongly orthogonally projective and that every subquotient has a projective resolution. It remains to prove that every strongly orthogonally projective subquotient is isomorphic to $\mathcal{N}_J \otimes \mathcal{H}$ for some $\mathcal{H} \otimes \ell_2$.

Let $\mathcal{E}$ be a projective subquotient of $\mathcal{N}_J \otimes \ell_2$. Since $\mathcal{E}$ is isomorphic to $K(\mathcal{E})$, then $K(\mathcal{E})$ is also strongly orthogonally projective. Since $K(\mathcal{E})$ is also a $*$-submodule, it follows from the proof of Theorem 4.8 that $K(\mathcal{E})$ is also a submodule and hence it is reducing. Recall that $\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha (L_\alpha \otimes I_{\ell_s})(L_\alpha \otimes I_{\ell_s})^* = P_0 \otimes I_{\ell_2}$. Then since $\mathcal{N}_J$ is $*$-invariant, $P_{\mathcal{N}_J \otimes \ell_2} \big[\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha (L_\alpha \otimes I_{\ell_s})(L_\alpha \otimes I_{\ell_s})^* \big] P_{\mathcal{N}_J \otimes \ell_2} = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha (V_\alpha \otimes I_{\ell_s})(V_\alpha \otimes I_{\ell_s})^* = P_{\mathcal{N}_J \otimes \ell_2} \big[P_0 \otimes I_{\ell_2} \big] P_{\mathcal{N}_J \otimes \ell_2}$. And since $K(\mathcal{E})$ is invariant under $V_\alpha \otimes I_{\ell_s}$ and $(V_\alpha \otimes I_{\ell_s})^*$, $P_{\mathcal{N}_J \otimes \ell_2} \big[P_0 \otimes I_{\ell_2} \big] P_{\mathcal{N}_J \otimes \ell_2}$ maps $K(\mathcal{E})$ to itself. Let $x = \sum_{\alpha \in \mathbb{F}_n^+} \delta_\alpha \otimes x_\alpha \in K(\mathcal{E})$. Then $P_{\mathcal{N}_J \otimes \ell_2} \big[P_0 \otimes I_{\ell_2} \big] P_{\mathcal{N}_J \otimes \ell_2} x = P_{\mathcal{N}_J} \delta_0 \otimes x_0 = \xi_0 \otimes x_0$, where $\xi_0 = P_{\mathcal{N}_J} \delta_0$. It is easy to see that the span of $V_\alpha \xi_0$, $\alpha \in \mathbb{F}_n^+$, is dense in $\mathcal{N}_J$ (use $\langle V_\alpha \xi_0, z \rangle = \langle \delta_\alpha, z \rangle$ to show that an element $z \in K(\mathcal{E})$ orthogonal to all $V_\alpha \xi_0$'s is zero). Then we obtain that $\mathcal{N}_J \otimes x_0 \subset K(\mathcal{E})$. Similarly, we get that $\mathcal{N}_J \otimes x_\alpha \subset K(\mathcal{E})$ for every $\alpha \in \mathbb{F}_n^+$. If we repeat this for all $x \in K(\mathcal{E})$ we conclude that $K(\mathcal{E}) = \mathcal{N}_J \otimes \mathcal{H}$, where $\mathcal{H}$ is the "right slice" of $K(\mathcal{E})$. ∎

5.3. THE SYMMETRIC FOCK SPACE. The symmetric Fock space $\mathcal{F}_+^2$ is the subspace of the Full Fock space $\mathcal{F}^2$ spanned by the vectors of the form $\sum_{\pi\in S_m}\delta_{\pi(\alpha)}$ for $m\in\mathbb{N}$, $\alpha\in\mathbb{F}_+^n$ with $|\alpha|=m$, and $\pi\in S_m$, where $\pi(\alpha)=\pi(g_{i_1}g_{i_2}\cdots g_{i_m})=g_{i_{\pi(1)}}g_{i_{\pi(2)}}\cdots g_{i_{\pi(m)}}$. $\mathcal{F}_+^2$ is also the orthogonal complement of the image of the commutator ideal (i.e., it is of the form $\mathcal{N}_J$ of the previous section where J is generated by $\varphi\otimes\psi-\psi\otimes\varphi$ for $\psi,\varphi\in F^\infty$), and as a result it satisfies Theorem 5.4. The commutant lifting theorem of the symmetric Fock space is an immediate consequence of [25].

It is more convenient to denote $z_i=\delta_i$ for $i\leqslant n$ and $1=\delta_0$. The product of two elements of $\mathcal{F}_+^2$ is defined by $p\cdot q=P_{\mathcal{F}_+^2}(p\otimes q)$. Since the product is commutative it is simpler to index the monomials with elements $k=(k_1,\dots,k_n)\in\mathbb{N}^n$ in the following way: $z^k=z_1^{k_1}z_2^{k_2}\cdots z_n^{k_n}$ with the convention that $z^0=1$. Then $\mathcal{F}_+^2$ has an orthogonal basis $\{z^k : k\in\mathbb{N}^n\}$ with $\|z^k\|_2^2=\frac{k_1!k_2!\cdots k_n!}{(k_1+k_2+\cdots+k_n)!}$.

The compressions of the left creation operators to the symmetric Fock space are the maps M_i, $i\leqslant n$, that multiply by z_i. Since they are commutative, their products are also indexed by elements $k=(k_1,\dots,k_n)\in\mathbb{N}^n$ in the following way: $M_k=M_1^{k_1}M_2^{k_2}\cdots M_n^{k_n}=M_{z^k}$.

According to Theorem 5.4, the strongly orthogonally projective submodules of $\mathcal{F}_+^2\otimes\ell_2$ are isomorphic to submodule of the form $\mathcal{F}_+^2\otimes\mathcal{H}$ for some $\mathcal{H}\subset\ell_2$. But as we saw in the previous subsection, this does not imply that they are exactly of this form. The next proposition shows that this is true in this case.

PROPOSITION 5.5. *A subquotient $\mathcal{E}\subset\mathcal{F}_+^2\otimes\ell_2$ is strongly orthogonally projective if and only if there exists $\mathcal{H}\subset\ell_2$ such that $\mathcal{E}=\mathcal{F}_+^2\otimes\mathcal{H}$.*

Proof. Let $\mathcal{E}\subset\mathcal{F}_+^2\otimes\ell_2$ be a strongly orthogonally projective subquotient with Poisson kernel K and $V_i=P_\mathcal{E}(M_i\otimes I_{\ell_2})|_\mathcal{E}, i\leqslant n$. According to Theorem 5.4, $K(\mathcal{E})=\mathcal{F}_+^2\otimes\widehat{\mathcal{H}}$ for some $\widehat{\mathcal{H}}\in\ell_2$. Hence for every $k\in\widehat{\mathcal{H}}$, $\|k\|_2=1$, we have that $K^*(1\otimes k)\in\mathcal{E}$ and $\|K^*(1\otimes k)\|_2=1$, and since K^* is a module map,

$$P_\mathcal{E}\big(M_{z_1z_2}\otimes I_{\ell_2}\big)K^*(1\otimes k)=V_1V_2K^*(1\otimes k)=K^*(z_1z_2\otimes h).$$

It is easy to see that $\|M_{z_1z_2}\|=\|M_{z_1z_2}1\|_2=\frac{1}{\sqrt{2}}$ and that $\|M_{z_1z_2}f\|_2=\frac{1}{\sqrt{2}}\|f\|_2$ implies that f is a multiple of 1 (notice that $M_{z_1z_2}$ maps the orthogonal basis $\{z^k : k\in\mathbb{N}^n\}$ to an orthogonal set and that it attains its norm only at 1). Then $\frac{1}{\sqrt{2}}=\|z_1z_2\otimes k\|_2=\|K^*(z_1z_2\otimes k)\|_2=\|P_\mathcal{E}\big(M_{z_1z_2}\otimes I_{\ell_2}\big)K^*(1\otimes k)\|_2\leqslant\|\big(M_{z_1z_2}\otimes I_{\ell_2}\big)K^*(1\otimes k)\|_2\leqslant\|M_{z_1z_2}\|=\frac{1}{\sqrt{2}}$. This implies that $\big(M_{z_1z_2}\otimes I_{\ell_2}\big)K^*(1\otimes k)\in\mathcal{E}$ and more importantly, that $K^*(1\otimes k)$ is of the form $1\otimes h$ for some $h\in\ell_2$. We define $\mathcal{H}=\{h\in\ell_2 : 1\otimes h=K^*(1\otimes k)\text{ for some }k\in\widehat{\mathcal{H}}\}$.

It remains to show that $z^k\otimes h\in\mathcal{E}$ for every $k\in\mathbb{N}^n$ and $h\in\mathcal{H}$. To see this, we need to observe that $\|M_{z^k}\|=\|M_{z^k}1\|_2=\|z^k\|_2$ (if z^k is a single power, then M_{z^k} can attain its norm in several vectors, but this is not important here). Let $h\in\mathcal{H}$, $\|h\|_2=1$ and find $h_1\in\widehat{\mathcal{H}}$ such that $1\otimes h=K^*(1\otimes h_1)$. Then

$$\begin{aligned}\|z^k\|_2&=\|K^*(z^k\otimes h_1)\|_2=\|P_\mathcal{E}\big(M_{z^k}\otimes I_{\ell_2}\big)K^*(1\otimes h_1)\|_2\\&=\|P_\mathcal{E}\big(M_{z^k}\otimes I_{\ell_2}\big)(1\otimes h)\|_2\leqslant\|\big(M_{z^k}\otimes I_{\ell_2}\big)(1\otimes h)\|_2\leqslant\|M_{z^k}\|=\|z^k\|_2.\end{aligned}$$

This implies that $z^k \otimes h = \big(M_{z^k} \otimes I_{\ell_2}\big)(1 \otimes h) \in \mathcal{E}$ and we now easily conclude that $\mathcal{E} = \mathcal{F}^2_+ \otimes \mathcal{H}$. ∎

COROLLARY 5.6. *If $\mathcal{E}$ is a subquotient of $\mathcal{F}^2_+ \otimes \ell_2$ that is not strongly orthogonally projective then all projective resolutions have infinite length.*

Proof. Suppose on the contrary that there exists a finite resolution

$$0 \longrightarrow \mathcal{F}^2_+ \otimes \mathcal{H}_k \xrightarrow{\Phi_k} \mathcal{F}^2_+ \otimes \mathcal{H}_{k-1} \xrightarrow{\Phi_{k-1}} \cdots$$
$$\cdots \longrightarrow \mathcal{F}^2_+ \otimes \mathcal{H}_1 \xrightarrow{\Phi_1} \mathcal{F}^2_+ \otimes \mathcal{H}_0 \xrightarrow{\Phi_0} \mathcal{E} \longrightarrow 0,$$

with partial isometric module maps Φ_i, $0 \leqslant i \leqslant k$. It follows that $\mathrm{Im}\Phi_k = \mathrm{Ker}\Phi_{k-1}$ is strongly orthogonally projective and since it has the form $\mathcal{F}^2_+ \otimes \mathcal{H}$ for some $\mathcal{H}$, then $(\mathrm{Ker}\Phi_{k-1})^\perp$ is also orthogonally projective. Since Φ_{k-1} is a partial isometry with initial space $(\mathrm{Ker}\Phi_{k-1})^\perp$, then $\mathrm{Im}\Phi_{k-1} = \mathrm{Ker}\Phi_{k-2}$ and $(\mathrm{Ker}\Phi_{k-2})^\perp$ are strongly orthogonally projective. Proceeding this way we obtain that for every $i, 0 \leqslant i \leqslant k-1$, $\mathrm{Ker}\Phi_i$ and $(\mathrm{Ker}\Phi_i)^\perp$ are strongly orthogonally projective. Since $\mathcal{E}$ is isomorphic to $(\mathrm{Ker}\Phi_0)^\perp$, we conclude that $\mathcal{E}$ is strongly orthogonally projective, and this contradicts the hypothesis. ∎

5.4. COMPLETE NEVANLINNA-PICK KERNELS. A function $K : X \times X \to \mathbb{C}$ is a positive definite kernel on the set X if for every finite set $x_1, \ldots, x_n \in X$, the matrix (x_{ij}) is positive semi-definite. Each kernel induces a family of functions $k_x : X \to \mathbb{C}$ defined by $k_x(y) = K(y, x)$. The reproducing kernel Hilbert space H_K is the Hilbert space spanned by the functions k_x's with inner product $\langle k_y, k_x\rangle = K(x, y)$. Hence the elements $f \in H_K$ are thought of as functions $f : X \to \mathbb{C}$ defined by $f(x) = \langle f, k_x\rangle$. A function $\phi : X \to \mathbb{C}$ defines an operator $(M_\phi)^*$ on the span of $\{k_x : x \in X\}$ by $(M_\phi)^* k_x = \overline{\phi(x)} k_x$. If this maps extends to a bounded linear operator on H_K, we say that $\phi \in M(K)$ is a multiplier. Equivalently, ϕ is a multiplier if and only if $\phi f \in H_K$ for every $f \in H_K$, where ϕf is the product of ϕ and f as functions on X. $M(K)$ is the multiplier algebra of H_K and the elements $\phi \in M(K)$ have norm $\|\phi\|_{M(K)} = \|M_\phi\|$. More generally, a function $\Phi : X \to B(\mathcal{H}_1, \mathcal{H}_2)$ is a multiplier from $H_K \otimes \mathcal{H}_1$ to $H_K \otimes \mathcal{H}_2$ if and only if the operator $(M_\Phi)^* k_x \otimes z_2 = k_x \otimes \Phi(x)^* z_2$, defined on the span of $\{k_x \otimes z_2 : x \in X, z_2 \in \mathcal{H}_2\}$, extends to a bounded operator on $H_K \otimes \mathcal{H}_2$. A kernel has the *complete Nevanlinna-Pick property* if for every finite set $x_1, \ldots, x_n \in X$ and $n \times n$ matrices $C_1, \ldots, C_n$, there exists $h \in M_n(M(K))$ such that $h(x_i) = C_i$ for $i \leqslant n$ if and only if the matrix $[(I - C_i C_j^*)\langle k_{x_j}, k_{x_i}\rangle]$ is positive semi-definite.

In [1] and in unpublished work, Agler reformulated the Nevanlinna-Pick interpolation this way. Quiggin ([31]) and McCullough ([16]) characterized the kernels with this property, and more recently, Agler and McCarthy ([2]) proved the remarkable result that an (irreducible) Nevanlinna-Pick kernel is the restriction of the kernel of the symmetric Fock space $\mathcal{F}^2_+$ to a subset of the ball $\mathbb{B}_n$. More precisely, they proved that there exist n (possibly infinite) and an injective function $g : X \to \mathbb{B}_n$ such that (after a renormalization)

$$K(x, y) = \frac{1}{1 - \langle g(x), g(y)\rangle}.$$

Let $J = \{\varphi \in F^\infty : \varphi(g(x)) = 0$ for every $x \in X\}$. Following the arguments of Theorem 3.10, it is easy to see that J is a w^*-closed 2-sided ideal of F^∞ and that $\mathcal{N}_J$ is the closed span of $\{z_{g(x)} : x \in X\}$. Since

$$\langle z_{g(x_1)}, z_{g(x_2)} \rangle = \frac{1}{1 - \langle g(x_2), g(x_1) \rangle} = K(x_2, x_1) = \langle k_{x_1}, k_{x_2} \rangle$$

we see that $\mathcal{N}_J$ is unitarily equivalent to the reproducing kernel Hilbert space H_K, and this equivalence is implemented by the map that sends $z_{g(x)}$ to k_x.

For simplicity, assume that $X \subset \mathbb{B}_n$. An element $\varphi \in F^\infty$ determines a function $\varphi : X \to \mathbb{C}$ defined by $\varphi(x) = \langle \varphi, z_x \rangle$ (recall that $X \subset \mathbb{B}_n$). The operator $T = P_{\mathcal{N}_J} L_\varphi|_{\mathcal{N}_J} : \mathcal{N}_J \to \mathcal{N}_J$ is bounded and satisfies $T^* z_x = \overline{\varphi(x)} z_x$. Hence the map $\varphi : X \to \mathbb{C}$ is a multiplier and $\|M_\varphi\| = \|P_{\mathcal{N}_J} L_\varphi|_{\mathcal{N}_J}\| = \|\varphi + J\|_{F^\infty/J}$. Conversely, if $\phi \in M(K)$, the operator $T : \mathcal{N}_J \to \mathcal{N}_J$ defined by $T^* z_x = \overline{\phi(x)} z_x$ for $x \in X$ is bounded and commutes with $P_{\mathcal{N}_J} R_i|_{\mathcal{N}_J}$ for $i \leqslant n$. By Corollary 3.8, there exists $\varphi \in F^\infty$ such that $P_{\mathcal{N}_J} L_\varphi|_{\mathcal{N}_J} = T$ and $\|\varphi\|_\infty = \|T\|$. The matricial case works the same way and we conclude that the space of multipliers $M(K)$ is unitarily equivalent to F^∞/J. Consequently, the results of the previous section apply to these examples. Furthermore, we note that since $M(K)$ is a quotient of F^∞, then it has the $\mathbb{A}_1(1)$-property (see [5]). Finally, since $M(K)$ is commutative, the compression of the left creation operators and right creation operators coincide and we do not need to distinguish between L-module maps or R-module maps, we call them simply module maps. Hence a map $T : M(K) \otimes \mathcal{H}_1 \to M(K) \otimes \mathcal{H}_2$ is a module map if and only if $TM_\phi \otimes I_{\mathcal{H}_1} = M_\phi \otimes I_{\mathcal{H}_2} T$ for every $\phi \in M(K)$ if and only if there exists a multiplier $\Phi : X \to B(\mathcal{H}_1, \mathcal{H}_2)$ such that $T = M_\Phi$.

6. APPLICATIONS AND FINAL REMARKS

6.1. Invariant subspaces of Nevanlinna-Pick kernels. In [17], McCullough and Trent characterized the subspaces of $H_K \otimes \mathcal{E}$ that are invariant under the maps $M_\phi \otimes I_\mathcal{E}$ for Nevanlinna-Pick kernels K. In this subsection we show that their results follow easily from our work, and then we prove that their conjecture is true.

McCullough and Trent proved that $\mathcal{M} \subset H_K \otimes \mathcal{E}$ is invariant under $M_\phi \otimes I_\mathcal{E}$ for $\phi \in M(K)$ if and only if there exist a Hilbert space $\mathcal{G}$ and a multiplier $\Phi : X \to B(\mathcal{G}, \mathcal{E})$ such that $M_\Phi : H_K \otimes \mathcal{G} \to H_K \otimes \mathcal{E}$ is a partial isometry with range $\mathcal{M}$. We sketch the proof of this result now. Suppose that $\mathcal{M}$ is invariant, and let $K : \mathcal{M} \to H_K \otimes \mathcal{M}$ be its Poisson kernel. Then the map $\iota_\mathcal{M} \circ K^* : H_K \otimes \mathcal{M} \to H_K \otimes \mathcal{E}$ is a module map because K^* is a module map (Lemma 5.1) and $\iota_\mathcal{M}$ is a module map ($\mathcal{M}$ is invariant), and it is a partial isometry with range $\mathcal{M}$. Since module maps between these spaces correspond to multiplier maps, there exists a multiplier $\Phi : X \to B(\mathcal{G}, \mathcal{E})$ such that $M_\Phi = \iota_\mathcal{M} \circ K^*$. The other direction is immediate because the image of a partial isometric module map is invariant.

McCullough and Trent also proved that if $\mathcal{M} \subset \mathcal{N} \subset H_K \otimes \mathcal{E}$ are invariant subspaces, $\Phi : X \to B(\mathcal{H}_1, \mathcal{E})$ and $\Psi : X \to B(\mathcal{H}_2, \mathcal{E})$ are multipliers satisfying M_Φ is a partial isometry with range $\mathcal{M}$ and M_Ψ is a partial isometry with range $\mathcal{N}$, then there exists a multiplier $\Gamma : X \to B(H_2, \mathcal{H}_1)$ such that M_Γ is a contraction and $\Phi = \Psi \circ \Gamma$. This follows from the fact that $H_K \otimes \mathcal{H}_2$ is strongly orthogonally projective. Indeed, since $\iota \circ M_\Phi : H_K \otimes \mathcal{H}_2 \to \mathcal{N}$ is a module map, and $M_\Psi : H_K \otimes \mathcal{H}_1 \to \mathcal{N}$ is a coisometric module map, then there exists a module map $T : H_K \otimes \mathcal{H}_2 \to H_K \otimes \mathcal{H}_1$ such that $\|T\| = \|\iota \circ M_\Phi\| = 1$,

$$\begin{array}{ccccc} H_K \otimes \mathcal{H}_2 & \xrightarrow{M_\Phi} & \mathcal{M} & \xrightarrow{\iota_\mathcal{M}} & H_K \otimes \mathcal{E} \\ \big\downarrow \exists\, T & & \big\downarrow \iota & & \| \\ H_K \otimes \mathcal{H}_1 & \xrightarrow{M_\Psi} & \mathcal{N} & \xrightarrow{\iota_\mathcal{N}} & H_K \otimes \mathcal{E} \end{array} .$$

The contractive module map T corresponds to a contractive multiplier $\Gamma : X \to B(\mathcal{H}_2, \mathcal{H}_1)$. McCullough and Trent showed that, in general, one cannot assume that the $M_\Gamma = T$ of the previous diagram is a partial isometry, and conjectured the following:

CONJECTURE 6.1. ([16]) *If $\mathcal{M} \subset \mathcal{N} \subset H_K \otimes \mathcal{E}$ are invariant subspaces and $\Psi : X \to B(\mathcal{H}_1, \mathcal{E})$ is a multipliers such that M_Ψ is a partial isometry with range $\mathcal{N}$, then there exist a Hilbert space $\mathcal{H}_3$ and a multiplier $\Gamma : X \to B(\mathcal{H}_3, \mathcal{H}_1)$ such that M_Γ is a partial isometry and $M_{\Psi\Gamma}$ is a partial isometry with range equal to $\mathcal{M}$.*

Proof. We claim that $\mathrm{Ker}(M_\Psi) \oplus (M_\Psi)^*(\mathcal{M})$ is an invariant subspace of $H_K \otimes \mathcal{H}_1$. To see this, we only need to check that whenever $x \in (M_\Psi)^*(\mathcal{M})$ and $\phi \in M(K)$, then $(M_\phi \otimes I_{\mathcal{H}_1})x \in \mathrm{Ker}(M_\Psi) \oplus (M_\Psi)^*(\mathcal{M})$, and this follows from

$$(M_\phi \otimes I_{\mathcal{H}_1})x = \big[(M_\phi \otimes I_{\mathcal{H}_1})x - (M_\Psi)^* M_\Psi (M_\phi \otimes I_{\mathcal{H}_1})x\big] + (M_\Psi)^* M_\Psi (M_\phi \otimes I_{\mathcal{H}_1})x.$$

Since M_Ψ is a partial isometry, the term inside the brackets belongs to the kernel of M_Ψ, and since M_Ψ is a module map, $M_\Psi(M_\phi \otimes I_{\mathcal{H}_1})x = (M_\phi \otimes I_\mathcal{E}) M_\Psi x \in \mathcal{M}$, and this implies that $(M_\Psi)^* M_\Psi (M_\phi \otimes I_{\mathcal{H}_1})x \in (M_\Psi)^*(\mathcal{M})$. Then $\mathrm{Ker}(M_\Psi) \oplus (M_\Psi)^*(\mathcal{M})$ has a Poisson kernel $K : \mathrm{Ker}(M_\Psi) \oplus (M_\Psi)^*(\mathcal{M}) \to H_K \otimes \ell_2$, and $K^* : H_K \otimes \ell_2 \to H_K \otimes \mathcal{H}_1$ is a partial isometry module map, and hence it corresponds to a multiplier $\Gamma : X \to B(\ell_2, \mathcal{H}_1)$ with $M_\Gamma = K^*$. Since $M_\Psi \circ M_\Psi$ has range $\mathcal{M}$, we finish the proof. ∎

This criterion can be used to characterize the invariant subspaces of quotients of the weighted Fock spaces. Namely, a subquotient $\mathcal{M}$ of $\mathcal{N}_J \otimes \ell_2$ with Poisson kernel $K : \mathcal{M} \to \mathcal{N}_J \otimes \mathcal{M}$ is a submodule if and only if the map $\iota_\mathcal{M} \circ K^* : \mathcal{N}_J \otimes \mathcal{M} \to \mathcal{N}_J \otimes \ell_2$ is a module map.

6.2. UNIQUENESS OF THE RESOLUTIONS. We start with the simplest case:

PROPOSITION 6.2. *If $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{F}^2(\omega_\alpha)$ are submodules of $\mathcal{F}^2(\omega_\alpha)$ for which $\mathcal{M}_1^\perp$ and $\mathcal{M}_2^\perp$ are isomorphic, then $\mathcal{M}_1 = \mathcal{M}_2$.*

Proof. Let $\mathcal{M}_1, \mathcal{M}_2$ be two submodules of $\mathcal{F}^2(\omega_\alpha)$ and suppose that there exists $u : \mathcal{M}_1^\perp \to \mathcal{M}_2^\perp$ such that u and u^{-1} are isometric module maps. By

Corollary 3.8, there exist $\varphi, \psi \in R^\infty(\omega_\alpha)$ satisfying: $\|R_\varphi\| = 1$, $P_{\mathcal{M}_2^\perp} R_\phi = u P_{\mathcal{M}_1^\perp}$, $\|R_\psi\| = 1$, and $P_{\mathcal{M}_1^\perp} R_\psi = u^{-1} P_{\mathcal{M}_2^\perp}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{F}^2(\omega_\alpha) & \overset{P_{\mathcal{M}_1^\perp}}{\longrightarrow} & \mathcal{M}_1^\perp & \longrightarrow & 0 \\
 & & & & \big\downarrow R_\varphi & & \big\downarrow u & & \\
0 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \mathcal{F}^2(\omega_\alpha) & \overset{P_{\mathcal{M}_2^\perp}}{\longrightarrow} & \mathcal{M}_2^\perp & \longrightarrow & 0 \\
 & & & & \big\downarrow R_\psi & & \big\downarrow u^{-1} & & \\
0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{F}^2(\omega_\alpha) & \overset{P_{\mathcal{M}_1^\perp}}{\longrightarrow} & \mathcal{M}_1^\perp & \longrightarrow & 0
\end{array}.$$

We first claim that for every $x \in \mathcal{M}_1^\perp$ and $y \in \mathcal{M}_2^\perp$, $u(x) = x \otimes \varphi$ and $u^{-1}(y) = y \otimes \psi$. To see this, let $x \in \mathcal{M}_1^\perp$. Since $P_{\mathcal{M}_2^\perp} R_\varphi x = u(x)$ and $\|x\| = \|u(x)\| = \|P_{\mathcal{M}_2^\perp} R_\varphi x\| \leqslant \|R_\varphi x\| \leqslant \|x\|$, it follows that $R_\varphi x \in \mathcal{M}_2^\perp$, and hence that $u(x) = R_\varphi(x) = x \otimes \varphi$. The other case is similar. We now claim that $\varphi \otimes \psi = \delta_0$. This follows from the fact that the product of two non-zero terms of $R^\infty(\omega_\alpha)$ is non-zero (to see this, just look at the non-zero coefficients of "smallest" length). Let $x \in \mathcal{M}_1^\perp$ be non-zero. Since $x = x \otimes \varphi \otimes \psi$, we get that $x \otimes (\varphi \otimes \psi - \delta_0) = 0$. Finally, we claim that there exist a_0, b_0 with $|a_0| = |b_0| = 1$ such that $\varphi = a_0 \delta_0$ and $\psi = b_0 \delta_0$. Since $\varphi, \psi \in \mathcal{F}^2(\omega_\alpha)$, there exists coefficients $(a_\alpha)_{\alpha \in \mathbb{F}_n^+}$ and $(b_\alpha)_{\alpha \in \mathbb{F}_n^+}$ such that

$$\varphi = a_0 \delta_0 + \sum_{|\alpha| \geqslant 1} a_\alpha \delta_\alpha, \quad \psi = b_0 \delta_0 + \sum_{|\alpha| \geqslant 1} b_\alpha \delta_\alpha,$$

and

$$\varphi \otimes \psi = (a_0 b_0)\delta_0 + \sum_{|\gamma| \geqslant 1} \Big[\sum_{\alpha\beta = \gamma} a_\alpha b_\beta \Big] \delta_\gamma.$$

Since $\|\varphi\|_2 \leqslant \|R_\varphi\| = 1$ and $\|\psi\|_2 \leqslant \|R_\psi\| = 1$, we have that $\omega_\alpha |a_\alpha|^2 \leqslant 1$ and $\omega_\alpha |b_\alpha|^2 \leqslant 1$ for each $\alpha \in \mathbb{F}_n^+$. In particular, $|a_0| \leqslant 1$ and $|b_0| \leqslant 1$. And since $a_0 b_0 = 1$ we get that $|a_0| = |b_0| = 1$. Then $1 \geqslant \|\varphi\|^2 = |a_0|^2 + \sum_{|\alpha| \geqslant 1} |a_\alpha|^2 \omega_\alpha = 1 + \sum_{|\alpha| \geqslant 1} |a_\alpha|^2 \omega_\alpha$, and this implies that $a_\alpha = 0$ for $|\alpha| \geqslant 1$. Similarly, we obtain that $b_\alpha = 0$ for $|\alpha| \geqslant 1$. Then R_φ and R_ψ are just multiples of the identity and chasing the diagrams we see that $\mathcal{M}_1 = \mathcal{M}_2$. ∎

COROLLARY 6.3. *Suppose that J is a w^*-closed to sided ideal of $F^\infty(\omega_\alpha)$. If $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{N}_J$ are submodules of $\mathcal{N}_J$ for which $\mathcal{M}_1^\perp$ and $\mathcal{M}_2^\perp$ are isomorphic, then $\mathcal{M}_1 = \mathcal{M}_2$.*

Proof. By $\mathcal{M}_1^\perp$ we mean $\mathcal{N}_J \ominus \mathcal{M}_1$ which is a $*$-submodule of $\mathcal{N}_J$. It is also a $*$-submodule of $\mathcal{F}^2(\omega_\alpha)$ because $\mathcal{N}_J$ is a $*$-submodule of $\mathcal{F}^2(\omega_\alpha)$. Then $\mathcal{F}^2(\omega_\alpha) \ominus \mathcal{M}_1^\perp$ and $\mathcal{F}^2(\omega_\alpha) \ominus \mathcal{M}_2^\perp$ are submodules of $\mathcal{F}^2(\omega_\alpha)$ with isomorphic orthogonal complement. By Proposition 6.2, they are equal. Since $\mathcal{M}_i = \mathcal{N}_J \cap [\mathcal{F}^2(\omega_\alpha) \ominus [\mathcal{N}_J \ominus \mathcal{M}_i]]$ for $i = 1, 2$, it follows that $\mathcal{M}_1 = \mathcal{M}_2$. ∎

Corollary 6.3 was proved by Arveson ([6]) for the symmetric Fock space. His proof used C^*-theory. Propositions 6.2 and 6.4 were proved by Popescu for the Full Fock space [30]. His proofs are based on the uniqueness of the characteristic function of a family of row contractions ([21]) (commuting or noncommuting). In the next subsection we show that Proposition 6.4 can be used to give an alternative proof of the uniqueness of the characteristic function in the particular case of C_0 contractions, and to define "characteristic functions" in weighted Fock spaces.

Proposition 6.1 was proved by Douglas and Foiaş [12] for the polydisc algebra $H^2(\mathbb{D}^n)$.

The conclusion of Proposition 6.2 cannot be true for submodules of $\mathcal{F}^2(\omega_\alpha)\otimes \ell_2$. If $\mathcal{M}$ is a submodule of $\mathcal{F}^2(\omega_\alpha)$ and $\mathcal{M}_1 = \mathcal{M}\otimes e_1$ (the Hilbert tensor product of $\mathcal{M}$ with the first vector basis of ℓ_2) and $\mathcal{M}_2 = \mathcal{M}\otimes e_2$, then $\mathcal{M}_1$ and $\mathcal{M}_2$ have isomorphic orthogonal complement but they are not equal. However, they are isomorphic via a map of the form $I_{\mathcal{F}^2(\omega_\alpha)}\otimes U : \mathcal{F}^2(\omega_\alpha)\otimes\ell_2 \to \mathcal{F}^2(\omega_\alpha)\otimes\ell_2$. Furthermore, if $\mathcal{M}_1 = \{0\} \subset \mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ and $\mathcal{M}_2 = \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}$ where $\mathcal{H}$ and $\mathcal{H}^\perp$ have infinite dimension, then $\mathcal{M}_1^\perp$ and $\mathcal{M}_2^\perp$ are isomorphic to $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$. In the next proposition we show that these are the only possibilities. Recall that the "right slice" of $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ is the smallest subspace $\mathcal{H}\subset\ell_2$ such that $\mathcal{E}\subset\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}$.

Proposition 6.4. *Suppose that $\mathcal{M}_1 \subset \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_1$ and $\mathcal{M}_2\subset\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_2$ are submodules for which $\mathcal{M}_1^\perp$ and $\mathcal{M}_2^\perp$ are isomorphic. Then if $\mathcal{H}_1$ is the right slice of $\mathcal{M}_1^\perp$ and $\mathcal{H}_2$ is the right slice of $\mathcal{M}_2^\perp$ (equivalently, if $\mathcal{M}_1$ and $\mathcal{M}_2$ do not have any nontrivial reducing subspaces), then $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic via a map of the form $I_{\mathcal{F}^2(\omega_\alpha)}\otimes U$ where $U : \mathcal{H}_1\to\mathcal{H}_2$ is unitary.*

Proof. Suppose that there exists $u : \mathcal{M}_1^\perp \to \mathcal{M}_2^\perp$ such that u and u^{-1} are isometric module maps. By Corollary 3.8, there exist $T_1 \in R^\infty(\omega_\alpha)\overline{\otimes}B(\mathcal{H}_1,\mathcal{H}_2)$ and $T_2\in R^\infty(\omega_\alpha)\overline{\otimes}B(\mathcal{H}_2,\mathcal{H}_1)$ such that $\|T_1\| = 1$, $P_{\mathcal{M}_2^\perp}T_1 = uP_{\mathcal{M}_1^\perp}$, $\|T_2\| = 1$, and $P_{\mathcal{M}_1^\perp}T_2 = u^{-1}P_{\mathcal{M}_2^\perp}$. Following the proof of Proposition 6.2, we see that $uz = T_1z$ and $uz' = T_2z'$ for every $z\in\mathcal{M}_1^\perp$ and $z'\in\mathcal{M}_2^\perp$, and this implies that $T_2T_1z = z$ for every $z\in\mathcal{M}_1^\perp$. For each $\alpha\in\mathbb{F}_n^+$, there exist $A_\alpha\in B(\mathcal{H}_1,\mathcal{H}_2)$ and $B_\alpha\in B(\mathcal{H}_2,\mathcal{H}_1)$ such that

$$T_1 = R_0\otimes A_0 + \sum_{|\alpha|\geqslant 1} R_\alpha\otimes A_\alpha \quad\text{and}\quad T_2 = R_0\otimes B_0 + \sum_{|\alpha|\geqslant 1} R_\alpha\otimes B_\alpha.$$

Moreover, $T_2T_1 = R_0\otimes B_0A_0 + \sum_{|\gamma|\geqslant 1} R_\delta\otimes\Big[\sum_{\beta\alpha=\gamma} B_\beta A_\alpha\Big]$. Since $\|T_1\| = \|T_2\| = 1$, it follows that $\|A_0\|\leqslant 1$ and $\|B_0\|\leqslant 1$. Let $z = \sum_\alpha \delta_\alpha\otimes x_\alpha \in\mathcal{M}_1^\perp$. We claim that $B_0A_0x_\beta = x_\beta$ for every $\beta\in\mathbb{F}_n^+$. If $x_\beta = 0$ there is nothing to prove, so assume that $x_\beta\neq 0$. Since $\mathcal{M}_1^\perp$ is invariant under $(L_\alpha\otimes I_{\ell_2})^*$, we have that $w = (L_\beta\otimes I_{\ell_2})^*z = \delta_0\otimes x_\beta + [\text{higher terms}]\in\mathcal{M}_1^\perp$. Then, since $T_2T_1w = \delta_0\otimes B_0A_0x_\beta + [\text{higher terms}]$ and $T_2T_1w = w$, we get that $B_0A_0x_\beta = x_\beta$. We will see now that $T_1 = I_{\mathcal{F}^2(\omega_\alpha)}\otimes A_0$.

Since $\|T_1\| = 1$, $\|A_0 x_\beta\| = \|x_\beta\|$, and $T_1(\delta_0 \otimes x_\beta) = \delta_0 \otimes A_0 x_\beta + \sum_{|\alpha|\geqslant 1} \delta_\alpha \otimes A_\alpha x_\beta$, then

$$\|T_1(\delta_0 \otimes x_\beta)\|^2 = \|x_\beta\|^2 + \sum_{|\alpha|\geqslant 1} \omega_\alpha \|A_\alpha x_\beta\|^2 \leqslant \|x_\beta\|^2.$$

Hence $A_\alpha x_\beta = 0$ for $|\alpha| \geqslant 1$. And since the span of the x_β's is dense in $\mathcal{H}_1$, we conclude that $A_\alpha = 0$ for $|\alpha| \geqslant 1$. Similarly we obtain that $B_\alpha = 0$ for $|\alpha| \geqslant 1$, and this implies that $A_0 : \mathcal{H}_1 \to \mathcal{H}_2$ is unitary. ∎

A surjective coisometric module map $\Phi : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H} \to \mathcal{E}$ is a *minimal resolution* if $\mathcal{H}$ is the "right-slice" of $\Phi^*(\mathcal{E})$. We will see that Proposition 6.4 implies easily that the Poisson kernel and minimal resolutions are essentially unique.

PROPOSITION 6.5. *Let $\mathcal{E}$ be a subquotient of $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ with Poisson kernel K and suppose that there exists a surjective coisometry module map $\Phi : \mathcal{F}^2(\omega_\alpha) \otimes \ell_2 \to \mathcal{E}$. Then there exists a partial isometry U such that for every $x \in \mathcal{E}$, $\Phi^*(x) = (I \otimes U)K(x)$. Furthermore, if $\Phi_1 : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1 \to \mathcal{E}$ and $\Phi_2 : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1 \to \mathcal{E}$ are two minimal resolutions of $\mathcal{E}$, there exists a unitary module map $V : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1 \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_2$ such that $\Phi_2 \circ V = \Phi_1$.*

Proof. It follows from Lemma 4.3 that $K^*(\mathcal{E})$ and $\Phi^*(\mathcal{E})$ are isomorphic to $\mathcal{E}$, and hence they are isomorphic to each other via the map $\Phi^* K^*$. Let $\mathcal{H}_1 \subset \mathcal{E}$ be the "right slice" of $K(\mathcal{E}) \subset \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{E}$ and $\mathcal{H}_2 \subset \ell_2$ the "right slice" of $\Phi^*(\mathcal{E}) \subset \mathcal{F}^2(\omega_\alpha)\otimes\ell_2$. By Proposition 6.4, there exists a unitary map $A_0 : \mathcal{H}_1 \to \mathcal{H}_2$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_1 & \xrightarrow{P_{K(\mathcal{E})}} & K(\mathcal{E}) & \xrightarrow{K^*} & \mathcal{E} \\ \big\downarrow{\scriptstyle I\otimes A_0} & & \big\downarrow{\scriptstyle \Phi^* K^*} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_2 & \xrightarrow{P_{\Phi^*(\mathcal{E})}} & \Phi^*(\mathcal{E}) & \xrightarrow{\Phi} & \mathcal{E} \end{array}.$$

Moreover, for every $x \in K(\mathcal{E})$, $(I \otimes A_0)(x) = \Phi^* K^*(x)$ and hence $\Phi^*(x) = \Phi^* K^*(K(x)) = (I \otimes A_0)K(x)$. We finish the first part of the proof by extending A_0 to a partial isometry $U : \mathcal{E} \to \ell_2$. The second part follows easily from the first one. ∎

In the Full Fock space, the uniqueness of minimal resolutions is due to Frazho ([14]), Bunce ([8]), and Popescu ([22]). Proposition 6.5 has an amusing corollary. If $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha)$ is a $*$-invariant subspace with Poisson kernel K, it has two natural resolutions: the orthogonal projection $P : \mathcal{F}^2(\omega_\alpha) \otimes \ell_2 \to \mathcal{E}$ and the adjoint of the Poisson kernel $K^* : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{E} \to \mathcal{E}$. Since P^* is just the inclusion map, K is essentially an inclusion map.

COROLLARY 6.6. *Let $\mathcal{E} \subset \mathcal{F}^2(\omega_\alpha)$ be a $*$-invariant subspace with Poisson kernel K. Then there exists $x_0 \in \mathcal{E}$ such that $K(x) = x \otimes_2 x_0$.*

Such a simple formula is not apparent from the definition of the Poisson kernel. However, once we know it is true, we can compute the kernel and find out that $x_0 = \frac{P_\mathcal{E}\delta_0}{\|P_\mathcal{E}\delta_0\|}$.

6.3. CHARACTERISTIC FUNCTIONS. Two module maps $\Phi : \mathcal{E}_2 \to \mathcal{E}_1$ and $\Psi : \mathcal{F}_2 \to \mathcal{F}_1$ are *unitarily equivalent* if and only if there exist unitary module maps $U_2 : \mathcal{E}_2 \to \mathcal{F}_2$ and $U_1 : \mathcal{E}_1 \to \mathcal{F}_1$ such that $U_1 \circ \Phi = \Psi \circ U_2$.

THEOREM 6.7. *Let $\mathcal{E}$ and $\mathcal{F}$ be subquotients of $\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ with minimal projective resolutions*

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\Phi_3} & \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_3 & \xrightarrow{\Phi_2} & \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_2 & \xrightarrow{\Phi_1} & \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_1 & \xrightarrow{\Phi_0} & \mathcal{E} & \longrightarrow & 0 \\ \cdots & \xrightarrow{\Psi_3} & \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{K}_3 & \xrightarrow{\Psi_2} & \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{K}_2 & \xrightarrow{\Psi_1} & \mathcal{F}^2(\omega_\alpha)\otimes\mathcal{K}_1 & \xrightarrow{\Psi_0} & \mathcal{F} & \longrightarrow & 0 \end{array}.$$

Then $\mathcal{E}$ is isomorphic to $\mathcal{F}$ if and only if Φ_1 is unitarily equivalent to Ψ_1.

Proof. Suppose first that $u:\mathcal{E}\to\mathcal{F}$ is an isomorphism. Then it follows from Proposition 6.5 that there exists a unitary $U_1:\mathcal{H}_1\to\mathcal{K}_1$ such that $u\circ\Phi_0=\Psi_0\circ(I_{\mathcal{F}^2(\omega_\alpha)}\otimes U_1)$. Hence $I_{\mathcal{F}^2(\omega_\alpha)}\otimes U_1:\mathrm{Ker}\Phi_0\to\mathrm{Ker}\Psi_0$ is an isomorphism. We apply Proposition 6.5 again to the minimal projective resolutions

$$\Phi_1:\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_2\to\mathrm{Ker}\Phi_0 \quad\text{and}\quad \Psi_1:\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{K}_2\to\mathrm{Ker}\Psi_0$$

and we find a unitary map $U_2:\mathcal{H}_2\to\mathcal{K}_2$ such that $(I_{\mathcal{F}^2(\omega_\alpha)}\otimes U_1)\circ\Phi_1=\Psi_1\circ(I_{\mathcal{F}^2(\omega_\alpha)}\otimes U_2)$. This implies that Φ_1 and Ψ_1 are unitarily equivalent. Of course this extends to all maps Φ_i and Ψ_i.

Suppose now that there exist unitary module maps $U_1:\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_1\to\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{K}_1$ and $U_2:\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_2\to\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{K}_2$ such that $U_1\circ\Phi_1=\Psi_1\circ U_2$. Since $\mathrm{Ker}\Phi_0=\mathrm{Im}\Phi_1$ and $\mathrm{Ker}\Psi_0=\mathrm{Im}\Psi_1$, we easily check that $U_1(\mathrm{Ker}\Phi_0)=\mathrm{Ker}\Psi_0$ and $U_1(\mathrm{Ker}\Phi_0)^\perp=(\mathrm{Ker}\Phi_0)^\perp$. We need to check that $U_1:(\mathrm{Ker}\Phi_0)^\perp\to(\mathrm{Ker}\Psi_0)^\perp$ is a module map with respect to the module structure of $(\mathrm{Ker}\Phi_0)^\perp$ and $(\mathrm{Ker}\Psi_0)^\perp$. This will finish the proof because $\mathcal{E}$ is isomorphic to $\Phi_0^*(\mathcal{E})=(\mathrm{Ker}\Phi_0)^\perp$ and $\mathcal{F}$ is isomorphic to $\Psi_0^*(\mathcal{F})=(\mathrm{Ker}\Psi_0)^\perp$ (Lemma 4.3). For $i\leqslant n$, let $V_i=P_{(\mathrm{Ker}\Phi_0)^\perp}(L_i\otimes I_{\mathcal{H}_1})|_{(\mathrm{Ker}\Phi_0)^\perp}$ and $T_i=P_{(\mathrm{Ker}\Psi_0)^\perp}(L_i\otimes I_{\mathcal{K}_1})|_{(\mathrm{Ker}\Psi_0)^\perp}$. We need to check that $U_1V_i=T_iU_1$ for $i\leqslant n$. Let $x\in(\mathrm{Ker}\Phi_0)^\perp$, then

$$\begin{aligned} U_1V_ix &= U_1P_{(\mathrm{Ker}\Phi_0)^\perp}(L_i\otimes I_{\mathcal{H}_1})x \\ &= U_1(L_i\otimes I_{\mathcal{H}_1})x-U_1P_{\mathrm{Ker}\Phi_0}(L_i\otimes I_{\mathcal{H}_1})x \\ &= (L_i\otimes I_{\mathcal{K}_1})U_1x-U_1(L_i\otimes I_{\mathcal{H}_1})P_{\mathrm{Ker}\Phi_0}x \\ &= (L_i\otimes I_{\mathcal{K}_1})U_1x-(L_i\otimes I_{\mathcal{K}_1})U_1P_{\mathrm{Ker}\Phi_0}x \\ &= (L_i\otimes I_{\mathcal{K}_1})U_1x-(L_i\otimes I_{\mathcal{K}_1})P_{\mathrm{Ker}\Psi_0}U_1x \\ &= (L_i\otimes I_{\mathcal{K}_1})U_1x-P_{\mathrm{Ker}\Psi_0}(L_i\otimes I_{\mathcal{K}_1})U_1x \\ &= P_{(\mathrm{Ker}\Psi_0)^\perp}(L_i\otimes I_{\mathcal{K}_1})U_1x=T_iU_1x. \end{aligned}$$

These equalities are based on the fact that $U_1:\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_1\to\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{K}_1$, $P_{\mathrm{Ker}\Phi_0}:\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_1\to\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_1$ and $P_{\mathrm{Ker}\Psi_0}:\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{K}_1\to\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{K}_1$ are module maps, and that $U_1P_{\mathrm{Ker}\Phi_0}=P_{\mathrm{Ker}\Psi_0}U_1P_{\mathrm{Ker}\Phi_0}$. ∎

In the Full Fock space, the map Φ_1 corresponds to Popescu's characteristic function ([21]; see also Theorem 2.1 of [30]).

6.4. HOMOLOGY. In this section we use ideas of Greene ([15]) to study the homology of some natural complexes associated to projective resolutions. Let $\mathcal{E}\subset\mathcal{F}^2(\omega_\alpha)\otimes\ell_2$ be a subquotient, and let

$$\cdots\xrightarrow{\Phi_3}\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_2\xrightarrow{\Phi_2}\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_1\xrightarrow{\Phi_1}\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{H}_0\xrightarrow{\Phi_0}\mathcal{E}\longrightarrow 0$$

be a projective resolution. Suppose that $J \subset F^\infty(\omega_\alpha)$ is a w^*-closed 2-sided ideal with $\mathcal{N}_J$ the orthogonal complement of the image of J. Since $\mathcal{N}_J$ is a $*$-L-submodule the orthogonal projection $\partial : \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_k \to \mathcal{N}_J \otimes \mathcal{N}_J$ is a module map, and since $\mathcal{N}_J$ is a $*$-R-submodule, for every $i \in \mathcal{N}$, there exists a module map $\Psi_i : \mathcal{N}_J \otimes \mathcal{H}_i \to \mathcal{N}_J \otimes \mathcal{H}_{i-1}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_i & \xrightarrow{\Phi_i} & \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_{i-1} \\ \downarrow \partial & & \downarrow \partial \\ \mathcal{N}_J \otimes \mathcal{H}_i & \xrightarrow{\Psi_i} & \mathcal{N}_J \otimes \mathcal{H}_{i-1} \end{array}.$$

These maps induce a complex

$$\cdots \quad \mathcal{N}_J \otimes \mathcal{H}_3 \xrightarrow{\Psi_3} \mathcal{N}_J \otimes \mathcal{H}_2 \xrightarrow{\Psi_2} \mathcal{N}_J \otimes \mathcal{H}_1 \xrightarrow{\Psi_1} \mathcal{N}_J \otimes \mathcal{H}_0$$

which in general is not exact. In particular, if J is the w^*-closed 2-sided ideal of elements of $F^\infty(\omega_\alpha)$ that "vanish" at zero, then we get the complex

$$\cdots \xrightarrow{\Psi_4} \mathcal{H}_3 \xrightarrow{\Psi_3} \mathcal{H}_2 \xrightarrow{\Psi_2} \mathcal{H}_1 \xrightarrow{\Psi_1} \mathcal{H}_0 \,.$$

There are many ideals J for which the homology of these complexes can be described. For simplicity, denote $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_i$ by P_i and $\mathcal{N}_J \otimes \mathcal{H}_i$ by C_i. Then we have the following commutative diagram:

$$\begin{array}{cccccccccccc} \cdots & \xrightarrow{\Phi_4} & P_3 & \xrightarrow{\Phi_3} & P_2 & \xrightarrow{\Phi_2} & P_1 & \xrightarrow{\Phi_1} & P_0 & \xrightarrow{\Phi_0} & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & & & \\ \cdots & \xrightarrow{\Psi_4} & C_3 & \xrightarrow{\Psi_3} & C_2 & \xrightarrow{\Psi_2} & C_1 & \xrightarrow{\Psi_1} & C_0 & & & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & 0 & & 0 & & & & \end{array} \tag{6.1}$$

Suppose that J is the w^*-closed two-sided ideal generated by the maps L_α for $|\alpha| \geqslant N$. Then $\mathcal{N}_J$ is the span of the δ_β's with $|\beta| < N$. We can complete the vertical arrows into short exact sequences in a natural way. Let E_N be the span of the δ_β's with $|\beta| = N$, and define

$$\partial : \left[\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_i\right] \otimes E_N \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_i \quad \text{by } \partial(x \otimes \delta_\beta) = (L_\beta \otimes I)(x).$$

It follows easily that

$$0 \longrightarrow \left[\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_i\right] \otimes E_N \xrightarrow{\partial} \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_i \xrightarrow{\partial} \mathcal{N}_J \otimes \mathcal{H}_i \longrightarrow 0$$

is a short exact sequence (notice that for every $i \leqslant n$ there exists a constant $c \geqslant 1$ such that $\frac{1}{c}\|\varphi\| \leqslant \|L_i\varphi\| \leqslant c\|\varphi\|$ for every $\varphi \in \mathcal{F}^2(\omega_\alpha)$). Denote $\left[\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}_i\right] \otimes E_N$ by Q_i, let $\widehat{\Phi}_i : Q_i \to Q_{i-1}$ be the map $\Phi_i \otimes I_{E_N}$, and define $\partial : \mathcal{E} \otimes E_N \to \mathcal{E}$ by $\partial(x \otimes \delta_\beta) = T_\beta(x)$, where the $T_i = P_{\mathcal{E}}(L_i \otimes I_{\ell_2})|_{\mathcal{E}}$'s are the maps associated with the subquotient $\mathcal{E}$. Then we have the following commutative diagram in which all columns (excepts perhaps $\partial : \mathcal{E}\otimes E_N \to \mathcal{E}$) and the first two rows are exact:

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\widehat{\Phi}_4} & Q_3 & \xrightarrow{\widehat{\Phi}_3} & Q_2 & \xrightarrow{\widehat{\Phi}_2} & Q_1 & \xrightarrow{\widehat{\Phi}_1} & Q_0 & \xrightarrow{\widehat{\Phi}_0} & \mathcal{E}\otimes E_N & \longrightarrow & 0\\
 & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\
\cdots & \xrightarrow{\Phi_4} & P_3 & \xrightarrow{\Phi_3} & P_2 & \xrightarrow{\Phi_2} & P_1 & \xrightarrow{\Phi_1} & P_0 & \xrightarrow{\Phi_0} & \mathcal{E} & \longrightarrow & 0\\
 & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & & & \\
\cdots & \xrightarrow{\Psi_4} & C_3 & \xrightarrow{\Psi_3} & C_2 & \xrightarrow{\Psi_2} & C_1 & \xrightarrow{\Psi_1} & C_0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & 0 & & & &
\end{array}.$$

Using standard arguments in homology we get

PROPOSITION 6.8.

$\mathrm{Ker}\Psi_1/\mathrm{Im}\Psi_2 \simeq \mathrm{Ker}(\partial : \mathcal{E}\otimes E_N \to \mathcal{E}), C_0/\mathrm{Im}\Psi_1 \simeq \mathcal{E}/\partial(\mathcal{E}\otimes E_N)$, *and*

$\mathrm{Ker}\Psi_i/\mathrm{Im}\Psi_{i+1} = 0$ *for* $i = 2, 3, \ldots$.

Sketch of the proof. We will show that $\mathrm{Ker}\Psi_2 = \mathrm{Im}\Psi_3$. Since $\mathrm{Im}\Psi_3 \subset \mathrm{Ker}\Psi_2$, we only need to show that $\mathrm{Ker}\Psi_2 \subset \mathrm{Im}\Psi_3$. Let $x_2 \in \mathrm{Ker}\Psi_2$. Lift x_2 to $y_2 \in P_2$ (i.e., $\partial y_2 = x_2$) and push it to $\Phi_2(y_2) \in P_1$. Since $\partial(\Phi_2(y_2)) = 0$, there exists $z_1 \in Q_1$ such that $\partial z_1 = \Phi_2(y_2)$. Since $\partial\widehat{\Phi}_1(z_1) = 0$ and since ∂ is one-to-one, $\widehat{\Phi}_1(z_1) = 0$ and hence, $z_1 = \widehat{\Phi}_2(z_2)$ for some $z_2 \in Q_2$. Now $\Phi_2\partial z_2 = \partial\widehat{\Phi}_2 z_2 = \partial z_1 = \Phi_2(y_2)$. Then $\Phi_2(y_2 - \partial z_2) = 0$, and hence, there exists $y_3 \in P_3$ such that $\Phi_3(z_3) = y_2 - \partial z_2$. Then $\Psi_3(\partial y_3) = \partial\Phi_3(y_3) = \partial(y_2 - \partial z_2) = \partial y_2 = x_2$, and hence $x_2 \in \mathrm{Im}\Psi_3$. The proof of $\mathrm{Ker}\Psi_i = \mathrm{Im}\Psi_{i+1}$ for $i = 3, 4, \ldots$ is identical.

We will show now that $\mathrm{Ker}\Psi_1/\mathrm{Im}\Psi_2 \simeq \mathrm{Ker}(\partial : \mathcal{E}\otimes E_N \to \mathcal{E})$. Let $x_1 \in \mathrm{Ker}\Psi_1$. Lift it to $y_1 \in P_1$ and push it to $\Phi_1 y_1 \in P_0$. Since $\partial\Phi_1 y_1 = 0$, $\Phi_1 y_1 = \partial z_0$ for some $z_0 \in Q_0$. Then $\widehat{\Phi}_0 z_0 \in \mathrm{Ker}(\partial : \mathcal{E}\otimes E_N \to \mathcal{E})$. We claim that $x_1 \mapsto \widehat{\Phi}_0 z_0$ is a well defined map. Indeed, if we lift $x_1 \in \mathrm{Ker}\Psi_1$ to $y_1' \in P_1$, push it to $\Phi_1 y_1' = \partial z_0'$ then $\widehat{\Phi}_0 z_0' \in \mathrm{Ker}(\partial : \mathcal{E}\otimes E_N \to \mathcal{E})$. Since $\partial(y_1 - y_1') = 0$ then $y_1 - y_1' = \partial z_1$ for some $z_1 \in Q_1$ and hence $\partial z_0 - \partial z_0' = \Phi_1 y_1 - \Phi_1 y_1' = \Phi_1\partial z_1 = \partial\widehat{\Phi}_1 z_1$. Since ∂ is one-to-one, then $z_0 = z_0' + \widehat{\Phi}_1 z_1$ and hence $\widehat{\Phi}_0 z_0 = \widehat{\Phi}_0(z_0' + \widehat{\Phi}_1 z_1) = \widehat{\Phi}_0 z_0'$, which shows that the map is well defined. Now if $x_1 \in \mathrm{Im}\Phi_2$, we can lift it to a $y_1 \in P_1$ that satisfies $\Phi_1 y_1 = 0$. Therefore, we get a map from $\mathrm{Ker}\Psi_1/\mathrm{Im}\Psi_2$ to $\mathrm{Ker}(\partial : \mathcal{E}\otimes E_N \to \mathcal{E})$. We will construct the inverse of this map now. Let $x \in \mathrm{Ker}(\partial : \mathcal{E}\otimes E_N \to \mathcal{E})$. Lift it to $z_0 \in Q_0$ and push it to $\partial z_0 \in P_0$. Since $\Phi_0\partial z_0 = 0$, $\partial z_0 = \Phi_1 y_1$ for some $y_1 \in P_1$. Then $\partial y_1 \in \mathrm{Ker}\Psi_1$. If we lift $x \in \mathrm{Ker}(\partial : \mathcal{E}\otimes E_N \to \mathcal{E})$ to $z_0' \in Q_0$, push it $\partial z_0' = \Phi_1 y_1'$, then $\partial y_1' \in \mathrm{Ker}\Psi_1$. Since $\widehat{\Phi}_0(z_0 - z_0') = 0$, then $z_0 - z_0' = \widehat{\Phi}_1 z_1$ for some $z_1 \in Q_1$ and then $\Phi_1 y_1 = \Phi_1 y_1' = \partial z_0 - \partial z_0' = \partial\widehat{\Phi}_1(z_1) = \Phi_1(\partial z_1)$. Hence $\Phi_1(y_1 - y_1' - \partial z_1) = 0$ and $y_1 - y_1' - \partial z_1 = \Phi_2(y_2)$ for some $y_2 \in P_2$. Then $\partial y_1 - \partial y_1' = \partial(y_1 - y_1' - \partial z_1) = \partial\Phi_2(y_2) = \Psi_2(\partial y_2) \in \mathrm{Im}\Psi_2$. This shows that the map $x \mapsto \partial y_1 + \mathrm{Im}\Psi_2$ from $\mathrm{Ker}(\partial : \mathcal{E}\otimes E_N \to \mathcal{E})$ to $\mathrm{Ker}\Psi_1/\mathrm{Im}\Psi_2$ is well defined. Since these two maps are clearly inverse to each other, we prove the result.

The proof of $C_0/\mathrm{Im}\Psi_1 \simeq \mathcal{E}/\partial(\mathcal{E}\otimes E_N)$ is similar. ∎

REMARK 6.9. Proposition 6.8 is similar to the work of D. Greene ([15]). He considers a "free" resolution of symmetric Fock spaces and, in the simplest example, looks at the diagram (6.1) with the ideal corresponding to $N = 1$. He completes the columns into exact sequences, but due to the commutativity of the operators, it cannot be done in one step (as it is done here). He uses the Koszul complex to accomplish this. Then he uses homological arguments to relate the homology of the "last" row to the homology of the "last" column.

6.5. FINAL REMARK. For simplicity, we decided to work with subquotients of $\mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ instead of working with representation of the unital norm closed algebra generated by $L_1, \dots, L_n$, which we denote by $\mathcal{A}_n(\omega_\alpha)$. A representation on this algebra is determined by an n-tuple of operators $T_1, \dots, T_n \in B(\mathcal{H})$. And since there are too many bounded representations, we would have to consider completely contractive ones. If $T_1, \dots, T_n \in B(\mathcal{H})$ induces a completely contractive representation on $\mathcal{A}_n(\omega_\alpha)$, then using Lemma 4.1 and Wittstock's Theorem we get

$$\sum_{|\alpha| \leqslant n} a_\alpha T_\alpha T_\alpha \geqslant 0.$$

But this still leads to too many representations, even in the Full Fock space. For example, if we allowed $T_1 T_1^* + \cdots + T_n T_n^* = I_{\mathcal{H}}$, we would have to consider the representations of Cuntz' algebras (since the Cuntz algebras are simple C^*-algebra which are not of type I, their representations cannot be classified up to unitary equivalence). We can remove the "spherical" representations if we require that

$$\text{SOT} - \lim_{N \to \infty} \sum_{|\gamma| > N} \Big[\sum_{\substack{\alpha\beta = \gamma \\ |\alpha| \leqslant N}} \frac{a_\beta}{\omega_\alpha} \Big] T_\gamma T_\gamma^* = 0,$$

which we called the C_0-condition in [4]. But then we showed in [4] that in this case there exists a Poisson kernel $K : \mathcal{H} \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$, and hence $(\mathcal{H} : T_1, \dots, T_n)$ is isomorphic to a $*$-submodule of $\mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$. Therefore, working with the subquotients of $\mathcal{F}^2(\omega_\alpha) \otimes \ell_2$ is equivalent to working with the completely contractive representations of $\mathcal{A}_n(\omega_\alpha)$ that satisfy the C_0-condition. Moreover, we showed in [4] that the C_0-completely contractive representations of $\mathcal{A}_n(\omega_\alpha)$ coincide with the C_0-completely representation of $F^\infty(\omega_\alpha)$.

We use this to give counter-examples to a question of Muhly and Solel ([18], p. 20) if we work with the restricted category of C_0-completely contractive representations. It follows from Theorem 5.4 that $\mathcal{N}_J$ is a strongly orthogonally projective module of $F^\infty(\omega_\alpha)/J$. However, Theorem 4.8 states that if $\mathcal{N}_J$ is strongly orthogonally projective in $F^\infty(\omega_\alpha)$ then $\mathcal{N}_J$ is invariant under $L_1, \dots, L_n$. For example, if J is the commutator ideal in the Full Fock space, $\mathcal{N}_J$ is the Symmetric Fock space, which is not invariant. Hence we obtain:

PROPOSITION 6.10. *Let J be a w^*-closed 2-sided ideal of $F^\infty(\omega_\alpha)$ such that $\mathcal{N}_J$ is not invariant under $L_1, \dots, L_n$. Then $\mathcal{N}_J$ is strongly orthogonally projective in the category of C_0-completely contractive representations of $F^\infty(\omega_\alpha)/J$, but it is not strongly orthogonally projective in the category of C_0-completely contractive representations of $F^\infty(\omega_\alpha)$.*

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