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CUNTZ-PIMSNER ALGEBRAS OF GROUP ACTIONS

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Abstract. We associate a $*$-bimodule over the group algebra to every self-similar group action on the space of one-sided sequences. Completions of the group algebra, which agree with the bimodule are investigated. This gives new examples of Hilbert bimodules and the associated Cuntz-Pimsner algebras. A direct proof of simplicity of these algebras is given. We show also a relation between the Cuntz algebras and the Higman-Thompson groups and define an analog of the Higman-Thompson group for the Cuntz-Pimsner algebra of a self-similar group action.

Keywords: *Self-similar group actions, one-sided shift, bimodules, Cuntz algebras, Cuntz-Pimsner algebras, Higman-Thompson groups, Grigorchuk group.*

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1. INTRODUCTION

The notion of a self-similar group action (fractal group or state-closed group) naturally appears in the theory of groups acting on rooted trees and groups defined by finite transducers. Study of such actions is motivated by examples of exotic groups which are easily defined through their self-similar action.

An example of such a group is the Grigorchuk 2-group. In [12] this group is introduced as a group of measure-preserving transformations of the unit interval. Probably it is the most simple example of a finitely generated infinite torsion group. The action of the Grigorchuk group on the interval has the self-similarity property: the restrictions of the action of any its element on the halfs of the interval is again an element of the group. This is used in a very short proof of the fact that it is an infinite torsion group. Later the same self-similarity was used to prove that the group has intermediate growth (see [13], [14]). It has other interesting properties such as just-infiniteness and finite width (see [15]).

Examples of self-similar groups of this sort include the Gupta-Sidki group ([18]), the just-nonsolvable torsion free group from [5] and other (see [15], [2] and [16] for more examples).

Other, independent topics, where self-similar actions appear (so far implicitly), are the rep-tilings of the Euclidean space and the numeration systems on $\mathbb{Z}^n$ (see the surveys [31], [32]). Some numeration systems originate from self-similar actions of the respective group $\mathbb{Z}^n$. This includes probably the most well known example of a self-similar group action: the adding machine, or the odometer. The paper [25] shows the relation between the self-similar actions of abelian groups and the numeration systems.

R. Grigorchuk, A. Żuk and L. Bartholdi in [17] and [2] used the self-similarity of some group actions to compute the spectra of the discrete Laplace operators of the Schreier graphs. The self-similarity of the action provides an operator recursion, which helps to compute the spectrum.

We show in our paper how these operator recursions can be interpreted in terms of the C^*-bimodules (or C^*-correspondences). So with every self-similar group action we associate a $*$-bimodule Φ over the group algebra $\mathbb{C}G$, which encodes the self-similarity.

We investigate the completions of the group algebra which agree with the self-similarity of the action. Among all such completions there exists a uniquely defined minimal algebra $\mathcal{A}_\Phi$, which is a homomorphic image of any other self-similar completion. Then the bimodule Φ becomes a C^*-bimodule over the algebra $\mathcal{A}_\Phi$.

The C^*-bimodules (correspondences) are important tools in the study of C^*-algebras and von Neumann algebras (see [8], [7], [21]).

The Cuntz-Pimsner algebra $\mathcal{O}_\Phi$, defined in [26], plays the role of a cross product by a bimodule. The Cuntz-Pimsner algebra generalizes the Cuntz algebras $\mathcal{O}_n$, which is the algebra generated by n isometries $S_1, \ldots, S_n$ such that $S_1S_1^* + S_2S_2^* + \cdots + S_nS_n^* = 1$. It also generalizes the Cuntz-Krieger algebras $\mathcal{O}_A$, which correspond to the bimodules over finite-dimensional commutative algebras. From the dynamical point of view the Cuntz-Krieger algebras are the algebras associated in a natural way with Markov shifts (see [10]).

So, self-similar group actions provide new examples of the Cuntz-Pimsner algebras. We prove that these algebras are simple (Theorem 8.3) and study its gauge invariant subalgebra (Section 7).

In the last section we investigate a subgroup $V_n(G)$ of the unitary group of the Cuntz-Pimsner algebra $\mathcal{O}_\Phi$. This group is defined as the group of unitaries which can be expressed as finite sums of products of the standard generators of the Cuntz algebra and the group elements.

We show that if we take only the products of the generators of the Cuntz algebra, then we get the well known Higman-Thompson group ($G_{n,1}$ in notation of [20]). It was one of the first examples of infinite simple finitely presented groups (it is simple for even n and has a simple subgroup of index 2 for odd n). This group (for the case $n = 2$) and its analogs were constructed by R. Thompson in 1965 during his studies in logics. They were used in [23] to construct an example of a finitely presented group with unsolvable word problem and in [30] for embeddings of groups into finitely presented simple groups. See the survey [6] for the properties of the Thompson groups. G. Higman in [20] generalized the Thompson groups and constructed a family of groups which he denoted $G_{n,r}$.

In the case of the Cuntz-Pimsner algebra related to a self-similar group action, the group $V_n(G)$ also has properties similar to the properties of the Higman-Thompson groups. We show in Theorem 9.11 that all the proper quotients of the group $V_n(G)$ are abelian and in Theorem 9.14 show a simple method to find its maximal abelian quotient. In many cases this maximal quotient is trivial, so the group $V_n(G)$ is simple.

The group $V_n(G)$ was introduced for the first time (using a different approach) by C. Röver for G equal to the Grigorchuk group (see [27]). He showed that in this case the group $V_n(G)$ is simple and finitely presented. In [28] he shows that this group is isomorphic to the abstract commensurator of the Grigorchuk group. The last property is based on the fact that the Grigorchuk group belongs to the class of the branch groups (see [15]).

2. SELF-SIMILAR GROUP ACTION

Let X be a finite set, which will be called an *alphabet*. By X^* we will denote the free monoid, generated by the set X. We write the elements of X^* as words $x_1x_2\ldots x_n$ including the empty word $\emptyset$. Thus $X^* = \bigcup\limits_{n\geqslant 0} X^n$, where $X^0 = \{\emptyset\}$ and X^n for $n \geqslant 1$ are Cartesian products. By $|v|$ we denote the length of the word v so that $v \in X^{|v|}$. By the symbol X^ω we denote the set of all infinite unilateral sequences (words) of the form $x_1x_2\ldots$, where $x_i \in X$ for every i.

If $v \in X^*$ and $w \in X^*$ (or $w \in X^\omega$), then the product $vw \in X^*$ (respectively $\in X^\omega$) is defined in the natural way. If A is a subset of X^ω and $v \in X^*$ is a finite word then $vA = \{vw : w \in A\}$.

The set X^ω is naturally identified with the Cartesian product $X^{\mathbb{N}}$. Hence it can be equipped with the topology of the direct (Tikhonov) product of discrete sets X. The basis of open sets in this topology is the collection of all *cylindrical sets* $a_1a_2\ldots a_nX^\omega$, where $a_1a_2\ldots a_n$ runs through X^*. The cylindrical sets are open and closed, thus the space X^ω is totally disconnected. It is also compact and without isolated points, thus is homeomorphic to the Cantor set.

The space X^ω has also a natural measure μ, which is the direct product of the uniform probabilistic measures on the set X. This measure is uniquely defined by the condition $\mu(a_1a_2\ldots a_nX^\omega) = |X|^{-n}$. The (unilateral) *shift* on the space X^ω is the map

$$\sigma : x_1x_2x_3\ldots \mapsto x_2x_3x_4\ldots,$$

which deletes the first letter of the infinite word.

Definition 2.1. A faithful action of a group G on the space X^ω is said to be *self-similar (state-closed)* if for every $g \in G$, and $x \in X$ there exist $h \in G$ and $y \in X$ such that for all $w \in X^\omega$ we have

$$g(xw) = yh(w). \tag{2.1}$$

We will write equation (2.1) formally as

$$g \cdot x = y \cdot h. \tag{2.2}$$

Since the action is faithful, the element h is defined uniquely.

Applying equation (2.1) several times we see that for every finite word $v \in X^*$ and every $g \in G$ there exist $h \in G$ and a word $u \in X^*$ such that $|u| = |v|$ and

$$g(vw) = uh(w) \tag{2.3}$$

for all $w \in X^\omega$. Hence, the self-similar action of the group G induces a natural action on the set X^*, where the word u is the image of v under the action of g, under the conditions of equation (2.3).

Since the beginning of length k of the word $g(w)$ depends only on the beginning of length k of the word w, any self-similar action is an action by homeomorphisms on the space X^ω. Moreover, the induced action on X^* is an action by automorphisms of the naturally constructed rooted tree with the set of vertices X^* and the space of ends X^ω (see [16] for more details). It follows also that every self-similar action preserves the measure μ on the space X^ω.

DEFINITION 2.2. A self-similar action is said to be *level-transitive* if the induced action on X^* is transitive on the sets X^n (note that these sets are invariant by definition).

The action is level-transitive if and only if it is minimal on X^ω, i.e., if all its orbits are dense (see [16]). In this paper all the self-similar actions are considered to be level-transitive. Formula (2.3) will be also written formally as

$$g \cdot v = u \cdot h. \tag{2.4}$$

We use the notation $g(v)$ to denote the image of the word v under the action of the element g, while the notation $g \cdot u$ is used in the formal expressions like (2.4).

EXAMPLE 2.3. Let $G \simeq \mathbb{Z}$ be an infinite cyclic group generated by a with the neutral element e and let $X = \{\mathit{0}, \mathit{1}\}$. Define a self-similar action of G on X^ω putting

$$a \cdot \mathit{0} = \mathit{1} \cdot e, \tag{2.5}$$

$$a \cdot \mathit{1} = \mathit{0} \cdot a. \tag{2.6}$$

Note that these equations define uniquely the action of a on the space X^ω.

The element a as a homeomorphism of the space X^ω is called the *adding machine*. This name originates from the fact that equalities (2.5), (2.6) imply that the action of the adding machine can be interpreted as addition of 1 to a dyadic number. More formally, we have $a(x_1x_2\ldots) = y_1y_2\ldots$ for $x_1x_2\ldots \in X^\omega$ if and only if

$$1 + (x_1 + x_2 2 + x_3 2^2 + x_4 2^3 + \cdots) = y_1 + y_2 2 + y_3 2^2 + y_4 2^3 + \cdots$$

in the ring of dyadics.

The constructed self-similar action of $\mathbb{Z}$ will be also called *adding machine*.

In the paper [25] self-similar actions of the group $\mathbb{Z}^n$ are investigated, which generalize the example of the adding machine. Such actions can be also characterized by the corresponding "numeration system" on $\mathbb{Z}^n$.

EXAMPLE 2.4. The group generated by transformations a, b of the space X^ω, for the alphabet $X = \{\mathit{0}, \mathit{1}\}$, defined inductively by the formulas

$$a \cdot \mathit{0} = \mathit{1} \cdot b, \quad b \cdot \mathit{0} = \mathit{0} \cdot b$$

$$a \cdot \mathit{1} = \mathit{0} \cdot a, \quad b \cdot \mathit{1} = \mathit{1} \cdot a.$$

This group is isomorphic to the "lamplighter group" $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ by the shift. See the papers [17] and [16] for the proofs of this isomorphism. In the paper [17] this representation was used to compute the spectrum of the lamplighter group.

Example 2.5. Many groups are defined directly by their action on the spaces X^ω (in most cases the language of rooted trees was used in the original definitions). These groups have exotic properties rare among the groups defined in other ways. The main tool in the study of such groups is their self-similarity and the actions on the rooted tree X^* and on the space X^ω. See the works [15], [18], [3], [2] for illustrations of such studies.

The most famous example of a group of this sort is the Grigorchuk group ([12]). This is the group defined over the alphabet $X = \{\mathit{0}, \mathit{1}\}$ and generated by four generators a, b, c, d such that

$$\begin{aligned} a \cdot \mathit{0} &= \mathit{1} \cdot 1, & a \cdot \mathit{1} &= \mathit{0} \cdot 1, \\ b \cdot \mathit{0} &= \mathit{0} \cdot a, & b \cdot \mathit{1} &= \mathit{1} \cdot c, \\ c \cdot \mathit{0} &= \mathit{0} \cdot a, & c \cdot \mathit{1} &= \mathit{1} \cdot d, \\ d \cdot \mathit{0} &= \mathit{0} \cdot 1, & d \cdot \mathit{1} &= \mathit{1} \cdot b, \end{aligned}$$

where 1 is the identity element of the group.

The Grigorchuk group is the simplest example of an infinite finitely generated torsion group (thus it is an answer to one of the Burnside problems). It is also the first example of a group of intermediate growth ([13]) which answers the Milnor problem. It has many other interesting properties such as just-infiniteness, finite width, etc. For more on the Grigorchuk group see [15] and the last chapter of [19]. In fact, the study of the self-similar actions in general were stimulated by the discoveries of the amazing properties of this group and its analogs.

3. SELF-SIMILARITY BIMODULE

Definition 3.1. Let A be a $*$-algebra. An A-bimodule Φ is a right A-module with an A-valued sesquilinear inner product and a $*$-homomorphism $\varphi : A \to \mathrm{End}(\Phi)$, where $\mathrm{End}(\Phi)$ is the $*$-algebra of adjointable endomorphisms of the right module Φ.

For $a \in A$ and $v \in \Phi$ we will write av instead of $\varphi(a)(v)$, so that the map φ is defining the left multiplication on the bimodule.

Let us fix some self-similar action of a group G on the space X^ω for the alphabet $X = \{\mathit{1}, \mathit{2}, \ldots, d\}$. Denote by Φ_{R} the free right module over the group algebra $\mathbb{C}G$ with the free basis identified with the alphabet X. So every element of the module Φ_{R} is a linear combination over $\mathbb{C}$ of the formal products of the form $x \cdot g$, where $x \in X$ and $g \in X$.

We consider the algebra $\mathbb{C}G$ as a $*$-algebra with the standard involution $(\alpha \cdot g)^* = \overline{\alpha} g^{-1}$, where $\alpha \in \mathbb{C}$, $g \in G$. Then the module Φ_{R} has a $\mathbb{C}G$-valued sesquilinear form defined by the equality

$$\left\langle \sum_{x \in X} x \cdot a_x \mid \sum_{y \in X} y \cdot b_y \right\rangle = \sum_{x \in X} a_x^* b_x,$$

where $a_x, b_y \in \mathbb{C}G$.

Equation (2.2) from the definition of self-similarity gives also a well defined structure of a left $\mathbb{C}G$-module on Φ_{R}. Namely, for any element $g \in G$ and any vector $x \in X$ from the basis of Φ_{R} we define the left product $g \cdot x = y \cdot h$, where $h \in G$ and $y \in X$ are such that $g(xw) = yh(w)$ for all $w \in X^\omega$. This multiplication is extended by linearity onto the whole module Φ_{R} and thus we get a map from G to $\mathrm{End}(\Phi_{\mathrm{R}})$. It is easy to see that this map extends to a morphism of algebras $\varphi : \mathbb{C}G \to \mathrm{End}(\Phi_{\mathrm{R}})$. Since the module Φ_{R} is free, the algebra $\mathrm{End}(\Phi_{\mathrm{R}})$ is isomorphic to the algebra $M_d(\mathbb{C}G) = M_d(\mathbb{C}) \otimes \mathbb{C}G$ of $d \times d$-matrices over the algebra $\mathbb{C}G$.

The morphism $\varphi : \mathbb{C}G \to M_d(\mathbb{C}G)$ is called the *linear recursion* of the self-similar action. The obtained $\mathbb{C}G$-bimodule Φ is called the *self-similarity* bimodule of the action. For instance, in the case of the adding machine action the self-similarity bimodule Φ is 2-dimensional as a right $\mathbb{C}\langle a\rangle$-module and the left multiplication on the generator a is the endomorphism of the right module defined by the matrix

$$\varphi(a) = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}.$$

The linear recursions of self-similar actions were used by R. Grigorchuk, A. Żuk and L. Bartholdi to compute the spectra of random walks on the Cayley graphs and on the Schreier graphs of the respective self-similar groups (see [17], [2]).

Suppose that ρ is a unitary representation of the group G on a Hilbert space H. Then the bimodule Φ defines a representation $\rho_1 = \Phi \otimes \rho$ of the group G on the space

$$\Phi \otimes H = \sum_{x \in X} x \otimes H,$$

where each $x \otimes H$ may be considered just as a copy of H with the natural isometry $T_x : H \to x \otimes H$ defined by $v \mapsto x \otimes v$. The representation ρ_1 acts on the space $\Phi \otimes H$ by the formula $\rho_1(g)(x \otimes v) = y \otimes (\rho(h)(v))$, where $g \in G$, $v \in H$, and $h \in G$, $y \in X$ are such that $g \cdot x = y \cdot h$.

A unitary representation ρ of the group G is said to be *self-similar* (or Φ-*invariant*) if $\Phi \otimes \rho$ is equivalent to ρ, i.e., if there exists an isometry $\psi : H \to \Phi \otimes H$ such that $\psi^{-1}(\Phi \otimes \rho)\psi = \rho$. More explicitly, the representation ρ is self-similar if there exists a decomposition of H into a direct sum $H = H_1 \oplus H_2 \oplus \cdots \oplus H_d$ and isometries $S_x : H \to H$ with the range equal to H_x, such that

$$\rho(g) S_x = S_y \rho(h) \tag{3.1}$$

whenever $g \cdot x = y \cdot h$.

EXAMPLE 3.2. Since the group G acts on the space X^ω be measure preserving transformations, we get a natural unitary representation ρ of the group G on the space $H = L^2(X^\omega)$. Since the set X^ω is a disjoint union $\bigcup_{x \in X} xX^\omega$, the space H is a direct sum of the spaces H_x of the functions with the support in xX^ω. We have a natural isometry $S_x : H \to H_x \subset H$ defined by the rule

$$S_x(f)(w) = \begin{cases} 0 & \text{if } w \notin xX^\omega, \\ \sqrt{|X|} \cdot f(w') & \text{if } w = xw'. \end{cases}$$

One checks directly that condition (3.1) holds, so the natural representation of G on the space $L^2(X^\omega)$ is self-similar.

EXAMPLE 3.3. We say that a set $\mathcal{M} \subseteq X^\omega$ is *self-similar* if $\mathcal{M} = \bigcup_{x\in X} x\mathcal{M}$.

Let $\mathcal{M} \subset X^\omega$ be a countable self-similar G-invariant set. Let $H = \ell^2(\mathcal{M})$. Then the space H is also a direct sum $H_1 \oplus H_2 \oplus \cdots \oplus H_d$, where H_x is the subspace spanned by the set $x\mathcal{M}$. The isometry $S_x : H \to H_x$ is the linear extension of the map $w \mapsto xw$. Since the set $\mathcal{M}$ is G-invariant, we have a natural permutation representation of the group G on the space H. It is also easy to check that this representation satisfies condition (3.1), thus is self-similar.

Suppose the representation ρ of the group G is self-similar and let S_x be the isometries for which (3.1) holds. Let $\mathbb{C}^*_\rho(G)$ be the completion of the group algebra $\mathbb{C}G$ with respect to the operator norm induced by the representation ρ. Let now Φ_ρ be the right $\mathbb{C}^*_\rho(G)$-module with the free basis X. Then formula (2.2) will define a left multiplication for the module Φ_ρ, which will give us a well defined $\mathbb{C}^*_\rho(G)$-bimodule structure on Φ_ρ due to condition (3.1). In fact, the bimodule Φ_ρ can be identified with the closed linear span of the operators $S_x \cdot \rho(g)$, $x \in X$, $g \in G$ in the space $\mathcal{B}(H)$ of bounded operators. The left and right multiplication by the elements of $\mathbb{C}^*_\rho$ will be defined by the natural rule

$$a \cdot \xi = \rho(a)\xi, \quad \xi \cdot a = \xi\rho(a),$$

which extends the original multiplication in the bimodule Φ due to formula (3.1).

Consequently, if the representation ρ is self-similar, then the $\mathbb{C}G$-bimodule Φ can be extended to a $\mathbb{C}_\rho(G)$-bimodule. We will often denote Φ_ρ also by Φ, when this does not lead to a confusion.

In general we adopt the following definition.

DEFINITION 3.4. A completion A of the algebra $\mathbb{C}G$ with respect to some C^*-norm is said to be *self-similar* if the self-similarity bimodule Φ extends to a Hilbert A-bimodule.

4. GENERIC POINTS OF X^ω

DEFINITION 4.1. Let G be a countable group acting by homeomorphisms on the space X^ω. A point $w \in X^\omega$ is *generic* with respect to $g \in G$ if either $w^g \neq w$ or there exists a neighborhood $U \ni w$ consisting of points fixed under the action of g. A point $w \in X^\omega$ is *G-generic* if it is generic with respect to every element of G.

So if the point w is G-generic, then it is moved or is fixed by an element $g \in G$ together with all the points of its neighborhood.

PROPOSITION 4.2. *For any countable group G acting by homeomorphism on the space X^ω the set of all G-generic points is comeager (in particular it is not empty).*

Proof. The set of generic points with respect to any homeomorphism g is an open dense set. Hence, the set of all G-generic points is an intersection of a countable number of open dense sets, and thus it is comeager. ∎

Denote by $G(w)$ the G-orbit of a point $w \in X^\omega$. Let $\ell^2(G(w))$ be the Hilbert space of all square-summable functions $G(w) \to \mathbb{C}$. We have the permutation representation π_w of G on $\ell^2(G(w))$. Let $\|\cdot\|_w$ be the operator norm on $\mathbb{C}G$ defined by the representation π_w.

PROPOSITION 4.3. *Let $w_1, w_2 \in X^\omega$ and suppose that w_1 is G-generic. Then for every $a \in \mathbb{C}G$ $\|a\|_{w_1} \leqslant \|a\|_{w_2}$.*

Proof. Let $a = \sum_{k=1}^{m} \gamma_k g_k \neq 0$, where $\gamma_k \in \mathbb{C}$, $g_k \in G$. For every $\varepsilon > 0$ there exists a nonzero element f of $\ell^2(G(w_1))$ with finite support such that $\|a(f)\| \geqslant (1-\varepsilon)\|a\|_{w_1} \cdot \|f\|$.

Let $f = \sum_{u \in S} \alpha_u u$, where $\alpha_u \in \mathbb{C}$, and $S \subset G(w_1)$ is a finite set. Here $u \in G(w_1)$ is identified with its characteristic function (i.e., with the respective element of $\ell^2(G(w_1))$). Then

$$a(f) = \sum_{k=1}^{m} \sum_{u \in S} \alpha_u \gamma_k \cdot g_k(u) = \sum_{v \in S'} \Big(\sum_{g_k(u)=v} \alpha_u \gamma_k \Big) v, \tag{4.1}$$

where $S' = \bigcup_{i=1}^{m} g_k(S)$. Every point $u \in S$ belongs to the orbit $G(w_1)$, thus there exists $h_u \in G$ such that $h_u(w_1) = u$.

Let $\mathcal{W}$ be the set of all the points $w \in X^\omega$ for which the equality $g_k h_u(w) = g_l h_v(w)$ is equivalent to the equality $g_k(u) = g_l(v)$ for all $u, v \in S$ and $1 \leqslant k, l \leqslant m$. The set $\mathcal{W}$ contains the point w_1, and since w_1 is G-generic, it contains a neighborhood of w_1. The action of G on X^ω is minimal, thus the orbit $G(w_2)$ also intersects the set $\mathcal{W}$.

Let $\widetilde{w} \in \mathcal{W} \cap G(w_2)$. For every $u \in S$ denote $\widetilde{u} = h_u(\widetilde{w}) \in G(w_2)$ and $\widetilde{f} = \sum_{u \in S} \alpha_u \widetilde{u} \in \ell^2(G(w_2))$. Note that $\|\widetilde{f}\| = \|f\|$. Then, for $u, v \in S$ we have $g_k(\widetilde{u}) = g_l(\widetilde{v})$ if and only if $g_k(u) = g_l(v)$, thus in the sum

$$a(\widetilde{f}) = \sum_{v \in S'} \Big(\sum_{g_k(\widetilde{u})=\widetilde{v}} \gamma_k \cdot \alpha_u \Big) \widetilde{v}$$

the coefficient of $\widetilde{v}$ is equal to the respective coefficient of v in the sum (4.1). Consequently,

$$\|a\|_{w_2} \|\widetilde{f}\| \geqslant \|a(\widetilde{f})\| = \|a(f)\| \geqslant (1-\varepsilon)\|a\|_{w_1} \|f\| = (1-\varepsilon)\|a\|_{w_1} \|\widetilde{f}\|,$$

hence

$$\|a\|_{w_2} \geqslant (1-\varepsilon)\|a\|_{w_1}$$

for every $\varepsilon > 0$, thus $\|a\|_{w_2} \geqslant \|a\|_{w_1}$. ∎

5. SELF-SIMILAR COMPLETIONS

Let us fix a self-similar action of a countable group G on the space X^ω. By A_w we denote the completion of the algebra $\mathbb{C}G$ with respect to the operator norm defined by the permutation representation of G on the orbit of the point $w \in X^*$. As an immediate corollary of Proposition 4.3, we get

THEOREM 5.1. *Let $w \in X^\omega$ be a G-generic point. Then, for every $u \in X^\omega$, the algebra A_w is a quotient of the algebra A_u. If u is also generic, then the algebras A_w and A_u are isomorphic.*

Let us denote the algebra A_w, where w is a generic point, by $\mathcal{A}_\Phi$.

THEOREM 5.2. *The algebra $\mathcal{A}_\Phi$ is self-similar. If A is another self-similar completion of $\mathbb{C}G$ then the identical map $G \to G$ extends to a surjective homomorphism of the C^*-algebras $A \to \mathcal{A}_\Phi$.*

Proof. For $m, n \geqslant 0$ and $g \in G$ denote by $\mathcal{G}_{g,m,n}$ the set of all the points $w \in X^\omega$ such that $vg(\sigma^n(w))$ is G-generic for every $v \in X^m$. (Recall that σ is the shift.)

From the definition and Proposition 4.2 it follows that the set $\mathcal{G}_{g,m,n}$ is comeager. Thus the intersection $\bigcap_{\substack{g\in G\\ m,n\in\mathbb{N}}} \mathcal{G}_{g,m,n}$ is also comeager and thus nonempty. Let w_0 belong to this intersection. Then all the points of the set $\mathcal{M} = \{vg(\sigma^n(w_0)) : g \in G,\ v \in X^*,\ n \geqslant 0\}$ are G-generic.

Note that the set $\mathcal{M}$ is self-similar, countable and G-invariant due to the self-similarity of the action. Thus $\mathcal{M}$ is a union of the G-orbits. The permutation representation π of $\mathbb{C}G$ on $\ell^2(\mathcal{M})$ is a countable direct sum of the permutation representations π_w of G on orbits of generic points, thus the completion of $\mathbb{C}G$ with respect to the norm defined by π is isomorphic to $\mathcal{A}_\Phi$, by Theorem 5.1. Thus we get a faithful Φ-invariant representation of the algebra $\mathcal{A}_\Phi$. Consequently, the algebra $\mathcal{A}_\Phi$ is self-similar (see Example 3.3).

Let A be another self-similar completion of the algebra $\mathbb{C}G$ with respect to a norm $\|\cdot\|$. Let $\|\cdot\|_0$ be the norm defined by a permutational representation of the group G on the orbit $G(w_0)$ of a generic point w_0. We have to prove that $\|a\| \geqslant \|a\|_0$ for every $a \in \mathbb{C}G$.

Let $a = \sum_{k=1}^{m} \gamma_k g_k \neq 0$, where $\gamma_k \in \mathbb{C}, g_k \in G$. We choose a nonzero vector $f \in \ell^2(G(w_0))$ with finite support S such that $\|a(f)\| \geqslant (1-\varepsilon)\|a\|_0 \cdot \|f\|$. Let $f = \sum_{u\in S} \alpha_u u$. We again have (as in the proof of Proposition 4.3)

$$a(f) = \sum_{v\in S'} \Big(\sum_{g_k(u)=v} \alpha_u \gamma_k \Big) v,$$

where $S' = \bigcup_{k=1}^{m} g_k(S)$. Let the elements $h_u \in G$ be such that for every $u \in S$ we have $h_u(w_0) = u$.

In the same way as during the proof of Proposition 4.3 let $\mathcal{W}$ be the set of all the points $w \in X^\omega$ for which the equality $g_k h_u(w) = g_l h_v(w)$ is equivalent to the equality $g_k(u) = g_l(v)$ for all $u, v \in S$ and $1 \leqslant k, l \leqslant m$. The set $\mathcal{W}$ contains a neighborhood of w_0, thus there exists a finite beginning $r \in X^*$ of the word $w_0 \in X^\omega$ such that every word $w \in X^\omega$ which begins with r belongs to $\mathcal{W}$. We take r sufficiently long, so that

$$g_k h_u(r) = g_l h_v(r) \Leftrightarrow g_k h_u(w_0) = g_l h_v(w_0) \Leftrightarrow g_k(u) = g_l(v).$$

Take any faithful representation ρ of the algebra A on a Hilbert space H. Then for any $n \in \mathbb{N}$ the representation $\rho_n = \Phi^{\otimes n} \otimes \rho$ is faithful and acts on the direct sum $H^{d^n} = \Phi^{\otimes n} \otimes H$. Every summand $x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n} \otimes H$ of the direct sum corresponds to the word $x_{i_1} x_{i_2} \ldots x_{i_n} \in X^n$. The representation ρ_n acts on the vectors from the summands according to the rule

$$\rho_n(g)(x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n} \otimes \xi) = y_{i_1} \otimes y_{i_2} \otimes \cdots \otimes y_{i_n} \otimes \rho(h)(\xi), \tag{5.1}$$

where $g \in G$, $\xi \in H$ and $g \cdot x_{i_1} x_{i_2} \ldots x_{i_n} = y_{i_1} y_{i_2} \ldots y_{i_n} \cdot h$. In particular, the summands of the direct sum $H^{d^n} = \Phi^{\otimes n} \otimes H$ are permuted by the elements of G in the same way as the points of X^n are permuted under the action of the group G on X^*.

Take $n = |r|$ and let $\widetilde{e}$ be a vector of norm 1, belonging to the summand of the direct sum H^{d^n}, which corresponds to the word r. We define now a vector

$$\widetilde{f} = \sum_{u \in S} \alpha_u \rho_n(h_u)(\widetilde{e}) \in H^{d^n}.$$

Every $\rho_n(h_u)(\widetilde{e})$ belongs to the summand of H^{d^n}, corresponding to the word $h_u(r)$. Since for different $u \in S$ the words $h_u(r)$ are different, $\rho_n(h_u)(\widetilde{e})$ are orthogonal and $\|\widetilde{f}\| = \sum_{u \in S} |\alpha_u|^2 = \|f\|$. The vectors $\rho_n(g_k h_u)(\widetilde{e})$ and $\rho_n(g_l h_v)(\widetilde{e})$ belong to the same summand of the direct sum H^{d^n} if and only if $g_k h_u(r) = g_l h_v(r)$. But from the choice of the word r it follows that in this case $g_k h_u(w) = g_l h_v(w)$ for all the infinite words w starting with r. But this means that $g_k h_u \cdot r = g_l h_v \cdot r = r' \cdot h$ for some $h \in G$ and $r' \in X^n$.

Then from equation (5.1) it follows that $\rho(g_k h_u)(\widetilde{e}) = \rho(g_l h_v)(\widetilde{e})$. Consequently, the vectors $\rho(g_k h_u)(\widetilde{e})$ and $\rho(g_l h_v)(\widetilde{e})$ for $u, v \in S$ are orthogonal if $g_k(u) \neq g_l(v)$, and coincide if $g_k(u) = g_l(v)$. Then $\|\rho(a)(\widetilde{f})\| = \|a(f)\|$ and, in the same way as in the proof of Proposition 4.3, we get the estimate

$$\|a\| = \|\rho(a)\| \geqslant \|a\|_0,$$

which finishes the proof. ∎

6. THE CUNTZ-PIMSNER ALGEBRA $\mathcal{O}_\Phi$

In [26] M. Pimsner associates to every bimodule Φ an algebra $\mathcal{O}_\Phi$, which generalizes the Cuntz algebra $\mathcal{O}_d$ (see [9]), and the Cuntz-Krieger algebra $\mathcal{O}_A$ from [10]. In the case of the self-similarity bimodule the Cuntz-Pimsner algebra $\mathcal{O}_\Phi$ can be defined in the following way.

Definition 6.1. Let Φ be the self-similarity bimodule of an action of a group G over the alphabet X. The *Cuntz-Pimsner algebra* $\mathcal{O}_\Phi$ is the universal C^*-algebra generated by the algebra $\mathcal{A}_\Phi$ and operators $\{S_x : x \in X\}$ satisfying the relations

$$S_x^* S_x = 1, \quad S_x^* S_y = 0 \text{ if } x \neq y, \tag{6.1}$$

$$\sum_{x\in X} S_x S_x^* = 1, \tag{6.2}$$

$$a S_x = \sum_{y\in X} S_y a_{x,y}, \tag{6.3}$$

where $a_{x,y} \in \mathcal{A}_\Phi$ are such that $a \cdot x = \sum_{y\in X} y \cdot a_{x,y}$, so $a_{x,y} = \langle y \mid a \cdot x\rangle$.

We will use the multi-index notation, so that for $v = x_1 x_2 \dots x_n \in X^*$ the operator S_v is equal to $S_{x_1} S_{x_2} \cdots S_{x_n}$ and for the empty word $\emptyset$ the operator $S_\emptyset$ is equal to 1. Then $g \cdot v = u \cdot h$ implies $g S_v = S_u h$ in the algebra $\mathcal{O}_\Phi$.

The *Cuntz algebra* $\mathcal{O}_d$ is the universal algebra generated by d isometries $S_1, S_2, \dots, S_d$ such that $\sum_{i=1}^{d} S_i S_i^* = 1$. This algebra is simple (see [9], [11]) and thus any isometries satisfying such a relation generate an algebra isomorphic to $\mathcal{O}_d$. In particular, since the operators $\{S_x : x \in X\}$ satisfy relations (6.1), (6.2), they generate a subalgebra of $\mathcal{O}_\Phi$, isomorphic to $\mathcal{O}_d$ for $d = |X|$. If $\rho : \mathcal{O}_\Phi \to \mathcal{B}(H)$ is a representation of the algebra $\mathcal{O}_\Phi$, then its restriction onto the subalgebra generated by $\mathcal{A}_\Phi$ is a Φ-invariant representation of the algebra $\mathcal{A}_\Phi$, due to (6.3). Conversely, if ρ is a Φ-invariant representation, then due to condition (3.1) we get a representation of the algebra $\mathcal{O}_\Phi$.

We say that two words $v, u \in X^*$ are *comparable* if one is a beginning of the other. It is easy to see that the words $v, u \in X^*$ are incomparable if and only if the cylindrical sets vX^ω and uX^ω are disjoint.

The following lemma is proved by direct application of equation (6.1).

Lemma 6.2. *For any $v, u \in X^*$ the product $S_v^* S_u$ is not equal to* 0 *if and only if the words u and v are comparable.*

If $v = uu_0$ for some $u_0 \in X^*$ then $S_v^* S_u$ is equal to $S_{u_0}^*$; if $u = vv_0$ then $S_v^* S_u = S_{v_0}$.

It follows from Lemma 6.2 that every element of the monoid, generated by the set $\{S_x, S_x^* : x \in X\}$ is either zero or is equal to a product of the form $S_u \cdot S_v^*$, where $u, v \in X^*$. It also follows from relation (6.3) that every element of the semigroup generated by $\{S_x, S_x^* : x \in X\} \cup G$ is either zero or equal to a product

$S_u g S_v^*$ for some $u, v \in X^*$ and $g \in G$. If $g \cdot x = y_x \cdot h_x$, then $g \cdot S_x S_x^* = S_{y_x} h_x S_x^*$ and using relation (6.2) we get

$$g = \sum_{x \in X} S_{y_x} h_x S_x^*. \tag{6.4}$$

For a fixed $n \in \mathbb{N}$, if $g \cdot v = u_v \cdot h_v$ where $v, u_v \in X^n$ and $h_v \in G$, then by the same arguments

$$g = \sum_{v \in X^n} S_{u_v} h_v S_v^*. \tag{6.5}$$

DEFINITION 6.3. Let $\mathcal{M}$ be a self-similar subset of X^ω. Let T_x be, for every $x \in X$, the transformation of the set $\mathcal{M}$ which maps a word $w \in X^\omega$ to the word xw. Then the transformation T_x induces an isometry S_x of the Hilbert space $\ell^2(\mathcal{M})$ and the isometries S_x satisfy the relations (6.1) and (6.2). Thus we get a representation of the Cuntz algebra $\mathcal{O}_d$. Such a representation is called a *permutation representation* of $\mathcal{O}_d$.

If the self-similar set $\mathcal{M}$ is G-invariant and contains only G-generic points, then the permutation representation of $\mathcal{O}_d$ together with the permutation representation of the group G on $\mathcal{M}$ define a representation of the algebra $\mathcal{O}_\Phi$, which will also be called a *permutation representation* of $\mathcal{O}_\Phi$ on $\ell^2(\mathcal{M})$.

We have constructed a G-invariant self-similar set of G-generic points in the beginning of the proof of Theorem 5.2.

The permutation representations of the Cuntz algebra and their relation with the wavelets, numeration systems and rep-tilings are studied in [4].

The transformations T_x generate an inverse semigroup, where the inverse transformation T_x^* is the partially defined transformation which deletes the first letter x; in other words, it is the shift, restricted to the set xX^ω. The inverse semigroup generated by G and $\{T_x : x \in X\}$ can be viewed as a semigroup analog of the Cuntz-Pimsner algebra $\mathcal{O}_\Phi$. As we have seen, every nonzero element of this semigroup is of the form $T_v g T_u^*$ for some $u, v \in X^*$ and $g \in G$. Here we also use the multi-index notation.

7. THE ALGEBRA $\mathcal{F}(\Phi)$

For $k \in \mathbb{N}$ denote by $\mathcal{F}_k$ the linear span over $\mathbb{C}$ of the products $S_v a S_u^*$, with $u, v \in X^k$ and $a \in \mathcal{A}_\Phi$.

If $v_1, u_1, v_2, u_2 \in X^k$, then, by Lemma 6.2, $S_{v_1} a_1 S_{u_1}^* \cdot S_{v_2} a_2 S_{u_2}^*$ is equal to $S_{v_1} a_1 a_2 S_{u_2}^*$ if $u_1 = v_2$ and to zero otherwise. Consequently, $\mathcal{F}_k$ is an algebra isomorphic to the algebra $M_{d^k}(\mathcal{A}_\Phi)$ of $d^k \times d^k$-matrices over the algebra $\mathcal{A}_\Phi$. This algebra coincides with $\mathrm{End}(\Phi_{\mathrm{R}}^{\otimes k})$.

By equation (6.4) we have inclusions $\mathcal{F}_k \subset \mathcal{F}_{k+1}$. Denote by $\mathcal{F}(\Phi)$ the closure of the union $\bigcup_{k \geqslant 1} \mathcal{F}_k$. The algebra $\mathcal{F}(\Phi)$ is isomorphic to the direct limit of the matrix algebras $M_{d^n}(\mathcal{A}_\Phi)$ with respect to the embeddings defined by the linear recursion φ. The set of all transformations of the form $T_u g T_v^*$, where $g \in G$, $u, v \in X^*$ and $|u| = |v|$, is an inverse semigroup, and the orbit of a point $w_0 \in X^\omega$

under the action of this semigroup is the set $\mathrm{Orb}(w_0) = \{ug(\sigma^n(w_0)) : g \in G,\ u \in X^*,\ |u| = n,\ n \in \mathbb{N}\}$. We will call this set the $\mathcal{F}(\Phi)$-*orbit* of the point w_0.

A *permutation representation* of the algebra $\mathcal{F}(\Phi)$ on the space $\ell^2(\mathrm{Orb}(w_0))$ is the representation π_{w_0} which acts on the basis vectors $w \in \mathrm{Orb}(w_0)$ by the rule $\pi_{w_0}(S_u g S_v^*)(w) = T_u g T_v^*(w)$. In particular, the restriction of a permutation representation of the Cuntz-Pimsner algebra $\mathcal{O}_\Phi$ to subalgebra $\mathcal{F}(\Phi)$ is a direct sum of permutation representations.

Proposition 7.1. *Suppose the point $w_0 \in X^\omega$ is such that all the points in the $\mathcal{F}(\Phi)$-orbits of all its shifts $\sigma^n(w_0)$ are G-generic. Then the permutation representation of $\mathcal{F}(\Phi)$ on the $\mathcal{F}(\Phi)$-orbit of w_0 is faithful.*

Proof. Let π_0 be the permutation representation on the space $\ell^2(\mathrm{Orb}(w_0))$. By definition of the algebra $\mathcal{A}_\Phi$, the representation π_0 is faithful on $\mathcal{A}_\Phi$. The set $\mathrm{Orb}(w_0)$ is equal to $\bigcup_{x\in X} x\mathrm{Orb}(\sigma(w_0))$. Thus the permutation representation π_0 of the algebra $\mathcal{A}_\Phi$ is equivalent to the representation $\Phi \otimes \pi_1$, where π_1 is the permutation representation of $\mathcal{A}_\Phi$ on the orbit $G(\sigma(w_0))$. The representation π_1 is faithful on $\mathcal{A}_\Phi$, so the representation $\pi_0 \approx \Phi \otimes \pi_1$ is faithful on $\mathcal{F}_1 = M_d(\mathcal{A}_\Phi)$.

The process can be continued and we get that for every $n \in \mathbb{N}$ the representation π_0 is equivalent to the representation $\Phi^{\otimes n} \otimes \pi_n$, where π_n is the permutation representation on the orbit $G(\sigma^n(w_0))$, thus the representation π_0 is faithful on $\mathcal{F}_n$. Since the algebra $\mathcal{F}(\Phi)$ is the union of the algebras $\mathcal{F}_n$, it follows that the permutation representation π_0 is faithful on $\mathcal{F}(\Phi)$. ∎

Every algebra $\mathcal{F}_k$ contains as a subalgebra the linear span F_k of the products of the form $S_v S_u^*$, where $v, u \in X^k$. The algebra F_n is isomorphic to the algebra M_{d^k} of $d^k \times d^k$-matrices over $\mathbb{C}$, where the products $S_v S_u^*$ are identified with the matrix units. We also have the inclusions $F_n \subset F_{n+1}$, which are the diagonal embeddings. Thus the closure of the union $\bigcup_{k\geqslant 1} F_k$ is the UHF algebra M_{d^∞}.

By analogy with the Cuntz algebra (see [9], [11]), we can define a strongly continuous (*gauge*) action Γ of the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ on $\mathcal{O}_\Phi$ by the rules

$$\Gamma_z(g) = g, \quad \Gamma_z(S_x) = zS_x$$

for all $g \in G$, $x \in X$ and $z \in \mathbb{T}$.

Then $\Gamma_z(S_v g S_u^*) = z^{|v|-|u|} S_v g S_u^*$ for $u, v \in X^*$, $g \in G$, and therefore the integral $\int \Gamma_z(S_v g S_u^*)\,\mathrm{d}z$ is equal to zero for $|v| \neq |u|$ and to $S_v g S_u^*$ for $|v| = |u|$, where $\mathrm{d}z$ is the normalized Lebesgue measure on the circle. So the map

$$\mathbb{M}_0(a) = \int \Gamma_z(a)\,\mathrm{d}z \tag{7.1}$$

is a conditional expectation onto the subalgebra $\mathcal{F}(\Phi)$. If $a \in \mathcal{O}_\Phi$ is positive and nonzero, then every $\Gamma_z(a)$ is also positive and $\Gamma_1(a) = a$, thus $\mathbb{M}_0(a) > 0$, so $\mathbb{M}_0$ is faithful.

A self-similar action of a group G is *recurrent* if for every element h of the group G there exists $g \in G$ such that $g \cdot x = x \cdot h$. The definition does not depend on the choice of the letter x and if the action is recurrent then for any two words v, u of equal length and any $h \in G$ there exists $g \in G$ such that $g \cdot v = u \cdot h$ (this is easily proved by induction on the length of the words). Thus, if the action is recurrent, the orbit of a point $w \in X^\omega$ under the action of G coincides with its $\mathcal{F}(\Phi)$-orbit.

THEOREM 7.2. *Suppose the self-similar action of G is recurrent and that the point $w_0 \in X^\omega$ and all its shifts $\sigma^n(w_0)$ are G-generic. Let π_1 be the permutation representation of G on the orbit of w_0 and let π_2 be the natural representation on $\ell^2(G(w_0))$ of the algebra $C(X^\omega)$ of continuous functions on X^ω. Then the algebra $\mathcal{F}(\Phi)$ is isomorphic to the C^*-algebra generated by $\pi_1(G) \cup \pi_2(C(X^\omega)) \subset \mathcal{B}(\ell^2(G(w_0)))$.*

Proof. Denote by $\mathcal{F}$ the algebra generated by $\pi_1(G) \cup \pi_2(C(X^\omega))$. Let π be the permutation representation of $\mathcal{F}(\Phi)$ on the orbit $G(w_0)$ (which coincides with the $\mathcal{F}(\Phi)$-orbit of w_0 in the recurrent case). Then $\pi(g) = \pi_1(g)$ for every $g \in G$ and $\pi(S_u S_u^*) = \pi_2(\mathbf{1}_u)$ for every $u \in X^*$, where $\mathbf{1}_u$ is the characteristic function of the cylindrical set uX^ω. The function $\mathbf{1}_u$ is continuous, since the cylindrical sets are closed and open. Therefore π defines an injective C^*-homomorphism from the subalgebra of $\mathcal{F}(\Phi)$, generated by G and $\{S_u S_u^* : u \in X^*\}$, to the algebra $\mathcal{F}$. So it remains to prove that this homomorphism is surjective and that the set $G \cup \{S_u S_u^* : u \in X^*\}$ generates the C^*-algebra $\mathcal{F}(\Phi)$.

The first assertion follows from the well known fact that the closed linear span of the characteristic functions $\mathbf{1}_u$ for $u \in X^*$ is equal to $C(X^\omega)$.

Since the action of the group G is level-transitive and recurrent, for any two words u, v of equal lengths and for any $h \in G$ there exists an element $g \in G$ such that $g \cdot u = v \cdot h$. Then $g S_u S_u^* = S_v h S_u^*$. Hence the closed linear span of all the products $g S_u S_u^*$ is equal to $\mathcal{F}(\Phi)$, so $G \cup \{S_u S_u^* : u \in X^*\}$ generates $\mathcal{F}(\Phi)$. ∎

EXAMPLE 7.3. For the case of the adding machine action the algebra $\mathcal{A}_\Phi$ is isomorphic to the algebra $C(\mathbb{T})$ with the linear recursion $C(\mathbb{T}) \to M_2(C(\mathbb{T}))$ coming from the double self-covering of the circle. Thus the algebra $\mathcal{F}(\Phi)$ in this case is the Bunce-Deddens algebra. Then Theorem 7.2 in this case is the well known fact that the Bunce-Deddens algebra is the cross-product algebra of the odometer action on the Cantor space X^ω (see [11]).

8. SIMPLICITY OF $\mathcal{O}_\Phi$

There are several general results on simplicity of the Cuntz-Pimsner algebras, which can be used to prove that the algebra $\mathcal{O}_\Phi$ is simple (see, for example [24], [22], [29]). But we prefer to give a direct proof of this fact, since the proof in this case is rather simple and it shows how the notion of a generic point works. We will use the following refinement of this notion.

DEFINITION 8.1. A point $w \in X^\omega$ is *strictly G-generic* if for any $v, u \in X^*$ and $g \in G$ the transformation $T_v g T_u^*$ either moves the point w, or fixes w together with every point in a neighborhood of w, or is not defined on w.

Recall (see Definition 6.3), that T_v is the transformation $w \mapsto vw$ and T_v^* is the partially defined inverse transformation. In particular, every strictly generic point is aperiodic, i.e., cannot be represented in the form $uvvvvv\ldots$, where $u, v \in X^*$. The aperiodic words were used in the original proof in [9] of the simplicity of the Cuntz algebra $\mathcal{O}_d$.

In the same way as for G-generic points, one can prove that the set of all strictly G-generic points is also comeager.

Lemma 8.2. *For any finite linear combination* $a = \sum_{i=1}^{m} \alpha_i S_{u_i} g_i S_{v_i}^*$, $\alpha_i \in \mathbb{C}$, $u_i, v_i \in X^*$, $g_i \in G$, *with* $\mathbb{M}_0(a) \neq 0$ *and for every* $\varepsilon > 0$ *there exists a partial isometry* $Q \in \mathcal{F}(\Phi)$ *such that*

$$Q^* a Q = Q^* \mathbb{M}_0(a) Q,$$
$$Q^* a Q \in F_m = M_{|X|^m}(\mathbb{C}) \quad \textit{for some } m \in \mathbb{N},$$

and

$$\|Q^* a Q\| \geqslant \|\mathbb{M}_0(a)\| - \varepsilon.$$

Proof. Let π be a permutation representation of the algebra $\mathcal{O}_\Phi$ on a countable self-similar set $\mathcal{M}$ of strictly generic points of X^ω. Let us take $\delta > 0$ such that $\delta \cdot \|\mathbb{M}_0(a)\| < \varepsilon$. There exists a nonzero vector $\xi \in \ell^2(\mathcal{M})$ with finite support such that $\|\pi(\mathbb{M}_0(a))(\xi)\| > (1-\delta)\|\pi(\mathbb{M}_0(a))\| \cdot \|\xi\|$. By Proposition 7.1, the representation π is faithful on $\mathcal{F}(\Phi)$, so $\|\pi(\mathbb{M}_0(a))\| = \|\mathbb{M}_0(a)\|$.

Let $\xi = \sum_{j=1}^{N} \gamma_j w_j$, where $w_j \in \mathcal{M}$ are identified with their characteristic functions. We will denote by $w_j^{(m)}$ the beginning of the length m of the word w_j. We may assume that all w_j belong to the same $\mathcal{F}(\Phi)$-orbit, so every w_j is equal to $T_{w_j^{(k)}} h_j T^*_{w_1^{(k)}}(w_1)$ for some $h_j \in G$ and a fixed k big enough. We take some extra w_j with zero coefficients so that conditions $|u_i| = |v_i|$ and $\pi(S_{u_i} g_i S^*_{v_i})(w_k) \notin \{w_j : j = 1, \ldots, N\}$ imply $\gamma_k = 0$.

Let the index i be such that $|u_i| \neq |v_i|$. Suppose that $T_{u_i} g_i T^*_{v_i}(w_{j_1}) = w_{j_2}$. Then

$$T_{u_i} g_i T^*_{v_i}(T_{w_{j_1}^{(k)}} h_{j_1} T^*_{w_1^{(k)}}(w_1)) = T_{w_{j_2}^{(k)}} h_{j_2} T^*_{w_1^{(k)}}(w_1). \tag{8.1}$$

The transformation $(T_{u_i} g_i T^*_{v_i})(T_{w_{j_1}^{(k)}} h_{j_1} T^*_{w_1^{(k)}})$ is equal to a transformation of the form $T_u g T_v^*$ with $|u| - |v| = |u_i| - |v_i|$. Since the point w_1 is strictly generic, equality (8.1) holds for all the points of a neighborhood of w_1. So for sufficiently big m for every $w \in w_1^{(m)} X^\omega$ we will have $T_u g T_v^*(w) = T_{w_{j_2}^{(k)}} h_{j_2} T^*_{w_1^{(k)}}(w)$. But the transformation $T_{w_{j_2}^{(k)}} h_{j_2} T^*_{w_1^{(k)}}$ preserves the measure of the subsets of $w_1^{(k)} X^\omega$, while the transformation $T_u g T_v^*$ multiplies the measure of every subset of vX^ω by $|X|^{|v|-|u|}$. This is a contradiction, so $T_{u_i} g_i T^*_{v_i}(w_{j_1}) \neq w_{j_2}$ when $|u_i| \neq |v_i|$.

Let now $T_{u_i} g_i T^*_{v_i}(w_{j_1}) \neq w_{j_2}$ (without regard on the lengths of the words u_i and v_i). Then, for m big enough, the sets $T_{u_i} g_i T^*_{v_i}(w_{j_1}^{(m)} X^\omega)$ and $w_{j_2}^{(m)} X^\omega$ are disjoint. Hence we get an equality

$$S^*_{w_{j_2}^{(m)}}(S_{u_i} g_i S^*_{v_i}) S_{w_{j_1}^{(m)}} = 0. \tag{8.2}$$

Note that equality (8.2) remains to be true if we take a bigger m. So we can find an m such that it holds for all j_1, j_2 and for all i such that $T_{u_i} g_i T^*_{v_i}(w_{j_1}) \neq w_{j_2}$; in particular, for all i such that $|u_i| \neq |v_i|$.

Let now i be such that $|u_i| = |v_i|$ and suppose $T_{u_i} g_i T^*_{v_i}(w_{j_1}) = w_{j_2}$. Then again equality (8.1) holds. Since the point w_1 is strictly generic, this equality holds for all points of a neighborhood of w_1. We may assume that this neighborhood is $w_1^{(m)}X^\omega$. This means that

$$(T_{u_i}g_iT_{v_i}^*)(T_{w_{j_1}^{(k)}}h_{j_1}T_{w_1^{(k)}}^*)(T_{w_1^{(m)}}T_{w_1^{(m)}}^*) = T_{w_{j_2}^{(m)}}h_{j_2}T_{w_1^{(m)}}^*(T_{w_1^{(m)}}T_{w_1^{(m)}}^*).$$

Thus we get the same relation in the Cuntz-Pimsner algebra $\mathcal{O}_\Phi$ (with all T_v changed by S_v). Note that $(S_{w_{j_i}^{(k)}}h_{j_i}S_{w_1^{(k)}}^*)(S_{w_1^{(m)}}S_{w_1^{(m)}}^*) = S_{w_{j_i}^{(m)}}\widetilde{h}_{j_i}S_{w_1^{(m)}}^*$, where $i=1,2$ and $\widetilde{h}_{j_i} = S_s^*h_{j_i}S_r \in G$ for the words $s, r \in X^*$ such that $w_1^{(m)} = w_1^{(k)}r$ and $w_{j_i}^{(m)} = w_{j_i}^{(k)}s$. Thus

$$\begin{aligned}(S_{w_1^{(m)}}\widetilde{h}_{j_2}^{-1}S_{w_{j_2}^{(m)}}^*)(S_{u_i}g_iS_{v_i}^*)(S_{w_{j_1}^{(m)}}\widetilde{h}_{j_1}S_{w_1^{(m)}}^*) &= (S_{w_1^{(m)}}\widetilde{h}_{j_2}^{-1}S_{w_{j_2}^{(m)}}^*)(S_{w_{j_2}^{(m)}}\widetilde{h}_{j_2}S_{w_1^{(m)}}^*)\\ &= S_{w_1^{(m)}}S_{w_1^{(m)}}^*.\end{aligned}$$

Multiplying the equality on the left by $S_{w_{j_2}^{(m)}}S_{w_1^{(m)}}^*$ and on the right by $S_{w_1^{(m)}}^*S_{w_{j_1}^{(m)}}$ we get

$$(S_{w_{j_2}^{(m)}}\widetilde{h}_{j_2}^{-1}S_{w_{j_2}^{(m)}}^*)(S_{u_i}g_iS_{v_i}^*)(S_{w_{j_1}^{(m)}}\widetilde{h}_{j_1}S_{w_{j_1}^{(m)}}^*) = S_{w_{j_2}^{(m)}}S_{w_{j_1}^{(m)}}^*. \tag{8.3}$$

Let now $Q = \sum_{j=1}^N S_{w_j^{(m)}}\widetilde{h}_jS_{w_j^{(m)}}^*$. Then from equality (8.2) it follows that $Q^*aQ = Q^*\mathbb{M}_0(a)Q$.

We may assume that the neighborhoods $w_j^{(m)}X^\omega$ are disjoint for different j. Then from (8.2) and (8.3) it follows that $Q^*aQ \in F_m = M_{|X|^m}(\mathbb{C})$ and that $\pi(S_{u_i}gS_{v_i}^*)(w_{j_1}) = w_{j_2}$ for $|u_i| = |v_i|$ is equivalent to $\pi(Q^*S_{u_i}gS_{v_i}^*Q)(\widetilde{w}_{j_1}) = \widetilde{w}_{j_2}$, where $\widetilde{w}_j = S_{w_j^{(m)}}S_{w_1^{(m)}}(w_1)$. Hence $\|\pi(\mathbb{M}_0(a))(\xi)\| = \|\pi(Q^*aQ)(\widetilde{\xi})\|$, where $\widetilde{\xi} = \sum_{j=1}^N \gamma_j\widetilde{w}_j$. Note that $\|\widetilde{\xi}\| = \|\xi\|$. Then we have

$$\begin{aligned}\|Q^*aQ\|\cdot\|\xi\| = \|\pi(Q^*aQ)\|\cdot\|\widetilde{\xi}\| &\geqslant \|\pi(Q^*aQ)(\widetilde{\xi})\|\\ &= \|\pi(\mathbb{M}_0(a))(\xi)\| \geqslant (1-\delta)\|\mathbb{M}_0(a)\|\cdot\|\xi\|,\end{aligned}$$

hence $\|Q^*aQ\| \geqslant (1-\delta)\|\mathbb{M}_0(a)\| > \|\mathbb{M}_0(a)\| - \varepsilon$. ∎

THEOREM 8.3. *The algebra $\mathcal{O}_\Phi$ is simple. Moreover, for any nonzero $x \in \mathcal{O}_\Phi$ there exist $p, q \in \mathcal{O}_\Phi$ such that $pxq = 1$.*

Proof. We follow the proof of the similar result for the Cuntz algebra $\mathcal{O}_d$ from [11]. The element x^*x is positive and nonzero, thus $\mathbb{M}_0(x^*x)$ is also positive and nonzero. Multiplying x by a scalar, we get $\|\mathbb{M}_0(x^*x)\| = 1$. Let y be a self-adjoint positive finite sum $\sum_{i=1}^m \alpha_iS_{u_i}g_iS_{v_i}^*$, $\alpha_i \in \mathbb{C}$, $u_i, v_i \in X^*$, $g_i \in G$, such that $\|x^*x - y\| < 1/4$. Then

$$\|\mathbb{M}_0(y)\| \geqslant \|\mathbb{M}_0(x^*x)\| - \|\mathbb{M}_0(x^*x-y)\| \geqslant 1 - \|x^*x-y\| > \frac{3}{4}.$$

By Lemma 8.2 there exists a partial isometry $Q \in \mathcal{F}(\Phi)$ such that $Q^*yQ = Q^*\mathbb{M}_0(y)Q \in M_{|X|^m}(\mathbb{C})$ for some $m \in \mathbb{N}$ and $\|Q^*yQ\| \geqslant \|\mathbb{M}_0(y)\| - 1/4 > 1/2$. Then Q^*yQ is a positive matrix, so it is diagonalizable and thus there exists a projection $P \in M_{|X|^m}(\mathbb{C})$ such that

$$PQ^*yQ = Q^*yQP = \|Q^*yQ\|P.$$

We can find a unitary $U \in M_{|X|^m}(\mathbb{C})$ such that $UPU^* = S_vS_v^*$ for any $v \in X^m$. Then for $Z = \|Q^*yQ\|^{-1/2}S_v^*UPQ^*$ we have $\|Z\| \leqslant \sqrt{2}$ and

$$\begin{aligned} ZyZ^* &= \|Q^*yQ\|^{-1}S_v^*UP(Q^*yQ)PU^*S_v \\ &= \|Q^*yQ\|^{-1}S_v^*U(\|Q^*yQ\|)PU^*S_v = S_v^*S_vS_v^*S_v = 1. \end{aligned}$$

Thus we have $\|1-Zx^*xZ^*\| = \|ZyZ^*-Zx^*xZ^*\| \leqslant \|Z\|^2\|y-x^*x\| \leqslant 2\cdot 1/4 = 1/2$. Thus Zx^*xZ^* is invertible. If b is its inverse, then $(bZx^*)xZ^* = 1$. ∎

9. THE HIGMAN-THOMPSON GROUPS

In this section we show a connection between the Cuntz algebra $\mathcal{O}_d$ and the Higman-Thompson group V_d and construct the analogs of the Higman-Thompson groups for the self-similar actions using the Cuntz-Pimsner algebras.

DEFINITION 9.1. A *table* over the alphabet X is a matrix of the form

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix}, \tag{9.1}$$

where $v_i, u_i \in X^*$ are such that the set X^ω is decomposed into disjoint unions

$$X^\omega = \bigsqcup_{i=1}^m v_iX^\omega = \bigsqcup_{i=1}^m u_iX^\omega,$$

i.e., for every infinite word $w \in X^\omega$ exactly one v_i and exactly one u_j is a prefix of w. Note that from this it follows that the words in one row of a table are incomparable.

Every table $t = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix}$ defines a homeomorphism $\bar{t}$ of the space X^ω by the rule

$$\bar{t}(v_iw) = u_iw.$$

It is easy to prove that the set of all homeomorphisms defined by the tables is a group.

DEFINITION 9.2. The group of the homeomorphisms of the space X^ω defined by tables is called the *Higman-Thompson group* and is denoted V_d, where $d = |X|$.

Two tables are said to be *equivalent* if they are obtained one from another by a permutation of the columns. A *split* of a table is a table obtained from the original one by several consecutive replacements of a column $\begin{pmatrix} v_i \\ u_i \end{pmatrix}$ by a matrix

$$\begin{pmatrix} v_ix_1 & v_ix_2 & \cdots & v_ix_d \\ u_ix_1 & u_ix_2 & \cdots & u_ix_d \end{pmatrix},$$

where $X = \{x_1, x_2, \ldots, x_d\}$. Obviously, a split of a table is again a table.

PROPOSITION 9.3. *Two tables t_1 and t_2 define equal elements of the Higman-Thompson group if and only if they have some equivalent splits.*

For odd d there exists an analog of the alternative group in V_d, which is defined in the following way. We fix some linear order on the alphabet X. Then it induces the lexicographic order on the set of all finite words X^*. Namely, if the words $v_1 = x_1x_2\dots x_n$ and $v_2 = y_1y_2\dots y_m$ are incomparable then $v_1 < v_2$ if and only if $x_k < y_k$ where k is such that $x_i = y_i$ for all $i < k$ but $x_k \neq y_k$. If the word v_1 is a beginning of v_2 and $v_1 \neq v_2$, then $v_1 < v_2$.

Any table defining an element of the Higman-Thompson group V_n is equivalent to a table $\begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix}$, in which the first row is an ascending sequence in the lexicographic order. Let $u_{i_1} < u_{i_2} < \cdots < u_{i_m}$ be the ascending permutation of $u_1, u_2, \dots, u_m$ with respect to the lexicographic ordering. We say that the table is *even* (*odd*) if the permutation $i_1, i_2, \dots, i_m$ is respectively even (odd). The parity of a table does not depend on the choice of the order on the alphabet X.

If the number d is odd, then any split of a table will have the same parity as the original table. From this it follows that the set of all the elements of the group V_d defined by even tables is a subgroup of index 2 in V_d. Let us denote this subgroup by V'_d. In the case of even d, we set $V'_d = V_d$.

The Higman-Thompson groups V_d have the following properties (see [20], [6], [30]).

THEOREM 9.4. *For every $d > 1$ the Higman-Thompson groups V_d and V'_d are finitely presented. The group V'_d is the only nontrivial normal subgroup of V_d and is simple.*

Let $\mathcal{M} \subseteq X^\omega$ be a countable self-similar set. We define a faithful permutation representation ρ of V_d on the space $H = \ell^2(\mathcal{M})$ by the natural rule

$$\rho(\bar{t})(f)(w) = f(\bar{t}^{-1}(w)),$$

where $\bar{t} \in V_d$, $f \in \ell^2(\mathcal{M})$ and $w \in \mathcal{M}$. Then for every table

$$t = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix},$$

defining an element $\bar{t}$ of the Higman-Thompson group V_d, the operator $\rho(\bar{t})$ is equal to the image of $S_{u_1}S_{v_1}^* + S_{u_2}S_{v_2}^* + \cdots + S_{u_m}S_{v_m}^* \in \mathcal{O}_d$ under the respective permutation representation of $\mathcal{O}_d$ on the space $\ell^2(\mathcal{M})$.

Since the permutation representations of the group V_d and of the algebra $\mathcal{O}_d$ on $\ell^2(\mathcal{M})$ are both faithful, the element $S_{u_1}S_{v_1}^* + S_{u_2}S_{v_2}^* + \cdots + S_{u_m}S_{v_m}^* \in \mathcal{O}_d$ of the Cuntz algebra depends only on the respective element $\bar{t}$ of the Higman-Thompson group (and not on the choice of the table t). Note that the splitting rule for the tables follows from relation (6.2).

We get thus

PROPOSITION 9.5. *The correspondence*

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix} \mapsto S_{u_1}S_{v_1}^* + S_{u_2}S_{v_2}^* + \cdots + S_{u_m}S_{v_m}^*$$

defines a faithful unitary representation of the Higman-Thompson group V_d in the Cuntz algebra $\mathcal{O}_d$.

So, we will identify the Higman-Thompson group V_d with its image in the Cuntz algebra.

The Higman-Thompson group can be defined in terms of Cuntz algebra generators in the following way.

PROPOSITION 9.6. *The Higman-Thompson group V_d coincides with the set of those elements of the Cuntz algebra $\mathcal{O}_d$ which are unitary and can be represented in the form $S_{u_1}S_{v_1}^* + S_{u_2}S_{v_2}^* + \cdots + S_{u_m}S_{v_m}^*$, where $u_i, v_i \in X^*$.*

Proof. We must prove that the sum $a = u_1v_1^* + u_2v_2^* + \cdots + u_mv_m^*$ is unitary only if $\begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix}$ is a table.

Let $\mathcal{M}$ be a countable self-similar subset of X^ω and let π be the respective permutation representation of $\mathcal{O}_d$. Suppose that there exists an infinite word $w \in X^\omega$ such that none of v_i is a beginning of w. Then there exists a beginning w_0 of the word w which is incomparable with every word v_i. Since the set $\mathcal{M}$ is self-similar, there exists a word $\widetilde{w} \in \mathcal{M}$, beginning with w_0. But then $\pi(a)(\widetilde{w}) = 0$, so a is not unitary.

Suppose now that there exists an infinite word $w \in X^\omega$ such that more than one word v_i is its beginning. Then by the same arguments, there exists a word $\widetilde{w} \in \mathcal{M}$ such that more than one v_i is a beginning of $\widetilde{w}$. But then $\pi(a)(\widetilde{w})$ is a sum of several unit vectors $w_i \in \mathcal{M}$, and so a is also nonunitary. Thus the words v_i satisfy the conditions of Definition 9.1.

To prove that the words u_i satisfy the conditions of the definition one argues in the same way for a^*. ∎

Let now G be a self-similar group over the alphabet X. We can define a subgroup of the unitary group of the algebra $\mathcal{O}_\Phi$, analogous to the Higman-Thompson group.

DEFINITION 9.7. $V_d(G)$ is the group of all the sums of the form $\sum_{i=1}^{m} S_{u_i}g_iS_{v_i}^*$, which are unitary in $\mathcal{O}_\Phi$, where $g_i \in G$ and $u_i, v_i \in X^*$.

PROPOSITION 9.8. *A sum $\sum_{i=1}^{m} S_{u_i}g_iS_{v_i}^*$ is unitary if and only if the sum $\sum_{i=1}^{m} S_{u_i}S_{v_i}^*$ is unitary, i.e., if and only if $\begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix}$ is a table.*

The proof of this proposition repeats the proof of Proposition 9.6.

Thus we get the following combinatorial definition of the group $V_d(G)$.

A *G-table* is an array of the form

$$t = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ g_1 & g_2 & \cdots & g_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix},$$

where the words v_i, u_i satisfy the conditions of Definition 9.1. The table t then corresponds to a transformation of the space X^ω mapping every word $v_i w \in X^\omega$ to the word $u_i g_i(w)$. The inverse of the element defined by the table t is defined by the table

$$t^{-1} = \begin{pmatrix} u_1 & u_2 & \cdots & u_m \\ g_1^{-1} & g_2^{-1} & \cdots & g_m^{-1} \\ v_1 & v_2 & \cdots & v_m \end{pmatrix}.$$

Equation (6.4) shows us the splitting rule for the tables. Namely, in every table we can replace a column $\begin{pmatrix} v \\ g \\ u \end{pmatrix}$ by the table, $\begin{pmatrix} vx_1 & vx_2 & \cdots & vx_d \\ h_1 & h_2 & \cdots & h_d \\ uy_1 & uy_2 & \cdots & uy_d, \end{pmatrix}$ where $g = \sum_{i=1}^{n} y_i h_i x_i^*$, i.e., $g \cdot x_i = y_i \cdot h_i$ and $X = \{x_1, x_2, \ldots, x_d\}$. In the same way as for the Higman-Thompson group, two tables define the same elements of the group V_d if and only if they have equivalent splittings.

We also get an analog of the subgroup V_d' inside the group $V_d(G)$. We say that a table $t = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ g_1 & g_2 & \cdots & g_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix}$ is even if the table $\begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix}$ is even. In particular, the tables of the form $\begin{pmatrix} \emptyset \\ g \\ \emptyset \end{pmatrix}$, defining the elements of the group G are considered to be even.

Then for odd d let $V_d'(G)$ be the group generated by the elements which can be defined by even tables. Note that the set of even tables is not always a group. In the case of even d we define $V_d'(G)$ to be equal to $V_d(G)$.

PROPOSITION 9.9. *Let d be odd. If the group G acts on the set X^1 by even permutations, then $V_d'(G)$ is a subgroup of index 2 in the group $V_d(G)$. Otherwise it coincides with $V_d(G)$.*

Proof. If the group G acts by even permutations on the set X^1, then a split of an even table is again an even table. Then the group $V_d'(G)$ coincides with the set of all elements defined by even tables and is a subgroup of index 2.

Suppose now that some element $h \in G \subset V_d'(G)$ acts on the set X^1 by an odd permutation. The element h is defined by an odd table $\begin{pmatrix} x_1 & x_2 & \cdots & x_d \\ h_1 & h_2 & \cdots & h_d \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}$, where $\{x_1, x_2, \ldots, x_d\} = X$ and $h \cdot x_i = y_i \cdot h_i$. Then $h \in V_d'(G)$ and

$$V_d'(G) \ni \begin{pmatrix} x_1 & x_2 & \cdots & x_d \\ y_1 & y_2 & \cdots & y_d \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_d \\ h_1 & h_2 & \cdots & h_d \\ y_1 & y_2 & \cdots & y_d \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & \cdots & x_d \\ h_1^{-1} & h_2^{-1} & \cdots & h_d^{-1} \\ x_1 & x_2 & \cdots & x_d \end{pmatrix}.$$

So, the group $V_d'(G)$ contains an element from $V_d \setminus V_d'$. This implies that $V_d \leqslant V_d'(G)$, since the group V_d' has index 2 in V_d.

Let g be an arbitrary element of $V_d(G)$ defined by $t = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ g_1 & g_2 & \cdots & g_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix}$. If the table t is even, then $g \in V_d'(G)$. If not, then g is a product of an element defined by the even table $\begin{pmatrix} v_1 & v_2 & v_3 & \cdots & v_m \\ g_1 & g_2 & g_3 & \cdots & g_m \\ u_2 & u_1 & u_3 & \cdots & u_m \end{pmatrix}$ and the element of V_d defined by the table $\begin{pmatrix} u_1 & u_2 & u_3 & \cdots & u_m \\ u_2 & u_1 & u_3 & \cdots & u_m \end{pmatrix}$ and thus also belongs to $V_d'(G)$. Hence, $V_d'(G) = V_d(G)$. ∎

For every nonempty word $r \in X^k$ and an element $f \in V_d(G)$ we denote

$$\Lambda_r(f) = S_r f S_r^* + (1 - S_r S_r^*) = S_r f S_r^* + \sum_{\substack{v \in X^k \\ v \neq r}} S_v S_v^*.$$

From the definition it follows directly that $\Lambda_r(f)$ is also an element of the group $V_d(G)$. Moreover, if f is defined by an even table, then $\Lambda_r(f)$ is also defined by an even table.

Lemma 9.10. *For every nonempty $r \in X^*$ the map $\Lambda_r : V_d(G) \to V_d(G)$ is an injective homomorphism. If r_1 and $r_2 \in X^*$ are incomparable, then for all $f_1, f_2 \in V_d(G)$ the elements $\Lambda_{r_1}(f_1)$ and $\Lambda_{r_2}(f_2)$ commute.*

Proof. Let $f_1, f_2 \in V_d(G)$, then

$$\begin{aligned} \Lambda_r(f_1)\Lambda_r(f_2) &= (S_r f_1 S_r^* + (1 - S_r S_r^*))(S_r f_2 S_r^* + (1 - S_r S_r^*)) \\ &= S_r f_1 S_r^* S_r f_2 S_r^* + (1 - S_r S_r^*) S_r f_2 S_r^* + S_r f_1 S_r^* (1 - S_r S_r^*) \\ &\qquad + (1 - S_r S_r^*)(1 - S_r S_r^*) \\ &= S_r f_1 f_2 S_r^* + (1 - S_r S_r^*) = \Lambda_r(f_1 f_2), \end{aligned}$$

since $S_r^* S_r = 1$, so $S_r f_1 S_r^* (1 - S_r S_r^*) = S_r f_1 S_r^* - S_r f_1 S_r^* S_r S_r^* = S_r f_1 S_r^* - S_r f_1 S_r^* = 0$; similarly $(1 - S_r S_r^*) S_r f_2 S_r^* = 0$, $(1 - S_r S_r^*)(1 - S_r S_r^*) = 1 - 2 S_r S_r^* + S_r S_r^* = 1 - S_r S_r^*$ and $S_r f_1 S_r^* S_r f_2 S_r^* = S_r f_1 f_2 S_r^*$.

If r_1 and r_2 are incomparable, then $S_{r_1}^* S_{r_2} = S_{r_2}^* S_{r_1} = 0$ by Lemma 6.2, so

$$\begin{aligned} \Lambda_{r_1}(f_1)\Lambda_{r_2}(f_2) &= (S_{r_1} f_1 S_{r_1}^* + (1 - S_{r_1} S_{r_1}^*))(S_{r_2} f_2 S_{r_2}^* + (1 - S_{r_2} S_{r_2}^*)) \\ &= S_{r_1} f_1 S_{r_1}^* S_{r_2} f_2 S_{r_2}^* + S_{r_1} f_1 S_{r_1}^* (1 - S_{r_2} S_{r_2}^*) \\ &\qquad + (1 - S_{r_1} S_{r_1}^*) S_{r_2} f_2 S_{r_2}^* + (1 - S_{r_1} S_{r_1}^*)(1 - S_{r_2} S_{r_2}^*) \\ &= S_{r_1} f_1 S_{r_1}^* + S_{r_2} f_2 S_{r_2}^* + (1 - S_{r_1} S_{r_1}^* - S_{r_2} S_{r_2}^*) \\ &= \Lambda_{r_2}(f_2)\Lambda_{r_1}(f_1). \end{aligned}$$ ∎

The group $V_d(G)$ was defined for the first time by C. Röver for the Grigorchuk group. In [27] he proved that in this case this group is finitely presented simple. Later, in [28] he proved that the group $V_2(G)$ for the Grigorchuk group is its abstract commensurizer.

Some of the results of C. Röver can be generalized for the group $V_d(G)$ in the case of an arbitrary self-similar group G.

Theorem 9.11. *All the proper quotients of the groups $V_d(G)$ and $V_d'(G)$ are abelian.*

Proof. We will use the following easy technical lemma (its analog is proved in [30]).

LEMMA 9.12. *A sequence of finite words $v_1, v_2, \ldots, v_m$ is a part of a row of a table if and only if the words v_i are pairwise incomparable.*

If $v_1, v_2, \ldots, v_m$ and $u_1, u_2, \ldots, u_m$ are two sets of pairwise incomparable words such that $v_1X^\omega \cup v_2X^\omega \cup \cdots \cup v_mX^\omega \neq X^\omega$ and $u_1X^\omega \cup u_2X^\omega \cup \cdots \cup u_mX^\omega \neq X^\omega$, then there exists an even table of the form $\begin{pmatrix} v_1 & v_2 & \cdots & v_m & \cdots \\ u_1 & u_2 & \cdots & u_m & \cdots \end{pmatrix}$.

We will prove Theorem 9.11 simultaneously for both groups $V_d(G)$ and $V_d'(G)$. So, when it is not important which group we consider ($V_d(G)$ or $V_d'(G)$), we denote the group under the consideration by V.

Let $g = \sum_{i=1}^{m} S_{u_i} g_i S_{v_i}^*$ be a nontrivial element of a normal subgroup N of the group V. The element g acts nontrivially on some point $w \in X^\omega$. Then there exists a neighborhood U of the point w such that the intersection $g(U) \cap U$ is empty. We may assume that the set U is of the form vX^ω for some $v \in X^*$. After possibly making several splittings using formula (6.5) and passing to a smaller cylindrical set U, we may assume that $v = v_i$ for some i. Then $g(U) = g(v_iX^\omega) = u_ig_i(X^\omega) = u_iX^\omega$. Now the condition that U and $g(U)$ do not intersect means that the words v_i and u_i are incomparable and thus $S_{v_i}^* S_{u_i} = S_{u_i}^* S_{v_i} = 0$, by Lemma 6.2. We may assume that $v_iX^\omega \cup u_iX^\omega \neq X^\omega$. Let us denote $u = u_i, v = v_i$ and $g_i = h$. We have $g = S_u h S_v^* + \widetilde{g}$, where $S_u^* \widetilde{g} = \widetilde{g} S_v = 0$, due to Lemma 6.2.

Let us call a pair of words $\{r, s\} \subset X^*$ an *incomplete antichain* if the words r and s are incomparable and $rX^\omega \cup sX^\omega \neq X^\omega$.

LEMMA 9.13. *For every $f \in V$ and for every incomplete antichain $\{r, s\} \subset X^*$, the element $S_s f S_s^* + S_r f^{-1} S_r^* + (1 - S_sS_s^* - S_rS_r^*) = \Lambda_s(f)\Lambda_r(f^{-1})$ belongs to N.*

Proof. Note that for every $r \in X^*$ the endomorphism Λ_r leaves the group $V_d'(G)$ invariant, i.e., $\Lambda_r(V_d'(G)) \subset V_d'(G)$, since it maps even tables to even tables and is an endomorphism of the group $V_d(G)$.

By Lemma 9.12, there exists an element p of the group V defined by a table, which contains the columns $\begin{pmatrix} u & v & \cdots \\ h^{-1} & 1 & \cdots \\ r & s & \cdots \end{pmatrix}$. So $p = S_r h^{-1} S_u^* + S_s S_v^* + \widetilde{p}$, where $\widetilde{p} S_u = \widetilde{p} S_v = S_r^* \widetilde{p} = S_s^* \widetilde{p} = 0$. Then

$$\begin{aligned} pgp^{-1} = pgp^* &= (S_r h^{-1} S_u^* + S_s S_v^* + \widetilde{p})(S_u h S_v^* + \widetilde{g})(S_u h S_r^* + S_v S_s^* + \widetilde{p}^*) \\ &= (S_r S_v^* + S_s S_v^* \widetilde{g} + \widetilde{p}\widetilde{g})(S_u h S_r^* + S_v S_s^* + \widetilde{p}^*) \\ &= S_r S_s^* + S_s S_v^* \widetilde{g} S_u h S_r^* + S_s S_v^* \widetilde{g} \widetilde{p}^* + \widetilde{p}\widetilde{g} S_u h S_r^* + \widetilde{p}\widetilde{g}\widetilde{p}^*. \end{aligned}$$

So, the group N contains an element $q = pgp^{-1}$ of the form $q = S_r S_s^* + \widetilde{q}$, where $S_r^* \widetilde{q} = \widetilde{q} S_s = 0$. We also have $\widetilde{q}^* \widetilde{q} = 1 - S_s S_s^*$, since $1 = q^* q = (S_s S_r^* + \widetilde{q}^*)(S_r S_s^* + \widetilde{q}) = S_s S_r^* S_r S_s^* + S_s S_r^* \widetilde{q} + \widetilde{q}^* S_r S_s + \widetilde{q}^* \widetilde{q} = S_s S_s^* + \widetilde{q}^* \widetilde{q}$.

Let us compute $\Lambda_r(f)^{-1} q^{-1} \Lambda_r(f) q \in N$ for arbitrary $f \in V$. We have

$$\begin{aligned} q^{-1}\Lambda_r(f)q &= (S_s S_r^* + \widetilde{q}^*)(S_r f S_r^* + 1 - S_r S_r^*)(S_r S_s^* + \widetilde{q}) \\ &= (S_s f S_r^* + \widetilde{q}^*)(S_r S_s^* + \widetilde{q}) \\ &= S_s f S_s^* + \widetilde{q}^* \widetilde{q} = S_s f S_s^* + (1 - S_s S_s^*) = \Lambda_s(f). \end{aligned}$$

So $\Lambda_r(f)^{-1} q^{-1} \Lambda_r(f) q = \Lambda_r(f^{-1})\Lambda_s(f)$, and the lemma is proved. ∎

In particular, the normal subgroup N contains a nontrivial element of the group V'_d. Consequently, it contains its normal closure in V'_d, which is equal to V'_d by Theorem 9.4.

Let H be the quotient of V by the subgroup N and let π be the canonical homomorphism onto H. By Lemma 9.13, if $\{r, s\}$ is an incomplete antichain then $\pi(\Lambda_r(f)) = \pi(\Lambda_s(f))$. But for any two nonempty words $r_1, r_2 \in X^*$ there exists a pair of words $s_1, s_2 \in X^*$ such that the pairs $\{r_1, s_1\}$, $\{s_1, s_2\}$ and $\{s_2, r_2\}$ are incomplete antichains. Then we have

$$\pi(\Lambda_{r_1}(f)) = \pi(\Lambda_{s_1}(f)) = \pi(\Lambda_{s_2}(f)) = \pi(\Lambda_{r_2}(f)).$$

So $\pi(\Lambda_r(f))$ does not depend on r, and all the elements of the form $\pi(\Lambda_r(f)) \in H$ commute by Lemma 9.10.

Suppose that the table $\begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ g_1 & g_2 & \cdots & g_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix}$, defining an element of V, is even. Let $h = \sum_{i=1}^{m} S_{u_i} S_{v_i}^*$ be the respective element of V'_d. Then $gh^{-1} = \sum_{i=1}^{m} S_{u_i} g_i S_{u_i}^* = \Lambda_{u_1}(g_1)\Lambda_{u_2}(g_2)\cdots\Lambda_{u_m}(g_m)$. Hence

$$\pi(g) = \pi(gh^{-1}) = \pi(\Lambda_{u_1}(g_1))\pi(\Lambda_{u_2}(g_2))\cdots\pi(\Lambda_{u_m}(g_m)).$$

So the elements $\pi(\Lambda_r(g))$ generate the image $\pi(V'_d(G))$; thus it is abelian. So the theorem is proved for the case $V = V'_d(G)$.

Suppose now that $V = V_d(G) \neq V'_d(G)$. In this case, by Proposition 9.9, the group G acts by even permutations on X^1 and every element of $V'_d(G)$ is defined only by even tables. Define

$$a = S_{x_1}S_{x_2}^* + S_{x_2}S_{x_1}^* + \sum_{\substack{x\in X\\ x\neq x_1\\ x\neq x_2}} S_x S_x^* = S_{x_1}S_{x_2}^* + S_{x_2}S_{x_1}^* + (1 - S_{x_1}S_{x_1}^* - S_{x_2}S_{x_2}^*).$$

Then a is defined by an odd table, so $a \notin V'_d(G)$. The group $\pi(V_d(G))$ is generated by the set $\pi(V'_d(G)) \cup \{\pi(a)\}$, so it is sufficient to prove that $\pi(a)$ commutes with every element of $\pi(V'_d(G))$. Let g be any element of $V'_d(G)$. It is defined by some even table $\begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ g_1 & g_2 & \cdots & g_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix}$, where the words v_i, u_i are all nonempty. Then $a^{-1}ga = aga$ is defined by an even table of the form $\begin{pmatrix} \widetilde{v}_1 & \widetilde{v}_2 & \cdots & \widetilde{v}_m \\ g_1 & g_2 & \cdots & g_m \\ \widetilde{u}_1 & \widetilde{u}_2 & \cdots & \widetilde{u}_m \end{pmatrix}$, where $\widetilde{v}_i$ and $\widetilde{u}_i$ are obtained from v_i and u_i by changing the initial letter x_1 (or x_2) to the letter x_2 (respectively x_1). If the word u_i does not begin neither with x_1 nor with x_2 then $u_i = \widetilde{u}_i$. So, as above,

$$\pi(g) = \pi(\Lambda_{u_1}(g_1))\pi(\Lambda_{u_2}(g_2))\cdots\pi(\Lambda_{u_m}(g_m)),$$
$$\pi(a^{-1}ga) = \pi(\Lambda_{\widetilde{u}_1}(g_1))\pi(\Lambda_{\widetilde{u}_2}(g_2))\cdots\pi(\Lambda_{\widetilde{u}_m}(g_m)).$$

But $\pi(\Lambda_{u_i}(g_i)) = \pi(\Lambda_{\widetilde{u}_i}(g_i))$, hence $\pi(a^{-1}ga) = \pi(g)$, so $\pi(a)$ commutes with the elements of the group $\pi(V'_d(G))$. ∎

So all the normal subgroups of $V_d(G)$ contain the commutator subgroup. The possible quotients of $V_d(G)$ (and thus its abelianization) are described in the next theorem.

THEOREM 9.14. *If d is even, then an abelian group H is a quotient of the group $V_d(G)$ if and only if there exists a surjective homomorphism $\pi : G \to H$ such that $g = \sum_{i=1}^{n} S_{y_i} h_i S_{x_i}^*$ implies $\pi(g) = \sum_{i=1}^{n} \pi(h_i)$.*

If d is odd, then an abelian group H is a quotient of the group $V_d(G)$ if and only if there exists an element $z \in H$ such that $z^2 = 1$ and a homomorphism $\pi : G \to H$ such that the group H is generated by $\pi(G) \cup \{z\}$, and $g = \sum_{i=1}^{n} S_{y_i} h_i S_{x_i}^*$ implies $\pi(g) = \sum_{i=1}^{n} \pi(h_i) + \mathrm{sign}(g)$, where $\mathrm{sign}(g) = z$ if g acts on X^1 by an odd permutation and $\mathrm{sign}(g) = 1$ otherwise.

Proof. Suppose d is even and let $\pi : V_d(G) \to H$ be a surjective homomorphism with a nontrivial kernel. Then the kernel contains V_d.

Let $g = \sum_{i=1}^{n} S_{y_i} h_i S_{x_i}^*$ be an arbitrary element of the group G and let $h = \sum_{i=1}^{m} S_{y_i} S_{x_i}^*$. Then we have

$$\pi(g) = \pi(gh^{-1}) = \sum_{i=1}^{n} \pi(\Lambda_{y_i}(h_i)). \tag{9.2}$$

$$\pi(\Lambda_x(g)) = \sum_{i=1}^{n} \pi(\Lambda_{xy_i}(h_i)), \quad \text{for any } x \in X. \tag{9.3}$$

We proved above that $\pi(\Lambda_r(f)) = \pi(\Lambda_s(f))$ for any nonempty $r, s \in X^*$. Hence equations (9.2), (9.3) imply that $\pi(g) = \pi(\Lambda_x(g)) = \sum_{i=1}^{n} \pi(h_i)$. It follows also that the elements $\pi(g)$ for $g \in G$ generate the whole group H.

Conversely, suppose that a surjective homomorphism $\pi : G \to H$ is such that $g = \sum_{i=1}^{n} S_{y_i} h_i S_{x_i}^*$ implies $\pi(g) = \sum_{i=1}^{n} \pi(h_i)$ for all $g \in G$. Let us extend it to the whole group $V_d(G)$ by the rule

$$\pi\Big(\sum_{i=1}^{m} S_{u_i} g_i S_{v_i}^*\Big) = \sum_{i=1}^{m} \pi(g_i).$$

From the condition on the homomorphism π and the splitting rule defining the elements of the group $V_d(G)$ it follows that this extension is well defined.

Suppose now that d is odd and let $\pi : V_d(G) \to H$ be a surjective homomorphism with a nontrivial kernel. Then the image of V_d under π is a group of order at most 2. Let z be equal to the nontrivial element of $\pi(V_d)$ if $|\pi(V_d)| = 2$ and to the identity otherwise. Then equation (9.2) transforms into

$$\pi(g) = \pi(gh^{-1}) = \sum_{i=1}^{n} \pi(\Lambda_{y_i}(h_i)) + \pi(h^{-1}),$$

but then $\pi(h^{-1}) = \mathrm{sign}(g)$. And similar arguments show that the homomorphism π must satisfy the conditions of the theorem.

Conversely, if π satisfies the conditions of the theorem then we can extend it to the whole group $V_d(G)$ by the rule

$$\pi\Big(\sum_{i=1}^{m} S_{u_i} g_i S_{v_i}^*\Big) = \mathrm{sign}(t) + \sum_{i=1}^{m} \pi(g_i),$$

where $\mathrm{sign}(t)$ is equal to z if the table $t = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ u_1 & u_2 & \cdots & u_m \end{pmatrix}$ is odd and to 1 otherwise. One can check directly that this extension is well defined and is a homomorphism. ∎

EXAMPLE 9.15. For the Grigorchuk group, generated by a, b, c, d over the alphabet $\{0, 1\}$, we have $a = S_0 S_1^* + S_1 S_0^*$, $b = S_0 a S_0^* + S_1 c S_1^*$, $c = S_0 a S_0^* + S_1 d S_1^*$ and $d = S_0 S_0^* + S_1 b S_1^*$. So if H is a quotient of the group $V_2(G)$, then there exists an epimorphism $\pi : G \to H$ such that $\pi(a) = 0, \pi(b) = \pi(a) + \pi(c) = \pi(c)$ and $\pi(c) = \pi(d)$. But then $\pi(b) = \pi(cd) = 2\pi(b)$, thus $\pi(a) = \pi(b) = \pi(c) = \pi(d) = 0$ and the group H is trivial, i.e., the group $V_2(G)$ is simple.

EXAMPLE 9.16. For the adding machine a we have $a = S_1 S_0^* + S_0 a S_1^*$. So, in this case we have an epimorphism $\pi : \langle a \rangle \to \mathbb{Z}$, which agrees with the recursion. Therefore, the abelianization of the group $V_2(\langle a \rangle)$ is $\mathbb{Z}$.

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