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GENERALIZED CESÀRO OPERATORS AND THE BERGMAN SPACE

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ABSTRACT. We investigate spectral properties of operators on L_a^2 of the form

$$\mathcal{C}_g(f)(z) = \frac{1}{z}\int\limits_0^z f(t)g(t)\mathrm{d}t.$$

We compute the spectrum when g is a rational function, as well as the essential spectrum and the Fredholm index. We also provide relations for these operators in the Calkin algebra.

KEYWORDS: *Cesàro, Fredholm, Calkin, Bergman, spectrum.*
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1. INTRODUCTION

The class of generalized Cesàro operators has been getting considerable attention recently. Pommerenke introduced the class in [11] by showing that

$$I_G(f)(z) = \int\limits_0^z f(t)G'(t)\mathrm{d}t \tag{1.1}$$

is bounded on H^2 if and only if $G \in$ BMOA. Aleman and Siskakis extend this result to H^p for $1 \leqslant p < \infty$ in [2]. Furthermore, they show that I_G is compact if and only if $G \in$ VMOA. In [1], Aleman and Cima show that the results holds for any $p > 0$.

A similiar boundedness property holds for the Bergman space L_a^2. Aleman and Siskakis show in [3] that I_G is bounded on L_a^2 if and only if $G \in \mathcal{B}$, the Bloch space. Likewise, there is a corresponding compactness result given: I_G is compact on L_a^2 if and only if $G \in \mathcal{B}_0$, the little Bloch space.

The Cesàro operator $\mathcal{C}$ is a paradigm of a noncompact operator in this class. Recall that $\mathcal{C}$ has the following form on L^2_a: if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in L^2_a$, then

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \frac{1}{n+1}\Big(\sum_{i=0}^{n} a_i\Big) z^n. \tag{1.2}$$

It is known that $\mathcal{C}$ is bounded on L^2_a. V.G. Miller and T.L. Miller show in [9] that $\sigma(\mathcal{C}) = \{z : |z-(1/2)| \leqslant (1/2)\}$. In contrast to the H^2 case where $\mathcal{C}$ is known to be subnormal (see [7]), they show that $\mathcal{C}$ on L^2_a is not hyponormal (see [10]).

Calculating Taylor series gives us the following integral representation:

$$\mathcal{C}(f)(z) = \frac{1}{z}\int_0^z \frac{f(t)}{1-t}\,\mathrm{d}t.$$

We define the following class of operators on L^2_a.

DEFINITION 1.1. Let g be analytic on the unit disk $\mathbb{D}$. The *generalized Cesàro operator on L^2_a with symbol g* is the map defined by

$$\mathcal{C}_g(f)(z) = \frac{1}{z}\int_0^z f(t)g(t)\mathrm{d}t.$$

Note that if S denotes multiplication by z on L^2_a and $G' = g$, then

$$I_G = S\mathcal{C}_g. \tag{1.3}$$

If we set

$$(\mathcal{B})^1 = \Big\{g : \mathbb{D} \to \mathbb{C} \;:\; \int_0^z g(t)\mathrm{d}t \in \mathcal{B}\Big\} \tag{1.4}$$

and

$$(\mathcal{B}_0)^1 = \Big\{g : \mathbb{D} \to \mathbb{C} \;:\; \int_0^z g(t)\mathrm{d}t \in \mathcal{B}_0\Big\} \tag{1.5}$$

then we can restate the boundedness and compactness conditions as:

(i) $\mathcal{C}_g$ is bounded on L^2_a if and only if $g \in (\mathcal{B})^1$.

(ii) $\mathcal{C}_g$ is compact on L^2_a if and only if $g \in (\mathcal{B}_0)^1$.

In [12], the fine spectrum for $\mathcal{C}_g$ on H^2 is computed when g is a rational symbol. Furthermore, certain products of generalized Cesàro operators are shown to be Hilbert-Schmidt. In the Bergman setting, we do not know if $\mathcal{C}$ is essentially normal. We present computations to overcome this obstacle. We also show that $S\mathcal{C} - \mathcal{C}S$ is Hilbert-Schmidt. This allows us to compute the fine spectrum for I_G as well.

For notational convenience, if β or β_j is notated, it is always assumed that $|\beta| = |\beta_j| = 1$. Furthermore, $\mathcal{C}_\beta = \mathcal{C}_{1/(1-\beta z)}$.

2. COMPACTNESS RESULTS FOR $\mathcal{C}_g$

We have several goals for this section. We remark that $\mathcal{C}_\beta \cong \mathcal{C}$ by the same computation in [12]. We sketch the proof. Define the operator $U_\beta : L_a^2 \to L_a^2$ by $U_\beta(f)(z) = f(\beta z)$. If we compute $U_\beta^* \mathcal{C}_{g_\beta} U_\beta$ and make the change of variables $s = \beta t$, then we get the desired result.

We show that $\mathcal{C}_{\beta_1}\mathcal{C}_{\beta_2} \in \mathcal{B}_2(L_a^2)$, the ideal of Hilbert-Schmidt operators on L_a^2. Since we do not have essential normality, we also show that $\mathcal{C}_{\beta_1}^*\mathcal{C}_{\beta_2} \in \mathcal{B}_2(L_a^2)$ and $\mathcal{C}_{\beta_1}\mathcal{C}_{\beta_2}^* \in \mathcal{B}_2(L_a^2)$.

The first condition will allow us to compute the spectrum of $\mathcal{C}_g$. The two remaining relations will give us the essential spectrum and the Fredholm index.

Before we proceed to the proofs, we show the following lemma based on the summation by parts.

Lemma 2.1. *If $\alpha \in \mathbb{Z}$, $B_{j-1} = 0$ and $B_k = \sum_{k=j}^{n} \beta^k$ for $k = j, \dots, n$, then*

$$\Big|\sum_{k=j}^{n} k^\alpha \beta^k\Big| \leqslant \frac{2}{|1-\beta|}(n^\alpha + |j^\alpha - n^\alpha|). \tag{2.1}$$

Proof. It is clear that $|B_k| \leqslant (2/|1-\beta|)$ for all k. We have the following computation:

$$\begin{aligned}\sum_{k=j}^{n} k^\alpha \beta^k &= \sum_{k=j}^{n} k^\alpha (B_k - B_{k-1})\\ &= \sum_{k=j}^{n} k^\alpha B_k - \sum_{k=j}^{n-1} (k+1)^\alpha B_k\\ &= n^\alpha B_n + \sum_{k=j}^{n-1} B_k(k^\alpha - (k+1)^\alpha).\end{aligned}$$

By taking absolute values, the lemma follows. ∎

Theorem 2.2. *$\mathcal{C}_{\beta_1}\mathcal{C}_{\beta_2} \in \mathcal{B}_2(L_a^2)$ for $\beta_1 \neq \beta_2$.*

Proof. As in [12], we reduce to the case when $\beta_2 = 1$. Relabel $\beta_1 = \beta$. It is easy to check that the matrix for $\mathcal{C}_\beta$ in the basis $\{\sqrt{n} z^{n-1}\}_{n=1}^\infty$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{\beta}{2\sqrt{2}} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{\beta^2}{3\sqrt{3}} & \frac{\beta\sqrt{2}}{3\sqrt{3}} & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.2}$$

If we write the above matrix for $\mathcal{C}_\beta$ in coordinates, we have the following representation for each β:

$$(\mathcal{C}_\beta)_{nj} = \begin{cases} 0 & n < j, \\ \frac{\beta^{n-j}\sqrt{j}}{n^{3/2}} & n \geqslant j. \end{cases} \tag{2.3}$$

Thus

$$(\mathcal{C}_\beta\mathcal{C})_{nj} = \begin{cases} 0 & n < j, \\ \beta^n \sum_{k=j}^n \frac{\overline{\beta}^k\sqrt{j}}{n^{3/2}k} & n \geqslant j\ . \end{cases} \tag{2.4}$$

We calculate the Hilbert-Schmidt norm of $\mathcal{C}_\beta\mathcal{C}$:

$$\begin{aligned}\|\mathcal{C}_\beta\mathcal{C}\|_{\mathrm{HS}}^2 &= \sum_{n,j} |(\mathcal{C}_\beta\mathcal{C})_{nj}|^2 \\ &= \sum_{n=1}^\infty \sum_{j=1}^n \frac{j}{n^3}\Big|\sum_{k=j}^n \frac{\overline{\beta}^k}{k}\Big|^2.\end{aligned}$$

Apply Lemma 2.1 with $\alpha = -1$ to conclude

$$\Big|\sum_{k=j}^n \frac{\overline{\beta}^k}{k}\Big|^2 \leqslant \Big(\frac{2}{|1-\beta|}\Big)^2\Big(\frac{1}{n} + \Big|\frac{1}{j} - \frac{1}{n}\Big|\Big)^2. \tag{2.5}$$

Therefore by (2.5) we have

$$\begin{aligned}\|\mathcal{C}_\beta\mathcal{C}\|_{\mathrm{HS}}^2 &\leqslant \sum_{n=1}^\infty \sum_{j=1}^n \frac{j}{n^3}\Big(\frac{2}{|1-\beta|}\Big)^2\Big(\frac{1}{n^2} + \frac{2}{n}\Big|\frac{1}{j} - \frac{1}{n}\Big| + \Big|\frac{1}{j} - \frac{1}{n}\Big|^2\Big) \\ &\leqslant \Big(\frac{2}{|1-\beta|}\Big)^2 \sum_{n=1}^\infty \sum_{j=1}^n \frac{1}{n^2}\Big(\frac{1}{n^2} + \frac{2}{n}\Big|1 - \frac{1}{n}\Big|\Big) \\ &= \Big(\frac{2}{|1-\beta|}\Big)^2 \sum_{n=1}^\infty \Big(\frac{1}{n^3} + \frac{2}{n^2}\Big) < \infty.\end{aligned}$$

Thus $\mathcal{C}_\beta\mathcal{C} \in \mathcal{B}_2(L_a^2)$. ∎

In order to complete our analysis, we need the next result.

THEOREM 2.3. *For $\beta_1 \neq \beta_2$, $\mathcal{C}_{\beta_1}^*\mathcal{C}_{\beta_2}, \mathcal{C}_{\beta_1}\mathcal{C}_{\beta_2}^* \in \mathcal{B}_2(L_a^2)$.*

Proof. Without loss of generality, assume that $\beta_2 = 1$ and relabel β_1 as β. The reduction is similar to that for Theorem 2.2.

We first will show that $\mathcal{C}_\beta^*\mathcal{C} \in \mathcal{B}_2(L_a^2)$.

From (2.2), we get that

$$(\mathcal{C}_\beta^*)_{nj} = \begin{cases} 0 & n > j, \\ \frac{\beta^{n-j}\sqrt{n}}{j^{3/2}} & j \geqslant n\ . \end{cases} \tag{2.6}$$

Thus, we have that

$$(\mathcal{C}_\beta^*\mathcal{C})_{nj} = \beta^n \sum_{k=\max\{n,j\}}^\infty \frac{\overline{\beta}^k\sqrt{nj}}{k^3}. \tag{2.7}$$

Therefore, we need to show that the following sum converges:

$$\sum_{n,j=1}^\infty \Big|\sum_{k=\max\{n,j\}}^\infty \frac{\overline{\beta}^k\sqrt{nj}}{k^3}\Big|^2. \tag{2.8}$$

We begin by eliminating the maximum that appears in the k index by rewriting the sum as follows:

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\Big|\sum_{k=\max\{n,j\}}^{\infty}\frac{\overline{\beta}^k\sqrt{nj}}{k^3}\Big|^2\\ =\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{n}(nj)\Big|\sum_{k=n}^{\infty}\frac{\overline{\beta}^k}{k^3}\Big|^2+\sum_{j=n+1}^{\infty}(nj)\Big|\sum_{k=j}^{\infty}\frac{\overline{\beta}^k}{k^3}\Big|^2\Big).\end{aligned}\tag{2.9}$$

From Lemma 2.1 with $\alpha=-3$ we know that

$$\Big|\sum_{k=n}^{\infty}\frac{\overline{\beta}^k}{k^3}\Big|^2\leqslant\Big(\frac{2}{|1-\beta|}\Big)^2\frac{1}{n^6}.\tag{2.10}$$

Completing the estimate using (2.10), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{n}(nj)\Big|\sum_{k=n}^{\infty}\frac{\overline{\beta}^k}{k^3}\Big|^2+\sum_{j=n+1}^{\infty}(nj)\Big|\sum_{k=j}^{\infty}\frac{\overline{\beta}^k}{k^3}\Big|^2\Big)\\ \leqslant\Big(\frac{2}{|1-\beta|}\Big)^2\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{n}\frac{j}{n^5}+\sum_{j=n+1}^{\infty}\frac{n}{j^5}\Big)\\ \leqslant\Big(\frac{2}{|1-\beta|}\Big)^2\sum_{n=1}^{\infty}\Big(\frac{n+1}{n^4}+\int_n^{\infty}\frac{n}{x^5}\,\mathrm{d}x\Big)\\ =\Big(\frac{2}{|1-\beta|}\Big)^2\sum_{n=1}^{\infty}\Big(\frac{n+1}{n^4}+\frac{1}{4n^3}\Big)<\infty.\end{aligned}$$

Thus $\mathcal{C}_\beta^*\mathcal{C}\in\mathcal{B}_2(L_a^2)$. Now we turn our attention to $\mathcal{C}_\beta\mathcal{C}^*$. As before, we give the following coordinate representation of the matrix for $\mathcal{C}_\beta\mathcal{C}^*$ in the same basis as above:

$$(\mathcal{C}_\beta\mathcal{C}^*)_{nj}=\sum_{k=1}^{\min\{n,j\}}\frac{\beta^{n-k}k}{(nj)^{3/2}}=\beta^n\sum_{k=1}^{\min\{n,j\}}\frac{\overline{\beta}^k k}{(nj)^{3/2}}.\tag{2.11}$$

We use a similar trick as in the above computation to eliminate the minimum that is in the k index.

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\Big|\sum_{k=1}^{\min\{n,j\}}\frac{\overline{\beta}^k k}{(nj)^{3/2}}\Big|^2\\ =\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{n}\frac{1}{(nj)^3}\Big|\sum_{k=1}^{j}\overline{\beta}^k k\Big|^2+\sum_{j=n+1}^{\infty}\frac{1}{(nj)^3}\Big|\sum_{k=1}^{n}\overline{\beta}^k k\Big|^2\Big).\end{aligned}\tag{2.12}$$

Using Lemma 2.1 for $\alpha=1$ we have

$$\Big|\sum_{k=1}^{n}k\overline{\beta}^k\Big|^2\leqslant\Big(\frac{2}{|1-\beta|}\Big)^2(n+|1-n|)^2.\tag{2.13}$$

Combining (2.13) and (2.12), we get

$$\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{n}\frac{1}{(nj)^3}\Big|\sum_{k=1}^{j}k\overline{\beta}^k\Big|^2+\sum_{j=n+1}^{\infty}\frac{1}{(nj)^3}\Big|\sum_{k=1}^{n}k\overline{\beta}^k\Big|^2\Big)$$
$$\leqslant\Big(\frac{2}{|1-\beta|}\Big)^2\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{n}\frac{1}{(nj)^3}(j+|1-j|)^2+\sum_{j=n+1}^{\infty}\frac{1}{(nj)^3}(n+|1-n|)^2\Big)$$
$$\leqslant\Big(\frac{2}{|1-\beta|}\Big)^2\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{n}\frac{(2j+1)^2}{n^3j^3}+\int_{n}^{\infty}\frac{(2n+1)^2}{n^3x^3}\,\mathrm{d}x\Big)$$
$$\leqslant\Big(\frac{2}{|1-\beta|}\Big)^2\sum_{n=1}^{\infty}\Big(\frac{9}{n^2}+\frac{(2n+1)^2}{2n^5}\Big)<\infty.$$

Therefore $\mathcal{C}_\beta\mathcal{C}^*\in\mathcal{B}_2(L^2_a)$. ∎

Let π denote the natural projection of $\mathcal{B}(L^2_a)$ onto the Calkin algebra $\mathcal{Q}(L^2_a)$. By Theorems 2.2 and 2.3 we have the following relations:

$$\pi(\mathcal{C}_\beta\mathcal{C})=0, \tag{2.14}$$

$$\pi(\mathcal{C}^*_\beta\mathcal{C})=0, \tag{2.15}$$

$$\pi(\mathcal{C}_\beta\mathcal{C}^*)=0. \tag{2.16}$$

3. THE FINE SPECTRUM OF $\mathcal{C}_g$

In this section, we prove the main result of the paper. We show that for a rational g that is analytic on $\mathbb{D}$, the spectrum of $\mathcal{C}_g$ is a union of disks and some isolated points. The boundary of each disk contains 0. This result is completely analogous to the H^2 case presented in [12]. We first designate some notation. Let $D(a)=\{z:|z-a|<|a|\}$. Let $\overline{D(a)}$ denote its closure and $\partial D(a)$ denote its boundary. Now we state the main result.

THEOREM 3.1. *For $g(z)=\sum\limits_{i=1}^{n}\frac{a_i}{1-\beta_i z}$ where β_i are distinct and $a_i\neq 0$, the following holds for $\mathcal{C}_g$ on L^2_a:*

(i) $\sigma(\mathcal{C}_g)=\bigcup\limits_{i=1}^{n}\overline{D(a_i/2)}\cup\Big(\Big\{\frac{g(0)}{k}\Big\}_{k=1}^{\infty}\setminus\bigcup\limits_{i=1}^{n}\overline{D(a_i/2)}\Big)$;

(ii) $\sigma_{\mathrm{e}}(\mathcal{C}_g)=\bigcup\limits_{i=1}^{n}\partial D(a_i/2)$;

(iii) *for* $\lambda\notin\sigma_{\mathrm{e}}(\mathcal{C}_g)$, $\mathrm{ind}(\mathcal{C}_g-\lambda)=-G(\lambda)$, *where* $G(z)=-\sum\limits_{i=1}^{n}\chi_{D(a_i/2)}$.

The proof will be based on several lemmas.

Lemma 3.2. *If $a_1, \dots, a_n$ are elements of a C^*-algebra $\mathcal{A}$ such that $a_i a_j = a_j a_i = 0$ and $a_i^* a_j = a_j a_i^* = 0$ for $i \neq j$ and $\alpha_i \in \mathbb{C}$, then*

$$\{0\} \cup \sigma\Big(\sum_{i=1}^{n} a_i\Big) = \bigcup_{i=1}^{n} \sigma(a_i). \tag{3.1}$$

We note that the $\{0\}$ in (3.1) is necessary since the sum of non-invertible elements can be invertible. For example, let P be a projection; $PP^\perp = P^\perp P = 0$, but $P + P^\perp = I$.

Proof. We omit the details. We note that under the above conditions, the C^*-subalgebra generated by $a_1, \dots, a_n$ is isomorphic to the direct sum of the C^*-subalgebras generated by the separate a_i. ∎

Lemma 3.3. (Miller-Miller)

$$\sigma_{\mathrm{e}}(\mathcal{C}) = \Big\{z : \Big|z - \frac{1}{2}\Big| = \frac{1}{2}\Big\}.$$

Proof. The proof of the lemma is due to V.G. Miller and T.L. Miller. The author received the proof in a private correspondence. It is reproduced here with permission from them. For the proof, let $\mathcal{C}|_{H^2}$ denote the Cesàro operator on H^2 and $\mathcal{C}$ denote the Cesàro operator on L_a^2.

Let i be the inclusion mapping $i : H^2 \to L_a^2$. The mapping i has dense range and intertwines $\mathcal{C}|_{H^2}$ and $\mathcal{C}$. The fact that 1 is a cyclic vector for $\mathcal{C}|_{H^2}$ (see [7]) implies that 1 is a cyclic vector for $\mathcal{C}$.

For each $\lambda \in \operatorname{int}\sigma(\mathcal{C})$, define

$$\psi_\lambda(z) = \frac{\mathrm{d}}{\mathrm{d}z}(z(1-z)^{(1-\lambda)/\lambda}).$$

Then by Proposition 2.1 (3) of [9], $\ker(\lambda - \mathcal{C}^*) = \operatorname{span}\{\psi_{\overline{\lambda}}\}$ and for each λ, the mapping

$$\lambda \to \langle f, \psi_{\overline{\lambda}}\rangle$$

is analytic on $\operatorname{int}\sigma(\mathcal{C})$. By definition, it follows that the set of analytic bounded point evaluations of $\mathcal{C}$ is precisely $\operatorname{int}\sigma(\mathcal{C})$ (see [8]). $\mathcal{C}$ has Bishop's property (β) (see [9] for definition). Thus Theorem 3.1 in [8] shows that the set of analytic point evaluations of $\mathcal{C}$ is $\operatorname{abpe}(\mathcal{C}) = \sigma(\mathcal{C})\backslash\sigma_{\mathrm{ap}}(\mathcal{C})$. Since $\mathcal{C} - \lambda$ has closed range with $\dim\ker(\lambda - \mathcal{C}^*) = 1$ for all $\lambda \in \operatorname{int}\sigma(\mathcal{C})$, it follows that

$$\sigma_{\mathrm{e}}(\mathcal{C}) = \sigma_{\mathrm{ap}}(\mathcal{C}) = \partial\sigma(\mathcal{C}). \quad ∎$$

The next lemma appears in [12]. We reproduce the proof here. Recall that $\mathcal{F}(\mathcal{H})$ denotes the set of Fredholm operators on the Hilbert space $\mathcal{H}$. Also recall that for $T, S \in \mathcal{F}(\mathcal{H})$ and K compact:

(i) $TS \in \mathcal{F}(\mathcal{H})$ and $\operatorname{ind}(TS) = \operatorname{ind}(T) + \operatorname{ind}(S)$;

(ii) $\operatorname{ind}(T + K) = \operatorname{ind}(T)$.

LEMMA 3.4. *If $T_1, \dots, T_n \in \mathcal{B}(\mathcal{H})$ such that T_iT_j is compact for $i \neq j$ and $\alpha_i \in \mathbb{C}$, then for $\lambda \notin \bigcup_{i=1}^{n} \sigma_{\mathrm{e}}(T_i)$, we have that*

$$\operatorname{ind}\Big(\sum_{i=1}^{n} \alpha_i T_i - \lambda\Big) = \sum_{i=1}^{n} \operatorname{ind}(\alpha_i T_i - \lambda). \tag{3.2}$$

Proof. We will proceed by induction.

Assume $\lambda \notin \sigma_{\mathrm{e}}(T_1) \cup \sigma_{\mathrm{e}}(T_2)$.

If $\lambda = 0$, then $T_1 \in \mathcal{F}(\mathcal{H})$ and $T_2 \in \mathcal{F}(\mathcal{H})$. So, $T_1T_2 \in \mathcal{F}(\mathcal{H})$. However, $\pi(T_1T_2) = 0$, and 0 is not invertible. Therefore, $0 \in \sigma_{\mathrm{e}}(T_1) \cup \sigma_{\mathrm{e}}(T_2)$. Thus, we can assume $\lambda \neq 0$.

We calculate $(T_1 - \lambda)(T_2 - \lambda) = T_1T_2 - \lambda(T_1 + T_2 - \lambda)$. Hence, we have $\frac{1}{\lambda}(T_1 - \lambda)(T_2 - \lambda) = \frac{1}{\lambda}T_1T_2 - T_1 - T_2 + \lambda$. Here, we note that $\sigma_{\mathrm{e}}(T_1 + T_2) \subseteq \sigma_{\mathrm{e}}(T_1) \cup \sigma_{\mathrm{e}}(T_2)$ since $T_1 + T_2 - \lambda$ is a compact perturbation of the Fredholm operator $(1/\lambda)(T_1 - \lambda)(T_2 - \lambda)$.

Computing, we see $\operatorname{ind}(T_1 + T_2 - \lambda) = \operatorname{ind}\Big(\frac{1}{\lambda}(T_1 - \lambda)(T_2 - \lambda)\Big)$. We have that

$$\operatorname{ind}\Big(\frac{1}{\lambda}(T_1 - \lambda)(T_2 - \lambda)\Big) = \operatorname{ind}\Big(\frac{1}{\lambda}I\Big) + \operatorname{ind}(T_1 - \lambda) + \operatorname{ind}(T_2 - \lambda).$$

Therefore,

$$\operatorname{ind}(T_1 + T_2 - \lambda) = \operatorname{ind}\Big(\frac{1}{\lambda}I\Big) + \operatorname{ind}(T_1 - \lambda) + \operatorname{ind}(T_2 - \lambda) = \operatorname{ind}(T_1 - \lambda) + \operatorname{ind}(T_2 - \lambda).$$

The proof for n operators now follows easily by induction. ∎

The final lemma is well-known, and we omit the details.

LEMMA 3.5. *If $T \in \mathcal{B}(\mathcal{H})$ has a lower triangular matrix representation in an orthnormal basis $\mathcal{B}$ with diagonal entries $D = \{\alpha_i\}_{i=0}^{\infty}$, then $\sigma_{\mathrm{p}}(T) \subseteq D$.*

Proof of Theorem 3.1. Since $g(z) = \sum_{i=1}^{n} \frac{a_i}{1-\beta_i z}$, we have that

$$\mathcal{C}_g = \sum_{i=1}^{n} a_i \mathcal{C}_{\beta_i}. \tag{3.3}$$

By (2.14), (2.15), (2.16), Lemma 3.2, and Lemma 3.3 we have that

$$\{0\} \cup \sigma_{\mathrm{e}}(\mathcal{C}_g) = \bigcup_{i=1}^{n} \sigma_{\mathrm{e}}(a_i \mathcal{C}_{\beta_i}) = \bigcup_{i=1}^{n} \partial D(a_i/2). \tag{3.4}$$

Notice that 0 is a limit point of the RHS in (3.4). Thus, $0 \in \sigma_{\mathrm{e}}(\mathcal{C}_g)$. Therefore,

$$\sigma_{\mathrm{e}}(\mathcal{C}_g) = \bigcup_{i=1}^{n} \partial D(a_i/2). \tag{3.5}$$

Likewise, we use Lemma 3.4 to conclude $\operatorname{ind}(\mathcal{C}_g - \lambda) = \sum_{i=1}^{n} \operatorname{ind}(a_i \mathcal{C}_{\beta_i} - \lambda) \quad \forall \lambda \notin \sigma_{\mathrm{e}}(\mathcal{C}_g)$.

For $\lambda \in D(a_i/2)$, we have $\operatorname{ind}(a_i\mathcal{C}_{\beta_i} - \lambda) = -1$. If $\lambda \notin \overline{D(a_i/2)}$, we have $\operatorname{ind}(a_i\mathcal{C}_{\beta_i} - \lambda) = 0$. Therefore, $\operatorname{ind}(a_i\mathcal{C}_{\beta_i} - \lambda) = -\chi_{D(a_i/2)}$. By Lemma 3.4, we know the index is additive for $\mathcal{C}_g$. Thus, for $\lambda \notin \sigma_{\mathrm{e}}(\mathcal{C}_g) = \bigcup_{i=1}^{n} \partial D(a_i/2)$, we have

$$\operatorname{ind}(\mathcal{C}_g - \lambda) = -\sum_{i=1}^{n} \chi_{D(a_i/2)} = -G(\lambda). \tag{3.6}$$

By (3.2), the only way for $\operatorname{ind}(\mathcal{C}_g - \lambda) = 0$ is for $\lambda \notin \overline{D(a_i/2)}$ $\forall i$.

Now, we investigate the points $\lambda \in \sigma(\mathcal{C}_g)$ such that $\operatorname{ind}(\mathcal{C}_g - \lambda) = 0$. First, note that this implies that $\lambda \in \sigma_{\mathrm{p}}(\mathcal{C}_g)$. Define $\mathcal{E} = \{\lambda \in \sigma(\mathcal{C}_g) : \operatorname{ind}(\mathcal{C}_g - \lambda) = 0\}$. Observe that $\mathcal{E} \cap \bigcup_{i=1}^{n} \overline{D(a_i/2)} = \emptyset$.

It is easy to check that the matrix for $\mathcal{C}_g$ in the standard basis of L_a^2 is lower triangular. Applying Lemma 3.5, we know that the only possible eigenvalues of $\mathcal{C}_g$ are the diagonal elements $\left\{\frac{g(0)}{k}\right\}_{k=1}^{\infty}$. From (2.6), we know that $\left\{\frac{g(0)}{k}\right\}_{k=1}^{\infty} \subseteq \sigma_{\mathrm{p}}(\mathcal{C}_g^*)$. Hence, $\mathcal{E} \subseteq \left\{\frac{g(0)}{k}\right\}_{k=0}^{\infty}$.

Thus,

$$\sigma(\mathcal{C}_g) = \bigcup_{i=1}^{n} \overline{D(a_i/2)} \cup \Big(\Big\{\frac{g(0)}{k}\Big\}_{k=1}^{\infty} \setminus \bigcup_{i=1}^{n} \overline{D(a_i/2)}\Big). \qquad \blacksquare$$

COROLLARY 3.6. *If $g(z) = \sum_{i=1}^{n} \frac{a_i}{1-\beta_i z}$, where β_i are distinct, $a_i \neq 0$, and $h \in (\mathcal{B}_0)^1$, then the only change to the conclusion of Theorem* 3.1 *is that*

$$\sigma(\mathcal{C}_{g+h}) = \bigcup_{i=1}^{n} \overline{D(a_i/2)} \cup \Big(\Big\{\frac{g(0)+h(0)}{k}\Big\}_{k=1}^{\infty} \setminus \bigcup_{i=1}^{n} \overline{D(a_i/2)}\Big).$$

Proof. $\mathcal{C}_h$ is a compact operator. Therefore, $\mathcal{C}_{g+h}$ is a compact perturbation of $\mathcal{C}_g$. Hence, the Fredholm index and the essential spectrum do not change.

As in the proof of Theorem 3.1, we get that

$$\mathcal{E} = \{\lambda \in \sigma(\mathcal{C}_{g+h}) : \operatorname{ind}(\mathcal{C}_{g+h} - \lambda) = 0\} = \Big\{\frac{g(0)+h(0)}{k}\Big\}_{k=1}^{\infty} \setminus \bigcup_{i=1}^{n} \overline{D(a_i/2)}. \qquad \blacksquare$$

4. RELATING I_G AND $\mathcal{C}_g$

This final section concerns the relationship between S, I_G, and $\mathcal{C}_g$. Specifically, we show that for a rational g, $S\mathcal{C}_g - \mathcal{C}_g S$ is Hilbert-Schmidt. Furthermore by (1.3) and $\mathcal{C}_g S = \mathcal{C}_{zg}$, we can determine the fine spectrum of I_G. We draw other consequences in the final corollary by relating Toeplitz operators on L_a^2 to $\mathcal{C}_g$.

THEOREM 4.1. *For a function g with $\int\limits_0^z g(t)\mathrm{d}t \in$ BMOA, $S\mathcal{C}_g - \mathcal{C}_g S \in \mathcal{B}_2(L_a^2)$. In particular, there is a compact operator K such that $S\mathcal{C}_g = \mathcal{C}_{zg} + K$.*

Proof. First note that $\int\limits_0^z g(t)\mathrm{d}t \in$ BMOA implies that $\int\limits_0^z g(t)\mathrm{d}t \in H^2$. Therefore if $g(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, then

$$\sum_{n=0}^{\infty} \Big(\frac{|a_n|}{n+1}\Big)^2 = M < \infty. \tag{4.1}$$

We compute matrices for $S\mathcal{C}_g$ and $\mathcal{C}_g S$, expressed in their coordinate representations:

$$(S\mathcal{C}_g)_{nj} = \begin{cases} \frac{a_{n-j-1}\sqrt{j}}{\sqrt{n}(n-1)} & n > j, \\ 0 & n \leqslant j. \end{cases} \tag{4.2}$$

$$(\mathcal{C}_g S)_{nj} = \begin{cases} \frac{a_{n-j-1}\sqrt{j}}{n^{3/2}} & n > j, \\ 0 & n \leqslant j. \end{cases} \tag{4.3}$$

By (4.2) and (4.3) we know

$$(S\mathcal{C}_g - \mathcal{C}_g S)_{nj} = \begin{cases} \frac{\sqrt{j}}{\sqrt{n}} \frac{a_{n-j-1}}{n(n-1)} & n > j, \\ 0 & n \leqslant j\ . \end{cases}$$

Now we calculate the Hilbert-Schmidt norm:

$$\begin{aligned} \|(S\mathcal{C}_g - \mathcal{C}_g S)\|_{HS}^2 &= \sum_{n=2}^{\infty}\sum_{j=1}^{n-1} \frac{\sqrt{j}}{\sqrt{n}} \Big(\frac{|a_{n-j-1}|}{n(n-1)}\Big)^2 \\ &\leqslant \sum_{n=2}^{\infty} \Big(\frac{1}{n-1}\Big)^2 \sum_{j=1}^{n-1} \Big(\frac{|a_{n-j-1}|}{n-j}\Big)^2 \\ &\leqslant \sum_{n=2}^{\infty} \Big(\frac{1}{n-1}\Big)^2 M < \infty. \qquad \blacksquare \end{aligned}$$

Notice that for $g(z) = 1/(1-\beta z)$ we have $zg(z) = \overline{\beta}(-1+1/(1-\beta z))$. Thus $\mathcal{C}_{zg}$ is a compact perturbation of $\mathcal{C}_{\overline{\beta}g}$. Furthermore we know that the spectrum of $S\mathcal{C}_g$ is the union of disks with the boundary circles containing 0. Notice that $0 \in \sigma_{\mathrm{p}}((S\mathcal{C}_g)^*)$ and $\mathcal{E} = \emptyset$.

In light of these comments we have the following theorem.

THEOREM 4.2. *Let $g(z) = \sum\limits_{i=1}^{n} \frac{a_i}{1-\beta_i z}$, where $\beta_i \in \mathbb{T}$ are distinct and $a_i \neq 0$. Let $h \in (\mathcal{B}_0)^1$. Then for $S\mathcal{C}_g$, the following are true:*

(i) $\sigma(S\mathcal{C}_{g+h}) = \bigcup\limits_{i=1}^{n} \overline{D(\overline{\beta_i} a_i/2)}$.

(ii) $\sigma_{\mathrm{e}}(S\mathcal{C}_{g+h}) = \bigcup\limits_{i=1}^{n} \partial D(\overline{\beta_i} a_i/2)$.

(iii) *If* $\lambda \notin \sigma_{\mathrm{e}}(S\mathcal{C}_{g+h})$ *and* $H = \sum_{i=1}^{n} \chi_{D(\overline{\beta_i} a_i/2)}$ *then* $\operatorname{ind}(\mathcal{C}_g - \lambda) = -H(\lambda)$.

We conclude our study of generalized Cesàro operators on L_a^2 with the following corollary.

COROLLARY 4.3. *If* g *is a rational function,* $\varphi \in A(\mathbb{D})$ *and* T_φ *is the analytic Toeplitz operator with symbol* φ*, then there is a compact operator* K *such that* $T_\varphi \mathcal{C}_g = \mathcal{C}_g T_\varphi + K = \mathcal{C}_{g\varphi} + K$.

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