J. OPERATOR THEORY
53:1(2005), 35–48

STATES WITH EQUIVALENT SUPPORTS

ESTEBAN ANDRUCHOW and ALEJANDRO VARELA

Communicated by William B. Arveson

ABSTRACT. Let $\mathcal{B}$ be a von Neumann algebra and X a C^* Hilbert $\mathcal{B}$-module. If $p \in \mathcal{B}$ is a projection, denote by $\mathcal{S}_p(X) = \{x \in X : \langle x, x\rangle = p\}$, the p-sphere of X. For φ a state of $\mathcal{B}$ with support p in $\mathcal{B}$ and $x \in \mathcal{S}_p(X)$, consider the state φ_x of $\mathcal{L}_{\mathcal{B}}(X)$ given by $\varphi_x(t) = \varphi(\langle x, t(x)\rangle)$. In this paper we study certain sets associated to these states, and examine their topologic properties. As an application of these techniques, we prove that the space of states of the hyperfinite II_1 factor $\mathcal{R}_0$, with support equivalent to a given projection $p \in \mathcal{R}_0$, regarded with the norm topology (of the conjugate space of $\mathcal{R}_0$), has trivial homotopy groups of all orders.

KEYWORDS: *State space, C^*-module.*

MSC (2000): 46L30, 46L05, 46L10, 46L08.

1. INTRODUCTION

A few years ago Popa and Takesaki ([10]) studied the homotopy theory of the unitary and automorphism groups of a factor in the weak topologies. They proved, for example, that the unitary group $U_{\mathcal{R}}$ of a II_1 factor belonging to a family that includes the hyperfinite factor $\mathcal{R}_0$, is contractible in the strong operator topology. Their results and techniques can be employed to prove (Theorem 3.4 below) that the set of partial isometries with fixed initial space of such a factor $\mathcal{R}$ has trivial homotopy groups when regarded with the strong operator topology.

Here we are interested in the set of (normal) states of a von Neumann algebra $\mathcal{B}$ which have equivalent supports. The states which have the same support form a convex set. There is a natural map relating partial isometries $v \in \mathcal{B}$ with initial space p, normal states φ with support *equal* to p, and normal states with support *equivalent* to p. Namely

$$v \times \varphi \mapsto \varphi(v^* \cdot v).$$

Clearly, all states with support equivalent to p arise in this manner. The purpose of this paper is the study of the properties of this map. Mainly, under which assumptions it is a fibration. And in the affirmative case, to use this fibration to describe the homotopy type of the sets involved.

First, there is the question of what is the right topology to consider in the set of partial isometries (among the various topologies available in $\mathcal{B}$). It turns out that the norm topology of $\mathcal{B}$ forces on the set of states, via the map above, a topology stronger than the norm topology for functionals. In order that this map induces the usual norm topology in the set of states (as the quotient topology), one has to consider on (the set of partial isometries of) $\mathcal{B}$ the strong operator topology. The paper by S. Popa and M. Takesaki ([10]) deals with the topologic properties of the unitary group in the weak topologies. There, Michael's theory of continuous selection ([7]) is used in a remarkable manner to obtain cross sections (i.e. fibration properties) for the quotient of the unitary groups of an inclusion of factors. We shall use their technique in our context.

In Section 2 of this paper we shall put our problem in the broader context of Hilbert C^*-modules. All the concepts above have their analogue in this general setting. We shall therefore state first the results valid in the general context.

In Section 3 we shall return again to our original situation, and shall be able to state the Theorem 3.4 described in the abstract, for the family of II_1 factors considered in [10], which encloses the hyperfinite factor $\mathcal{R}_0$. Namely, that if $\mathcal{B}$ is a separable factor of type II_1 such that the tensor product $\mathcal{B} \otimes B(K)$ (K a separable Hilbert space) admits a one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$ scaling the trace of $\mathcal{B} \otimes B(K)$, i.e. $\tau \circ \theta_s = \mathrm{e}^{-s}\theta_s$, $s \in \mathbb{R}$, with τ a faithful semifinite normal trace in $\mathcal{B} \otimes B(K)$, then, for any projection $p \in \mathcal{B}$, the sets

$$\{v \in \mathcal{B} : v^* v = p\}$$

and

$$\{\varphi(v^* \cdot v) : \varphi \text{ with support } p, v^* v = p\}$$

have trivial homotopy groups of all orders.

Let us introduce some notation. Throughout this paper X will denote a Hilbert C^*-module over $\mathcal{B}$. For a fixed projection $p \in \mathcal{B}$, let

$$\mathcal{S}_p(X) = \{x \in X : \langle x, x\rangle = p\}$$

be the p-sphere of X. If φ is a normal state of $\mathcal{B}$ with support projection p, and x is an element of $\mathcal{S}_p(X)$, then one obtains a new state φ_x, defined on $\mathcal{L}_{\mathcal{B}}(X)$, the C^*-algebra of adjointable operators on X, by means of

$$\varphi_x(t) = \varphi(\langle x, t(x)\rangle).$$

If the module is selfdual, then φ_x is clearly a normal state of the von Neumann algebra $\mathcal{L}_{\mathcal{B}}(X)$. We shall call these states *modular vector states*. We consider the following sets of modular vector states. First, for a fixed φ,

$$\mathcal{O}_\varphi = \{\varphi_x : x \in \mathcal{S}_p(X)\}.$$

Denote by $\Sigma_p(\mathcal{B})$ the set of normal states of $\mathcal{B}$ with support equal to p, and by

$$\Sigma_{p,X} = \{\psi_x : x \in \mathcal{S}_p(X), \psi \in \Sigma_p(\mathcal{B})\},$$

the set of all modular vector states associated to the projection p.

As is usual notation, if $x, y \in X$, $\theta_{x,y}$ denotes the "rank one" operator given by $\theta_{x,y}(z) = x\langle y, z\rangle$. Note that if $x \in \mathcal{S}_p(X)$, then $e_x = \theta_{x,x}$ is a projection in $\mathcal{L}_\mathcal{B}(X)$. A straightforward computation shows that e_x is the support of φ_x, and that all states with support (equivalent to) e_x are of this form.

Also note that if $\varphi, \psi \in \Sigma_p(\mathcal{B})$ and $x, y \in \mathcal{S}_p(X)$, then $\varphi_x = \psi_y$ if and only if there exists a unitary $u \in p\mathcal{B}p$ such that $y = xu$ and $\psi = \varphi \circ \mathrm{Ad}(u)$. These elementary facts are proved in [3].

2. PURIFICATION OF $\Sigma_{p,X}$

There is a natural representation for $\mathcal{L}_\mathcal{B}(X)$, studied in [8] and [11], in which all the states $\varphi_x \in \Sigma_{p,X}$ turn out to be induced by vectors in the Hilbert space of this representation. Let $X \otimes H$ be the algebraic tensor product, where H is a Hilbert space on which $\mathcal{B}$ acts. We will choose H as the space of a standard representation of $\mathcal{B}$. Recall the fact that for such a standard representation there exists a cone $\mathcal{P}$, called the positive standard cone, with many remarkable properties. Among them, any positive normal functional in $\mathcal{B}$ is implemented by a unique vector in this cone. In the vector space $X \otimes H$ consider the semidefinite positive form given by $[x \otimes \xi, y \otimes \eta] = (\xi, \langle x, y\rangle \eta)$, where $(\cdot,\cdot)$ is the inner product of H. Denote by $Z = \{z \in X \otimes H : [z, z] = 0\}$, and let $\mathcal{H}$ be the Hilbert space obtained as the completion of $(X \otimes H)/Z$. The representation $\rho : \mathcal{L}_\mathcal{B}(X) \to B(\mathcal{H})$ is given by $\rho(t)([x \otimes \xi]) = [t(x) \otimes \xi]$.

LEMMA 2.1. *In the representation ρ, the state $\varphi_x \in \Sigma_{p,X}$ is implemented by the (class of the) vector $x \otimes \xi$, where ξ is the unique vector in the positive cone $\mathcal{P}$ which implements φ ($\varphi(a) = \omega_\xi(a) = (a\xi, \xi)$), that is*

$$\varphi_x(t) = [\rho(t)(x \otimes \xi), x \otimes \xi], \quad t \in \mathcal{L}_\mathcal{B}(X).$$

Proof. Straightforward: $[\rho(t)(x \otimes \xi), x \otimes \xi] = (\xi, \langle t(x), x\rangle \xi) = (\langle x, t(x)\rangle \xi, \xi)$ $= \varphi(\langle x, t(x)\rangle) = \varphi_x(t)$. ∎

In order to simplify the exposition, we shall restrict ourselves to the case $p = 1$. This is in fact not significant, since the general case can be easily reduced to this situation (note that $\mathcal{S}_p(X)$ is the unit sphere of the $p\mathcal{B}p$ module Xp). The unit vectors of the cone implementing the faithful states of $\mathcal{B}$ are the vectors which are cyclic and separating for $\mathcal{B}$. Let us denote

$$\mathcal{A}(X) = \{[x \otimes \xi] : x \in \mathcal{S}_1(X), \xi \in \mathcal{P} \text{ cyclic and separating for } \mathcal{B}, \|\xi\| = 1\} \subset \mathcal{H}.$$

LEMMA 2.2. *Let $x, y \in \mathcal{S}_1(X)$ and $\xi, \eta \in \mathcal{P}$ unit, cyclic and separating; then the elements $x \otimes \xi$ and $y \otimes \eta$ induce the same element in $\mathcal{A}(X)$ only if $x = y$ and $\xi = \eta$. In other words, there is a bijection*

$$\mathcal{S}_1(X) \times \Sigma_1(\mathcal{B}) \leftrightarrow \mathcal{A}(X), \quad (x, \varphi) \mapsto x \otimes \xi.$$

Proof. Suppose that $x \otimes \xi \sim y \otimes \eta$, with x, y, ξ, η as above. Then

$$0 = [x \otimes \xi - y \otimes \eta, x \otimes \xi - y \otimes \eta] = 2 - 2\mathrm{Re}((\xi, \langle x, y\rangle \eta)).$$

That is, $(\xi, \langle x, y\rangle\eta) = 1$. Since ξ, η are unital and $\|\langle x, y\rangle\| \leqslant 1$, by the Cauchy-Schwarz inequality this implies that $\langle x, y\rangle\eta = \lambda\xi$, for $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Again, using that ξ, η are unital, this implies that $\lambda = 1$, i.e. $\langle x, y\rangle\eta = \xi$. On the other hand, the states induced in $\mathcal{L}_{\mathcal{B}}(X)$ by the vectors $[x \otimes \xi]$ and $[y \otimes \eta]$ via the representation ρ were shown to be φ_x and ψ_y, where φ, ψ are the states induced in $\mathcal{B}$ by ξ, η, respectively, as shown in the lemma above. By the last remark in the introduction, $\varphi_x = \psi_y$ implies that there exists $u \in U_{\mathcal{B}}$ such that $y = xu$ and $\psi = \varphi \circ \mathrm{Ad}(u)$. So $\langle x, y\rangle\eta = \xi$ translates into $u\eta = \xi$. The other identity $\psi = \varphi \circ \mathrm{Ad}(u)$ can also be interpreted in terms of these vectors in the cone $\mathcal{P}$. Namely, the unique vector in the cone associated to the state $\varphi \circ \mathrm{Ad}(u)$ is $u^* J u^* J\xi$, where J denotes the modular conjugation of the standard representation $\mathcal{B} \subset B(H)$. Indeed, clearly $u^* J u^* J\xi \in \mathcal{P}$, and $(au^* J u^* J\xi, u^* J u^* J\xi) = (Ju^* J au^*\xi, Ju^* Ju^*\xi) = (au^*\xi, u^*\xi) = (uau^*\xi, \xi) = \varphi(uau^*)$. Therefore, by the uniqueness condition (on vectors in the cone inducing states), it follows that $\eta = u^* J u^* J\xi$. Combining this with $u\eta = \xi$ yields

$$\xi = Ju^* J\xi = Ju^*\xi, \quad \text{i.e. } \xi = u^*\xi.$$

This implies that u^* acts as the identity operator on $\mathcal{B}'\xi$, which is dense in H, because ξ is cyclic for $\mathcal{B}'$. Therefore $u = 1$. Then $x = y$ and $\xi = \eta$. ∎

These two lemmas state that the map

$$\wp_1 : \mathcal{S}_1(X) \times \Sigma_1(\mathcal{B}) \to \Sigma_{1,X}, \quad \wp_1(x, \varphi) = \varphi_x,$$

in this representation looks like

$$\vec{\wp}_1 : \mathcal{A}(X) \to \Omega_{\mathcal{A}(X)}, \quad \vec{\wp}_1([x \otimes \xi]) = \omega_{[x\otimes\xi]},$$

where $\omega_{[x\otimes\xi]}$ is the vector state induced by $[x \otimes \xi] \in \mathcal{H}$, and $\Omega_{\mathcal{A}(X)}$ is the space of all such states with symbols in $\mathcal{A}(X)$. What one gains by taking this standpoint is that $\mathcal{A}(X)$ has a natural topology, as a subset of the Hilbert space $\mathcal{H}$. The set $\Omega_{\mathcal{A}(X)} \sim \Sigma_{1,X}$ is therefore endowed with the quotient topology induced by $\mathcal{A}(X)$ and $\vec{\wp}_1$. The fibre of this map is a copy of the unitary group $U_{\mathcal{B}}$ of $\mathcal{B}$. The next result examines how the unitary group $U_{\mathcal{B}}$ appears inside $\mathcal{A}(X)$ and which is its relative topology. By the above result, we can omit the brackets when dealing with classes of elementary tensors of the form $x \otimes \xi$ in $\mathcal{A}(X) \subset \mathcal{H}$ ($x \in \mathcal{S}_1(X)$, ξ unit, cyclic and separating in $\mathcal{P}$). Also, note that the vectors $x \otimes \xi \in \mathcal{A}(X)$ are cyclic for ρ, but not separating in general.

In what follows, we shall make the assumption that the module X is *selfdual* ([8]).

PROPOSITION 2.3. *Given a fixed element $x \otimes \xi \in \mathcal{A}(X)$, the fibre $\vec{\wp}_1^{-1}(\omega_{x\otimes\xi})$ is the set $\{xu \otimes u^*Ju^*J\xi : u \in U_{\mathcal{B}}\}$ which is in one to one correspondence with $U_{\mathcal{B}}$. The relative topology induced on $U_{\mathcal{B}}$ by this bijection is the strong operator topology.*

Proof. If $y \otimes \eta$ lies in the fibre $\vec{\wp}_1^{-1}(\omega_{x\otimes\xi})$, then $\omega_{x\otimes\xi} = \omega_{y\otimes\eta}$, or $\varphi_x = \psi_y$, where as in the previous lemma φ and ψ are the states of $\mathcal{B}$ associated to the vectors ξ and η. Again, this implies that there exists a unitary in $U_{\mathcal{B}}$ such that $y = xu$ and $\eta = u^*Ju^*J\xi$. Then $y \otimes \eta = xu \otimes u^*Ju^*J\xi$. Now suppose that a net $xu_\alpha \otimes u_\alpha^*Ju_\alpha^*J\xi$ converges to $xu \otimes u^*Ju^*J\xi$ in the Hilbert space topology (of $\mathcal{H}$). This implies that

$$\|xu_\alpha \otimes u_\alpha^*Ju_\alpha^*J\xi - xu \otimes u^*Ju^*J\xi\|^2 = 2 - 2\mathrm{Re}((u_\alpha^*Ju_\alpha^*J\xi, \langle xu_\alpha, xu\rangle u^*Ju^*J\xi)) \to 0$$

with α. In other words,

$$(u_\alpha^*Ju_\alpha^*J\xi, u_\alpha^*Ju^*J\xi) = (u_\alpha^*\xi, u^*\xi) \to 1.$$

Equivalently, $(uu_\alpha^*\xi, \xi) \to 1$. This implies that $\|(u - u_\alpha)\xi\| \to 0$ in the Hilbert space norm (of H). Now let $a' \in \mathcal{B}'$; then

$$\|(u - u_\alpha)a'\xi\| = \|a'(u - u_\alpha)\xi\| \leqslant \|a'\|\|(u - u_\alpha)\xi\| \to 0.$$

That is, $u_\alpha v \to uv$ in a dense subset of vectors $v \in H$. Since u, u_α, being unitaries, are bounded in norm, this implies strong operator convergence of u_α to u. The converse implication is straightforward. ∎

The tensor product $(X \otimes H)/Z$ is a $\mathcal{B}$-bimodule tensor product, in the sense that for any $b \in \mathcal{B}$, $x \in X$ and $v \in H$, one has $xb \otimes v$ equivalent to $x \otimes bv$. Then the elements $xu \otimes u^*Ju^*J\xi$ in the fibre of $\omega_{x\otimes\xi} = \varphi_x$ can be parametrized $x \otimes Ju^*J\xi = x \otimes Ju^*\xi$ for $u \in U_{\mathcal{B}}$. We prefer the first presentation because the vector $Ju^*\xi$ does not belong to $\mathcal{P}$. However the latter clarifies the action of $U_{\mathcal{B}}$ on $\mathcal{A}(X)$; namely, the right action

$$(x \otimes \xi) \bullet u = x \otimes Ju^*J\xi.$$

Note that it is indeed a right action: $(x \otimes \xi) \bullet vu = x \otimes J(vu)^*J\xi = x \otimes Ju^*JJv^*J\xi = ((x \otimes \xi) \bullet v) \bullet u$.

The sphere $\mathcal{S}_1(X)$ and the set $\Sigma_1(\mathcal{B})$ of faithful states of $\mathcal{B}$ lie inside $\mathcal{A}(X)$. Pick fixed elements $x_0 \in \mathcal{S}_1(X)$ and $\xi_0 \in \mathcal{P}$ unit, cyclic and separating, inducing the state φ_0. The following maps are one to one:

$$\mathcal{S}_1(X) \to \{x \otimes \xi_0 : x \in \mathcal{S}_1(X)\} \subset \mathcal{A}(X), \quad x \mapsto x \otimes \xi_0,$$

and

$$\Sigma_1(\mathcal{B}) \to \{x_0 \otimes \xi : \xi \in \mathcal{P} \text{ unit, cyclic and separating}\}, \quad \varphi \mapsto x_0 \otimes \xi,$$

where ξ is the vector in the cone associated to φ.

PROPOSITION 2.4. *The first bijection endows $\mathcal{S}_1(X)$ with the relative topology induced from $\mathcal{H}$, which is given by the following: a net x_α converges to x if and only if $\varphi_0(\langle x_\alpha - x, x_\alpha - x\rangle) \to 0$, or equivalently*

$$|x_\alpha - x| \to 0$$

in the strong operator topology of $\mathcal{B} \subset B(H)$. The sphere $\mathcal{S}_1(X) \subset X$ is closed in this topology.

The second bijection is a homeomorphism when $\Sigma_1(\mathcal{B})$ is regarded with the norm topology and $\{x_0 \otimes \xi : \xi \in \mathcal{P}$ unit, cyclic and separating$\} \subset \mathcal{H}$ is regarded with the Hilbert space norm of $\mathcal{H}$.

Proof. The second statement is straightforward, because $\|x_0 \otimes \xi - x_0 \otimes \eta\|^2 = 2 - 2\mathrm{Re}(\xi, \eta) = \|\xi - \eta\|^2$ and the well known fact that the topology of the distance between the vectors in $\mathcal{P}$ yields a topology which is equivalent to the one given by the norm of the induced states in the conjugate space. Let $x_\alpha \otimes \xi_0$ be a net, and $x \otimes \xi_0$ an element in $\mathcal{A}(X)$. Then $\|x_\alpha \otimes \xi_0 - x \otimes \xi_0\|^2 = 2 - 2\mathrm{Re}(\xi_0, \langle x_\alpha, x\rangle \xi_0) = 2 - 2\mathrm{Re}(\varphi_0(\langle x_\alpha, x\rangle)) = \varphi_0(\langle x_\alpha - x, x_\alpha - x\rangle)$. Next we check that the convergence of the net in the sense described is equivalent to convergence to zero of $|x_\alpha - x|$ in the strong topology, where, as is usual notation, $|y| = \langle y, y\rangle^{1/2}$ for $y \in X$. Since φ_0 is implemented by the vector ξ_0, $\varphi_0(\langle x_\alpha - x, x_\alpha - x\rangle) = \||x_\alpha - x|\xi_0\|^2$, and therefore convergence in the strong topology implies convergence in the former sense. Suppose now that $\||x_\alpha - x|\xi_0\| \to 0$, and take $a' \in \mathcal{B}'$. Then $\||x_\alpha - x|a'\xi_0\| = \|a'|x_\alpha - x|\xi_0\| \leqslant \|a'\|\||x_\alpha - x|\xi_0\| \to 0$. The set $\{a'\xi_0 : a' \in \mathcal{B}'\}$ is dense in H, and the operators $|x_\alpha - x|$ have bounded norms, therefore $|x_\alpha - x|$ tends strongly to zero.

Let us prove now that the sphere $\mathcal{S}_1(X) \subset X$ is closed in this topology. First note that this topology, on norm bounded sets, is induced by the seminorms $n_\nu(x) = (\langle x, x\rangle \nu, \nu)$, $\nu \in H$, $\|\nu\| = 1$ ([8]). Then it suffices to see that if $x_\alpha \to x$ with $x_\alpha \in \mathcal{S}_1(X)$, then $x \in \mathcal{S}_1(X)$. Now, $(\langle x, x\rangle \nu, \nu) = 1$. Indeed, if $\omega_\nu(a) = (a\nu, \nu)$, then $(\langle x_\alpha - x, x\rangle \nu, \nu) = \omega_\nu(\langle x_\alpha - x, x\rangle) \leqslant \omega_\nu(\langle x_\alpha - x, x_\alpha - x\rangle)^{1/2} = n_\nu(x_\alpha - x)^{1/2}$, i.e. $(\langle x_\alpha, x\rangle \nu, \nu) \to (\langle x, x\rangle \nu, \nu)$. Therefore

$$0 \leftarrow (\langle x_\alpha - x, x_\alpha - x\rangle \nu, \nu) = 1 + ((\langle x, x\rangle - \langle x_\alpha, x\rangle - \langle x, x_\alpha\rangle)\nu, \nu) \to 1 - (\langle x, x\rangle \nu, \nu).$$

Since this is true for all unit vectors $\nu \in H$, it follows that $x \in \mathcal{S}_1(X)$. ∎

REMARK 2.5. Since X is selfdual, it is a conjugate space ([8]). The result above shows that the topology of $\mathcal{S}_1(X)$ induced by the Hilbert space norm of $\mathcal{H}$ coincides with the w*-topology of $X \supset \mathcal{S}_1(X)$. Indeed, it was shown in [8] that a net $x_\alpha \to x$ in the w*-topology if and only if $\varphi(\langle x_\alpha, y\rangle) \to \varphi(\langle x, y\rangle)$ for all $y \in X$, $\varphi \in \mathcal{B}_*^+$. This clearly implies that $\varphi(\langle x_\alpha - x, x_\alpha - x\rangle) \to 0$, which is the topology considered in the lemma (here the fact $\langle x, x\rangle = \langle x_\alpha, x_\alpha\rangle = 1$ is crucial). Conversely,

$$\varphi(\langle x_\alpha - x, y\rangle) \leqslant \varphi(\langle x_\alpha - x, x_\alpha - x\rangle)^{1/2} \varphi(\langle y, y\rangle)^{1/2}$$

yields the other implication.

We have examined the topologies induced on $\mathcal{S}_1(X)$ and $\Sigma_1(\mathcal{B})$ by the described inclusions on $\mathcal{A}(X)$. We have seen above that $\mathcal{A}(X) \sim \mathcal{S}_1(X) \times \Sigma_1(\mathcal{B})$. These facts alone however do not imply that $\mathcal{A}(X)$ is homeomorphic to $\mathcal{S}_1(X) \times \Sigma_1(\mathcal{B})$ in the product topology (of the w*-topology and the norm topology respectively). The next result shows that this is the case.

THEOREM 2.6. *The bijection*

$$\mathcal{S}_1(X) \times \Sigma_1(\mathcal{B}) \to \mathcal{A}(X)\,, \quad (x, \varphi) \mapsto x \otimes \xi,$$

is a homeomorphism when $\mathcal{S}_1(X) \times \Sigma_1(\mathcal{B})$ is endowed with the product topology of the w**-topology of $\mathcal{S}_1(X)$ and the norm topology of $\Sigma_1(\mathcal{B})$.*

Proof. By the Proposition 2.4 above, it is clear that if $x_\alpha \to x$ in $\mathcal{S}_1(X)$ and $\varphi_\beta \to \varphi$ in $\Sigma_1(\mathcal{B})$, then $x_\alpha \otimes \xi_\beta \to x \otimes \xi$, where ξ_β, ξ are the vectors in the positive cone inducing φ_β, φ. In the other direction, suppose that $x_\alpha \otimes \xi_\alpha \to x \otimes \xi$ in $\mathcal{A}(X)$. This means that $(\langle x, x_\alpha\rangle \xi_\alpha, \xi) \to 1$. Then, since $\|\langle x, x_\alpha\rangle \xi_\alpha\| \leqslant 1$, it follows that $\|\langle x, x_\alpha\rangle\xi_\alpha - \xi\|^2 = 1 + \|\langle x, x_\alpha\rangle\xi_\alpha\|^2 - 2\mathrm{Re}(\langle x, x_\alpha\rangle\xi_\alpha, \xi) \to 0$ and similarly $\|\langle x_\alpha, x\rangle\xi - \xi_\alpha\| \to 0$. Then we get

$$\langle x, x_\alpha\rangle\xi_\alpha - \xi \to 0 \quad \text{and} \quad \langle x_\alpha, x\rangle\xi - \xi_\alpha \to 0. \tag{2.1}$$

This implies that

$$\omega_{\langle x, x_\alpha\rangle\xi_\alpha} - \omega_\xi = \varphi_\alpha\left(\langle x_\alpha, x\rangle \;\cdot\; \langle x, x_\alpha\rangle\right) - \varphi(\,\cdot\,) \to 0. \tag{2.2}$$

Using (2.1) it follows that $J\langle x_\alpha, x\rangle J\langle x, x_\alpha\rangle\xi_\alpha - \xi_\alpha \to 0$ and so

$$\omega_{J\langle x_\alpha, x\rangle J\langle x, x_\alpha\rangle\xi_\alpha} - \omega_{\xi_\alpha} = \omega_{J\langle x_\alpha, x\rangle J\langle x, x_\alpha\rangle\xi_\alpha} - \varphi_\alpha \to 0. \tag{2.3}$$

Note that for every $a \in \mathcal{B}$

$$\begin{aligned} \omega_{J\langle x_\alpha, x\rangle J\langle x, x_\alpha\rangle\xi_\alpha}(a) &= (aJ\langle x_\alpha, x\rangle J\langle x, x_\alpha\rangle\xi_\alpha, J\langle x_\alpha, x\rangle J\langle x, x_\alpha\rangle\xi_\alpha) \\ &= (\langle x_\alpha, x\rangle a\langle x, x_\alpha\rangle\xi_\alpha, J\langle x, x_\alpha\rangle\langle x_\alpha, x\rangle J\xi_\alpha). \end{aligned} \tag{2.4}$$

But

$$\begin{aligned} \|J\langle x, x_\alpha\rangle\langle x_\alpha, x\rangle J\xi_\alpha - \xi_\alpha\| \leqslant{}& \|J\langle x, x_\alpha\rangle\langle x_\alpha, x\rangle J\xi_\alpha - J\langle x, x_\alpha\rangle\langle x_\alpha, x\rangle J\langle x_\alpha, x\rangle\xi\| \\ &+ \|J\langle x, x_\alpha\rangle\langle x_\alpha, x\rangle J\langle x_\alpha, x\rangle\xi - \xi_\alpha\|. \end{aligned}$$

It is easy to prove that the first term on the right hand side of the last inequality tends to zero using (2.1). The second term is equal to $\|J\langle x, x_\alpha\rangle\langle x_\alpha, x\rangle J\langle x_\alpha, x\rangle J\xi - \xi_\alpha\| = \|JJ\langle x_\alpha, x\rangle J\langle x, x_\alpha\rangle\langle x_\alpha, x\rangle\xi - \xi_\alpha\|$, which also tends to zero again using (2.1). Therefore $J\langle x, x_\alpha\rangle\langle x_\alpha, x\rangle J\xi_\alpha - \xi_\alpha \to 0$, and using (2.4) we get that

$$\omega_{J\langle x_\alpha, x\rangle J\langle x, x_\alpha\rangle\xi_\alpha} - \varphi_\alpha\left(\langle x_\alpha, x\rangle \cdot \langle x, x_\alpha\rangle\right) \to 0. \tag{2.5}$$

Finally, using (2.2), (2.3) and (2.5) it follows that $\varphi_\alpha \to \varphi$ in norm.

This last convergence is equivalent to $\xi_\alpha \to \xi$ in H. Then $(\langle x, x_\alpha\rangle\xi_\alpha, \xi) \to 1$ implies that $(\langle x, x_\alpha\rangle\xi, \xi) \to 1$. Then $\varphi(|x - x_\alpha|) \to 0$, i.e. $x_\alpha \to x$ in the w*-topology. ∎

COROLLARY 2.7. *The space $\mathcal{A}(X)$ is homotopically equivalent to the sphere $\mathcal{S}_1(X)$ with the* w**-topology.*

Proof. Recall that $\Sigma_1(\mathcal{B})$ is convex. ∎

Now we focus on the map

$$\vec{\wp}_1 : \mathcal{A}(X) \to \Omega_{\mathcal{A}(X)}, \quad \vec{\wp}_1(x \otimes \xi) = \omega_{x\otimes\xi}.$$

In order to see if this map is a fibration, we shall look for local cross sections. A powerful tool to state the existence of cross sections is Michael's theory of continuous selections ([7]). An example of the use of this theory in the context of operator algebras is the paper by S. Popa and M. Takesaki ([10]). To use Michael's theorem one must check first that the set function $\omega_{z\otimes\xi} \mapsto \vec{\wp}_1^{-1}(\omega_{z\otimes\xi})$, which assigns to each point in the base space the fibre over it, is *lower semicontinuous* ([7]).

REMARK 2.8. In our context lower semicontinuity means that, for any $r > 0$, and $x \otimes \xi \in \mathcal{A}(X)$, the set $\{\omega_{y\otimes\eta} : \|y \otimes Ju^*\eta - x \otimes \xi\| < r\}$ is open in $\Omega_{\mathcal{A}(X)}$. In other words, for a state $\omega_{y\otimes\eta}$ close to $\omega_{x\otimes\xi}$ one should find an element $y \otimes Ju^*\eta$ in the fibre of $\omega_{y\otimes\eta}$ at distance less than r to the fibre of $\omega_{x\otimes\xi}$. We have not specified yet the topology of this set $\Omega_{\mathcal{A}(X)}$. Lower semicontinuity implies that whatever topology one chooses, it must be stronger than the quotient topology given by $\vec{\wp}_1$. Indeed, two states in $\Omega_{\mathcal{A}(X)}$ are close in this quotient topology if and only if there are elements in their fibres which are close in $\mathcal{A}(X)$.

On the other hand, this quotient topology is stronger than the norm topology. Recall Bures metric for states ([4]), defined as the infimum of the distances between vectors inducing the states, taken over all possible representations where the two states are vector states. The topology induced by Bures metric on the state space is equivalent to the norm topology. This raises the question of whether this two topologies, the norm topology and the one induced by this purification, coincide in $\Omega_{\mathcal{A}(X)}$ $(= \Sigma_{1,X})$.

THEOREM 2.9. *The quotient and the norm topology coincide in $\Omega_{\mathcal{A}(X)}$.*

Proof. It was noted that the quotient topology is stronger than the norm topology. Let us check the converse statement. Let $\omega_{y_n\otimes\eta_n}$ be a sequence in $\Omega_{\mathcal{A}(X)}$ converging to $\omega_{x\otimes\xi}$ in norm. Testing convergence in operators of the form $\theta_{y_n a, y_n}$, for $a \in \mathcal{B}$, yields

$$\begin{aligned}\|a\|\|\omega_{y_n\otimes\eta_n} - \omega_{x\otimes\xi}\| &= \|\theta_{y_n a,y_n}\|\|\omega_{y_n\otimes\eta_n} - \omega_{x\otimes\xi}\| \\ &\geqslant |\omega_{y_n\otimes\eta_n}(\theta_{y_n a,y_n}) - \omega_{x\otimes\xi}(\theta_{y_n a,y_n})|.\end{aligned}$$

Note that $\omega_{y_n\otimes\eta_n}(\theta_{y_n a,y_n}) = (a\eta_n, \eta_n)$ and $\omega_{x\otimes\xi}(\theta_{y_n a,y_n}) = (a\langle y_n, x\rangle\xi, \langle y_n, x\rangle\xi)$. This implies that

$$\|\omega_{\eta_n} - \omega_{\langle y_n,x\rangle\xi}\| \to 0 \quad \text{as } n \to \infty.$$

In particular, testing this difference at $1 \in \mathcal{B}$, implies $(\langle x, y_n\rangle\langle y_n, x\rangle\xi, \xi) \to 1$. Therefore,

$$\|\langle x, y_n\rangle\langle y_n, x\rangle\xi - \xi\|^2 = 1 + \|\langle x, y_n\rangle\langle y_n, x\rangle\xi\|^2 - 2\mathrm{Re}(\langle x, y_n\rangle\langle y_n, x\rangle\xi, \xi) \to 0.$$

Coming back to ω_{η_n} and $\omega_{\langle y_n, x\rangle\xi}$, note that the vectors η_n belong to the cone $\mathcal{P}$, but not necessarily the vectors $\langle y_n, x\rangle\xi$. However $\delta_n = \langle y_n, x\rangle J\langle y_n, x\rangle\xi = \langle y_n, x\rangle J\langle y_n, x\rangle J\xi \in \mathcal{P}$ and we shall see that $\omega_{\delta_n} - \omega_{\langle y_n, x\rangle\xi} \to 0$ in norm. Indeed, note that

$$\begin{aligned}\omega_{\delta_n}(a) &= (a\langle y_n, x\rangle J\langle y_n, x\rangle\xi, \langle y_n, x\rangle J\langle y_n, x\rangle\xi)\\ &= (a\langle y_n, x\rangle J\langle x, y_n\rangle\langle y_n, x\rangle J\xi, \langle y_n, x\rangle\xi),\end{aligned}$$

and therefore

$$\begin{aligned}|(a\langle y_n, x\rangle\xi, \langle y_n, x\rangle\xi) - (a\delta_n, \delta_n)| &= |(a\langle y_n, x\rangle(\xi - J\langle x, y_n\rangle\langle y_n, x\rangle J\xi), \langle y_n, x\rangle\xi)|\\ &\leqslant \|a\|\|\xi - J\langle x, y_n\rangle\langle y_n, x\rangle\xi\|,\end{aligned}$$

which tends to zero. Combining these results one obtains that $\|\omega_{\delta_n} - \omega_{\eta_n}\| \to 0$. Now, because the vectors δ_n, η_n lie in $\mathcal{P}$, and the fact that norm convergence of vector states with symbols in $\mathcal{P}$ implies norm convergence of those symbols, one has that $\|\delta_n - \eta_n\| \to 0$ in H. In other words,

$$\mathrm{Re}(\langle y_n, x\rangle\xi, J\langle x, y_n\rangle J\eta_n) \to 1.$$

Suppose now that the states $\omega_{y_n\otimes\eta_n}$ do not converge to $\omega_{x\otimes\xi}$ in the quotient topology of $\Omega_{\mathcal{A}(X)}$. This means that the fibres of these states are not near in H, i.e., there exists a subsequence $y_{n_k} \otimes \eta_{n_k}$ such that $\|x \otimes \xi - y_{n_k} \otimes Ju^*\eta_{n_k}\| \geqslant d > 0$ for all $u \in U_{\mathcal{B}}$. Or equivalently,

$$\mathrm{Re}(\langle y_{n_k}, x\rangle\xi, Ju^*\eta_{n_k}) \leqslant 1 - \frac{d^2}{2} \quad \text{for all } u \in U_{\mathcal{B}}.$$

Clearly this inequality is preserved by taking convex combinations of unitaries $u \in U_{\mathcal{B}}$ (and leaving everything else fixed), as well as by taking norm limits of such combinations. It follows, using the Russo-Dye theorem, that for $a \in \mathcal{B}$, $\|a\| \leqslant 1$,

$$\mathrm{Re}(\langle y_{n_k}, x\rangle\xi, JaJ\eta_{n_k}) \leqslant 1 - \frac{d^2}{2}.$$

This clearly contradicts the inequality above, as seen by taking $a = \langle x, y_{n_k}\rangle$ for appropriate k. ∎

The next result uses part of the proof of Lemma 3 of [10].

THEOREM 2.10. *If $\mathcal{B}$ is a separable factor of type* II_1 *such that the tensor product $\mathcal{B} \otimes B(K)$ (K a separable Hilbert space) admits a one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$ scaling the trace of $\mathcal{B} \otimes B(K)$, i.e. $\tau \circ \theta_s = \mathrm{e}^{-s}\theta_s$, $s \in \mathbb{R}$, with τ a faithful semifinite normal trace in $\mathcal{B} \otimes B(K)$, then the map*

$$\vec{\wp}_1 : \mathcal{A}(X) \to \Omega_{\mathcal{A}(X)}, \quad \vec{\wp}_1(x \otimes \xi) = \omega_{x\otimes\xi},$$

admits a (global) continuous cross section when $\Omega_{\mathcal{A}(X)}$ is endowed with the norm topology.

Proof. In this case, since $\mathcal{B}$ is finite, $U_{\mathcal{B}}$ is complete in the strong (= strong*) operator topology ([12]). Moreover, Popa and Takesaki proved in [10] that $U_{\mathcal{B}}$ admits a geodesic structure in the sense of Michael ([7]). It has been already remarked that the set function $\omega_{x\otimes\xi} \mapsto \{xu \otimes u^* Ju^* J\xi : u \in U_{\mathcal{B}}\}$ is lower semicontinuous in the norm topology. Therefore Theorem 5.4 of [7] applies, and $\vec{\wp}_1$ has a continuous cross section. ∎

COROLLARY 2.11. *If $\mathcal{B}$ is a II_1 factor satisfying the conditions of Theorem 2.10, then for all $n \geqslant 0$, $x \in \mathcal{S}_1(X)$, $\varphi = \omega_\xi \in \Sigma_1(\mathcal{B})$,*

$$\pi_n(\Omega_{\mathcal{A}(X)}, \omega_{x\otimes\xi}) = \pi_n(\mathcal{S}_1(X), x),$$

where $\Omega_{\mathcal{A}(X)}$ is considered with the norm topology, and $\mathcal{S}_1(X)$ with the w**-topology.*

Proof. In [10] it was proven that the unitary group $U_{\mathcal{B}}$ of such a factor is contractible in the ultra strong operator topology, and therefore also in the strong operator topology. The result follows using the above theorem, recalling that the fibre of the fibration $\vec{\wp}_1$ is $U_{\mathcal{B}}$ with this topology. ∎

In [10] it is noted that remarkable examples of II_1 factors enjoy this property (of having a one parameter group of automorphisms that scale the trace when tensored with an infinite type I factor), for example the hyperfinite II_1 factor $\mathcal{R}_0$.

3. STATES OF THE HYPERFINITE II_1 FACTOR

We will apply the results of the previous section to obtain our main result, namely, that the set of states of $\mathcal{R}_0$, or more generally, of a factor satisfying the hypothesis of Theorem 2.10, having support equivalent to a given projection p, considered with the norm topology, has trivial homotopy groups of all orders.

There is a first result which can be obtained directly from the previous section. If $\mathcal{R}$ is a factor as in Theorem 2.10, and $p \in \mathcal{R}$ is a proper projection, put $X = \mathcal{R}p$ and $\mathcal{B} = p\mathcal{R}p$. Clearly $\mathcal{B}$ is a factor which also verifies the hypothesis of Theorem 2.10. Note that $\langle X, X\rangle = \mathrm{span}\{px^*yp : x, y \in \mathcal{R}p\} = p\mathcal{R}p = \mathcal{B}$ in this case. Therefore by 2.2 of [9], $\{\theta_{x,y} : x, y \in X\}$ spans an ultraweakly dense two sided ideal of $\mathcal{L}_{\mathcal{B}}(X)$. On the other hand, it is clear that $\mathcal{R} \subset \mathcal{L}_{\mathcal{B}}(X)$ as left multipliers, and also that $\theta_{x,y} \in \mathcal{R}$, for $x, y \in X = \mathcal{R}p$. Indeed, $\theta_{x,y}(z) = x\langle y, z\rangle = xpy^*z$, i.e. left multiplication by $xpy^* \in \mathcal{R}$. Therefore $\mathcal{L}_{\mathcal{B}}(X) = \mathcal{R}$. In particular, if $x \in \mathcal{S}_1(X)$, $e_x = \theta_{x,x} = xpx^*$ which is equivalent to $px^*xp = \langle x, x\rangle = p$ in $\mathcal{L}_{\mathcal{B}}(X)$. The set $\Sigma_{1,X} = \Omega_{\mathcal{A}(X)}$ equals, then the set of states of $\mathcal{R}$ with support (unitarily) equivalent to p. Note that this set is (arcwise) connected in the norm topology. Indeed, if $\mathcal{B}$ is finite, $\mathcal{S}_1(X)$ is connected ([1]). It was remarked that $\Sigma_1(\mathcal{B})$ is convex.

Using the (onto) map $\wp_1$,

$$\wp_1 : \mathcal{S}_1(X) \times \Sigma_1(\mathcal{B}) \to \Sigma_{1,X}$$

it follows that $\Sigma_{1,X}$ is connected.

Applying Theorem 2.10 in this situation implies the following:

COROLLARY 3.1. *Let $\mathcal{R}$ be a factor as in Theorem 2.10, and $p \in \mathcal{R}$ an arbitrary projection. The set of normal states of $\mathcal{R}$ with support equivalent to p considered with the norm topology has the same homotopy groups as the set*

$$\mathcal{S}_p(\mathcal{R}) = \{v \in \mathcal{R} : v^* v = p\} \subset \mathcal{R}$$

regarded with the (relative) ultraweak topology.

Proof. In this case $\mathcal{S}_1(X)$ clearly equals $\mathcal{S}_p(\mathcal{R})$ above, and the topology is the w* (i.e.) ultraweak topology of $\mathcal{R}$. If $p = 0$ the statement is trivial. If $p = 1$ it follows from the strong operator contractibility of $U_{\mathcal{R}}$ for such $\mathcal{R}$ proved in [10]. The case of a proper projection follows from 2.10 and the above remark. ∎

If $p = 0, 1$, then $\mathcal{S}_p(\mathcal{R})$ is contractible (if $p = 1$, $\mathcal{S}_p(\mathcal{R}) = U_{\mathcal{R}}$). A natural question would be if $\mathcal{S}_p(\mathcal{R})$ is contractible for proper $p \in \mathcal{R}$.

We need the following elementary fact:

LEMMA 3.2. *Let $\mathcal{M} \subset B(H)$ be a finite von Neumann algebra, and let $a_n \in \mathcal{M}$ be such that $\|a_n\| \leqslant 1$ and $a_n^* a_n$ tends to 1 in the strong operator topology. Then there exist unitaries u_n in $\mathcal{M}$ such that $u_n - a_n$ converges strongly to zero.*

Proof. Consider the polar decomposition $a_n = u_n |a_n|$, where u_n can be chosen unitaries because $\mathcal{M}$ is finite. Note that $|a_n| \to 1$ strongly. Indeed, since $\|a_n\| \leqslant 1$, $a_n^* a_n \leqslant (a_n^* a_n)^{1/2}$. Therefore, for any unit vector $\xi \in H$, $1 \geqslant (|a_n|\xi, \xi) \geqslant (a_n^* a_n \xi, \xi) \to 1$. Therefore

$$\|(a_n - u_n)\xi\|^2 = \|u_n(|a_n| - 1)\xi\|^2 \leqslant \||a_n|\xi - \xi\|^2 = 1 + (a_n^* a_n \xi, \xi) - 2(|a_n|\xi, \xi),$$

which tends to zero. ∎

In [1] it was proven that for a fixed $x_0 \in \mathcal{S}_1(X)$ the map $\pi_{x_0} : U_{\mathcal{L}_{\mathcal{B}}(X)} \to \mathcal{S}_1(X)$ given by $\pi_{x_0}(U) = U(x_0)$ is onto when $\mathcal{B}$ is finite. In that paper it was considered with the norm topologies. Here we shall regard it with the weak topologies and in the particular case at hand, namely $X = \mathcal{R}p$ and $\mathcal{B} = p\mathcal{R}p$ with $\mathcal{R}$ as above. Then, choosing $x_0 = p \in \mathcal{S}_1(X) = \mathcal{S}_p(\mathcal{R})$, the mapping π_p is

$$\pi_p : U_{\mathcal{R}} \to \mathcal{S}_p(\mathcal{R}), \quad \pi_p(u) = up.$$

THEOREM 3.3. *If $\mathcal{R}$ is a factor satisfying the hypothesis of Theorem 2.10, then the map π_p above is a trivial principal bundle, when $U_{\mathcal{R}}$ is regarded with the strong operator topology and $\mathcal{S}_p(\mathcal{R})$ is regarded with the ultraweak topology. The fibre is (homeomorphic to) the unitary group of $q\mathcal{R}q$, where $q = 1 - p$, again with the strong operator topology.*

Proof. The key of the argument is again Lemma 3 of [10]. In that result it is shown that the homogeneous space $U_{\mathcal{R}}/U_{\mathcal{M}}$ admits a global continuous cross section, where $\mathcal{M} \subset \mathcal{R}$ are factors, with $\mathcal{M}$ satisfying the hypothesis of Theorem 2.10, and their unitary groups are endowed with the strong operator topology. In our situation, the fibre of π_p (over p) is the set $\{u \in U_{\mathcal{R}} : up = p\} = \{qwq + p : qwq \in U_{q\mathcal{R}q}\} = U_{q\mathcal{R}q} \times \{p\}$. The fibre is the unitary group of the factor $q\mathcal{R}q$, which also verifies the hypothesis of Theorem 2.10. Indeed, $q\mathcal{R}q \simeq \mathcal{R}$. Therefore in order to prove our result it suffices to show that in $\mathcal{S}_p(\mathcal{R})$ the ultraweak topology (equal to the weak operator topology) coincides with the quotient topology induced by the map π_p. In other words, that the bijection

$$U_{\mathcal{R}}/U_{q\mathcal{R}q} \times \{p\} \to \mathcal{S}_p(\mathcal{R}), \quad [u] \to up$$

is a homeomorphism in the mentioned topologies. It is clearly continuous. It suffices to check continuity of the inverse at the point p. Suppose that u_α is a net of unitaries in $U_{\mathcal{R}}$ such that $u_\alpha p$ converges weakly to p. Then we claim that there are unitaries $qw_\alpha q$ in $q\mathcal{R}q$ such that $qw_\alpha q + p - u_\alpha$ converges strongly to zero, which would end the proof. This amounts to saying that there exists unitaries $qw_\alpha q$ verifying that

$$\mathrm{Re}((qw_\alpha q + p)\xi, u_\alpha \xi) \to \|\xi\|^2$$

for all $\xi \in H$. Now since $u_\alpha p \to p$, one has $u_\alpha p\xi \to p\xi$, the former limit is equivalent to the following

$$\mathrm{Re}(qw_\alpha q\xi, u_\alpha q\xi) \to \|q\xi\|^2.$$

Again, $u_\alpha p \to p$ strongly (and the fact that $\mathcal{R}$ is finite), imply that $qu_\alpha p$, $pu_\alpha q$, $qu_\alpha^* p$ and $pu_\alpha^* q$ all converge to zero strongly. Using that u_α are unitaries, these facts imply that $qu_\alpha^* qu_\alpha q \to q$ strongly. Using the lemma above, for the algebra $\mathcal{M} = q\mathcal{R}q$, and $a_\alpha = qu_\alpha q$, it follows that there exist unitaries $qw_\alpha q$ in $q\mathcal{R}q$ such that $qw_\alpha q - qu_\alpha q$ converges to zero strongly. Since $pu_\alpha q$ also tends to zero, it follows that

$$qw_\alpha q - u_\alpha q = qw_\alpha q - qu_\alpha q - pu_\alpha q \to 0$$

strongly. Clearly this last limit proves our claim. ∎

Our main result then follows easily

THEOREM 3.4. *Let $\mathcal{R}$ be a factor satisfying the hypothesis of Theorem 2.10, and let p be a projection in $\mathcal{R}$. Then both $\mathcal{S}_p(\mathcal{R})$ with the ultraweak topology, and the set of normal states of $\mathcal{R}$ with support equivalent to p with the norm topology, have trivial homotopy groups of all orders.*

Proof. By the Theorem 3.3, $\mathcal{S}_p(\mathcal{R})$ has trivial homotopy groups, since it is the base space of a fibration with contractible space and contractible fibre. The same consequence holds for the set of normal states with support equivalent to p, using the Corollary 3.1. ∎

REMARK 3.5. Consider now the restriction of the fibration $\mathcal{A}(X) \to \Omega_{\mathcal{A}(X)}$ to the subset $\{\omega_{x\otimes\xi_0} : x \in \mathcal{S}_1(X)\} \subset \Omega_{\mathcal{A}(X)}$, for a fixed unit, cyclic and separating vector ξ_0, i.e.

$$\{x \otimes \xi_0 : x \in \mathcal{S}_1(X)\} \simeq \mathcal{S}_1(X) \to \{\omega_{x\otimes\xi_0} : x \in \mathcal{S}_1(X)\}, \quad x \otimes \xi_0 \mapsto \omega_{x\otimes\xi_0},$$

which is again a fibration with the relative topologies. Note that the latter set is in one to one correspondence with $\mathcal{O}_\varphi$, where $\varphi = \omega_{\xi_0}$. Therefore one obtains the map $\sigma : \mathcal{S}_1(X) \to \mathcal{O}_\varphi$, $\sigma(x) = \varphi_x = \omega_{x\otimes\xi_0}$. It follows that this map is a fibration. The fibre is equal to $U_{\mathcal{R}^\varphi}$ in the strong operator topology, where $\mathcal{R}^\varphi$ is the von Neumann algebra fixed by the modular group of φ, or centralizer algebra of φ ([3]).

One can consider this fibration σ in the particular case $X = \mathcal{B} = \mathcal{R}$, for $\mathcal{R}$ as above, to obtain the following:

COROLLARY 3.6. *Let φ be a faithful normal state of a factor $\mathcal{R}$ as in Theorem* 2.10. *Then the map*

$$\sigma : U_{\mathcal{R}} \to \mathcal{U}_\varphi = \{\varphi \circ \mathrm{Ad}(u) : u \in U_{\mathcal{R}}\}, \quad \sigma(u) = \varphi \circ \mathrm{Ad}(u),$$

is a fibration when the unitary group $U_{\mathcal{R}}$ is considered with the strong operator topology and the unitary orbit $\mathcal{U}_\varphi$ of φ is considered with the norm topology. The fibre is the unitary group $U_{\mathcal{R}^\varphi}$ of the centralizer of φ also with the strong operator topology. Moreover, for $n \geqslant 0$ one has

$$\pi_{n+1}(\mathcal{U}_\varphi, \varphi) = \pi_n(U_{\mathcal{R}^\varphi}, 1).$$

Proof. When $X = \mathcal{R}$ is a finite von Neumann algebra, then $\mathcal{S}_1(X)$ is $U_{\mathcal{R}}$ and $\mathcal{O}_\varphi$ is the unitary orbit of φ. $\mathcal{S}_1(X) = U_{\mathcal{R}}$ is endowed with the ultraweak topology, which coincides in $U_{\mathcal{R}}$ with the strong operator topology. The rest of the corollary follows using that in this case σ is (the restriction) of a fibration, and again ([10]) that for such factors $\mathcal{R}$ the unitary group is contractible in the strong operator topology. ∎

When $n = 0$, since $U_{\mathcal{R}^\varphi}$ is connected, one obtains that $\mathcal{U}_\varphi$ is simply connected in the norm topology. A related result was obtained in [2], where it was shown that $\mathcal{U}_\varphi$ is simply connected in the quotient topology ($U_{\mathcal{B}}/U_{\mathcal{B}^\varphi}$) for any von Neumann algebra $\mathcal{B}$.

Let $p_1, \ldots, p_n$ be pairwise orthogonal projections in $\mathcal{R}$ such that $p_1 + \cdots + p_n = 1$ and put $h = r_1 p_1 + \cdots + r_n p_n$, where r_i are positive real numbers such that $r_i \neq r_j$ if $i \neq j$ and $\tau(h) = 1$. Consider the state $\varphi = \tau(h\,\cdot)$. Then clearly $\mathcal{R}^\varphi = p_1\mathcal{R}p_1 \oplus \cdots \oplus p_n\mathcal{R}p_n$. Now $U_{p_i\mathcal{R}p_i}$ is contractible in the strong operator topology, and therefore $U_{\mathcal{R}^\varphi}$ is contractible. It follows that the unitary orbit $\mathcal{U}_\varphi$ (with the norm topology) has trivial homotopy groups of all orders for such φ. Consider this other example: let $\mathcal{A} \subset \mathcal{R}$ be a maximal abelian sub (von Neumann) algebra; then there exists a normal faithful state φ of $\mathcal{R}$ such that $\mathcal{R}^\varphi = \mathcal{A}$. Clearly, since $\mathcal{R}$ is of type II_1, $\mathcal{A}$ has no atomic projections. It follows that $\mathcal{A} \simeq L^\infty(0, 1)$.

It is fairly elementary to see that $U_{L^\infty(0,1)}$ is contractible in the ultraweak, i.e. w*-topology. It follows that also for such states φ, $\mathcal{U}_\varphi$ (in the norm topology) has trivial homotopy groups of all orders. We would like to know if this holds for any faithful normal state of $\mathcal{R}$.

REFERENCES

[1] E. ANDRUCHOW, G. CORACH, D. STOJANOFF, Projective space of a C^*-module, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **4**(2001), 289–307.

[2] E. ANDRUCHOW, A. VARELA, Homotopy of state orbits, *J. Operator Theory* **48**(2002), 419–430.

[3] E. ANDRUCHOW, A. VARELA, C^*-modular vector states, *Integral Equations Operator Theory*, to appear.

[4] D. BURES, An extension of Kakutani's theorem on infinite product measures to the tensor product of semifinite w*-algebras, *Trans. Amer. Math. Soc.* **135**(1969), 199–212.

[5] R.V. KADISON, J.R. RINGROSE, *Fundamentals of the Theory of Operator Algebras*, vol. II, Academic Press, New York 1986.

[6] E.C. LANCE, *Hilbert C^*-Modules. A Toolkit for Operator Algebraists*, London Math. Soc. Lecture Notes Ser., vol. 210, Cambridge Univ. Press, Cambridge 1995.

[7] E. MICHAEL, Convex structures and continuous selections, *Canadian J. Math.* **11**(1959), 556–575.

[8] W.L. PASCHKE, Inner product modules over B^*-algebras, *Trans. Amer. Math. Soc.* **182**(1973), 443–468.

[9] W.L. PASCHKE, Inner product modules arising from compact automorphism groups of von Neumann algebras, *Trans. Amer. Math. Soc.* **224**(1976), 87–102.

[10] S. POPA, M. TAKESAKI, The topological structure of the unitary and automorphism groups of a factor, *Comm. Math. Phys.* **155**(1993), 93–101.

[11] M.A. RIEFFEL, Induced representations of C^*-algebras, *Adv. Math.* **13**(1974), 176–257.

[12] M. TAKESAKI, *Theory of Operator Algebras*. I, Springer Verlag, New York 1979.

E. ANDRUCHOW, INSTITUTO DE CIENCIAS, UNIVERSIDAD NACIONAL DE GRAL. SARMIENTO, J.M. GUTIERREZ ENTRE J.L. SUAREZ Y VERDI, (1613) LOS POLVORINES, ARGENTINA

E-mail address: eandruchow@ungs.edu.ar

A. VARELA, INSTITUTO DE CIENCIAS, UNIVERSIDAD NACIONAL DE GRAL. SARMIENTO, J.M. GUTIERREZ ENTRE J.L. SUAREZ Y VERDI, (1613) LOS POLVORINES, ARGENTINA

E-mail address: avarela@ungs.edu.ar

Received November 22, 2002.