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GENERALIZED TOEPLITZ OPERATORS, RESTRICTIONS TO INVARIANT SUBSPACES AND SIMILARITY PROBLEMS

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ABSTRACT. Our purpose is to investigate the asymptotic properties of an operator T on an invariant subspace $E \in \mathrm{Lat}(T)$ and on $E^{\perp}$ using the generalized Toeplitz operators associated with T. We show how the relative properties may be used in order to give a general result linking the behaviour of T on E and on $E^{\perp}$ with the possibility for T to be similar to a scalar multiple of a contraction. Some applications are indicated. In particular, one of our results implies that there is no hope to construct a power bounded operator of Foguel type that is not similar to a contraction and such that for every $x \in H\backslash\{0\}$ the sequence $(T^n)_{n\geqslant 0}$ does not converge to 0. We also study the asymptotic and spectral properties of these operators of Foguel type.

KEYWORDS: *Toeplitz operators, invariant subspaces, similarity problems.*

MSC (2000): Primary 47A15, 47B35; Secondary 47C15.

1. INTRODUCTION AND PRELIMINARIES

Let H_1, H_2 be a separable complex Hilbert spaces and $B(H_1, H_2)$ be the Banach space of all continuous, linear operators from H_1 into H_2; we abbreviate $B(H, H)$ to $B(H)$. The ultra-weak topology of $B(H)$ is the weak* topology (in the sequel we will shorten weak* to w*) that comes from the well known duality $B(H) = (C_1(H))^*$ where $C_1(H)$ is the Banach space of trace class operators on H endowed with the trace norm (see [16]). We will denote by $\mathcal{LI}(H_1, H_2)$ (respectively $\mathcal{RI}(H_1, H_2)$) the set of left invertible (respectively right invertible) operators from H_1 to H_2 and set $\mathcal{LI}(H) = \mathcal{LI}(H, H)$ (respectively $\mathcal{RI}(H) = \mathcal{RI}(H, H)$). Let us introduce the *Moore-Penrose left* inverse T_{l} and the *Moore-Penrose right* inverse T_{r} of an operator T acting on H when of course they exist:

(i) The *Moore-Penrose left inverse* of T, when of course it exists, is defined to satisfy $T_{\mathrm{l}}T = I$ and $TT_{\mathrm{l}} = P_{\mathrm{Im}T}$ where $P_{\mathrm{Im}T}$ is the orthogonal projection onto $\mathrm{Im}T$ (which is necessarily closed in this case).

(ii) The *Moore-Penrose right inverse* of T, when of course it exists, is defined by setting $TT_{\mathrm{r}} = I$ and $T_{\mathrm{r}}T = P_{\mathrm{Im}T^*}$ where $P_{\mathrm{Im}T^*}$ is the orthogonal projection onto $\mathrm{Im}T^*$.

The *reduced minimum* modulus of a nonzero operator $T \in B(H, H')$ is defined by $\gamma(T) = \inf\{\|Tx\| : x \in H, d(x, \ker T) = 1\}$.

We say that two operators S and T are *quasisimilar* if there exists two injective operators A and B with dense ranges such that:

$$\left\{ \begin{array}{ll} SA = AT & (1) \\ TB = BS & (2) \end{array} \right. . \tag{1.1}$$

Recall also that an operator T is *similar* to an other operator R if there exists an invertible operator A such that $T = ARA^{-1}$. Let T be an operator similar to a contraction. We will denote by $C_{\mathrm{sim}}(T)$ the optimal constant of similarity to a contraction, which is given by $C_{\mathrm{sim}}(T) = \inf\{\|A\|\|A^{-1}\| : \|ATA^{-1}\| \leqslant 1\}$. An operator $T \in B(H)$is said to be *power bounded* if the sequence $(T^n)_{n\geqslant 0}$ is bounded in the algebra $B(H)$ (notation $T \in PWB(H)$). We also recall that an operator $T \in B(H)$ is *polynomially bounded* (notation $T \in PB(H)$) if there exists $M \in [1, +\infty[$ such that

$$\|p(T)\| \leqslant M \sup_{|z|=1} |p(z)| \tag{1.2}$$

for every polynomial $p \in \mathbb{C}[X]$. We will denote by M_T the optimal constant in (1.2). An operator $T \in B(H)$ is said to be *completely polynomially bounded* if there exists a real constant $C \geqslant 1$ such that:

$$\|[p_{i,j}(T)]_{1\leqslant i,j\leqslant n}\| \leqslant C \sup\{\|[p_{i,j}(z)]_{1\leqslant i,j\leqslant n}\| : |z| = 1\} \tag{1.3}$$

for every positive integer n and for every $n \times n$ matrix of polynomials $[p_{i,j}]_{1\leqslant i,j\leqslant n} \in \mathcal{M}_{n,n}(\mathbb{C}[X])$. Recall that $[p_{i,j}(T)]_{1\leqslant i,j\leqslant n}$ denotes the $n \times n$ matrix which, as usual, acts on the direct sum of n copies of H and whose coefficients are some operators. One of the main interest of the completely polynomially bounded operators comes from *Paulsen criterion* [42] which asserts that:

$T \in B(H)$ is similar to a contraction if and only if
T is completely polynomially bounded.

Moreover, if C_T is the optimal constant in (1.3), then we have $C_{\mathrm{sim}}(T) = C_T$.

A von Neumann algebra acting on H is by definition an ultra-weakly closed *-subalgebra of $B(H)$. Such a von Neumann algebra M is finite if it admits a faithful normal trace τ, which means that τ is a ultra-weakly continuous linear functional on M satisfying:

(1) $\tau(AB) = \tau(BA)$ for any $A, B \in M$;

(2) for any positive element A in M, we have $\tau(A) \geqslant 0$ and $\tau(A) = 0 \Rightarrow A = 0$.

Let E and F be two Hilbert spaces; as usual $[A_{i,j}]_{1\leqslant i,j\leqslant n} \in \mathcal{M}_{n,n}(B(E,F))$ denotes the $n \times n$ matrix which acts from the orthogonal sum of n copies of E into the orthogonal sum of n copies of F (its entries are operators acting from E into F) and its norm is the norm of the associated operator. Let Ψ be a linear mapping from $B(E,F)$ into itself; we define $\Psi_n : \mathcal{M}_{n,n}(B(E,F)) \to \mathcal{M}_{n,n}(B(E,F))$ by $\Psi_n([A_{i,j}]_{1\leqslant i,j\leqslant n}) = [\Psi(A_{i,j})]_{1\leqslant i,j\leqslant n}$. We call Ψ completely bounded (respectively completely contractive) if $\sup\limits_{n\geqslant 1} \|\Psi_n\| < +\infty$ (respectively if $\sup\limits_{n\geqslant 1} \|\Psi_n\| \leqslant 1$). If $E = F$, $\mathcal{M}_{n,n}(B(E))$ inherits a unique structure of von Neumann algebra and a map Ψ from $B(E)$ into itself is called completely positive if Ψ_n is positive for all n.

In similarity problems, the idea of using limits in a Banach meaning comes from B. Sz.-Nagy ([40]). In the sequel, we frequently use this idea. Recall that a Banach limit $\mathcal{L}$ is a state (e.g. $\|\mathcal{L}\| = \mathcal{L}(\mathbf{1}) = 1$) acting on the classical space l^∞ of all complex bounded sequences and satisfying $\mathcal{L}((u_{n+1})) = \mathcal{L}((u_n))$. We will denote by $\mathcal{B}$ the weakly compact convex set of all Banach limits. If $(u_n) \in l^\infty$, recall that we have $\mathcal{L}((u_n)) \in [q'((u_n)), q((u_n))]$ where the functionals q' and q are defined by

$$q'((u_n)) = \sup\Big\{\liminf_{k\to+\infty}\Big(\frac{1}{m}\sum_{i=1}^{m} u_{n_i+k}\Big) : m \in \mathbb{N}, n_1,\ldots,n_m \in \mathbb{N}\Big\}$$

and

$$q((u_n)) = \inf\Big\{\limsup_{k\to+\infty}\Big(\frac{1}{m}\sum_{i=1}^{m} u_{n_i+k}\Big) : m \in \mathbb{N}, n_1,\ldots,n_m \in \mathbb{N}\Big\}.$$

A bounded sequence $(u_n)_{n\geqslant 0}$ is said to be *almost convergent* to a complex number l if and only if

$$\lim_{n\to+\infty}\sup_{k\in\mathbb{N}}\Big|\frac{1}{n+1}\sum_{i=k}^{k+n} u_i - l\Big| = 0.$$

Lorentz ([38]), proved that $(u_n)_{n\geqslant 0}$ is almost convergent to l if and only if for every Banach limit $\mathcal{L}$ we have $l = \mathcal{L}((u_n))$ (notation: $u_n \xrightarrow{\mathrm{a}} l$). A sequence $(u_n)_{n\geqslant 0}$ is strongly almost convergent to l if and only if the sequence $(|u_n - l|)_{n\geqslant 0}$ is almost convergent to 0 (notation: $u_n \xrightarrow{\mathrm{sa}} l$).

DEFINITION 1.1. A map $p : \mathbb{N} \to]0, +\infty)$ is called a *gauge* if there exists $c_p > 0$ such that the sequence $p(n+1)/p(n)$ is strongly almost convergent to c_p. Moreover, if the sequence $c_p^n/p(n)$ strongly almost converges to 1, we say that p is a *regular gauge*. We say that p is *almost regular* if $q'((p(n)^{-2}c_p^{2n})_{n\geqslant 0}) > 0$.

We will say that a sequence $(T_n)_{n\geqslant 0}$, acting on a Banach space, is dominated by a gauge p if there exists a positive number $C \geqslant 1$ such that the inequality $\|T_n\| \leqslant Cp(n)$ holds for every positive integer n. We follow ([28]) in saying that

$(T_n)_{n\geqslant 0}$ is compatible with a gauge p if in addition the sequence $(\|T_n\|/p(n))_{n\geqslant 0}$ does not almost converge to 0. An operator T is compatible with p if the sequence $(T^n)_{n\geqslant 0}$ is compatible with p. For some recent contributions in this area, we refer the reader to [8], [28], [29], [31], [30], [32], [33], [34], and [35].

One of the aim of this paper is to study the relationship between invariant subspaces, asymptotic behaviour and similarity problems. To this aim, we consider the following classes introduced by L. Kerchy ([28]). These are the following:

$$C_{1,\cdot}(p) = \{T \in B(H) : \forall x \in H - \{0\}, \|T^n x\| p(n)^{-1} \overset{\text{a}}{\nrightarrow} 0\}$$

and

$$C_{\cdot,1}(p) = \{T \in B(H) : \forall x \in H - \{0\}, \|T^{*n} x\| p(n)^{-1} \overset{\text{a}}{\nrightarrow} 0\}.$$

The class $C_{1,1}(p)$ is equal to $C_{1,\cdot}(p) \cap C_{\cdot,1}(p)$. Let us remind to the reader that an operator $T \in PWB(H)$ is called $C_{1,\cdot}$, in the well known terminology of B. Sz.-Nagy and C. Foiaş, if $\inf\limits_{n\geqslant 0} \|T^n x\| > 0$ for every non zero $x \in H$. Observe that, for a power bounded operator T, the relation $\inf\limits_{n\geqslant 0} \|T^n x\| = 0$ is equivalent to $\lim\limits_{n\to+\infty} \|T^n x\| = 0$. It can easily seen that $C_{1,\cdot}(1)$ coincides with the set of $C_{1,\cdot}$ power bounded operators.

The starting point of a large area in operator theory is a result obtained by B. Sz.-Nagy in [40]. More precisely, he proved that an invertible operator T on a Hilbert space is similar to a unitary if and only if the sequence $(\|T^n\|_{n\in\mathbb{Z}})$ is bounded. This result leads to the following question: Is the boundedness of the sequence $(\|T^n\|)_{n\geqslant 0}$ sufficient for the similarity of T to a contraction? Foguel answered negatively. The operators constructed by Foguel have the following matrix representation:

$$\begin{bmatrix} S^* & R \\ 0 & S \end{bmatrix},$$

where S is the usual shift on the Hardy space H^2. Halmos refined the conjecture by replacing the hypothesis of power boundedness for the operator T with the stronger assumption that T is polynomially bounded. It was settled in the negative by Pisier in [44]. In order to produce the first polynomially bounded operators which are not similar to contractions, G. Pisier used operators which have the above representation taking for S a shift of infinite multiplicity ([44]; see also [15], [36]). Nevertheless Pisier's counter-examples are not in the class $C_{1,1}$ nor in the class $C_{1,\cdot}$. We refer also to [3] for related results.

Let F be a Hilbert space and denote by S the shift on the Hardy space $H^2(F)$. In the sequel we denote by *operators of Foguel type* the operators whose matrix representation has one of the following forms:

$$\text{(I)} \begin{bmatrix} S^* & R \\ 0 & S \end{bmatrix}, \quad \text{(II)} \begin{bmatrix} S^* & R \\ 0 & S^* \end{bmatrix}, \quad \text{(III)} \begin{bmatrix} S & R \\ 0 & S \end{bmatrix}, \quad \text{(IV)} \begin{bmatrix} S & R \\ 0 & S^* \end{bmatrix}.$$

Remark that an operator of type (I) and (II) can not be in the class $C_{1,\cdot}$.

A well known result of B. Sz.-Nagy asserts that if T is a power bounded operator in the class $C_{1,1} = C_{1,1}(1)$, then T is quasi-similar to an unitary operator. The above result leads to the following question (see [27]):

Is every power bounded operator in the class $C_{1,1}$ similar to a contraction?

A result of B. Sz.-Nagy and C. Foiaş ([43]) asserts that two quasi-similar unitaries are necessarily unitarily similar. When T belongs to a finite von Neumann algebra, it suffices that T satisfy (1) of (1.1) to insure that T is similar to a unitary ([11]; for stronger results see also [8]).

Now, let us mention an other result which show the deep relationships between invariant subspaces, asymptotic behaviour and similarity problems. Recall that one of our first motivations goes back to the theorem in [10] (Theorem 8, p. 334) about the invariant subspace problem for operators in $C_{1,\cdot}$.

REMARK 1.2. We can easily check that if the same result ([10]) holds for operators similar to a contraction T of class $C_{1,\cdot}$ and belonging to the von Neumann algebra generated by T then it would solve positively the invariant subspace problem for all contractions of class $C_{1,\cdot}$ (for instance, see [33]). We refer the reader to [12] and [33] for related results in this area. This theorem gives more than the existence of non trivial invariant subspaces, it shows the importance of the class of operators with an invariant subspace on which their compression is similar to an isometry.

Let us now give some results which will be very useful in the sequel. We first state the well known *criterion of Douglas* ([17]) about ranges and factorizations of operators (see also [14] for more informations).

THEOREM 1.3 (Douglas criterion). *Let $A, B \in B(H)$. Then the following conditions are equivalent:*

(i) $\mathrm{Im}(B) \subseteq \mathrm{Im}(A)$.

(ii) *There exists $Z \in B(H)$ such that $B = AZ$.*

(iii) *There exists a positive number δ such that $BB^* \leqslant \delta AA^*$.*

Moreover, in this case there exists a unique solution R of the equation $AZ = B$ such that $\ker Z = \ker A$, $\mathrm{Im}(Z) \subseteq \overline{\mathrm{Im}(A^*)}$. *This solution is called the reduced solution and we have* $\|Z\|^2 = \inf\{\delta : BB^* \leqslant \delta AA^*\}$.

Secondly, we give a general operator Cauchy-Schwartz inequality (it seems that the first kind of such inequality is due to U. Haagerup cf. [22]).

PROPOSITION 1.4. *Let (Ω, μ) be a measurable space and H_i, $i = 1, 2$, be a separable Hilbert space. Assume that the applications $t \to A_t \in B(H_1)$ and $t \to B_t \in B(H_2)$ are such that $t \to \|A_t x\|^2$ and $t \to \|B_t y\|^2$ are μ integrable for every pair $(x, y) \in H_1 \times H_2$. Then we have the following operator Cauchy-Schwarz inequality:*

$$\Big\| \int_\Omega A_t^* B_t \mathrm{d}\mu(t) \Big\| \leqslant \sqrt{\Big\| \int_\Omega A_t^* A_t \mathrm{d}\mu(t) \Big\|} \sqrt{\Big\| \int_\Omega B_t^* B_t \mathrm{d}\mu(t) \Big\|} \tag{1.4}$$

(the operators defined by integrals are well defined in a Bochner sense).

Proof. First, applying the Banach Steinhaus theorem we get that the operators defined by integrals are well defined in a Bochner sense. Letting $(x, y) \in H_1 \times H_2$ we have

$$\begin{aligned}\Big|\int_\Omega \langle B_t y \mid A_t x\rangle \mathrm{d}\mu(t)\Big| &\leqslant \int_\Omega \|B_t y\|\|A_t x\|\mathrm{d}\mu(t)\\ &\leqslant \sqrt{\int_\Omega \langle A_t^* A_t x \mid x\rangle \mathrm{d}\mu(t)}\sqrt{\int_\Omega \langle B_t^* B_t y \mid y\rangle \mathrm{d}\mu(t)}\\ &= \sqrt{\Big\langle\Big[\int_\Omega A_t^* A_t \mathrm{d}\mu(t)\Big]x \mid x\Big\rangle}\sqrt{\Big\langle\Big[\int_\Omega B_t^* B_t \mathrm{d}\mu(t)\Big]y \mid y\Big\rangle}\\ &\leqslant \sqrt{\Big\|\int_\Omega A_t^* A_t \mathrm{d}\mu(t)\Big\|}\sqrt{\Big\|\int_\Omega B_t^* B_t \mathrm{d}\mu(t)\Big\|}\|x\|\|y\|.\end{aligned}$$

The desired inequality follows immediately. ∎

REMARK 1.5. (i) If $\Omega = \{1, \dots, n\}$ and $\mu = \sum_{i=1}^{n} \delta_n$, we obtain the following Cauchy Schwarz inequality:

$$\Big\|\sum_{k=1}^{n} A_k^* B_k\Big\| \leqslant \sqrt{\Big\|\sum_{k=1}^{n} A_k^* A_k\Big\|}\sqrt{\Big\|\sum_{k=1}^{n} B_k^* B_k\Big\|},$$

where $A_1, \dots, A_n \in B(H_3, H_2)$ and $B_1, \dots, B_n$ are in $B(H_1, H_2)$.

(ii) When $t \to x_t \in L^2(\Omega, H_1)$ (respectively $t \to y_t \in L^2(\Omega, H_2)$), we deduce from (1.4) the useful inequality

$$\Big\|\int_\Omega x_t \otimes y_t \mathrm{d}\mu(t)\Big\| \leqslant \sqrt{\Big\|\int_\Omega x_t \otimes x_t \mathrm{d}\mu(t)\Big\|}\sqrt{\Big\|\int_\Omega y_t \otimes y_t \mathrm{d}\mu(t)\Big\|}$$

(set $A_t = 1 \otimes x_t \in B(H_1, C)$ and $B_t = 1 \otimes y_t \in B(H_2, C)$).

2. ASYMPTOTIC BEHAVIOUR AND GENERALIZED TOEPLITZ OPERATORS

2.1. GENERALIZED TOEPLITZ OPERATORS. Assume that T_1 and T_2 are two operators dominated by a gauge p, $T_i \in B(H_i)$. If $\mathcal{L}$ is a Banach limit, let us introduce the operator $E^{pq}_{\mathcal{L},T_1,T_2}$, acting on $B(H_2, H_1)$, by

$$\langle E^{pq}_{\mathcal{L},T_1,T_2}(X)x \mid y\rangle = \mathcal{L}(\{\langle T_1^n X T_2^n x \mid y\rangle p(n)^{-1} q(n)^{-1}\}_{n\geqslant 0})$$

for any $(x, y) \in H_2 \times H_1$. The following proposition summarizes some useful properties of this operator.

PROPOSITION 2.1. *Let (T_1, T_2) be a pair of operators acting on two separable Hilbert spaces (respectively H_1 and H_2). Assume that T_1 (respectively T_2) is dominated by a gauge p (respectively a gauge q). Then, for any Banach limit $\mathcal{L}$, we have:*

(i) *$E^{pq}_{\mathcal{L},T_1,T_2}$ is a completely bounded map. It is a completely contractive map when $\sup_{n\geqslant 0}\{\|T_1^n\|p(n)^{-1}\}\sup_{n\geqslant 0}\{\|T_2^n\|q(n)^{-1}\} \leqslant 1$ and it is a completely positive map when $T_1 = T_2^*$.*

(ii) *$E^{pq}_{\mathcal{L},T_1,T_2}(T_1XT_2) = c_pc_qE^{pq}_{\mathcal{L},T_1,T_2}(X)$ for any $X \in B(H_2, H_1)$.*

(iii) *If A (respectively B) commute T_1 (respectively T_2), then we have $E^{pq}_{\mathcal{L},T_1,T_2}(AXB) = AE^{pq}_{\mathcal{L},T_1,T_2}(X)B$ for any $X \in B(H_2, H_1)$.*

(iv) *$T_1E^{pq}_{\mathcal{L},T_1,T_2}(X)T_2 = c_pc_qE^{pq}_{\mathcal{L},T_1,T_2}(X)$ for any $X \in B(H_2, H_1)$.*

(v) *If T_1 (respectively T_2) is compatible with p (respectively with q), there exists $\rho_{\mathcal{L}}(p, q) \in [0, 1]$ such that $E^{pq}_{\mathcal{L},T_1,T_2} \circ E^{pq}_{\mathcal{L},T_1,T_2} = \rho_{\mathcal{L}}(p, q)E^{pq}_{\mathcal{L},T_1,T_2}$. When p and q are two regular gauges, we have $\rho_{\mathcal{L}}(p, q) = 1$ (hence $E^{pq}_{\mathcal{L},T_1,T_2}$ is a projection).*

(vi) *If T_1 and T_2 act on the same space H, then for any $x, y \in H$ and any $X, Y \in B(H)$ we have:*

$$|\langle E^{pq}_{\mathcal{L},T_1,T_2}(XY)x \mid y\rangle| \leqslant \sqrt{\langle E^{q^2}_{\mathcal{L},T_2^*,T_2}(Y^*Y)x \mid x\rangle}\sqrt{\langle E^{p^2}_{\mathcal{L},T_1,T_1^*}(XX^*)y \mid y\rangle}. \tag{2.1}$$

REMARK 2.2. (i) If $T \in B(H)$ is compatible with a gauge p, then the spectral radius $r(T)$ satisfies $r(T) = c_p$ (see [28]).

(ii) Assume that there is no non zero solution X of the equation $T_1^*XT_1 = c_p^2X$ (T_1^* generalized Toeplitz operators) or a non zero solution of the equation $T_2YT_2^* = Y$, then using Proposition 2.1 (vi) we have necessarily $E^{pq}_{\mathcal{L},T_1,T_2} = 0$.

Proof of Proposition 2.1 (i). Let $[X_{i,j}]_{1\leqslant i,j\leqslant m}$ be a $m \times m$ matrix whose coefficients are operators in $B(H_2, H_1)$, $x_1, \ldots, x_n$ be vectors in H_2 and $y_1, \ldots, y_n$ be vectors in H_1; we have

$$\begin{aligned}
&\left|\left\langle \begin{bmatrix} E^{pq}_{\mathcal{L},T_1,T_2}(X_{1,1}) & \cdots & E^{pq}_{\mathcal{L},T_1,T_2}(X_{1,m}) \\ \vdots & & \vdots \\ E^{pq}_{\mathcal{L},T_1,T_2}(X_{m,1}) & \cdots & E^{pq}_{\mathcal{L},T_1,T_2}(X_{m,m}) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \middle| \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \right\rangle\right| \\
&= \left|\mathcal{L}\Big(\sum_{i,j=1}^{m}\{\langle T_1^nX_{i,j}T_2^nx_j \mid y_i\rangle p(n)^{-1}q(n)^{-1}\}_{n\geqslant 0}\Big)\right| \\
&\leqslant \|[X_{i,j}]_{1\leqslant i,j\leqslant m}\|\mathcal{L}\left(\sqrt{\sum_{j=1}^{m}\|x_j\|^2\|T_2^n\|^2q(n)^{-2}}\sqrt{\sum_{j=1}^{m}\|y_j\|^2\|T_1^{*n}\|^2p(n)^{-2}}\right) \\
&\leqslant \sup_{n\geqslant 0}\{\|T_1^n\|p(n)^{-1}\}\sup_{n\geqslant 0}\{\|T_2^n\|q(n)^{-1}\}\|[X_{i,j}]_{1\leqslant i,j\leqslant m}\|\sqrt{\sum_{j=1}^{m}\|x_j\|^2}\sqrt{\sum_{j=1}^{m}\|y_j\|^2}.
\end{aligned}$$

It follows that $E^{pq}_{\mathcal{L},T_1,T_2}$ is a completely bounded map, it is completely contractive when $\sup\limits_{n\geqslant 0}\{\|T_1^n\|p(n)^{-1}\}\sup\limits_{n\geqslant 0}\{\|T_2^n\|q(n)^{-1}\}\leqslant 1$. The positivity of $\mathcal{L}$ implies immediately that $E^{p^2}_{\mathcal{L},T_1^*,T_1}$ is completely positive.

(ii) Given any pair (x,y) in $H_2\times H_1$, we have

$$\begin{aligned}\langle E^{pq}_{\mathcal{L},T_1,T_2}(T_1XT_2)x\mid y\rangle &= \mathcal{L}(\{\langle T_1^{n+1}XT_2^{n+1}x\mid y\rangle p(n)^{-1}q(n)^{-1}\}_{n\geqslant 0})\\ &= \mathcal{L}\Big(\Big\{\frac{\langle T_1^{n+1}XT_2^{n+1}x\mid y\rangle}{p(n+1)q(n+1)}\frac{p(n+1)}{p(n)}\frac{q(n+1)}{q(n)}\Big\}_{n\geqslant 0}\Big).\end{aligned}$$

Since p and q are two gauges, we see that the sequence $(|p(n+1)/p(n)q(n+1)/q(n)-c_pc_q|)_{n\geqslant 0}$ almost converges to 0. By the Lemma 1 from [28], we get

$$\begin{aligned}\langle E^{pq}_{\mathcal{L},T_1,T_2}(T_1XT_2)x\mid y\rangle &= c_pc_q\mathcal{L}(\{\langle T_1^{n+1}XT_2^{n+1}x\mid y\rangle p(n+1)^{-1}q(n+1)^{-1}\}_{n\geqslant 0})\\ &= c_pc_q\langle E^{pq}_{\mathcal{L},T_1,T_2}(X)x\mid y\rangle,\end{aligned}$$

and (ii) follows.

(iii) Let A (respectively B) be an operator in $B(H_1)$ (respectively $B(H_2)$) commuting with T_1 (respectively T_2). For any $(x,y)\in H_2\times H_1$, we have

$$\begin{aligned}\langle E^{pq}_{\mathcal{L},T_1,T_2}(AXB)x\mid y\rangle &= \mathcal{L}(\{\langle T_1^nAXBT_2^nx\mid y\rangle p(n)^{-1}q(n)^{-1}\}_{n\geqslant 0})\\ &= \mathcal{L}(\{\langle T_1^nXT_2^nBx\mid A^*y\rangle p(n)^{-1}q(n)^{-1}\}_{n\geqslant 0})\\ &= \langle E^{pq}_{\mathcal{L},T_1,T_2}(X)Bx\mid A^*y\rangle.\end{aligned}$$

This establishes the formula.

(iv) follows immediately from (ii) and (iii).

(v) Let $(x,y)\in H_2\times H_1$, using (iv), we get

$$\begin{aligned}\langle E^{pq}_{\mathcal{L},T_1,T_2}(E^{pq}_{\mathcal{L},T_1,T_2}(X))x\mid y\rangle &= \mathcal{L}(\{\langle T_1^nE^{pq}_{\mathcal{L},T_1,T_2}(X)T_2^nx\mid y\rangle p(n)^{-1}q(n)^{-1}\}_{n\geqslant 0})\\ &= \mathcal{L}([c_p^nc_q^np(n)^{-1}q(n)^{-1}]_{n\geqslant 0})\langle E^{pq}_{\mathcal{L},T_1,T_2}(X)x\mid y\rangle\\ &= \rho_{\mathcal{L}}(p,q)\langle E^{pq}_{\mathcal{L},T_1,T_2}(X)x\mid y\rangle,\end{aligned}$$

by setting $\rho_{\mathcal{L}}(p,q)=\mathcal{L}([c_p^nc_q^np(n)^{-1}q(n)^{-1}]_{n\geqslant 0})$. Since T_1 (respectively T_2) is compatible with p (respectively with q), the formulas $c_p=\inf\{p(n)^{1/n}:n\in\mathbb{N}\}$ and $c_q=\inf\{q(n)^{1/n}:n\in\mathbb{N}\}$ are valid (see Proposition 1 of [28]), we immediately deduce that $\rho_{\mathcal{L}}(p,q)\in[0,1]$. In particular, when p and q are two regular gauges and $H_1=H_2$, we get $E^{pq}_{\mathcal{L},T_1,T_2}$ is a projection.

(vi) Since every Banach limit $\mathcal{L}$ is a positive state on l^∞, we have the following Cauchy-Schwarz inequality:

$$|\mathcal{L}(\{u_n\}_{n\geqslant 0}\{v_n\}_{n\geqslant 0})|\leqslant\sqrt{\mathcal{L}(\{u_n^2\}_{n\geqslant 0})}\sqrt{\mathcal{L}(\{v_n^2\}_{n\geqslant 0})},\tag{2.2}$$

where $(u_n)_{n\geqslant 0}$, $(v_n)_{n\geqslant 0}\in l^\infty$.

Let X, Y be two operators in $B(H)$ and (x, y) be a pair of elements in H. Applying inequality (2.1), we obtain

$$\begin{aligned}
&|\langle E^{pq}_{\mathcal{L},T_1,T_2}(XY)x \mid y\rangle| \\
&= |\mathcal{L}(\{\langle T_1^n XYT_2^n x \mid y\rangle p(n)^{-1}q(n)^{-1}\}_{n\geqslant 0})| \\
&\leqslant \mathcal{L}(\{[\|YT_2^n x\| p(n)^{-1}][\|X^* T_1^{*n} y\| q(n)^{-1}]\}_{n\geqslant 0}) \\
&\leqslant \sqrt{\mathcal{L}(\{\langle T_2^{*n} Y^* YT_2^n x \mid x\rangle q(n)^{-2}\}_{n\geqslant 0})}\sqrt{\mathcal{L}(\{\langle T_1^n XX^* T_1^{*n} y \mid y\rangle p(n)^{-2}\}_{n\geqslant 0})} \\
&= \sqrt{\langle E^{q^2}_{\mathcal{L},T_2^*,T_2}(Y^*Y)x \mid x\rangle}\sqrt{\langle E^{p^2}_{\mathcal{L},T_1,T_1^*}(XX^*)y \mid y\rangle}.
\end{aligned}$$

This completes the proof. ∎

Let (H_1, H_2) be a pair of separable Hilbert spaces and $(T_1, T_2) \in B(H_1) \times B(H_2)$. Assume that T_1 (respectivelyT_2) is dominated by a gauge p (respectively a gauge q). We define the set $\mathcal{T}_{p,q}(T_1, T_2)$ of (T_1, T_2, p, q)-Toeplitz operators by setting

$$\mathcal{T}_{p,q}(T_1, T_2) = \{X \in B(H) : T_1 X T_2 = c_p c_q X\}$$

and we write $\mathcal{T}_p(T) = \mathcal{T}_{p,p}(T^*, T)$ for short. We will denote by $\tau_{p,q}(T_1, T_2)$ the set of canonical (T_1, T_2, p, q)-Toeplitz operators defined by

$$\tau_{p,q}(T_1, T_2) = \{E^{pq}_{\mathcal{L},T_1,T_2}(I) : \mathcal{L} \in \mathcal{B}\}.$$

Note that $\tau_{p,q}(T_1, T_2)$ is a weak compact convex set. For simplicity, we write $\tau_p(T)$ for $\tau_{p,p}(T^*, T)$. The next proposition summarizes some useful properties of the set $\tau_{p,q}(T_1, T_2)$.

Mention first the following useful properties:

(i) The canonical (T^*, T, p^2) Toeplitz operators are always positive.

(ii) Assume that p is almost regular, then an operator T belongs to the class $C_{1,\cdot}(p)$ if and only if there exists a one to one canonical (T^*, T, p^2) Toeplitz operator (in this case they are all one to one).

(iii) An operator T is similar to a scalar multiple of an isometry if and only if there exists an invertible canonical (T^*, T, p^2) Toeplitz operator (in this case they are all invertible).

PROPOSITION 2.3. *Let T be an operator acting on a separable Hilbert space H. Assume that T is dominated by an almost regular gauge p. Then we have:*

(i) *the range of $\sqrt{X}$ is the same for all $X \in \tau_p(T)$, we will denote it by $\mathcal{E}_T$;*

(ii) *the range of any positive (T^*, T, p^2)-Toeplitz operator is included in $\mathcal{E}_T$.*

REMARK 2.4. Let T be an operator satisfying the assumptions of the previous proposition and $X, Y \in \tau_p(T)$. Since $\sqrt{X}$ and $\sqrt{Y}$ have the same range, applying *Douglas criterion*, we can see that there exists an invertible operator A such that $X = A^*YA$.

Proof. (i) Let $X, Y \in \tau_p(T)$ and $\mathcal{L}$ be a Banach limit such that $X = E^{p^2}_{\mathcal{L},T^*,T}(I)$. By Proposition 2.1 (vi), we have

$$\begin{aligned}\rho_{\mathcal{L}}(p,p)|\langle Yx \mid y\rangle| &= |\langle E^{p^2}_{\mathcal{L},T^*,T}(Y)x \mid y\rangle| \\ &\leqslant \sqrt{\langle E^{p^2}_{\mathcal{L},T^*,T}(I)x \mid x\rangle}\sqrt{\langle E^{p^2}_{\mathcal{L},T^*,T}(Y^2)y \mid y\rangle},\end{aligned}$$

for any $x, y \in H$. Since p is almost regular, we have $\rho_{\mathcal{L}}(p,p) \neq 0$ ($\mathcal{L}(p(n)^{-2}c_p^{2n}) \in [q'((p(n)^{-2}c_p^{2n})_{n\geqslant 0}), q((p(n)^{-2}c_p^{2n})_{n\geqslant 0})]$) and

$$\|\sqrt{Y}x\|^2 \leqslant \|E^{p^2}_{\mathcal{L},T^*,T}(Y^2)\|\|\sqrt{X}x\|^2.$$

Applying Douglas criterion, we see that $\mathrm{Im}(\sqrt{Y}) \subseteq \mathrm{Im}(\sqrt{X})$ and interchanging X and Y we obtain the equality.

(ii) is proved in a similar way. ∎

2.2. LOWER AND UPPER T-TOEPLITZ OPERATORS. In order to work with power bounded operators it is interesting to study the operators satisfying

$$T^*XT \leqslant X \tag{2.3}$$

where X is a positive operator acting on H. More generally, if $T \in B(H)$, we will say that a positive operator X satisfying (2.3) is a *lower T-Toeplitz* operator and we will denote by $\mathcal{T}_{\inf}(T)$ the set of all such operators. For instance, notice that T is similar to a contraction if and only if there exists an invertible element in $\mathcal{T}_{\inf}(T)$. Observe also that a non zero *lower T-Toeplitz* which is not injective produces a non trivial invariant subspace for T, that is $\ker X$. Analogously, we will say that a positive operator X is a *upper T-Toeplitz* operator if it satisfies

$$T^*XT \geqslant X. \tag{2.4}$$

We will denote by $\mathcal{T}_{\sup}(T)$ the set of solutions of (2.4). Concerning *lower T-Toeplitz* operators, we have the following proposition of stability.

PROPOSITION 2.5. *Let $T \in PWB(H)$ and X be a* lower T-Toeplitz *operator. Then, for any absolutely converging series $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$ with $\sum\limits_{n=0}^{+\infty} |a_n| < +\infty$ mapping the closed unit disc into itself, we have $X \in \mathcal{T}_{\inf}(f(T))$. Moreover, if $T \in PB(H)$ (respectively an absolutely continuous polynomially bounded operator), then the assertion $X \in \mathcal{T}_{\inf}(f(T))$ is valid for any f in the unit ball of the disc algebra $A(\mathbb{D})$ (respectively in the unit ball of the Hardy algebra H^∞).*

Proof. (i) Applying *Douglas criterion*, we get a contraction C such that $\sqrt{X}T = C\sqrt{X}$. It follows that we have $\sqrt{X}f(T) = f(C)\sqrt{X}$. Hence,

$$f(T^*)Xf(T) \leqslant \sqrt{X}f(C)^*f(C)\sqrt{X} \leqslant X.$$

The last inequality comes from the classical von Neumann one for a contraction. Using the $A(\mathbb{D})$ functional calculus for a polynomially bounded operator and taking into account the H^∞ functional calculus for absolutely continuous polynomially bounded operators ([39]), we get the remaining assertions. ∎

The following proposition shows the usefulness of *upper T-Toeplitz* operators for similarity to an isometry.

PROPOSITION 2.6. *Let $T \in PWB(H)$. If there exists an invertible* upper T-Toeplitz *operator X, then T is similar to an isometry.*

Proof. Let X be an invertible *upper T-Toeplitz* operator. We first observe that we have $T^{*n}XT^n \geqslant X$ for any non negative integer n. Therefore, we obtain $E^1_{\mathcal{L},T^*,T}(X) \geqslant X$ for any Banach limit $\mathcal{L}$. Let $\mathcal{L}$ be a Banach limit and consider the generalized T-Toeplitz operator $Y = E^1_{\mathcal{L},T^*,T}(X)$, then we see clearly that $Y^{1/2}TY^{-(1/2)}$ is an isometry. This ends the proof. ∎

2.3. ABEL TYPE SUMMABILITY AND GENERALIZED TOEPLITZ OPERATORS. The following result, which is of independent interest, enables us to link resolvent properties to generalized Toeplitz operators.

THEOREM 2.7 (Abel type characterization of almost convergence). *Consider $(u_n)_{n\geqslant 0}$ a bounded sequence of complex numbers. Then the following assertions are equivalent:*

(i) $(u_n)_{n\geqslant 0}$ *is almost convergent to a number l;*

(ii) *we have*

$$\limsup_{r\to 1} \sup_{k\geqslant 0} \Big|(1-r)r^{-k}\sum_{n=k}^{+\infty} r^n u_n - l\Big| = 0.$$

Proof. (i)⇒(ii) Assume that (ii) is not satisfied, then there exists a sequence $(r_p)_{p\geqslant 0}$ increasing to 1 and a sequence of integers $(n_p)_{p\geqslant 0}$ such that $\Big|(1-r_p)r_p^{-n_p} \cdot \sum_{k=n_p}^{+\infty} r_p^k u_k - l\Big| \geqslant \rho$ for a strictly positive number ρ. For instance, we may assume that

$$(1-r_p)r_p^{-n_p}\sum_{k=n_p}^{+\infty} r_p^k u_k \geqslant l+\rho \tag{2.5}$$

for any p. Let us consider a non trivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$. We define a linear functional $\mathcal{L}$ on the space l^∞ by setting

$$\mathcal{L}((v_n)_{n\geqslant 0}) = \lim_{\mathcal{U}}\Big((1-r_p)r_p^{-n_p}\sum_{k=n_p}^{+\infty} r_p^k v_k\Big).$$

Let us verify that $\mathcal{L}$ is a Banach limit. It is obvious that $\mathcal{L}$ is well defined, positive and such that $\mathcal{L}(\mathbf{1}_{l^\infty}) = 1$. It remains to prove that $\mathcal{L}((v_{n+1})_{n\geqslant 0}) = \mathcal{L}((v_n)_{n\geqslant 0})$

for any sequence $(v_n)_{n\geqslant 0}$ in l^∞. It follows from the following inequality:

$$\Big|(1-r_p)r_p^{-n_p}\sum_{k=n_p}^{+\infty} r_p^k v_{k+1}-(1-r_p)r_p^{-n_p}\sum_{k=n_p}^{+\infty} r_p^k v_k\Big| \leqslant 2(1-r_p)\|(v_n)_{n\geqslant 0}\|_\infty .$$

On the one hand, we get from (2.5) that $\mathcal{L}((u_n)_{n\geqslant 0}) \geqslant l+\rho > l$. On the other hand, since $(u_n)_{n\geqslant 0}$ is almost convergent to a number l we have obviously $\mathcal{L}((u_n)_{n\geqslant 0}) = l$. Therefore, we get a contradiction and the implication (i)⇒(ii) is proved.

(ii)⇒(i) Given $\epsilon > 0$, we get from (ii) that there exists $r_0 > 0$ such that for any $r > r_0$, we have

$$l-\frac{\varepsilon}{2} \leqslant (1-r)\sum_{k=0}^{+\infty} r^k u_{k+n} \leqslant l+\frac{\varepsilon}{2}$$

for any $n \geqslant 0$. Since the sequence $(u_n)_{n\geqslant 0}$ is bounded, we can find N such that

$$l-\varepsilon \leqslant (1-r)\sum_{n=0}^{N} r^k u_{k+n} \leqslant l+\varepsilon. \tag{2.6}$$

Let $\mathcal{L}$ be a Banach limit, it follows from (2.6) that

$$-\varepsilon \leqslant (1-r)\sum_{n=0}^{N} r^k \mathcal{L}((u_{k+n})_{n\geqslant 0}) = (1-r^{N+1})\mathcal{L}((u_n)_{n\geqslant 0}) \leqslant l+\varepsilon.$$

If r goes to 1, we obtain $l-\varepsilon \leqslant \mathcal{L}((u_n)_{n\geqslant 0}) \leqslant l+\varepsilon$. Since ε is arbitrary, it gives $\mathcal{L}((u_n)_{n\geqslant 0}) = l$ and the proof of the Theorem 2.7 is complete. ∎

REMARK 2.8. In the same manner we can give a short proof of Lorentz's result ([38]).

In particular, we get

COROLLARY 2.9. *Let $(u_n)_{n\geqslant 0}$ be a bounded sequence which is almost convergent to a number l, then we have*

$$\lim_{r\to 1}(1-r)r^{-n}\sum_{k=n}^{+\infty} r^k u_k = l$$

for any positive integer n.

Let T be an operator dominated by a gauge p. We define the function $\Phi_{T,p}$ on $\mathbb{D}$ by setting

$$\Phi_{T,p}(z) = \sum_{n=0}^{+\infty} \overline{z}^n p(n)^{-1} T^n.$$

Notice that we have $\Phi_{T,1}(z) = (I-\overline{z}T)^{-1}$ when $p(n)=1$ for any n.

The next result links the function $\Phi_{T,p}$ (hence the resolvent when $p(n)=1$ for any n) to the generalized Toeplitz operators.

PROPOSITION 2.10. *Let T be an operator dominated by a gauge p. Then all weak limit points of the operators*

$$X_r = (1-r^2)\int_0^{2\pi} \Phi_{T,p}(re^{it})^* \Phi_{T,p}(re^{it})dm(t)$$

are generalized Toeplitz operators when r goes to 1. In particular, for every power bounded operator T the weak limit points of

$$(1-r^2)\int_0^{2\pi} (I - re^{it}T^*)^{-1}(I - re^{-it}T)^{-1}dm(t)$$

satisfy the equation $T^*XT = X$.

Proof. Since T is dominated by the gauge p, there exists positive number M such that $\|T^n\| \leqslant Mp(n)$. Observe that

$$\begin{aligned}&c_p^2 X_r - T^* X_r T \\ &\quad = c_p^2(1-r^2)p(0)^{-2} + (1-r^2)\Big[\sum_{n=0}^{+\infty}(c_p^2 - r^{-2}p(n)^2 p(n-1)^{-2})r^{2n}\frac{T^{*n}T^n}{p(n)^2}\Big].\end{aligned}$$

Hence

$$\begin{aligned}&\|c_p^2 X_r - T^* X_r T\| \\ &\leqslant c_p^2(1-r^2)p(0)^{-2} + (1-r^2)M\sum_{n=0}^{+\infty}|c_p^2 - r^{-2}p(n)^2p(n-1)^{-2}|r^{2n} \\ &\leqslant (1-r^2)\Big[c_p^2 p(0)^{-2} + M\sum_{n=0}^{+\infty}\Big|c_p^2 - \frac{p(n)^2}{p(n-1)^2}\Big|r^{2n} + \frac{M}{r^2}(1-r^2)^2\sum_{n=0}^{+\infty}\frac{p(n)^2}{p(n-1)^2}r^{2n}\Big] \\ &\leqslant (1-r^2)[c_p^2 p(0)^{-2} + Mr^{-2}\sup\{(p(n)^2p(n-1)^{-2}) : n \geqslant 0\}] \\ &+M(\sup\{(p(n)^2p(n-1)^{-2}) : n \geqslant 0\} + c_p)(1-r^2)\sum_{n=0}^{+\infty}|c_p - p(n)p(n-1)^{-1}|r^{2n}.\end{aligned}$$

Since p is a gauge, we know that the sequence $\left(\left|c_p - \frac{p(n)}{p(n-1)}\right|\right)_{n\geqslant 0}$ is almost convergent to 0 and using the previous corollary we obtain

$$\lim_{r\to 1}(c_p^2 X_r - T^* X_r T) = 0.$$

The desired result follows immediately. ∎

2.4. DECOMPOSITION AND FACTORIZATION OF A CANONICAL TOEPLITZ OPERATOR. The next step in our study consists in the decomposition and in the factorization of a canonical Toeplitz operator X associated with an operator T and with respect to an invariant subspace of T. This factorization, which is of independent interest, is crucial for the sequel. Let T be an operator dominated by a regular gauge p and $X \in \tau_p(T)$. The polar decomposition of $\sqrt{X}T$ provides a unique partial isometry U whose support coincides with the one of X and which satisfies

$$\sqrt{X}T = c_p U \sqrt{X}. \tag{2.7}$$

We will denote by $P_X = I - UU^*$ the orthogonal projection of H onto $\ker U^*$.

THEOREM 2.11. *Let T be an operator dominated by a gauge p. If $E \in \mathrm{Lat}(T)$, we write $T_1 = P_E T P_E$ and $T_2 = P_{E^\perp} T P_{E^\perp}$ where P_E (respectively $P_{E^\perp}$) denotes the orthogonal projection of H onto E (respectively $P_{E^\perp}$).*

(i) *Every operator $X \in \tau_p(T)$ can be uniquely decomposed, with respect to the orthogonal decomposition $H = E \oplus E^\perp$, under the following form:*

$$X = \begin{pmatrix} \sqrt{X_1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & A \\ A^* & B \end{pmatrix} \begin{pmatrix} \sqrt{X_1} & 0 \\ 0 & I \end{pmatrix} \tag{2.8}$$

where $X_1 \in \tau_p(T_1)$ and with the condition $\mathrm{Im}A \subseteq \overline{\mathrm{Im}(\sqrt{X_1})}$. *Moreover, the operator*

$$R_{X,\mathcal{L}} = \begin{pmatrix} I & A \\ A^* & B - E^{p^2}_{\mathcal{L},T_2^*,T_2}(I) \end{pmatrix}$$

is positive for any Banach limit $\mathcal{L}$ such that $X = E^{p^2}_{\mathcal{L},T,T}(I)$.

(ii) *The operator $\Delta_T(X) = B - A^*A$ can be uniquely decomposed in the following way:*

$$\Delta_T(X) = \Delta_T(X)_1 + \Delta_T(X)_2$$

where $\Delta_T(X)_1$ is a c_p^2 generalized T_2 Toeplitz and $\Delta_T(X)_2$ is a c_p^2 generalized lower T_2 Toeplitz such that $c_p^{-2n} T_2^{*n} \Delta_T(X)_2 T_2^n$ strongly converge to 0.

(iii) *Every operator $Y \in \tau_p(T^*)$ can be uniquely decomposed, with respect to the orthogonal decomposition $H = E \oplus E^\perp$, in the following form:*

$$Y = \begin{pmatrix} I & 0 \\ 0 & \sqrt{Y_2} \end{pmatrix} \begin{pmatrix} D & C \\ C^* & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \sqrt{Y_2} \end{pmatrix} \tag{2.9}$$

where $Y_2 \in \tau_p(T_2^*)$ and with the condition $\mathrm{Im}C \subseteq \overline{\mathrm{Im}(\sqrt{Y_2})}$. *Moreover the operator*

$$R'_{Y,\mathcal{L}} = \begin{pmatrix} B - E^{p^2}_{\mathcal{L},T_1,T_1^*}(I) & C \\ C^* & I \end{pmatrix}$$

is positive for any Banach limit $\mathcal{L}$ such that $Y = E^{p^2}_{\mathcal{L},T^*,T}(I)$.

(iv) *The operator $\Delta'_T(Y) = D - CC^*$ can be uniquely decomposed in the following way:*

$$\Delta'_T(Y) = \Delta'_T(Y)_1 + \Delta'_T(Y)_2$$

where $\Delta'_T(Y)_1$ is a c_p^2 generalized T_1^* Toeplitz and $\Delta'_T(Y)_2$ is a c_p^2 generalized lower T_1^* Toeplitz such that $c_p^{-2n} T_2^{*n} \Delta_T(X)_2 T_2^n$ strongly converges to 0.

REMARK 2.12. (i) The operator $\Delta_T(X)_1$ belongs to $\tau_p(T_2)$. There exists $L_T(X) \in B(E^\perp, E)$ such that the operator $L_0 = X_1 R + A_0 T_2$ can be factorized in $L_0 = \sqrt{X_1} L_T(X)$ and such that

$$\Delta_T(X)_2 = \sum_{n=0}^{+\infty} c_p^{-2n} T_2^{*n} L_T(X)^* P_{X_1} L_T(X) T_2^n \qquad \text{(strong convergence).}$$

(ii) The operator $\Delta'_T(Y)_1$ belongs to $\tau_p(T_1^*)$. There exists $L'_T(Y) \in B(E^\perp, E)$ such that the operator $L_0^{'} = T_1 C_0 + R Y_2$ can be factorized in $L_0^{'} = L'_T(Y)\sqrt{Y_2}$ and such that

$$\Delta'_T(Y)_2 = \sum_{n=0}^{+\infty} c_p^{-2n} T_1^n L'_T(Y) P_{Y_2} L_T^{'}(Y)^* T_1^{*n} \qquad \text{(strong convergence).}$$

Proof of Theorem 2.11. (i) With respect to the orthogonal decomposition $H = E \oplus E^\perp$, we have $T^n = \begin{bmatrix} T_1^n & R_n \\ 0 & T_2^n \end{bmatrix}$.

Let $X \in \tau_p(T)$ and $\mathcal{L}$ be a Banach limit such that $X = E^{p^2}_{\mathcal{L},T^*,T}(I)$. Then we have

$$X = \begin{bmatrix} X_1 & A_0 \\ A_0^* & B \end{bmatrix} \tag{2.10}$$

with $X_1 = E^{p^2}_{\mathcal{L},T_1^*,T_1}(I) \in \tau_p(T_1)$, $A_0 = \mathcal{L}(\{T_1^{*n} R_n p(n)^{-2}\}_{n \geqslant 0})$ and $B = \mathcal{L}(\{[R_n^* R_n + T_2^{*n} T_2^n] p(n)^{-2}\}_{n \geqslant 0})$. Using (2.7), we can write $\sqrt{X_1} T_1 = c_p U_1 \sqrt{X_1}$ where U_1 is a partial isometry whose support coincide with the one of X_1.

Since T is an operator dominated by a gauge p, we see that $\|R_n\| \leqslant p(n)$. Hence we obtain

$$\begin{aligned} |\langle A_0^* x \mid y\rangle| &= |\mathcal{L}(\{\langle R_n^* T_1^n x \mid y\rangle p(n)^{-2}\}_{n \geqslant 0})| \leqslant |\mathcal{L}(\{\|T_1^n x\| \|R_n y\| p(n)^{-2}\}_{n \geqslant 0})| \\ &\leqslant \sqrt{\mathcal{L}(\{\|T_1^n x\|^2 p(n)^{-2}\}_{n \geqslant 0}} \sqrt{\mathcal{L}(\{\|R_n x\|^2 p(n)^{-2}\}_{n \geqslant 0}} \qquad \text{(using (2.1)),} \end{aligned}$$

and so

$$|\langle A_0^* x \mid y\rangle|^2 \leqslant \langle X_1 x \mid x\rangle \langle (B - E^{p^2}_{\mathcal{L},T_2^*,T_2}(I)) y \mid y\rangle. \tag{2.11}$$

Consequently, there exists $\gamma > 0$ such that $A_0 A_0^* \leqslant \gamma X_1$. Applying *Douglas criterion*, we see that there exists a unique operator $A \in B(H)$ such that

$$A_0 = \sqrt{X_1} A,\ \ker A = \ker A_0,\ \mathrm{Im} A \subseteq \overline{\mathrm{Im}(\sqrt{X_1})} \text{ and } A_0 = \sqrt{X_1} A. \tag{2.12}$$

Now carrying the equation (2.12) in the equality (2.10) proves the decomposition (2.8) of the theorem. The unicity of this decomposition is immediate. By virtue of (2.11) and (2.12), we get $|\langle \sqrt{X_1} x \mid A y\rangle|^2 \leqslant \|\sqrt{X_1} x\|^2 \langle (B - E^{p^2}_{\mathcal{L},T_2^*,T_2}(I)) y \mid y\rangle$. Since the image of A is included in $\overline{\mathrm{Im}(\sqrt{X_1})}$, the previous inequality yields

$\|Ay\|^2 \leqslant \sqrt{\langle (B - E^{p^2}_{\mathcal{L},T_2^*,T_2}(I))y \mid y\rangle}$. The above inequality proves that the operator $R_{X,\mathcal{L}}$ is also positive.

(ii) Since $X \in \tau_p(T)$, obviously we have $T^*XT = c_pX$. The translation of the previous equality with respect to $E \in \mathrm{Lat}(T)$ and $E^\perp$ gives

$$(2.13)\quad c_p^2A_0 = T_1^*X_1R + T_1^*A_0T_2, \quad c_p^2B = R^*X_1R + R^*A_0T_2 + T_2^*A_0^*R + T_2^*BT_2.$$

The first equality of (2.13) implies that $c_p^2A_0 = T_1^*L_0$ with $L_0 = X_1R + A_0T_2$. Hence we obtain:

$$(2.14)\qquad X_1R = L_0 - c_p^{-2}T_1^*L_0T_2.$$

Observe that the equation (2.14) implies that for any integer n

$$L_0 - c_p^{-(2n+2)}T_1^{*n+1}L_0T_2^{n+1} = \sum_{k=0}^{n} c_p^{-2k}T_1^{*k}X_1RT_2^k.$$

Therefore, in the case where $\tau_{pq}(T_1, T_2) = \{0\}$ (for instance when $T_1 \in C_{\cdot,0}(p)$ or $T_2 \in C_{0,\cdot}(p)$), the operator L_0 is entirely determined by (2.14). Indeed, $\sum\limits_{k=0}^{n}(1/c_p^{2k}) \cdot T_1^{*k}X_0RT_2^k$ almost weakly converges to L_0.

To study the general case, it may be seen that the operator L_0 can be factorized in the following way:

$$(2.15)\qquad L_0 = X_1R + A_0T_2 = X_1R + \sqrt{X_1}AT_2 = \sqrt{X_1}L_T(X)$$

with

$$(2.16)\qquad L_T(X) = \sqrt{X_1}R + AT_2.$$

If we carry the equations (2.15) and (2.16) in the first equality of (2.13) we obtain:

$$(2.17)\qquad c_p^2\sqrt{X_1}A = c_p\sqrt{X_1}U_1^*\sqrt{X_1}R + c_p\sqrt{X_1}U_1^*AT_2.$$

Since the range of $c_pA - U_1^*\sqrt{X_1}R - U_1^*AT_2$ is included in $\mathrm{Im}(\sqrt{X_1})$, it follows from (2.17) that

$$(2.18)\qquad c_pA = U_1^*\sqrt{X_1}R - U_1^*AT_2 = U_1^*L_T(X).$$

Using the second equality of (2.13) we get:

$$(2.19)\qquad c_p^2(B - A^*A) = T_2^*(B - A^*A)T_2 + L_T(X)^*P_{X_1}L_T(X)$$

where P_{X_1} denotes the operator $I - U_1 U_1^*$ and hence is the orthogonal projection of H_1 onto $\ker(U_1^*)$. Indeed we have

$$\begin{aligned}
&c_p^2(B - A^*A) \\
&\quad = R^*X_1R + R^*\sqrt{X_1}AT_2 + T_2^*A^*\sqrt{X_1}R + T_2^*BT_2 - L_T(X)^*U_1U_1^*L_T(X) \\
&\quad = R^*X_1R + R^*\sqrt{X_1}AT_2 + T_2^*A^*\sqrt{X_1}R + T_2^*BT_2 \\
&\qquad - (R^*\sqrt{X_1} + T_2^*A^*)(I - P_{X_1})(\sqrt{X_1}R + AT_2) \\
&\quad = T_2^*(B - A^*A)T_2 + L_T(X)^*P_{X_1}L_T(X).
\end{aligned}$$

Therefore, for any integer n, we have:

$$\begin{aligned}
&B - A^*A \\
&\quad = c_p^{-(2n+2)}T_2^{*n+1}(B - A^*A)T_2^{n+1} + \sum_{k=0}^{n} c_p^{-2k}T_2^{*k}L_T(X)^*P_{X_1}L_T(X)T_2^k. \qquad (2.20)
\end{aligned}$$

Since the operator $\Delta_T(X) = B - A^*A$ is positive (since X is positive, it follows from (i)), that $\Big(\sum_{k=0}^{n}(1/c_p^{2k})T_2^{*k}L_T(X)^*P_{X_1}L_T(X)T_2^k\Big)_{n\geqslant 0}$ is strongly convergent. Let us denote by $\triangle_T(X)_2$ the sum of the previous series, it is easily seen that $T_2^{*k}\triangle_T(X)_2T_2^k \leqslant c_p^2 \triangle_T(X)_2$ and that $c_p^{-2n}T_2^{*n}\Delta_T(X)_2T_2^n$strongly converges to 0. Hence, we deduce that $c_p^{-2n-2}T_2^{*n+1}(B - A^*A)T_2^{n+1}$ is also strongly convergent to an operator belonging to $\mathcal{T}_p(T)$ which will be denoted by $\triangle_T(X)_1$. The uniqueness of a such decomposition $\Delta_T(X) = \triangle_T(X)_1 + \triangle_T(X)_2$ follows immediately. In the same manner we can prove (iii) and (iv). This ends the proof of the theorem. ∎

We are now ready to look more closely at the links between the asymptotic behaviour of the sequence $(T^n)_{n\geqslant 0}$ in regards with the invariant subspace E of T.

COROLLARY 2.13. *Let T be an operator dominated by an almost regular gauge p. If $E \in \mathrm{Lat}(T)$, we write $T_1 = P_ETP_E$ and $T_2 = P_{E^\perp}TP_{E^\perp}$, where P_E (respectively $P_{E^\perp}$) denotes the orthogonal projection of H onto E (respectively $E^\perp$).*

(i) *Let $X \in \tau_p(T)$ and $Y \in \tau_p(T^*)$. The following equivalences hold:*

(i1) $T \in C_{1,\cdot}(p) \Leftrightarrow T_1 \in C_{1,\cdot}(p)$ *and* $\ker(\Delta_T(X)) \cap A^{-1}(\mathrm{Im}(\sqrt{X_1})) = \{0\}$;

(i2) $T \in C_{\cdot,1}(p) \Leftrightarrow T_2^* \in C_{\cdot,1}(p)$ *and* $\ker(\Delta'_T(Y)) \cap C^{*(-1)}(\mathrm{Im}(\sqrt{Y_2})) = \{0\}$.

Moreover if p is almost regular, we have

(i'1) $T \in C_{1,\cdot}(p) \Leftrightarrow T_1 \in C_{1,\cdot}(p)$ *and* $\ker(\Delta_T(X)) \cap A^{-1}(\mathcal{E}_{T_1}) = \{0\}$;

(i'2) $T \in C_{\cdot,1}(p) \Leftrightarrow T_2^* \in C_{\cdot,1}(p)$ *and* $\ker(\Delta'_T(Y)) \cap C^{*(-1)}(\mathcal{E}_{T_2^*}) = \{0\}$.

In particular, we have:

(i″1) $T_1 \in C_{1,\cdot}(p)$ *and* $\ker(\Delta_T(X)) = \{0\} \Rightarrow T \in C_{1,\cdot}(p)$;

(i″2) $T_2^* \in C_{\cdot,1}(p)$, *and* $\ker(\Delta'_T(Y)) = \{0\} \Rightarrow T \in C_{\cdot,1}(p)$.

(ii) *If T_1 and T_2 belong to the class $C_{1,\cdot}(p)$, then the operator T belongs to the class $C_{1,\cdot}(p)$.*

(iii) *If* T_1 *and* T_2 *belong to the class* $C_{\cdot,1}(p)$*, then the operator* T *belongs to the class* $C_{\cdot,1}(p)$*.*

Proof. (i) Assume that $T \in C_{1,\cdot}(p)$. It is clear that this forces T_1 to belong to the class $C_{1,\cdot}(p)$. Let y be an element of the space $\ker(\Delta_T(X)) \cap A^{-1}(\mathrm{Im}(\sqrt{X_1})$, then there exists $x \in H$ such that $Ay = \sqrt{X_1}(-x)$. It follows that

$$X\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X_1 x + \sqrt{X_1}Ay \\ A^*\sqrt{X_1}x + By \end{pmatrix} = \begin{pmatrix} \sqrt{X_1}(\sqrt{X_1}x + Ay) \\ (B - A^*A)y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently we have $y = 0$.

Conversely, assume that $T_1 \in C_{1,\cdot}(p)$ and $\ker(\Delta_T(X)) \cap A^{-1}(\mathrm{Im}(\sqrt{X_1}) = \{0\}$. If $x \oplus y \in \ker(X)$, we have

$$X_1 x + \sqrt{X_1}Ay = 0 \tag{2.21}$$

$$A^*\sqrt{X_1}x + By = 0. \tag{2.22}$$

Since $T_1 \in C_{1,\cdot}(p)$, we deduce from (2.21) that $\sqrt{X_1}x + Ay = 0$ and carrying this equality in the equation (2.22) we obtain $0 = (B - A^*A)y = \Delta_T(X)y$. It follows that the vector y belongs to the subspace $\ker(\Delta_T(X)) \cap A^{-1}(\mathrm{Im}(\sqrt{X_1})$, and hence $y = 0$. As X_1 is a positive and injective, we get that $x = 0$ which ends the proof of the first part of (i). In the same manner we can prove (i2). If p is almost regular, we have $\mathrm{Im}(\sqrt{X_1}) = \mathcal{E}_{T_1}$ and $\mathrm{Im}(\sqrt{Y_2}) = \mathcal{E}_{T_2^*}$ and the assertion (i'1) (respectively (i'2)) follows immediately. The assertion (i"1) (respectively (i"2) comes from (i1) (respectively from (i2)).

(ii) Let $\mathcal{L}$ be a Banach limit such that $X = E^{p^2}_{\mathcal{L},T^*,T}(I)$. By Theorem 2.11, the operator $R_{X,\mathcal{L}}$ is positive, it implies that the operator $(B - A^*A) - E^{p^2}_{\mathcal{L},T_2^*,T_2}(I)$ is positive. Since T_2 belongs to the class $C_{1,\cdot}(p)$, $E^{p^2}_{\mathcal{L},T_2^*,T_2}(I)$ is injective, we get that $(B - A^*A) = \Delta_T(X)$ is injective and we get the desired result from (i'1). The proof of (iii) runs as before. This completes the proof of Corollary 2.13. ∎

REMARK 2.14. Let T be an operator satisfying the assumptions of the previous corollary, then $\ker(\Delta_T(X)_1)$ and $\ker(\Delta_T(X)_2)$ are invariant subspaces for T_2 and $\ker(\Delta_T(X)_2)$ is precisely the orthogonal complement of the subspace $\bigvee\{T_2^{*k}L^*(\ker T_1^*) : k \geqslant 0\}$.

3. SIMILARITY PROBLEMS AND INVARIANT SUBSPACES

The main result of this section (Theorem 3.3) sheds some light on how the similarity of $T \in B(H)$ to a "nice operator" restricted to an invariant subspace $E \subseteq H$ can "propagate" so that T is similar to a scalar multiple of a contraction to the whole space H. It takes also into account the asymptotic behaviour of the restriction of T to E. Before giving this result, first begin with two useful propositions.

PROPOSITION 3.1. *Let $T \in B(H)$ be an operator acting on a Hilbert space H and let $E \in \mathrm{Lat}(T)$ be such that $T_1 = T_{|E} \in \mathcal{LI}(E)$. Assume that T_1 admits a left inverse S_1 such that $\sup\limits_{n\geqslant 0}(\|S_1^n\|\|T_2^n\|) < +\infty$. Then the following assertions are equivalent:*

(i) $\sup\limits_{n\geqslant 0}(\|S_1^n R_{n+1}\|) < +\infty$;

(ii) $\sup\limits_{n\geqslant 0}\Big(\Big\|\sum\limits_{k=0}^{n} S_1^k R T_2^k\Big\|\Big) < +\infty$;

(iii) *there exists an operator K in $B(E^\perp, E)$such that $R = K - S_1 K T_2$.*

Proof. (i)⇒(ii) We have

$$S_1^n R_{n+1} = \sum_{k=0}^{n} S_1^{n-k} S_1^k T_1^k R T_2^{n-k} = \sum_{k=0}^{n} S_1^{n-k} R T_2^{n-k}.$$

The desired implication follows immediately.

(ii)⇒(iii) Assume that we have $\sup\limits_{n\geqslant 0}\Big(\Big\|\sum\limits_{k=0}^{n} S_1^k R T_2^k\Big\|\Big) < +\infty$. Considering a Banach limit $\mathcal{L}$, we define an operator $K \in B(E^\perp, E)$ by setting $K = \mathcal{L}\Big(\sum\limits_{k=0}^{n} S_1^k R T_2^k\Big)$. Then we have

$$S_1 K T_2 = \mathcal{L}\Big(\sum_{k=0}^{n} S_1^{k+1} R T_2^{k+1}\Big) = \mathcal{L}\Big(\sum_{k=0}^{n+1} S_1^k R T_2^k - R\Big) = K - R$$

and the desired equality is obtained.

(iii)⇒(i) If there exists an operator K such that $R = K - S_1 K T_2$, then we have $S_1^n R_{n+1} = \sum_{k=0}^{n} S_1^k R T_2^k = K - S_1^{n+1} K T_2^{n+1}$. Thus, we get that

$$\sup_{n\geqslant 0}\Big(\Big\|\sum_{k=0}^{n} S_1^k R T_2^k\Big\|\Big) < \|K\|\Big(1 + \sup_{k\geqslant 0}(\|S_1^k\|\|T_2^k\|)\Big) < +\infty.$$

This ends the proof. ∎

For any f in the algebra $A(\mathbb{T})$ of absolutely convergent series on $\mathbb{T} = \{z \in C : |z| = 1\}$, denote by $\widetilde{f}$ the function defined by $z\widetilde{f}(z) = f(z) - f(0)$ where z belongs to the closed unit disc $\overline{\mathbb{D}}$. Let $T \in PWB(H)$ and let $E \in \mathrm{Lat}(T)$, with respect to the orthogonal decomposition $H = E \oplus E^\perp$ the operator $f(T)$ has the following form

$$f(T) = \begin{bmatrix} f(T_1) & R(f) \\ 0 & f(T_2) \end{bmatrix}.$$

The next proposition shows how to compute $R(f)$ from $\widetilde{f}(T_1), \widetilde{f}(T_2)$ and R. It will be very useful, for instance it will enable us to apply Paulsen criterion in the sequel.

PROPOSITION 3.2. *Let* $T \in PWB(H)$ *and let* $E \in \mathrm{Lat}(T)$ *be such that* $T_1 = T_{|E} \in \mathcal{LI}(E)$. *Assume that*

(i) $\sup_{n \geqslant 0}(\|S_1^n R_n\|) < +\infty$;

(ii) $\sup_{n \geqslant 0}(\|S_1^n\|\|T_2^n\|) < +\infty$,

where S_1 *denotes the* Moore Penrose left inverse *of* T_1. *Then, for any* $f \in A(\mathbb{T})$, *the partial sums* $\sum_{k=0}^{n} S_1^k[\widetilde{f}(T_1)R - S_1RT_2\widetilde{f}(T_2)]T_2^k$ *almost weakly converge to* $R(f)$.

Proof. (i)⇒(ii) First notice that $R(f)$ satisfies the equation

$$T_1R(f) - R(f)T_2 = f(T_1)R - Rf(T_2) \tag{3.1}$$

which is linear and continuous with respect to f. In fact it suffices to prove (3.1) for $f(z) = z^n$, which is clear from the expression of $R_n = R(z^n)$. It follows from the equality (3.1) that

$$R(f) - S_1R(f)T_2 = \widetilde{f}(T_1)R - S_1RT_2\widetilde{f}(T_2). \tag{3.2}$$

If $n \in \mathbb{N}$, the equation (3.2) yields to:

$$R(f) = S_1^{n+1}R(f)T_2^{n+1} + \sum_{k=0}^{n} S_1^k[\widetilde{f}(T_1)R - S_1RT_2\widetilde{f}(T_2)]T_2^k. \tag{3.3}$$

To prove the lemma it is then sufficient to show that the sequence $S_1^nR(f)T_2^n$ is almost weakly convergent to 0 when n tends to ∞. Assume that it is not the case. Then, by Lorentz's result ([38]), we see that there exists a Banach limit $\mathcal{L}$ for which we have $\mathcal{L}(S_1^nR(f)T_2^n) \neq 0$. Consider the operator $E^M_{\mathcal{L},S_1,T_2}$ acting on $B(H_2, H_1)$ where $M = \sup_{n \geqslant 0}(\|S_1^n\|\|T_2^n\|)$. Since $\sup_{n \geqslant 0}(\|S_1^nR_n\|) < +\infty$, we know by Proposition 3.1 that there exists an operator K in $B(H_2, H_1)$ such that $R = K - S_1KT_2$. Thus, we obtain

$$E^M_{\mathcal{L},S_1,T_2}(R) = E^M_{\mathcal{L},S_1,T_2}(K) - E^M_{\mathcal{L},S_1,T_2}(S_1KT_2) = E^M_{\mathcal{L},S_1,T_2}(K) - E^M_{\mathcal{L},S_1,T_2}(K) = 0.$$

If there exists $p \geqslant 1$ such that $E^M_{\mathcal{L},S_1,T_2}(R_p) = 0$, we observe that

$$\begin{aligned} E^M_{\mathcal{L},S_1,T_2}(R_{p+1}) &= E^M_{\mathcal{L},S_1,T_2}(T_1^pR + R_pT_2) \\ &= E^M_{\mathcal{L},S_1,T_2}(T_1^pR) + E^M_{\mathcal{L},S_1,T_2}(R_p)T_2 = E^M_{\mathcal{L},S_1,T_2}(T_1^pR). \end{aligned}$$

Using Proposition 2.1 (ii) and (iv), we see that

$$E^M_{\mathcal{L},S_1,T_2}(T_1^pR) = E^M_{\mathcal{L},S_1,T_2}(S_1^p(T_1^pR)T_2^p) = E^M_{\mathcal{L},S_1,T_2}(RT_2^p) = E^M_{\mathcal{L},S_1,T_2}(R)T_2^p = 0.$$

Consequently, one can prove by induction that $E^M_{\mathcal{L},S_1,T_2}(R_n) = 0$ for every integer n, which obviously implies that $E^M_{\mathcal{L},S_1,T_2}(R(f)) = 0$ and which is absurd. This ends the proof of the proposition. ∎

We can now formulate the strongest result of this section. This theorem will enable us to prove the main result of Section 4.

THEOREM 3.3. *Let $T \in B(H)$ be an operator which is dominated by a gauge p satisfying $\sup_{n\geqslant 0}\{c_p^{-n}p(n)\} < +\infty$ and let $E \in \mathrm{Lat}(T)$ such that $c_p^{-1}T_1 = c_p^{-1}T_{|E} \in \mathcal{LI}(E)$ and such that $c_p^{-1}T_2^* = c_p^{-1}T^*_{|E^\perp}$ is a contraction. Assume that there exists $X_1 \in \tau_p(T_1)$ such that $\sqrt{X_1} \in \mathcal{L}I(\ker S_1, E)$ and that*

(i) $M_1 = \sup_{n\geqslant 0}(\|S_1^n P_E T^n P_{E^\perp}\|) < +\infty$;

(ii) $M_2 = \sup_{n\geqslant 0}(\|S_1^n\|\|T_2^n\|) < +\infty$;

(iii) $M_3 = \sum_{k\neq l} \|P_{\ker(T_1^*)}T_1^{*k}T_1^l P_{\ker(T_1^*)}\| c_p^{-(k+l)} < +\infty$;

where S_1 denotes the Moore-Penrose left inverse *of T_1. Then there exists an invertible operator J in $B(H)$ such that $\|J^{-1}TJ\| \leqslant c_p$.*

Before giving the proof of Theorem 3.3, first begin with a few comments on the hypothesis of the theorem.

REMARK 3.4. (i) The hypothesis $c_p^{-1}T^*_{|E^\perp}$ *is similar to a contraction* is minimal. Indeed if it is not the case, then there exists an operator A of Foguel type such that $c_p^{-1}T^*_{|E^\perp} = A$ and thus, obviously, $c_p^{-1}T$ can not be similar to a contraction.

(ii) The existence of $X_1 \in \tau_p(T_1)$ such that $\sqrt{X_1} \in \mathcal{LI}(\ker S_1, E)$ is automatically fulfilled when $\ker S_1$ is finite dimensional.

(iii) The condition $c_p^{-1}T_{|E}$ *is similar to an injective contraction with closed range* cannot be relaxed. Indeed if we suppose that is no so longer so, we can show that $c_p^{-1}T$ is not necessarily similar to a contraction, given that the counterexample introduced by Foguel, says $\widetilde{T}$, is such that its restriction to one of its invariant subspace is a coisometry.

(iv) When T is compatible with a gauge p, we have $c_p = r(T)$ (see [28]).

(v) The proof gives an estimate of the constant $\|J\|\|J^{-1}\|$.

Proof of Theorem 3.3. Since $\sup_{n\geqslant 0}\{c_p^{-n}p(n)\} < +\infty$, we observe that replacing T if necessary by $c_p^{-1}T$ we may assume that T is power bounded. We first show that the assumptions of our theorem are stable under similarity. Assume that $T_1' = AT_1A^{-1}$ where A is an invertible operator in $B(E)$. We easily see that the operator $S_1' = AS_1A^{-1}$ is a left inverse of T_1'. Choose a Banach limit $\mathcal{L}$ such that $X_1 = E^{p^2}_{\mathcal{L},T_1,T_1}(I)$. Since $\sqrt{X_1} \in \mathcal{LI}(\ker S_1, E)$, we know that there exists a positive number ρ such that $\langle X_1 x \mid x\rangle \geqslant \rho\|x\|^2$ for every $x \in \ker S_1$. Let us consider the operator $X_1' = E^{p^2}_{\mathcal{L},T_1',T_1'}(I)$, then for any $y = Ax \in \ker S_1' = A(\ker S_1)$ we have

$$\begin{aligned}\langle X_1' y \mid y\rangle = \langle A^{-1}E^{p^2}_{\mathcal{L},T_1,T_1}(A^2)A^{-1}Ax \mid Ax\rangle &\geqslant \|A^{-1}\|^{-2}\langle E^{p^2}_{\mathcal{L},T_1,T_1}(I)x \mid x\rangle\\ &\geqslant \|A^{-1}\|^{-2}\rho\|x\|^2 \geqslant \rho\|A^{-1}\|^{-2}\|A\|^{-2}\|y\|^2.\end{aligned}$$

It follows immediately that $\sqrt{X_1'} \in \mathcal{LI}(\ker S_1', E)$. It is clear that the other hypotheses are fulfilled by T_1'. It is easy to check that the hypotheses are still fulfilled if we replace T_2 by an operator similar to T_2. Hence, from now on we will assume that T_1 and T_2 are two contractions.

We now consider the operators $Y_n(f)$ (which appear in Proposition 3.2) defined by:

$$Y_n(f) = \sum_{k=0}^{n} S_1^k[\widetilde{f}(T_1)R - S_1RT_2\widetilde{f}(T_2)]T_2^k. \tag{3.4}$$

Recall that it follows from Proposition 3.1 that $R = K - S_1KT_2$. Thus we can rewrite the first sum of (3.4) as follows

$$\begin{aligned}\sum_{k=0}^{n} S_1^k\widetilde{f}(T_1)RT_2^k &= \sum_{k=0}^{n} S_1^k\widetilde{f}(T_1)KT_2^k - \sum_{k=0}^{n} S_1^{k+1}\widetilde{f}(T_1)T_1S_1KT_2^{k+1} \\ &= \widetilde{f}(T_1)K - S_1^{n+1}\widetilde{f}(T_1)KT_2^{n+1} + \sum_{k=1}^{n+1} S_1^k\widetilde{f}(T_1)PKT_2^k,\end{aligned}$$

where P denotes the orthogonal projection on $\ker(T_1^*)$. In the same way we can rewrite the second sum of (3.4)

$$\sum_{k=0}^{n} S_1^{k+1}R\widetilde{f}(T_2)T_2^{k+1} = S_1K\widetilde{f}(T_2)T_2 - S_1^{n+2}K\widetilde{f}(T_2)T_2^{n+2}.$$

Therefore, we obtain

$$\begin{aligned}Y_n(f) = \widetilde{f}(T_1)K &- S_1K\widetilde{f}(T_2)T_2 - S_1^{n+1}\widetilde{f}(T_1)KT_2^{n+1} \\ &+ S_1^{n+2}K\widetilde{f}(T_2)T_2^{n+2} + \sum_{k=1}^{n+1} S_1^k\widetilde{f}(T_1)PKT_2^k.\end{aligned} \tag{3.5}$$

The next step is to define the right context in which we will be able to apply *Paulsen criterion*. Let N be a positive integer. If F is a Hilbert space, denote by F_N the N-amplification of F, that is

$$F_N = \underbrace{F \oplus \cdots \oplus F}_{(N \text{ copies})}.$$

If F, G are some Hilbert spaces and $B \in B(F, G)$, we shall denote by B_N the operator of $B(F_N, G_N)$ defined by:

$$B_N = \underbrace{B \oplus \cdots \oplus B}_{(N \text{ copies})}.$$

Now consider a matrix $A_N = [f_{i,j}]_{1\leqslant i,j\leqslant N}$ which belongs to the algebra $\mathcal{M}_N$ of all the $N \times N$ matrices with coefficients in $A(\mathbb{T})$ and equipped with its unique norm of C^*-algebra. Denote by $A_N(T) = [f_{i,j}(T)]_{1\leqslant i,j\leqslant N}$ the matrix associated with A_N

via the functional calculus in T. Clearly $A_N(T)$ belongs to $B(H_N)$. With respect to the orthogonal decomposition $H = E \oplus E^{\perp}$, notice that

$$\sum_{i=0}^{N}\sum_{j=0}^{N}\left\langle f_{i,j}(T)\begin{bmatrix} x_j \\ y_j \end{bmatrix} \mid \begin{bmatrix} x'_i \\ y'_i \end{bmatrix}\right\rangle$$
$$= \sum_{i=0}^{N}\sum_{j=0}^{N}\langle f_{i,j}(T_1)x_j \mid x'_i\rangle + \sum_{i=0}^{N}\sum_{j=0}^{N}\langle R(f_{i,j})y_j \mid x'_i\rangle + \sum_{i=0}^{N}\sum_{j=0}^{N}\langle f_{i,j}(T_2)y_j \mid y'_i\rangle.$$

We can then easily deduce that:

$$\|A_N(T)\|_{B(H_N)} \leqslant 2\|[f_{i,j}]\|_{\mathcal{M}_N(A(\mathbb{T}))} + \|[R(f_{i,j})]\|_{B(E_N^{\perp}:E_N)}. \tag{3.6}$$

Therefore the control of the norm of $A_N(T)$ depends on the control of the norm of the matrices $[R(f_{i,j})]$. It follows from Proposition 3.2, that the control of the norm of the matrices $[Y_n(f_{i,j})]$ will give an estimate of the norm of $[R(f_{i,j})]$. The equation (3.5) implies that:

$$\begin{aligned}\|[Y_n(f_{i,j})]\|_{B(E_N^{\perp};E_N)} = \|&[\widetilde{f}_{i,j}(T_1)]K_N + (S_1)_N^{n+1}K_N[\widetilde{f}_{i,j}(T_2)](T_2)_N^{n+1} \\ &- (S_1)_N K_N[\widetilde{f}_{i,j}(T_2)](T_2)_N - (S_1)_N^{n+1}[\widetilde{f}_{i,j}(T_1)]K_N(T_2)_N^{n+1} \\ &+ \sum_{k=1}^{n+1}(S_1)_N^k[\widetilde{f}_{i,j}(T_2)_{i,j}(T_1)]\|.\end{aligned}$$

It follows that

$$\begin{aligned}&\|[Y_n(f_{i,j})]\|_{B(E_N^{\perp};E_N)} \\ &\qquad \leqslant (1+3\sup\|S_1^n\|\|T_2^n\|)\|K\|\|[\widetilde{f}_{i,j}]\|_{\mathcal{M}_N(A(\mathbb{T}))} + \|[Z_n(f_{i,j})]\|_{B(E_N^{\perp};E_N)} \\ &\qquad \leqslant 2(1+3\sup\|S_1^n\|\|T_2^n\|)\|K\|\|[f_{i,j}]\|_{\mathcal{M}_N(A(\mathbb{T}))} + \|[Z_n(f_{i,j})]\|_{B(E_N^{\perp};E_N)},\end{aligned}$$

where $Z_n(f) = \sum\limits_{k=1}^{n+1} S_1^k\widetilde{f}(T_1)PKT_2^k$.

Applying the operator Cauchy-Schwartz inequality (Proposition 1.4 and Remark 1.5 (i)), we obtain

$$\begin{aligned}&\|[Z_n(f_{i,j})]\|_{B(E_N^{\perp};E_N)} \\ &\qquad \leqslant \sqrt{\|\sum_{k=1}^{n+1}(S_1)_N^k[\widetilde{f}_{i,j}(T_1)]P_N[\widetilde{f}_{i,j}(T_1)]^*(S_1^*)_N^k\|_{B(E_N)}} \\ &\qquad\quad \times \sqrt{\|\sum_{k=1}^{n+1}(T_2^*)_N^k K_N^* P_N K_N(T_2)_N^k\|_{B(E_N^{\perp})}}.\end{aligned} \tag{3.7}$$

In order to estimate the first square root of the right member of (3.7), we need the next lemma.

LEMMA 3.5. *Let $[f_{i,j}]_{1\leqslant i,j\leqslant N}$ be a $N\times N$ matrix with coefficients in $A(\mathbb{T})$ with $N\in\mathbb{N}^*$, $T_1\in\mathcal{LI}(E,E)$ and S_1 a left inverse of T_1. For every n, we have the following estimate*

$$\Big\|\sum_{k=0}^{n}(S_1)_N^k[f_{i,j}(T_1)]P_N[f_{i,j}(T_1)]^*(S_1^*)_N^k\Big\|_{B(E_N)}\leqslant\|[f_{i,j}]\|^2_{\mathcal{M}_N(A(\mathbb{T}))}\tag{3.8}$$

where P is the orthogonal projection on the kernel of T_1^*.

Proof. Using if necessary a straightforward passage to the limit, we are reduced to proving the lemma when all $f_{i,j}$ are polynomials.

Let p be a polynomial of degree m. Consider the operator

$$W_n(p)=\sum_{k=0}^{n}S_1^kp(T_1)Pp(T_1)^*S_1^{*k},$$

then for any $n\geqslant m$, we have

$$\begin{aligned}W_n(p)&=\sum_{k=0}^{n}\sum_{i=0}^{m}\sum_{j=0}^{m}a_i\bar{a}_jS_1^kT_1^iPT_1^{*j}S_1^{*k}\\&=\sum_{k=0}^{n}\sum_{i=k}^{m}\sum_{j=k}^{m}a_i\bar{a}_jr^{i+j}S_1^kT_1^iPT_1^{*j}S_1^{*k}\qquad(S_1^lP=0=PS_1^{*l}\text{ for any }l\geqslant1)\\&=\sum_{k=0}^{n}p_k(T_1)Pp_k(T_1)^*\end{aligned}\tag{3.9}$$

where $p_k(z)=\sum\limits_{i=k}^{m}a_iz^{i-k}$. If $r\in]0,1[$, we will denote by p_r the polynomial $p(rz)$. Write $m=\max(d^\circ(f_{i,j}))$. Let $n\geqslant m$ and $r\in]0,1[$, using (3.9) we get

$$\begin{aligned}W_{n,r}&=\sum_{k=0}^{n}(S_1)_N^k[f_{i,j}(rT_1)]P_N[f_{i,j}(rT_1)]^*(S_1)_N^{k*}\\&=[W_n((f_{i,j})_r)]_{1\leqslant i,j\leqslant N}=\sum_{k=0}^{m}[(f_{i,j})_k(rT_1)P][(f_{i,j})_k(rT_1)P]^*.\end{aligned}$$

We now consider the operator kernel $K_\alpha(T_1),\alpha<1$, defined by

$$K_\alpha(T_1)=(I-\bar{\alpha}T_1)^{-1}+(I-\alpha T_1^*)^{-1}-I=(I-\bar{\alpha}T_1)^{-1}(I-r^2T_1T_1^*)(I-\alpha T_1^*)^{-1}.$$

Let us denote by $D_{T_1^*}(r)$ the operator $\sqrt{I-r^2T_1T_1^*}$, $0<r<1$. Choose an orthonormal basis $(e_l)_{l\in\Lambda}$ in $\ker(T_1^*)$, then for every $x=(x_1,\dots,x_N)$, we have

$$\begin{aligned}\langle W_{n,r}x\mid x\rangle&=\sum_{q=0}^{m}\|[(f_{k,l})_q(rT_1)P]^*x\|^2=\sum_{q=0}^{m}\|P[(f_{k,l})_q(rT_1)]^*x\|^2\\&=\sum_{q=0}^{m}\sum_{l=1}^{N}\sum_{j\in\Lambda}\Big|\sum_{k=1}^{N}\langle(f_{k,l})_q(rT_1)^*x_k\mid e_j\rangle\Big|^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{q=0}^{m}\sum_{l=1}^{N}\sum_{j\in\Lambda}\Big|\sum_{k=1}^{N}\int_0^{2\pi}(f_{k,l})_q(\mathrm{e}^{\mathrm{i}t})\langle K_{r\mathrm{e}^{\mathrm{i}t}}(T_1)e_j \mid x_k\rangle\Big|^2\\
&= \sum_{q=0}^{m}\sum_{l=1}^{N}\sum_{j\in\Lambda}\Big|\sum_{k=1}^{N}\int_0^{2\pi}(f_{k,l})_q(\mathrm{e}^{\mathrm{i}t})\langle (I-r\mathrm{e}^{-\mathrm{i}t}T_1)^{-1}e_j \mid x_k\rangle \mathrm{d}m(t)\Big|^2\\
&= \sum_{q=0}^{m}\sum_{l=1}^{N}\sum_{j\in\Lambda}\Big|\sum_{k=1}^{N}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}qt}f_{k,l}(\mathrm{e}^{\mathrm{i}t})\langle (I-r\mathrm{e}^{-\mathrm{i}t}T_1)^{-1}e_j \mid x_k\rangle \mathrm{d}m(t)\Big|^2\\
&= \sum_{q=0}^{m}\sum_{l=1}^{N}\sum_{j\in\Lambda}\Big|\sum_{k=1}^{N}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}qt}f_{k,l}(\mathrm{e}^{\mathrm{i}t})\langle (I-r\mathrm{e}^{-\mathrm{i}t}T_1)^{-1}D_{T_1^*}(r)e_j \mid x_k\rangle \mathrm{d}m(t)\Big|^2\\
&= \sum_{q=0}^{m}\sum_{l=1}^{N}\sum_{j\in\Lambda}\Big|\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}qt}\Big\langle e_j \mid \sum_{k=1}^{N}\overline{f_{k,l}(\mathrm{e}^{\mathrm{i}t})}D_{T_1^*}(r)(I-r\mathrm{e}^{\mathrm{i}t}T_1^*)^{-1}x_k\Big\rangle \mathrm{d}m(t)\Big|^2\\
&\leqslant \sum_{l=1}^{N}\sum_{j\in\Lambda}\int_0^{2\pi}\Big|\Big\langle e_j \mid \sum_{k=1}^{N}\overline{f_{k,l}(\mathrm{e}^{\mathrm{i}t})}D_{T_1^*}(r)(I-r\mathrm{e}^{\mathrm{i}t}T_1^*)^{-1}x_k\Big\rangle\Big|^2 \mathrm{d}m(t)\\
&\leqslant \int_0^{2\pi}\sum_{l=1}^{N}\Big\|\sum_{k=1}^{N}\overline{f_{k,l}(\mathrm{e}^{\mathrm{i}t})}D_{T_1^*}(r)(I-r\mathrm{e}^{\mathrm{i}t}T_1^*)^{-1}x_k\Big\|^2 \mathrm{d}m(t)\\
&= \int_0^{2\pi}\left\|\overline{[f_{k,l}(\mathrm{e}^{it})]}_{1\leqslant k,l\leqslant N}\begin{bmatrix}\sqrt{I-r^2T_1T_1^*}(I-r\mathrm{e}^{\mathrm{i}t}T_1^*)^{-1}x_1\\ \vdots \\ \sqrt{I-r^2T_1T_1^*}(I-r\mathrm{e}^{\mathrm{i}t}T_1^*)^{-1}x_N\end{bmatrix}\right\|^2 \mathrm{d}m(t)\\
&\leqslant \|[f_{k,l}]_{1\leqslant k,l\leqslant N}\|^2_{\mathcal{M}_N(A(slT))}\int_0^{2\pi}\sum_{k=1}^{N}\langle K_{r\mathrm{e}^{\mathrm{i}t}}(T_1)x_k \mid x_k\rangle \mathrm{d}m(t)\\
&= \|[f_{k,l}]_{1\leqslant k,l\leqslant N}\|^2_{\mathcal{M}_N(A(\mathbb{T}))}\|x\|^2.
\end{aligned}
$$

We get the estimate (3.8) by letting $r\to 1$. It ends the proof of Lemma 3.5. ∎

Given that the operator $\sum\limits_{k=1}^{n+1}(T_2^*)^k_N K_N^* P_N K_N (T_2)^k_N$ is diagonal, it follows that:

$$
\Big\|\sum_{k=1}^{n+1}(T_2^*)^k_N K_N^* P_N K_N (T_2)^k_N\Big\|_{B(E_N^\perp)} = \Big\|\sum_{k=1}^{n+1}T_2^{*k}K^*PKT_2^k\Big\|_{B(E^\perp)}. \tag{3.10}
$$

We have

$$R_{n+1} = \sum_{k=0}^{n} T_1^{n-k} R T_2^k = \sum_{k=0}^{n} T_1^{n-k}(K - S_1 K T_2) T_2^k$$

$$= T_1^n K - S_1 K T_2^{n+1} + \sum_{k=1}^{n} T_1^{n-k} PK T_2^k. \tag{3.11}$$

We denote by $X_1 = \operatorname{strong} \lim(T_1^{*k} T_1^k)$. Note that we have

$$\begin{aligned}
&\sum_{k=1}^{n} T_2^{*k} K^* P X_1 P K T_2^k \\
&\qquad \leqslant \sum_{k=1}^{n} T_2^{*k} K^* P T_1^{*n-k} T_1^{n-k} P K T_2^k \\
&\qquad = \Big| \sum_{k=1}^{n} T_1^{n-k} P K T_2^k \Big|^2 - \sum_{1 \leqslant k \neq l \leqslant n} T_2^{*k} K^* P T_1^{*n-k} T_1^{n-l} P K T_2^l \qquad (3.12)\\
&\qquad \leqslant [(\|K\| + \|S_1 K\| + \sup\{\|R_n\|\})^2 + \|K\|^2 M_3] I \quad \text{(using (3.11) and (iii))}.
\end{aligned}$$

Since $\sqrt{X_1} \in \mathcal{LI}(\ker S_1, E)$ and $\operatorname{Im}(Q) = \ker S_1$, we see that $\gamma(\sqrt{X_1} \mid_E) > 0$. Hence we get from (3.10) and (3.12)

$$\begin{aligned}
&\Big\| \sum_{k=1}^{n+1} (T_2^*)_N^k K_N^* P_N K_N (T_2)_N^k \Big\|_{B(E_N^{\perp})} \\
&\qquad \leqslant [\gamma(\sqrt{X_1} \mid_E)]^{-1} [(\|K\| + \|S_1 K\| + \sup\{\|R_n\|\})^2 + \|K\|^2 M_3] \qquad (3.13)\\
&\qquad \leqslant [\gamma(\sqrt{X_1} \mid_E)]^{-1} [(2 + \|S_1\|)^2 M_1^2 + M_1^2 M_3].
\end{aligned}$$

Finally, using Lemma 3.5 and the above inequality (3.13) we obtain

$$\|[R(f_{i,j})]\|_{B(E_N^{\perp}; E_N)} \leqslant 2 \|[f_{i,j}]\|_{\mathcal{M}_N(A(\mathbb{T}))} M_1 \left[(1 + 3M_2) + \sqrt{\frac{(2 + \|S_1\|)^2 + M_3}{\gamma(\sqrt{X_1} \mid_E)}} \right].$$

The above estimate in the equation (3.6), gives:

$$\begin{aligned}
&\|[f_{i,j}(T)]\|_{B(H_N)} \\
&\quad \leqslant 2 \|[f_{i,j}]\|_{\mathcal{M}_N(A(\mathbb{T}))} \Big[1 + M_1 \Big[(1 + 3M_2) + \sqrt{\gamma(\sqrt{X_1} \mid_E)^{-1} [(2 + \|S_1\|)^2 + M_3]} \Big] \Big].
\end{aligned}$$

The *Paulsen criterion* implies the existence of a contraction $\widetilde{T}$ and an invertible positive operator J satisfying

$$T = J \widetilde{T} J^{-1}$$

with the following estimate

$$\|J\| \|J^{-1}\| \leqslant 2 \Big[1 + M_1 \Big[(1 + 3M_2) + \sqrt{\gamma(\sqrt{X_1} \mid_E)^{-1} [(2 + \|S_1\|)^2 + M_3]} \Big] \Big].$$

The proof of Theorem 3.3 is complete. ∎

THEOREM 3.6. *Let $T \in B(H)$ be an operator and let $E \in \mathrm{Lat}(T)$ be such that both $r(T)^{-1}T_1 = r(T)^{-1}T_{|E}$ and $r(T)^{-1}T_2^* = r(T)^{-1}T_{|E^\perp}^*$ are similar to contractions. Assume that T_1 admit a left inverse S_1 and that $\sum\limits_{n=0}^{+\infty} \|S_1^n\|\|T_2^n\| < +\infty$. Then there exists an invertible operator J in M such that $\|JTJ^{-1}\| \leqslant r(T)$.*

Proof. First observe that the convergence of the series $\sum\limits_{n=0}^{+\infty} \|S_1^n\|\|T_2^n\|$ implies that the assumptions of Propositions 3.1 and 3.2 are satisfied. Therefore, the first part of the proof runs as before, the only difference being in the estimate of $\|[Z_n(f_{i,j})]\|_{B(E_N^\perp;E_N)}$. Here , we obtain

$$\|[Z_n(f_{i,j})]\|_{B(E_N^\perp;E_N)} \leqslant \Big[\sum_{n=0}^{+\infty} \|S_1^n\|\|T_2^n\|\Big]\|K\|,$$

where $K = \sum\limits_{k=0}^{+\infty} S_1^k R T_2^k$. Applying again Paulsen criterion we get the desired result. ∎

The following result may be proved in the same way as Theorem 3.3. It takes also into account the asymptotic behaviour of the restriction of T to an invariant subspace E.

THEOREM 3.7. *Let $T \in B(H)$ be an operator which is dominated by a gauge p satisfying $\sup\limits_{n\geqslant 0}\{c_p^{-n}p(n)\} < +\infty$. Assume that $c_p^{-1}T_1 = c_p^{-1}T_{|E} \in \mathcal{LI}(E)$ is similar to a contraction and that $c_p^{-1}T_2^* = c_p^{-1}T_{|E^\perp}^* \in PB(H)$. If there exists $X_1 \in \tau_p(T_1)$ such that $\sqrt{X_1} \in \mathcal{LI}(\ker S_1, E)$ and that*

(i) $M_1 = \sup\limits_{n\geqslant 0}(\|S_1^n P_E T^n P_{E^\perp}\|) < +\infty$;

(ii) $M_2 = \sup\limits_{n\geqslant 0}(\|S_1^n\|\|T_2^n\|) < +\infty$;

(iii) $M_3 = \sum\limits_{k\neq l} \|P_{\ker(T_1^*)} T_1^{*k} T_1^l P_{\ker(T_1^*)}\| p(k)^{-1} p(l)^{-1} < +\infty$;

where S_1 denotes the Moore-Penrose left inverse *of T_1, then the operator $r(T)^{-1}T$ is polynomially bounded and we have*

$$M_{r(T)^{-1}T} \leqslant C_{\mathrm{sim}}(T_1)\Big[1 + M_{T_2} + 2M_1\Big[1 + M_2 + 2M_2 M_{T_2} + 2\sqrt{\gamma(\sqrt{X_1}\ |_E)^{-1}[(2+\|S_1\|)^2 + M_3]}\ \Big]\Big].$$

4. SEVERAL APPLICATIONS

We present now several applications of the previous results.

4.1. AN IMPORTANT PARTICULAR CASE. When the compression of the operator to an invariant subspace is a scalar multiple of an isometry Corollary 2.13 gives a useful criterion for T being in the class $C_{1,\cdot}$.

COROLLARY 4.1. *Let $T \in B(H)$ be an operator which is dominated by a gauge p and let $E \in \mathrm{Lat}(T)$ be such that $c_p^{-1}T_1 = c_p^{-1}T_{|E}$ is similar to an isometry and such that $c_p^{-1}T_2^* = c_p^{-1}T^*_{|E^\perp}$ is a contraction. Let $X \in \tau_p(T)$, then we have*

$$T \in C_{1,\cdot}(p) \Leftrightarrow T_1 \in C_{1,\cdot}(p) \quad \textit{and} \quad \ker(\Delta_T(X) = \{0\}.$$

Proof. Since $c_p^{-1}T_1$ is similar to an isometry, we see that X_1 is invertible. Thus we have $\mathrm{Im}(\sqrt{X_1}) = H$ and we obtain the desired result by Corollary 2.13 (i1). ∎

At the start, the following result was the main motivation in similarity problems for operators of Foguel type ([7], [9]).

COROLLARY 4.2. *Let $T \in B(H)$ be an operator which is dominated by a gauge p satisfying $\sup\limits_{n\geqslant 0}\{c_p^{-n}p(n)\} < +\infty$. Let $E \in \mathrm{Lat}(T)$ be such that $c_p^{-1}T_1 = c_p^{-1}T_{|E}$ is similar to an isometry and such that $c_p^{-1}T_2^* = c_p^{-1}T^*_{|E^\perp}$ is similar to a contraction. Then there exists an invertible operator A in $B(H)$ such that $c_p^{-1}T$ is similar to a contraction. In particular, if $c_p^{-1}T_1 = V$ is an isometry and $c_p^{-1}T_2^*$ is a contraction, we can find an invertible operator J such that*

$$\|J^{-1}TJ\| \leqslant c_p \quad \textit{and} \quad \|J\|\|J^{-1}\| \leqslant 2 + \inf_{X\in\tau_p(T)}\{4\|L_T(X)\| + \sqrt{\|\Delta_T(X)_2\|}\}.$$

Before giving the proof of Corollary 4.2, first begin with a few comments on the hypothesis of the theorem.

REMARK 4.3. (i) Observe that, in this situation, the similarity problem does not depend on the multiplicity of the isometry V although the multiplicity is very important when $T_1 = V^*$ is a coisometry (See Pisier counterexample! ([44])).

(ii) The hypothesis $c_p^{-1}T^*_{|E^\perp}$ *is similar to a contraction* is minimal. Indeed if it is not the case, then there exists an operator A of Foguel type such that $c_p^{-1}T^*_{|E^\perp} = A$ and thus, obviously, $c_p^{-1}T$ can not be similar to a contraction.

(iii) When p is a constant gauge (i.e. $T \in PWB(H)$) the similarity to a contraction was given in [7] and [9]. See also [19] (with different technics) where T is assumed to be polynomially bounded and the isometry V to be of multiplicity one.

(iv) Observe that the previous result gives more details about the similarity to a scalar multiple of a contraction, it provides an estimate of the constant of similarity involving operators naturally associated with T. These operators lead to various connections with function theory in [13].

Poof of Corollary 4.2. By similarity, we may assume that $c_p^{-1}T_1 = V$ is a isometry and $c_p^{-1}T_2^*$ is a contraction. Then we have $S_1 = c_p^{-1}V^*$. If $k > l$, we have

$$P_{\ker(T_1^*)}T_1^{*k}T_1^l P_{\ker(T_1^*)} = (I - VV^*)c_p^2(V^{*k-l} - V^{*k-l}VV^*) = 0.$$

The same conclusion can be drawn for the case $k > l$, finally we get $M_3 = 0$. On the other hand, we have

$$\|S_1^n P_E T^n P_{E^\perp}\| \leqslant c_p^{-n}\|T^n\| \leqslant Cc_p^{-n}p(n) \leqslant C\sup_{n\geqslant 0}\{c_p^{-n}p(n)\} < +\infty.$$

Hence we get $M_1 = \sup_{n\geqslant 0}(\|S_1^n P_E T^n P_{E^\perp}\|) < +\infty$. Analysis similar to the previous one shows that $M_1 = \sup_{n\geqslant 0}(\|S_1^n\|\|T_2^n\|) < +\infty$. Note that $\tau_p(T_1) = \{I\}$ and obviously $I \in \mathcal{LI}(\ker V^*, E)$. Hence, we may apply Theorem 3.3 to obtain that $c_p^{-1}T$ is similar to a contraction.

In order to get the estimate, we proceed as in the proof of Theorem 3.3. We observe that we can choose $K = L_T(X)$ for any X in $\tau_p(T)$ and we apply the operator Cauchy-Schwartz inequality in a such way that we have

$$\begin{aligned}
&\|[Z_n(f_{i,j})]\|_{B(E_N^\perp;E_N)}\\
&\leqslant \sqrt{\Big\|\sum_{k=1}^{n+1} r(T)^{2k}(S_1)_N^k[f_{i,j}(T_1)]P_N[f_{i,j}(T_1)]^*(S_1^*)_N^k\Big\|_{B(E_N)}}\\
&\qquad\times\sqrt{\Big\|\sum_{k=1}^{n+1}\frac{1}{c_p^{2k}}(T_2^*)_N^k L_T(X)_N^* P_N L_T(X)_N (T_2)_N^k\Big\|_{B(E_N^\perp)}}\\
&\leqslant \sqrt{\Big\|\sum_{k=1}^{n+1} V_N^{*k}[f_{i,j}(T_1)]P_N[f_{i,j}(T_1)]^* V_N^k\Big\|_{B(E_N)}}\sqrt{\|\varDelta_T(X)\|} \text{ (use Remark 2.12 (i))}\\
&= \sqrt{\|V_N^{*n+1}[f_{i,j}(T_1)][f_{i,j}(T_1)]^*V_N^{n+1} - [f_{i,j}(T_1)][f_{i,j}(T_1)]^*\|_{B(E_N)}}\sqrt{\|\varDelta_T(X)\|}\\
&\leqslant \|[f_{i,j}]\|_{\mathcal{M}_N(A(\mathbb{T}))}\sqrt{\|\varDelta_T(X)\|}.
\end{aligned}$$

It ends the proof of Corollary 4.2. ∎

COROLLARY 4.4. *Let $T \in B(H)$ be an operator which is compatible with a regular gauge p satisfying $\sup_{n\geqslant 0}\{c_p^{-n}p(n)\} < +\infty$. Let $E \in \mathrm{Lat}(T)$ such that $VN(T_1)$ is a finite von Neumann algebra, T_1 is a $C_{1,\cdot}(p)$ and $c_p^{-1}T_2^* = c_p^{-1}T^*_{|E^\perp}$ is similar to a contraction, then there exists an invertible operator J such that $\|J^{-1}TJ\| \leqslant c_p$.*

Proof. Since the von Neumann algebra is finite and $T_1 \in C_{1,\cdot}(p)$, we know by [8] that $c_p^{-1}T_1$ is similar to a unitary operator ($T_1 \in C_{1,\cdot}(p) \to r(T) = r(T_1) = c_p$; see [28]). Therefore, we may assume (up to a similarity) that $c_p^{-1}T_1$ is a unitary operator and that $c_p^{-1}T_2^*$ is a contraction and we can apply the previous corollary. ∎

COROLLARY 4.5 (B. Sz.-Nagy [41]). *Let $T \in PWB(H)$ be a compact operator, then T is similar to a contraction.*

Proof. We consider the invariant subspace E associated with the spectrum which lies on the torus. It is clear that $VN(T_1)$ is a finite von Neumann algebra (E is finite dimensional), $T_1 \in C_{1,\cdot}$. On the other hand, we easily see that $r(T_2) < 1$, and thus that T_2 is similar to a contraction, by Rota's theorem. Now, it suffices to apply the previous corollary in order to obtain the desired result. ∎

COROLLARY 4.6. *Let $T \in B(H)$ be an operator which is dominated by a gauge p satisfying $\sup\limits_{n\geqslant 0}\{c_p^{-n}p(n)\} < +\infty$. Let $E \in \mathrm{Lat}(T)$ be such that $c_p^{-1}T_1 = V$ is an isometry and such that $c_p^{-1}T_2^* = c_p^{-1}T^*_{|E^\perp} \in PB(H)$. Then the following conditions are equivalent:*

(i) $c_p^{-1}T \in PWB(H)$;

(ii) $c_p^{-1}\, T \in PB(H)$.

Moreover, if one of the above conditions is satisfied, we have the following estimate

$$M_{c_p^{-1}T} \leqslant 3 + \Big[1 + \sup_{n\geqslant 0}(\|T_2^n\|)\Big]M_{T_2} + \Big(1 + \sup_{n\geqslant 0}(\|T_2^n\|\Big) + 2\sup_{n\geqslant 0}((\|T_2^n\|)M_{T_2})\sup_{n\geqslant 0}(\|P_E T^n P_{E^\perp}\|).$$

REMARK 4.7. If p is constant and $V = S$ is the usual shift of multiplicity one, the above equivalence of (i) and (ii) is contained in [20] and the former result is attributed to C. Foiaş and J.P. Williams.

4.2. COMPRESSION AND SIMILARITY.

The next result is concerned with a similarity result about the sequence of powers of T.

COROLLARY 4.8. (i) *Let $T \in B(H)$ be an injective operator with closed range. Assume that $R \in B(H)$ is such that there exists $\rho \in]0,1[$ such that*

$$\underline{\lim}\|R^n - T^n\|^{1/n}\rho^{-(1/n)} \langle \sup\{r(S)^{-1} : S \in B(H) \text{ and } ST = I\}. \tag{4.1}$$

Then there exists a sequence $(P_n)_{n\geqslant 0}$, where each P_n is a projection on the range of T^n and an invertible operator A such that $P_nAR^nA^{-1} = T^n$. In particular, if $\ker T^*$ is of dimension 1, then R is similar to a rank one perturbation of T.

(ii) *Let T be an isometry and $R \in B(H)$ satisfying the following inequality*

$$\underline{\lim}\|R^n - T^n\|^{1/n}\rho^{-(1/n)} < 1 \tag{4.2}$$

where $\rho \in]0,1[$. Then R is similar to an isometry.

Proof. (i) By assumption, we can choose a left inverse S of T satisfying the following inequality

$$\underline{\lim}\|R^n - T^n\|^{1/n}\rho^{-(1/n)} < r(S)^{-1}.$$

By the above result, we see that there exists a strictly increasing sequence of integers $(n_k)_{k\geqslant 0}$ such that

$$\|R^{n_k} - T^{n_k}\|\|S^{n_k}\| \leqslant \rho < 1.$$

Replacing if necessary $(n_k)_{k\geqslant 0}$ by a subsequence, we may assume that

$$\lim_{k\to+\infty}\|R^{n_k} - T^{n_k}\|\|S^{n_k}\| = l < 1.$$

Let us choose an ultrafilter $\mathcal{U}$ containing the Frechet filter associated with $(n_k)_{k\geqslant 0}$. We can define a Banach limit $\mathcal{L}$ by setting

$$\mathcal{L}((u_n)_{n\geqslant 0}) = \lim_{\mathcal{U}}\Big(\frac{u_0 + \cdots + u_n}{n+1}\Big).$$

Considering the operator $A = E^1_{\mathcal{L},S,R}(I)$, we can see that

$$\|A - I\| \leqslant \mathcal{L}(\|R^n - T^n\|\|S^n\|) = l < 1.$$

From the above it follows that A is an invertible operator satisfying the equation $SAR = A$ (Proposition 2.1 (iv)). Set $P_n = T^nS^n$, since S is a left inverse of T, it is easy to check that P_n is a projection (not necessarily orthogonal) on the range of T^n. To deduce the desired conclusion, observe that we have by construction $P_nAR^n = T^nA$.

(ii) We first observe that the condition (4.2) implies that R is power bounded. The first part of the proof follows by the same method as in (i), the only difference being that we can take precisely for S the adjoint of the isometry T. Consequently, we note that P_1 is the orthogonal projection on the range of T. Now, we deduce that

$$A^*A = R^*A^*P_1AR \leqslant R^*A^*AR.$$

It follows that A^*A is an upper R Toeplitz operator which is invertible. Applying Proposition 2.6, we conclude that R is similar to an isometry. This ends the proof of the corollary. ∎

REMARK 4.9. (i) When $T \in \mathcal{L}I(H)$ is a Fredholm operator we know from [2] that $\lim\gamma(T^n)^{1/n} = \sup\{r(S)^{-1} : ST = I\}$.

(ii) If we take for T the usual shift on the Hardy space H^2, we see that R is similar to a rank one perturbation of T which is similar to T. For a recent account of the treatment of rank one perturbation of the usual shift we refer the reader to [13].

(iii) We mention that Corollary 3.1 in [1] is actually a consequence of the assertion (ii).

4.3. CRITERION FOR SIMILARITY TO AN ISOMETRY. The next result relies the properties of $\Phi_{T,p}$ with the possibility to be similar to a scalar multiple of an isometry.

COROLLARY 4.10. *Let $T \in B(H)$ be a operator which is compatible with a regular gauge p. Then the following conditions are equivalent:*

(i) *there exists $\rho > 0$ such that $(1-r^2)\int_0^{2\pi} \|\Phi_{T,p}(r\mathrm{e}^{\mathrm{i}t})x\|^2 \mathrm{d}m(t) \geqslant \rho\|x\|^2$ for any $x \in H$;*

(ii) *the operator T is similar to some scalar multiple of an isometry in $B(H)$.*

In particular, when $T \in PWB(H)$, we get that T is similar to an isometry in $B(H)$ if and only if there exists $\rho > 0$ such that

$$(1-r^2)\int_0^{2\pi} \|(I - r\mathrm{e}^{-\mathrm{i}t}T)^{-1}x\|^2 \mathrm{d}m(t) \geqslant \rho\|x\|^2$$

for any $x \in H$.

Proof. (i)⇒(ii) Considering any weak limit point of the operators X_r in Proposition 2.10, we see that there exists a generalized T-Toeplitz operator X such that

$$\langle Xx \mid x\rangle \geqslant \rho\|x\|^2$$

for any $x \in H$. Hence, X is invertible and satisfies the equation $T^*XT = c_p^2 X$. Clearly, we obtain that the operator $c_p^{-1}X^{-(1/2)}RX^{1/2}$ is an isometry in $B(H)$.

(i)⇒(ii) Conversely, assume that $T = \alpha AUA^{-1}$ where A is an invertible operator, U is an isometry in $B(H)$ and α is a non zero complex number. We first note that we have necessarily $\alpha = r(T)\mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in \mathbb{R}$. Therefore, without lost of generality we may assume that $\alpha = r(T)$. For any $x \in H$ we have

$$\begin{aligned}
&(1-r^2)\int_0^{2\pi} \|\Phi_{T,p}(r\mathrm{e}^{\mathrm{i}t})x\|^2 \mathrm{d}m(t) \\
&\qquad = (1-r^2)\sum_{n=0}^{+\infty} r^{2n}p(n)^{-2}\langle T^{*n}T^n x \mid x\rangle \\
&\qquad = (1-r^2)\sum_{n=0}^{+\infty} r^{2n}p(n)^{-2}\langle r(T)^{2n}A^{*-1}U^{*n}A^*AU^nA^{-1}x \mid x\rangle \qquad (4.3)\\
&\qquad \geqslant \|A\|^2\|A^{-1}\|^2(1-r^2)\|x\|^2\sum_{n=0}^{+\infty} r^{2n}p(n)^{-2}r(T)^{2n}.
\end{aligned}$$

Since T is compatible with the regular gauge p, we know that we have necessarily $r(T) = c_p$. Now applying Theorem 2.7, we see that there exists r_0 such that

$$(1 - r^2) \sum_{n=0}^{+\infty} r^{2n} p(n)^{-2} r(T)^{2n} \geqslant \frac{1}{2}$$

for any $r \geqslant r_0$. With (4.3) we get easily the desired condition. This ends the proof of the proposition. ∎

4.4. ABEL TYPE SUMMABILITY AND AN OPERATOR KERNEL FOR $C_{0,\cdot}$ OPERATORS OF CLASS C_ρ. Let T be an operator dominated by a regular gauge with its spectrum included in the closed unit disc. For any r with $0 \leqslant r < 1$ and $t \geqslant 0$, we consider the operator kernel

$$K^{0,\cdot}_{r,t}(T) = (I - r\mathrm{e}^{\mathrm{i}t} T^*)^{-1}(I - T^*T)(I - r\mathrm{e}^{-\mathrm{i}t} T)^{-1}.$$

We denote by $E_{0,\cdot}(T)$ the subspace of H where the iterates of T converge to 0. When T is of class C_ρ, $\rho > 0$, so that it has a unitary ρ dilation on some Hilbert space $\mathcal{H} \supseteq H$, then there exists a sesquilinear map $(x, y) \to \mu_{x,y}$, from $H \times H$ into the Banach space $ca(\mathbb{T})$ of all complex measures on $\mathbb{T}$, which gives spectral scalar measures for T (with $|\mu_{x,y}| \leqslant (2\rho - 1)\|x\|\|y\|$) that satisfy

$$\langle p(T)x \mid y\rangle = \int_0^{2\pi} p(\mathrm{e}^{\mathrm{i}t}) \mathrm{d}\mu_{x,y}(t) \tag{4.4}$$

for any $x, y \in H$ and any polynomial p. If T is absolutely continuous, then there exists a functional calculus $\Phi_T : H^\infty \to B(H)$, $\Phi_T(f) = f(T)$, $f \in H^\infty$, which extends the polynomial functional calculus. The map Φ_T is a weak*-continuous algebra homomorphism (norm-decreasing when T is an absolutely continuous contraction).

COROLLARY 4.11. *Let T be an operator belonging to a finite von Neumann algebra. Assume that T is dominated by a regular gauge p with its spectrum included in the closed unit disc. Then there exists a unique canonical generalized Toeplitz operator S_T,* $\ker S_T = E_{0,\cdot}(T)$, *and the integrals*

$$(1 - r^2) \int_0^{2\pi} \langle \Phi_{T,p}(r\mathrm{e}^{\mathrm{i}t})x \mid \Phi_{T,p}(r\mathrm{e}^{\mathrm{i}t})y\rangle \mathrm{d}m(t)$$

almost converge to $\langle S_T x \mid y\rangle$ for any $(x, y) \in H^2$. In particular, when $T \in PWB(H)$, we have

$$\lim_{r\to 1}(1 - r^2) \int_0^{2\pi} \|(I - r\mathrm{e}^{-\mathrm{i}t} T)^{-1} x\|^2 \mathrm{d}m(t) = \langle S_T x \mid x\rangle$$

for any $x \in H$.

Proof. (i) We define the functional ϕ_n, $n \in \mathbb{N}$, acting on the finite von Neumann algebra M generated by T as follows

$$\phi_n(X) = T^{*n} X T^n$$

for any $X \in M$. Since p is regular, the sequence $c_p^{2n} p(n)^{-2}$ is almost convergent to 1. By Theorem 2.4 (iii) of [8], we obtain $\phi_n(X)p(n)^{-2}$ is weakly almost convergent for any $X \in M$. In particular, it implies that $\phi_n(I)p(n)^{-2}$ is weakly almost convergent to an operator S_T. We immediately deduce that $\tau_p(T) = \{S_T\}$. Applying Proposition 2.10, we get the other desired results. ∎

COROLLARY 4.12. *Let $T \in B(H)$ be an absolutely continuous operator of class C_ρ. For any $(x, y) \in E_{0,\cdot}(T)^2$ the one parameter family of functions $t \to \langle K^{0,\cdot}_{r,t}(T)x \mid y\rangle$ converge in $L^1(\mathbb{T}, \mathrm{d}m)$ to the function $t \to \mathrm{d}\mu_{x,y}/\mathrm{d}m$ when r goes to 1. In particular, when T is of class $C_{0,\cdot}$ we can use the kernel $K^{0,\cdot}_{r,t}(T)$ instead of the kernel $K_{r,t}(T)$.*

Proof. Since T is of class C_ρ, we have $K_{r,t}(T) + (\rho - 1)I \geqslant 0$ ([10]). Using the properties of radial limits of positive harmonic functions (see for instance [26]) and (4.4) we see that

$$\lim_{r\to 1} \int_0^{2\pi} \Big| \frac{\mathrm{d}\mu_{x,y}}{\mathrm{d}m}(t) - \langle K_{r,t}(T)x \mid y\rangle \Big| \mathrm{d}m(t) = 0.$$

Thus, it remains to prove that

$$\lim_{r\to 1} \int_0^{2\pi} |\langle K^{0,\cdot}_{r,t}(T)x \mid y\rangle - \langle K_{r,t}(T)x \mid y\rangle| \mathrm{d}m(t) = 0 \tag{4.5}$$

for any $(x, y) \in E_{0,\cdot}(T)^2$. Now, observe that

$$\begin{aligned}
&\int_0^{2\pi} |\langle K^{0,\cdot}_{r,t}(T)x \mid y\rangle - \langle K_{r,t}(T)x \mid y\rangle| \mathrm{d}m(t)\\
&\quad = (1 - r^2) \int_0^{2\pi} |\langle (I - r\mathrm{e}^{\mathrm{i}t} T^*)^{-1} T^* T (I - r\mathrm{e}^{-\mathrm{i}t} T)^{-1} x \mid y\rangle| \mathrm{d}m(t)\\
&\quad \leqslant (1 - r^2) \sqrt{\int_0^{2\pi} \|(I - r\mathrm{e}^{-\mathrm{i}t} T)^{-1} T x\|^2 \mathrm{d}m(t)} \sqrt{\int_0^{2\pi} \|(I - r\mathrm{e}^{-\mathrm{i}t} T)^{-1} T y\|^2 \mathrm{d}m(t)}. \qquad (4.6)
\end{aligned}$$

By Proposition 2.10, we know that every point limit of

$$(1 - r^2) \int_0^{2\pi} \|(I - r\mathrm{e}^{-\mathrm{i}t} T)^{-1} T x\|^2 \mathrm{d}m(t)$$

is of the form $\langle Xx \mid x\rangle$ where X satisfies the equation $T^*XT = X$. Consequently, we have

$$|\langle Xx \mid x\rangle| \leqslant \|X\|\|T^n x\|^2.$$

Since T is of class C_ρ, the sequence $\|T^n x\|$ is convergent ([18]) and the limit is necessarily 0 because $x \in E_{0,\cdot}(T)$. Combining this fact with (4.6), we obtain the desired property (4.5). The rest of the proof is then an immediate consequence. ∎

REMARK 4.13. When T is a contraction the spectral scalar measure is positive, that is $\mu_{x,x}$ is a positive measure for any $x \in H$.

5. OPERATORS OF FOGUEL TYPE AND OPERATORS IN THE CLASS $C_{1,\cdot}$

The remainder of this section is devoted to the study of the iterates of the operators of Foguel type. First we study the particular case where the restriction of T^* is similar to an isometry giving a similarity result. Second we state some properties which are of inner interest and which exceed the context of operator of Foguel type. If $T \in PB(H)$ and $E \in \mathrm{Lat}(T)$ we study the asymptotic and spectral properties of the components of T with respect to the orthogonal sum $H = E \oplus E^\perp$. Third, we show how to use the informations we obtain in order to study the operators of Foguel type and we attempt to motiving that the notions we have introduced are, in a sense, the best adapted to our problem.

5.1. SIMILARITY. The next result precise the similarity when the restriction of T^* is similar to an isometry.

PROPOSITION 5.1. *Let $T \in PWB(H)$ and $E \in \mathrm{Lat}(T)$ such that $T \mid E$ is similar to an isometry and $T^* \mid E^\perp$ is similar to a coisometry. Then the operator T is similar to an isometry.*

REMARK 5.2. Proposition 5.1 implies in particular that Foguel operators of type (III) are similar to isometries. Notice that the adjoint of an operator of type (II) is in fact conjugate by means of the involution $J : x_1 \oplus x_2 \to x_2 \oplus x_1$ to an operator of type (III). Thus, we deduce that Foguel operators of type (II) are similar to coisometries.

Proof of Proposition 5.1. Once more, with respect to the orthogonal decomposition $H = E \oplus E^\perp$ the operator T can be decomposed under the following form:

$$T = \begin{bmatrix} T_1 & R \\ 0 & T_2 \end{bmatrix}.$$

Given a Banach limit $\mathcal{L}$, we consider the T-Toeplitz operator $X = E^{p^2}_{\mathcal{L},T^*,T}(I)$. Since T_1 is similar to an isometry the associated canonical Toeplitz operator $X_1 =$

$E^{p^2}_{\mathcal{L},T_1^*,T_1}(I)$ is invertible. Then the operator $J = X_1^{-(1/2)} \oplus I$ is well defined. Under the notations of Section 3, we obtain

$$J^{-1}XJ = \begin{bmatrix} I & A \\ A^* & B \end{bmatrix}.$$

It follows from Theorem 3.3, that the operator $B - E^{p^2}_{\mathcal{L},T_2^*,T_2}(I)$ is positive. Moreover, since the operator T_2 is similar to an isometry, the T_2 Toeplitz $E^{p^2}_{\mathcal{L},T_2^*,T_2}(I)$ is positive and invertible. An immediate consequence is that the operator $B - A^*A$ is invertible. Therefore the operator X is invertible and moreover we get an explicit formula for the inverse of X, namely:

$$X^{-1} = \begin{pmatrix} X_1^{-1} + X_1^{-(1/2)}A(B - A^*A)A^*X_1^{-(1/2)} & -X_1^{-(1/2)}A(B - A^*A) \\ -(B - A^*A)A^*X_1^{-(1/2)} & (B - A^*A)^{-1} \end{pmatrix}.$$

Since X is a T-Toeplitz operator, the operator $W = X^{1/2}TX^{-(1/2)}$ is an isometry. ∎

5.2. LINKS WITH THE OPERATORS OF FOGUEL TYPE. As it was already observed the operators which are the analogous of the operators of Foguel type and which can produce operators in class $C_{1,\cdot}$ are operators of type (III) and (IV). The operator of type (III) are similar to isometries as shown in the previous section. Therefore we will concentrate our study to the operators of type (IV) which belong to $PWB(H)$. This study reveals that such operators are really relevant. Indeed, we prove that those operators can be expressed by means of an operator valued function.

We first study the membership of the operator of type (IV) to the set of power bounded operators. First notice that for operator of type (IV), the operator $L_T(X)$ (respectively $L'_T(Y)$) which appears in Remark 2.12 is uniquely determined. Moreover, T is uniquely determined by R in this case, therefore we will denote this operator defined on $H^2(F)$ by $L(R)$ (respectively $L'(R)$); observe that the link between $L(R)$ and $L'(R)$ is given by the relation $L'(R) = L(R^*)^*$. Conversely, given an operator L, consider the operator of type (IV) which is associated with L, taking for R the operator $L - S^*LS^*$. It follows that:

$$T = \begin{bmatrix} S^n & R_n \\ 0 & S^{*n} \end{bmatrix}$$

where R_n is defined by the formula $R_n = S^{n-1}L(R) - S^*L(R)S^{*n-1} + \sum\limits_{i=1}^{n-1} S^{i-1} \cdot PL(R)S^{*n-i}$. If Y_n denotes the operator $\sum\limits_{i=0}^{n} S^iPLS^{*n-i}$, an easy calculus yields:

$$Y_n^*Y_n = \sum_{k=0}^{n} S^kL(R)^*PL(R)S^{*k} \tag{5.1}$$

where P denotes the orthogonal projection on $\ker(S^*)$. It may be seen that T is a power-bounded operator if and only if the above series is weak convergent in $B(H)$, then it implies its strong convergence. Now observe that

$$(B - A^*A) - S(B - A^*A)S^* = L(R)^*PL(R) = JJ^*$$

with $J \in B(H, F)$. Theorem 1.14 in [45] enables us to factorize the operator $B - A^*A = ZZ^*$ where $Z \in \{S\}'$. Since Z commutes with S, it can be identified with an operator field of $H^\infty(D, B(F))$. In our case Z can be explicitly defined (up to an isometry of $B(F)$) by $Z = \sum\limits_{k=0}^{+\infty} S^k L(R)^* P S^{*k}$. If $(\varepsilon_n)_{n\in\Lambda}$ is an orthonormal basis of F, we can write, a.e.

$$Z = \sum_{n\in\Lambda} (L(R)^*\varepsilon_n) \otimes \varepsilon_n$$

In the same way we obtain:

$$D - CC^* = Z'Z'^* \quad \text{with } Z' = \sum_{k=0}^{+\infty} S^k L(R^*)^* P S^{*k}.$$

If n is a positive integer, set

$$Z_n = \sum_{k=0}^{n} S^k L(R)^* P S^{*k} \text{ and } Z'_n = \sum_{k=0}^{n} S^k L(R^*)^* P S^{*k}.$$

The next proposition summarizes the above statements.

PROPOSITION 5.3. *Let T be an operator of Foguel type* (IV) *and X be a canonical T-Toeplitz operator. Then the operator T is power-bounded if and only if one of the sequences $(Z_n)_{n\geqslant 0}$ or $(Z'_n)_{n\geqslant 0}$ is bounded in $H^\infty(D, B(F))$.*

REMARK 5.4. (i) Similar result occurs in a different context [6].

(ii) The boundedness of the sequence $(Z_n)_{n\geqslant 0}$ is equivalent to the boundedness of the sequence $(Z'_n)_{n\geqslant 0}$.

Let $(\varepsilon_k)_{k\in\Lambda}$ be an orthonormal sequence of F. A simple calculus shows that the series given by (5.1) strongly converges if and only if there exists a constant $M > 0$ and a set $\mathcal{N}$ of Lebesgue measure equal to 0 such that:

$$\Big\| \sum_{k\in\Lambda} \alpha_k L^* \varepsilon_k(z) \Big\| \leqslant M \sqrt{\sum_{k\in\Lambda} |\alpha_k|^2}\ , \quad \forall z \in \mathbb{T}\backslash\mathcal{N}, \forall \alpha = (\alpha_k) \in l^2(\Lambda). \tag{5.2}$$

In the particular case where F is of finite dimension, the inequality (5.2) yields a simple characterization of the membership of T to $PWB(H)$ in terms of L directly.

PROPOSITION 5.5. *If F is of finite dimension, then $T \in PWB(H)$ if and only if the image by L^* of the unit ball of F is a bounded set of $H^\infty(F)$.*

REMARK 5.6. In the particular case where V is the usual shift on H^2, Proposition 5.5 implies that $T \in PWB(H)$ if and only if $L^*(1) \in H^\infty$.

We can now give a complete characterization of the operators of type (IV) which belong to the class $C_{1,\cdot}$ (respectively $C_{\cdot,1}$). For this purpose, we use our last statement and Corollary 4.1.

PROPOSITION 5.7. *Let T be an operator of type* (IV) *whose powers are bounded. Then T belongs to the class $C_{1,\cdot}$ (respectively $C_{\cdot,1}$) if and only if Z (respectively Z') is an outer function and the closure of $Z(\mathrm{e}^{\mathrm{i}t})H^2(F)$ is equal to $H^2(F)$ (respectively the closure of $Z'(\mathrm{e}^{\mathrm{i}t})H^2(F)$ is equal).*

REMARK 5.8. Notice that $B - A^*A = D - CC^*$ if R is selfadjoint. Therefore, in this case, $T \in C_{1,1}$ as soon as $T \in C_{1,\cdot}$.

Proof of Proposition 5.7. The identity $B - A^*A = ZZ^*$ implies that:

$$f \in \ker(B - A^*A) \Leftrightarrow Z(\mathrm{e}^{\mathrm{i}t})^* f(\mathrm{e}^{\mathrm{i}t}) = 0 \quad \text{a.e} \Leftrightarrow f(\mathrm{e}^{\mathrm{i}t}) \in (Z(\mathrm{e}^{\mathrm{i}t})H^2(F))^{\perp}.$$

Using Corollary 4.1, we obtain $T \in C_{1,\cdot}$ if and only if Z is an outer function which verifies $\overline{Z(\mathrm{e}^{\mathrm{i}t})H^2(F)} = H^2(F)$. ∎

5.3. SPECTRAL PROPERTIES OF OPERATORS OF FOGUEL TYPE (IV). This section is devoted to the study of the spectrum of operators of Foguel type (IV). If S is the usual shift on $H^2(F)$ and if λ is a point of the open unit disc D, we denote by E_λ the kernel of $S^* - \lambda I$ and by P_λ the orthogonal projection of H onto E_λ.

PROPOSITION 5.9. *Let T be an operator of type* (IV)*. The spectrum of T, $\sigma(T)$, is contained in the unit disc. Moreover, a point λ of the open unit disc belongs to $\sigma(T)$ if and only if $P_\lambda R P_\lambda \notin GL(E_{\overline{\lambda}}, E_\lambda)$.*

The proof of Proposition 5.9 relies essentially on the next lemma. Let $E \in \mathrm{Lat}(T)$ and let

$$T = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \tag{5.3}$$

be the matrix of T with respect to the orthogonal decomposition $H = E \oplus E^{\perp}$. It is of interest to clarify the links between $\sigma(T)$ and $\sigma(A)$, $\sigma(B)$. Let us recall the following notations:

(i) We will use the symbol A_{l} to denote the *Moore-Penrose left inverse* of A when of course it exists.

(ii) We will use the symbol B_{r} to denote the *Moore-Penrose right inverse* of B when of course it exists.

LEMMA 5.10. *Let T be an operator of type* (5.3)*. Then T is invertible if and only if:*

(i) *the operator A^* is surjective;*
(ii) *the operator B is surjective;*
(iii) $\dim\ker A^* = \dim\ker B = d$*.*

Moreover, if $d \neq 0$, the compression Q_1XQ_2 of X, defined on $\ker B$ *and whose image is a subset of* $\ker A^*$, *must be invertible. Finally if R denotes the operator defined in an obvious way by $R = 0$ if $d = 0$ and $R = [Q_1XQ_2]^{-1}$ if $d \neq 0$, the inverse of T is given by the matrix*

$$T^{-1} = \begin{bmatrix} A_l - A_lXR & A_lXRXB_r - A_lXB_r \\ R & B_r - RXB_r \end{bmatrix}.$$

Proof. We can derive the first part of Lemma 5.10 from [25], the second part is left to the reader (for more details see [7]). ∎

REMARK 5.11. In the particular case where T is an operator of Foguel of type (IV) with S the usual shift on H^2, it may be seen that

$$P_\lambda RP_\lambda = \overline{h(\lambda)}(1 - |\lambda|^2)(1 - \lambda z)^{-1} \otimes (1 - \overline{\lambda}z)^{-1}$$

with $h = L^*(1)$. In this context, Proposition 5.9 implies that a point λ of the open unit disc belongs to the spectrum of T if and only if $h(\lambda) = 0$. Once more, we see that the function $h = L^*(1)$ parameterizes the operators of Foguel type (IV) (see also Remark 5.6).

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