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THE FINE STRUCTURE OF THE KASPAROV GROUPS. III: RELATIVE QUASIDIAGONALITY

CLAUDE L. SCHOCHET

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ABSTRACT. In this paper we identify $QD(A, B)$, the quasidiagonal classes in $KK_1(A, B)$, in terms of $K_*(A)$ and $K_*(B)$, and we use these results in various applications. Here is our central result:

Let $\widetilde{\mathcal{N}}$ denote the category of separable nuclear C^*-algebras which satisfy the Universal Coefficient Theorem. Suppose that $A \in \widetilde{\mathcal{N}}$ and A is quasidiagonal relative to B. Then there is a natural isomorphism

$$QD(A, B) \cong \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0.$$

Thus, for $A \in \widetilde{\mathcal{N}}$ quasidiagonality of KK-classes is indeed a topological invariant.

KEYWORDS: *Kasparov KK-groups, quasidiagonality, relative quasidiagonality, Universal Coefficient Theorem,* Pext.

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1. INTRODUCTION: QUASIDIAGONALITY AND *KK*-THEORY

This is the third in a series of papers in which the topological structure of the Kasparov groups, systematically studied first by Salinas, is developed and put to use. The first two papers [21] and [22] are devoted to general structural results and serve as the theoretical background for the present work, which centers about quasidiagonality. From the point of view of [21] and [22], this is an exploration of the closure of zero in the Kasparov groups, which we have termed the *fine structure* subgroup.

Quasidiagonality was defined by P.R. Halmos [12] in 1970. A bounded operator on Hilbert space is *quasidiagonal* if it is a compact perturbation of a block-diagonal operator. This soon was generalized to C^*-algebras. Quasidiagonality is thus a finite dimensional approximation property. It is not well understood.

L.G. Brown, R.G. Douglas, and P.A. Fillmore ([6]) first recognized that the study of quasidiagonality for operators and for C^*-algebras might be approached by topological methods. They topologized their functor $\operatorname{Ext}(X)$ (which is known now to be isomorphic to the Kasparov group $KK_1(C(X), \mathbb{C})$) and announced that the closure of zero corresponded to the quasidiagonal extensions. L.G. Brown pursued this theme, particularly in [4] (cf. Section 6 and Section 7).

Salinas ([15]) studied the topology on the Kasparov group $KK_1(A, B)$ and showed that this topology is related to relative quasidiagonality. The quasidiagonal classes $QD(A, B)$ (defined precisely in Section 2) constitute a certain subgroup of KK-theory:

$$QD(A, B) \subseteq KK_1(A, B).$$

If A is in the category $\widetilde{\mathcal{N}}$ of separable nuclear C^*-algebras which satisfy the Universal Coefficient Theorem, then more can be said. The UCT is a natural short exact sequence

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK_*(A, B) \xrightarrow{\gamma} \operatorname{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

which splits unnaturally and thus computes $KK_*(A, B)$ in terms of $K_*(A)$ and $K_*(B)$. In particular, it identifies a canonical subgroup of $KK_*(A, B)$, namely

$$\operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \overset{\delta}{\cong} \operatorname{Ker}(\gamma) \hookrightarrow KK_*(A, B).$$

Henceforth we generally suppress mention of the map δ. (Note that the map δ has degree one and so the elements of degree zero in the group $\operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$, denoted by $\operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$, are contained in $KK_1(A, B)$.)

Salinas has shown in 5.1 of [15], that if A is quasidiagonal relative to B then

$$QD(A, B) \subseteq \operatorname{Ker}(\gamma)$$

and in fact (by Theorem 5.2(a) of [15]) as reformulated by M. Dădârlat (private communication) that if $A \in \widetilde{\mathcal{N}}$ so that the UCT holds and identifies

$$\operatorname{Ker}(\gamma) \cong \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$$

then

$$QD(A, B) \subseteq \operatorname{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$$

where $\operatorname{Pext}^1_{\mathbb{Z}}(G, H)$ is the subgroup of $\operatorname{Ext}^1_{\mathbb{Z}}(G, H)$ consisting of pure extensions. (A subgroup $H \subseteq J$ is said to be *pure* if for each $n \in \mathbb{N}$,

$$nH = H \cap nJ$$

and an extension of abelian groups

$$0 \to H \to J \to G \to 0$$

is said to be *pure* if H is a pure subgroup of J. If H is a direct summand of J then H is a pure subgroup, but the most interesting cases involve non-split pure extensions. For example, tJ, the torsion subgroup of J, is always a pure subgroup of J but it is not necessarily a direct summand of J: see Section 53 of [11], and [23].)

A topological space is *polonais* if it is separable, complete, and metric. If it is a topological group then we insist that the metric be invariant. A *pseudopolonais* group is a separable topological group with invariant pseudometric whose Hausdorff quotient group is polonais.

In the first two papers in this series ([21],[22]) we demonstrated the following facts:

1.1. There is a natural structure of a pseudopolonais group on $KK_*(A, B)$ ([21], 6.2).

1.2. The Kasparov pairing is jointly continuous with respect to this topology, provided that all C^*-algebras which appear in the first variable are K-nuclear ([21], 6.8).

1.3. If $K_*(A)$ is finitely generated then $KK_*(A, B)$ is polonais ([21], 6.2).

1.4. The index map

$$\gamma : KK_*(A, B) \to \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B))$$

is continuous. If $\mathrm{Im}(\gamma)$ is closed (e.g., if γ is onto), then γ is an open map. If γ is an algebraic isomorphism then it is an isomorphism of topological groups ([21], 7.4).

1.5. The Universal Coefficient Theorem short exact sequence is a sequence of pseudopolonais groups and each of the splittings of the UCT constructed in [14] is a topological splitting ([22], 4.5).

1.6. If $A \in \widetilde{\mathcal{N}}$ then there is a natural isomorphism

$$Z_*(A, B) \cong \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$$

where $Z_*(A, B)$ denotes the closure of zero in the group $KK_*(A, B)$ ([22], 3.3).

1.7. If $A \in \widetilde{\mathcal{N}}$ then it has an associated KK-filtration diagram for (A, B) which is functorial into the category of pseudopolonais groups [22]. (We proved 1.6 and 1.7 in [22] for $A \in \mathcal{N}$. However, the results hold for $A \in \widetilde{\mathcal{N}}$ as may easily be seen by inspection.) In particular, the Milnor and Jensen sequences take values in this category and are natural with respect to both A and B.

The following theorem is the most important result in this paper; all of our applications flow from it. Salinas ([15]) proved the theorem under certain stringent assumptions on A and $K_*(A)$ which we have removed.

THEOREM 2.3. *Suppose that $A \in \widetilde{\mathcal{N}}$ and A is quasidiagonal relative to B. Then there is a natural isomorphism of topological groups*

$$QD(A, B) \cong \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$$

regarded as topological subgroups of $KK_*(A, B)$ via the canonical inclusion δ in the UCT.

Note as an immediate consequence of this theorem that quasidiagonality of extensions is a topological invariant for $A \in \widetilde{\mathcal{N}}$, answering the relative form of a question of D. Voiculescu ([24], [25]). For instance (see Theorem 3.5), if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B\otimes\mathcal{K} & \longrightarrow & E_\tau & \longrightarrow & A & \longrightarrow & 0 \\ & & \Big\downarrow 1 & & \Big\downarrow e_\tau & & \Big\downarrow \tau & & \\ 0 & \longrightarrow & B\otimes\mathcal{K} & \longrightarrow & .\mathcal{L}(H_B) & \longrightarrow & \mathcal{Q}(H_B) & \longrightarrow & 0 \end{array}$$

is an essential extension classified by τ such that $A \in \widetilde{\mathcal{N}}$ and A is quasidiagonal relative to B, then $e_\tau(E_\tau)$ is B-quasidiagonal if and only if both of the following topological conditions hold:

(1) $\gamma(\tau) = 0 : K_*(A) \to K_*(B)$; and
(2) $\tau \in \bigcap_n n\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0 = \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$.

If $K_*(A)$ is torsionfree then condition (2) is automatically satisfied, so $e_\tau(E_\tau)$ is B-quasidiagonal if and only if $\gamma(\tau) = 0$. Thus when $A \in \widetilde{\mathcal{N}}$ with $K_*(A)$ torsionfree, the index invariant γ is a complete obstruction to relative quasidiagonality.

The remainder of the paper is organized as follows.

In Section 2 the definitions of quasidiagonality are recalled and Theorem 2.3 is established.

Theorem 2.3 has several corollaries which are developed in Section 3. Here is one. Suppose that $f : A \to B$ so that

$$[f] \in KK_0(A,B) \xrightarrow{\beta^A} KK_1(SA,B)$$

where β^A denotes the Bott periodicity isomorphism in the first variable. When is $\beta^A([f])$ a quasidiagonal class? It is easy to show that the following are necessary conditions:

(1) The induced homomorphism $f_* : K_*(A) \to K_*(B)$ is trivial; and,
(2) The associated short exact sequence

$$0 \to K_*(SB) \to K_*(Cf) \to K_*(A) \to 0$$

is pure exact, where Cf denotes the mapping cone of f.

It is shown that (for $A \in \widetilde{\mathcal{N}}$) these conditions are also sufficient. If $K_*(A)$ is torsionfree then (2) is automatic, and so $\beta^A([f])$ is a quasidiagonal class if and only if $f_* = 0$. The section concludes with Theorem 3.5, which demonstrates the topological nature of relative quasidiagonality in a concrete manner.

The remaining four sections are devoted to applications.

Section 4 is devoted to the application of some of the theory of infinite abelian groups to obtain results on quasidiagonality.

In Section 5 we answer a question raised by L.G. Brown ([4]) concerning the relation between quasidiagonality and the kernel of the map

$$\theta^* : \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A)_t, K_*(B)),$$

induced by the natural inclusion of $K_*(A)_t$, the torsion subgroup of A, into $K_*(A)$.

Section 6 deals with another result of L.G. Brown. Brown constructed ([4]) an operator T which was not quasidiagonal but such that $T \oplus T$ was quasidiagonal. In Section 7 we analyze such phenomena.

Section 7 presents a converse to a theorem of Davidson, Herrero, and Salinas which deals with conditions under which the quasidiagonality of $A/\mathcal{K}$ implies the quasidiagonality of A.

We require nuclearity for two reasons. First, we need at least K-nuclearity so that the Kasparov product will be continuous (cf. 1.2). Second, we apparently need nuclearity in order to satisfy the hypotheses of Salinas's result identifying the quasidiagonal elements with the closure of zero in the Kasparov groups (cf. 2.2). It seems possible that if one restricts attention to extensions that have additive inverses so that the identification

$$\mathrm{Ext}_*(A,B) \;\cong\; KK_*(A,B)$$

holds, then his result might generalize to the K-nuclear setting.

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In this paper all C^*-algebras are assumed separable with the obvious exceptions of multiplier algebras $\mathcal{M}(B \otimes \mathcal{K})$ and their quotients. Whenever we speak of quasidiagonal classes in $KK_1(A,B)$ it is understood that B is separable, $B \otimes \mathcal{K}$ has a countable approximate unit consisting of projections, (Dădârlat calls this property *stably unital*. Note that if B is unital and $\{p_i\} \subset \mathcal{K}$ is a countable approximate unit for $\mathcal{K}$ consisting of projections, then $\{1 \otimes p_i\}$ is a countable approximate unit for $B \otimes \mathcal{K}$ consisting of projections. Thus if B is unital then $B \otimes \mathcal{K}$ is stably unital.) and A is quasidiagonal relative to B.

An isomorphism of topological groups is an algebraic isomorphism which is a homeomorphism of topological spaces. We use the topologists' notation for graded abelian groups. For example,

$$\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0 = \mathrm{Pext}^1_{\mathbb{Z}}(K_0(A), K_0(B)) \oplus \mathrm{Pext}^1_{\mathbb{Z}}(K_1(A), K_1(B)).$$

2. QUASIDIAGONALITY

Our description of the various definitions of quasidiagonality leans heavily upon the remarkable survey paper of Nathaniel P. Brown ([7]) and upon the paper of Marius Dădârlat ([9]). We are most grateful to them for clarifying these issues.

Halmos ([12]) introduced the notion of a quasidiagonal operator. A bounded linear operator on a separable Hilbert space $T \in \mathcal{L}(H)$ is a *block diagonal* operator if there exists a countable approximate unit consisting of projections, that is, an increasing sequence of finite rank projections $P_1 \leqslant P_2 \leqslant P_3 \cdots$ converging to the identity in the strong operator topology, which is central with respect to T:

$$P_nT - TP_n = 0 \quad \forall\, n.$$

An operator $T \in \mathcal{L}(H)$ is *quasidiagonal* if there exists a countable approximate unit consisting of projections $\{P_n\}$ which is quasicentral with respect to T:

$$\lim_{n\to\infty} \|P_nT - TP_n\| = 0.$$

The sum of a compact operator and a block diagonal operator is quasidiagonal, and Halmos proved that in fact every quasidiagonal operator has this form.

The concept extends to C^*-algebras as follows. Suppose that B is a separable C^*-algebra. Let $H_B = B \otimes H$. We write

$$\mathcal{L}(H_B) = \mathcal{M}(B \otimes \mathcal{K}) \quad \text{and} \quad \mathcal{Q}(H_B) = \mathcal{M}(B \otimes \mathcal{K})/B \otimes \mathcal{K}.$$

A separable subset $E \subset \mathcal{L}(H_B)$ is called a *B-quasidiagonal set* if there exists a countable approximate unit $\{p_n\}$ of $B \otimes \mathcal{K}$ consisting of projections $p_1 \leqslant p_2 \leqslant p_3 \cdots$ which is quasicentral with respect to each $a \in E$:

$$\lim_{n\to\infty} \|p_na - ap_n\| = 0 \quad \forall\, a \in E.$$

This definition is not correct if E is not separable; see [7] for the correct definition. Dădârlat calls this "quasidiagonal", but we prefer to keep track of B.

A representation $\rho : A \to \mathcal{L}(H_B)$ of a separable C^*-algebra A is said to be a *B-quasidiagonal representation* if the set $\rho(A)$ is a B-quasidiagonal set. If a separable C^*-algebra A has a faithful essential (that is, the induced homomorphism $A \xrightarrow{\rho} \mathcal{L}(H_B) \to \mathcal{Q}(H_B)$ is faithful, or, equivalently, $\rho(A) \cap (B \otimes \mathcal{K}) = \{0\}$) and absorbing B-quasidiagonal representation, then A is said to be a *B-quasidiagonal* C^**-algebra*.

Note that a set is $\mathbb{C}$-quasidiagonal if and only if there exists a countable approximate unit consisting of projections in $\mathcal{K}$ such that each operator in the set is quasicentral with respect to the countable approximate unit. In other words, each operator must be quasidiagonal in the classical sense, and there must be a countable approximate unit consisting of projections which works for every operator. We write *quasidiagonal* rather than $\mathbb{C}$*-quasidiagonal*.

Salinas ([15], 4.3) shows that if A is quasidiagonal (say via a quasidiagonal representation ρ) and B has a countable approximate unit consisting of projections then A is B-quasidiagonal, since we may easily construct the requisite countable approximate unit in $B \otimes \mathcal{K}$ which is quasicentral with respect to the representation

$$A \xrightarrow{\rho} \mathcal{L}(H) \cong \mathcal{M}(\mathcal{K}) \to \mathcal{M}(B \otimes \mathcal{K}) \cong \mathcal{L}(H_B).$$

Every commutative C^*-algebra is quasidiagonal — this is a consequence of the spectral theorem. It is easy to show that any AF algebra is also quasidiagonal. As subalgebras of quasidiagonal algebras are obviously quasidiagonal, it follows that any C^*-algebra which embeds in an AF algebra is itself quasidiagonal. For example, this implies that the irrational rotation C^*-algebras are quasidiagonal.

The unitalization of a quasidiagonal C^*-algebra is quasidiagonal, as is the product and minimal tensor product of quasidiagonal C^*-algebras. Quasidiagonality does *not* pass to quotients in general. We return to this point in Section 7.

Any quasidiagonal Fredholm operator must have Fredholm index zero. Halmos used this fact to show that the unilateral shift is not quasidiagonal. Then any C^*-algebra containing the shift cannot be quasidiagonal. More generally, quasidiagonal C^*-algebras must be stably finite — they and matrix rings over them may not contain proper isometries.

Voiculescu proved ([25]) that if A and B are homotopy equivalent C^*-algebras then if one of them is quasidiagonal then the other must be as well. Thus for instance CA, the cone on any C^*-algebra, is quasidiagonal, being homotopy equivalent to 0, and the suspension SA is quasidiagonal, as it is a subalgebra of CA. Salinas extended this to the B-quasidiagonality setting. We state his result formally as part of Proposition 2.1.

Blackadar and Kirchberg ([3], [4]) introduce the class of NF algebras and demonstrate that this class coincides with the class of separable nuclear quasidiagonal C^*-algebras.

Suppose that an injection

$$\tau : A \to \mathcal{Q}(H_B)$$

classifies an extension of C^*-algebras. Taking pullbacks yields the corresponding extension

$$0 \to B \otimes \mathcal{K} \to E_\tau \to A \to 0$$

together with the canonical faithful representation

$$e_\tau : E_\tau \to \mathcal{L}(H_B).$$

We say that this extension is a *quasidiagonal extension* if e_τ is a B-quasidiagonal representation. Equivalently, the extension is quasidiagonal if there is an approximate unit consisting of projections in $B \otimes \mathcal{K}$ which is quasicentral in E_τ. Salinas shows that this property depends only upon the equivalence class

$$[\tau] \in KK_1(A, B)$$

and hence it makes sense to speak of a *quasidiagonal class* in $KK_1(A, B)$.

Suppose given an essential extension

$$0 \to B \otimes \mathcal{K} \to E_\tau \to A \to 0$$

together with the canonical faithful representation $e_\tau : E_\tau \to \mathcal{L}(H_B)$, and suppose given a $*$-homomorphism $f : A' \to A$ which is an injection. Then the induced extension $f^*\tau$ is obtained by the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B\otimes\mathcal{K} & \longrightarrow & E_{f^*\tau} & \longrightarrow & A' & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow{\tilde f} & & \Big\downarrow{f} & & \\ 0 & \longrightarrow & B\otimes\mathcal{K} & \longrightarrow & E_\tau & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with $\tilde{f}$ also an injection. Then $e_{f^*\tau} : E_{f^*\tau} \to \mathcal{L}(H_B)$ is given by the composition

$$E_{f^*\tau} \xrightarrow{\tilde{f}} E_\tau \xrightarrow{e_\tau} \mathcal{L}(H_B).$$

If e_τ is a B-quasidiagonal representation then it is clear from the construction that the canonical faithful representation $e_{f^*\tau}$ is also B-quasidiagonal. Using Salinas's result (last part of Proposition 2.1) we can see that it is not necessary to stipulate that f be an injection, once it is established that the homomorphism

$$f^* : KK_*(A,B) \to KK_*(A',B)$$

is continuous.

A separable nuclear C^*-algebra A is said to be *quasidiagonal relative to B* if the class of the trivial extension $0 \in KK_1(A,B)$ is quasidiagonal. If A is quasidiagonal relative to B then trivial extensions are quasidiagonal, obviously, but there may be no other quasidiagonal classes. For example, if $K_*(A)$ is a direct sum of cyclic groups then Theorem 2.3 implies that every quasidiagonal extension is trivial. We discuss such matters in some length in later sections.

N. Brown remarks ([7], 8.2) that it is possible to have an extension

$$0 \to B\otimes\mathcal{K} \to E \to A \to 0$$

with E quasidiagonal relative to B without either B or A being quasidiagonal. However, given a separable C^*-algebra $D \subset \mathcal{L}(H)$ then D is a quasidiagonal set if and only if the extension

$$0 \to \mathcal{K} \to C^*\{D,\mathcal{K}\} \to C^*\{D,\mathcal{K}\}/\mathcal{K} \to 0$$

is a quasidiagonal extension, where $C^*\{D,\mathcal{K}\}$ denotes the C^*-algebra generated by D and by $\mathcal{K}$ in $\mathcal{L}(H)$. We expand upon this remark as follows.

PROPOSITION 2.1. *Let A and B be separable nuclear C^*-algebras. Then the following are equivalent:*

(i) *A is quasidiagonal relative to B;*

(ii) *A is B-quasidiagonal.*

If these hold then they also hold for $*$-subalgebras of A. In addition, if A is B-quasidiagonal and A is homotopy equivalent to A' then A' is B-quasidiagonal.

Proof. Suppose first that A is quasidiagonal relative to B. Then there exists an essential extension τ representing $0 \in KK_1(A,B)$ and a commuting classifying diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B\otimes\mathcal{K} & \longrightarrow & E_\tau & \longrightarrow & A & \longrightarrow & 0 \\ & & \big\downarrow 1 & & \big\downarrow e_\tau & & \big\downarrow \tau & & \\ 0 & \longrightarrow & B\otimes\mathcal{K} & \longrightarrow & \mathcal{L}(H_B) & \longrightarrow & \mathcal{Q}(H_B) & \longrightarrow & 0 \end{array}$$

and the map τ lifts to a $*$-homomorphism $\tilde{\tau} : A \to \mathcal{L}(H_B)$ since $[\tau] = 0$. By assumption e_τ is a B-quasidiagonal representation, and hence $e_\tau(E_\tau)$ is a B-quasidiagonal set. The representation $\tilde{\tau}$ has range contained in $e_\tau(E_\tau)$ and hence $\tilde{\tau}(A)$ is a B-quasidiagonal set. Thus $\tilde{\tau}$ is an essential B-quasidiagonal representation of A. So A is B-quasidiagonal.

In the other direction, suppose that A is B-quasidiagonal. Then there exists an essential B-quasidiagonal representation $\tilde{\tau} : A \to \mathcal{L}(H_B)$. Let

$$E = C^*\{\tilde{\tau}(A), B\otimes\mathcal{K}\}$$

and let $e : E \to \mathcal{L}(H_B)$ denote the natural inclusion. Then e is a B-quasidiagonal representation and

$$0 \to B\otimes\mathcal{K} \to E \to A \to 0$$

is an essential quasidiagonal extension which is split by the map $\tilde{\tau}$. Thus A is quasidiagonal relative to B.

If the properties hold then one uses the fact pointed out previously that the restriction of a B-quasidiagonal extension to a subalgebra of A is again a B-quasidiagonal extension. The final statement was established by Salinas in Theorem 4.8 of [15]. ∎

Let $QD(A, B)$ denote the set of quasidiagonal classes of $KK_1(A, B)$. This set is non-empty if and only if A is quasidiagonal relative to B, by Theorem 4.4 of [15]. Salinas has shown that the quasidiagonal elements may be described in terms of the topology on $KK_1(A, B)$. For $B = \mathcal{K}$ the following theorem was established by L.G. Brown in p. 63, Remark 1 of [4].

THEOREM 2.2. ([15], Theorem 4.4) *If A is quasidiagonal relative to B then there is a natural isomorphism*

$$Z_1(A, B) \cong QD(A, B). \tag{2.1}$$

Of course this implies that $QD(A, B)$ is a subgroup of $KK_*(A, B)$.

Here is the principal result of this paper.

THEOREM 2.3. *Suppose that $A \in \widetilde{\mathcal{N}}$, B is a separable C^*-algebra, and A is quasidiagonal relative to B. Then there is a natural isomorphism of topological groups*

$$QD(A, B) \cong \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$$

regarded as subgroups of $KK_1(A, B)$ via the canonical inclusion δ in the UCT.

Proof. We have isomorphisms of topological groups

$$QD(A,B) \;\cong\; Z_1(A,B) \cong \operatorname{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0,$$

by (2.1), and by Theorem 1.6, respectively. This completes the proof. ∎

COROLLARY 2.4. *Suppose that $A \in \widetilde{\mathcal{N}}$ and that A is quasidiagonal. Then*

$$QD(A,\mathcal{K}) \;\cong\; \operatorname{Pext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}).$$

Theorem 2.3 and Corollary 2.4 may be used readily since much is known about computing the Pext groups shown; see [23].

REMARK 2.5. The group $QD(A,B)$ may well have torsion. For instance, suppose that $A \in \widetilde{\mathcal{N}}$, A is quasidiagonal and $K_*(A)$ is torsionfree but not free. Then, since

$$\operatorname{Pext}^1_{\mathbb{Z}}(G,H) \cong \operatorname{Ext}^1_{\mathbb{Z}}(G,H)$$

whenever G is torsionfree,

$$QD(A,\mathcal{K}) \;\cong\; \operatorname{Pext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \;\cong\; \operatorname{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z})$$

which is always an uncountable, divisible group (cf. Theorem 9.7 of [23], due to C.U. Jensen.)

Now choose $A \in \widetilde{\mathcal{N}}$ with $K_0(A) = \mathbb{Z}_p$, the integers localized at p. Then

$$QD(A,\mathcal{K}) \;\cong\; \mathbb{Q}^{\aleph_0} \oplus \mathbb{Z}(p^\infty)$$

where $\mathbb{Z}(p^\infty)$ denotes the p-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$. This point was overlooked in Corollary 5.4 of [15]. In that paper the primary interest was the case with $K_*(B)$ torsionfree and $K_*(A)$ finitely generated. If G is finitely generated then $\operatorname{Pext}^1_{\mathbb{Z}}(G,H) = 0$ for all H, so if $A \in \widetilde{\mathcal{N}}$ with $K_*(A)$ finitely generated then every quasidiagonal extension of the form

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0$$

is trivial in $KK_1(A,B)$.

Recall that a group is *algebraically compact* if it is a direct summand in every group that contains it as a pure subgroup. Equivalently, it is algebraically compact if it is algebraically a direct summand in a group which admits a compact topology. Examples include compact groups, divisible groups, and bounded groups. A group is algebraically compact if and only if it is of the form

$$D \oplus \prod_p D^p$$

where D is divisible and for each prime p the group D^p is the completion in the p-adic topology of the direct sum of cyclic p-groups and groups of p-adic integers. For any sequence of abelian groups $\{G_i\}$ the group $\prod G_i / \oplus G_i$ is algebraically compact and its structure is known.

COROLLARY 2.6. *Suppose that $A \in \widetilde{\mathcal{N}}$, A is quasidiagonal relative to B, and either*

(i) $K_*(A)$ *is a direct sum of (finite and/or infinite) cyclic groups; or*

(ii) $K_*(B)$ *is algebraically compact.*

Then any essential quasidiagonal extension

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0$$

is a trivial extension.

Proof. If G is a direct sum of cyclic groups then $\mathrm{Pext}^1_{\mathbb{Z}}(G,H) = 0$ for all groups H. Dually, if H is algebraically compact then $\mathrm{Pext}^1_{\mathbb{Z}}(G,H) = 0$ for all groups G. The result is then immediate from Theorem 2.3. ∎

COROLLARY 2.7. *Suppose that $A \in \widetilde{\mathcal{N}}$, A is quasidiagonal relative to B, $K_*(A)$ is torsionfree, and we are given an essential extension*

$$\tau : 0 \to B \otimes \mathcal{K} \to E \to A \to 0.$$

Suppose further that the connecting homomorphism

$$K_*(A) \to K_{*-1}(B)$$

is trivial. Then the extension is quasidiagonal. If $K_*(A)$ is a direct sum of (finite and/or infinite) cyclic groups then the extension is trivial.

Proof. The fact that the connecting homomorphism vanishes implies that the class of τ lies in the group $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$. Since $K_*(A)$ is torsionfree we know [23] that

$$\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0 \;=\; \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$$

and then we have

$$[\tau] \in \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0 \;=\; QD(A,B)$$

as required. If $K_*(A)$ is a direct sum of (finite and/or infinite) cyclic groups then $\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), H) = 0$ for any abelian group H and so $Q(A,B) = 0$ and the extension is trivial. ∎

REMARK 2.8. The group $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ is either zero or huge. For example:

(i) (Warfield) If both G and H are countable groups with $\mathrm{Pext}^1_{\mathbb{Z}}(G,H) \neq 0$ and H torsionfree, then $\mathrm{Pext}^1_{\mathbb{Z}}(G,H)$ has $\mathbb{Q}^{\aleph_0}$ as a direct summand. It also may have torsion.

(ii) (R. Baer) If G is a torsionfree group and H is a countable torsion group then

$$\mathrm{Pext}^1_{\mathbb{Z}}(G,H) = \mathbb{Q}^n$$

with either $n = 0$ or $n \geqslant \aleph_0$.

For proofs and references to these facts and for many more examples, please see [23].

3. HOMOMORPHISMS AND SPLIT MORPHISMS

In this section the identification of the quasidiagonal elements of Theorem 2.3 is made concrete. For instance, we answer a simple question. Suppose that $f : A \to B$. Then when is the canonically associated class $\beta^A[f] \in KK_1(SA, B)$ a quasidiagonal class? There are certain easy algebraic necessary conditions, and we show that (for $A \in \widetilde{\mathcal{N}}$) these conditions are sufficient. Finally, we demonstrate how to deal with an extension given explicitly.

In applications many of the most interesting classes in KK-theory come from $*$-homomorphisms $f : A \to B$. The class $[f] \in KK_0(A, B)$ determines the structure of $KK_*(A, B)$ as a module over the ring $KK_*(A, A)$ and hence has special importance.

Let

$$\tau_{\mathbb{C}} : 0 \to \mathbb{C} \otimes \mathcal{K} \to \mathcal{T} \to S\mathbb{C} \to 0$$

denote the universal Toeplitz extension which generates the group

$$KK_1(S\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}.$$

We may tensor the extension with B to obtain the extension

$$\tau_B : 0 \to B \otimes \mathcal{K} \to B \otimes \mathcal{T} \to SB \to 0.$$

Note that $\gamma(\tau_B) = 1 : K_*(B) \to K_*(B)$ and hence $\gamma(\tau_B) = 0$ if and only if $K_*(B) = 0$, which would imply that $KK_*(B, B) = 0$ if $B \in \widetilde{\mathcal{N}}$. Thus if $B \in \widetilde{\mathcal{N}}$ then the extension τ_B is B-quasidiagonal if and only if it is trivial, and this only happens when $K_*(B) = 0$.

There is a commuting diagram

$$\begin{array}{ccc} KK_0(\mathbb{C}, \mathbb{C}) & \xrightarrow{\beta^{\mathbb{C}}} & KK_1(S\mathbb{C}, \mathbb{C}) \\ \downarrow \iota_B & & \downarrow \iota_B \\ KK_0(B, B) & \xrightarrow{\beta^B} & KK_1(SB, B) \end{array}$$

where β^A represents the Bott isomorphism in the first variable of KK and ι_B the canonical structural map. Then $\tau_B = \iota_B(\tau_{\mathbb{C}})$ (by construction) $= \iota_B \beta^{\mathbb{C}}([1_{\mathbb{C}}])$ (since this is the universal Toeplitz extension) $= \beta^B \iota_B([1_{\mathbb{C}}])$ (since the diagram commutes) $= \beta^B([1_B])$.

The diagram

$$\begin{array}{ccc} KK_0(B, B) & \xrightarrow{\beta^B} & KK_1(SB, B) \\ \downarrow f^* & & \downarrow (Sf)^* \\ KK_0(A, B) & \xrightarrow{\beta^A} & KK_1(SA, B) \end{array}$$

commutes by the naturality of the Bott isomorphism, and hence we conclude that

$$\beta^A([f]) = \beta^A f^*([1_B]) = (Sf)^* \beta^B([1_B]) = (Sf)^* \tau_B \in KK_1(SA, B).$$

Thus we may represent the class $\beta^A([f])$ as the pullback of the extension τ_B by Sf, namely

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B\otimes\mathcal{K} & \longrightarrow & E & \longrightarrow & SA & \longrightarrow & 0 \\ & & \Big\downarrow{\cong} & & \Big\downarrow & & \Big\downarrow{Sf} & & \\ 0 & \longrightarrow & B\otimes\mathcal{K} & \longrightarrow & B\otimes\mathcal{T} & \longrightarrow & SB & \longrightarrow & 0 \end{array}.$$

(If Sf is not mono then as usual we add on a trivial extension so that the resulting Busby map classifying the pullback is mono.) So, here is a canonical extension associated to the map f. The C^*-algebra SA is always quasidiagonal and so the trivial extension added on to make Sf mono was quasidiagonal. Thus it makes sense to ask when the class $\beta^A([f])$ is quasidiagonal.

Note that $\gamma(\beta^A[f]) = (\beta^A)^*\gamma(f) = (\beta^A)^* f_*$, where

$$(\beta^A)^* : \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(K_*(SA), K_*(B)),$$

and since $(\beta^A)^*$ is an isomorphism (shifting parity, of course), we see that

$$\gamma(\beta^A[f]) = 0 \quad \text{if and only if} \quad f_* = 0 : K_*(A) \to K_*(B).$$

In order to state the answer, we must first introduce the mapping cone of f. Recall that the *mapping cone* Cf is the C^*-algebra

$$Cf = \{(\xi, a) \in B[0,1] \oplus A : \xi(0) = 0, \xi(1) = f(a)\}$$

with associated mapping cone sequence (cf. [18]) $0 \to SB \to Cf \to A \to 0$ that has equivalence class $\beta_B[f] \in KK_1(A, SB)$ where β_B represents Bott periodicity in the second variable of KK. Note that the diagram

$$\begin{array}{ccc} KK_0(A,B) & \xrightarrow{\beta^A} & KK_1(A,SB) \\ \Big\uparrow{f_*} & & \Big\uparrow{f_*} \\ KK_0(A,A) & \xrightarrow{\beta^A} & KK_1(A,SA) \end{array}$$

commutes and hence

$$\beta_B([f]) = \beta_B f_*([1_A]) = (Sf)_*\beta_A([1_A]),$$

where $\beta_A([1_A])$ is the class of the mapping cone of the identity map $1 : A \to A$ which has the form

$$0 \to SA \to CA \to A \to 0.$$

This class is KK-invertible with KK-inverse the Toeplitz class $\tau_A \in KK_1(SA, A)$. This fact is the core case in the proof of Bott periodicity in the KK-context.

PROPOSITION 3.1. *Suppose that $A \in \widetilde{\mathcal{N}}$. Let $f : A \to B$ be a $*$-homomorphism. Then the class $\beta^A[f] \in KK_1(SA, B)$ is quasidiagonal if and only if both of the following conditions hold:*

(i) *the induced homomorphism $f_* : K_*(A) \to K_*(B)$ is trivial; and,*

(ii) *the associated short exact sequence*

$$0 \to K_*(SB) \to K_*(Cf) \to K_*(A) \to 0$$

is pure exact.

If $K_*(A)$ is torsionfree then condition (ii) *is automatically satisfied, so that $\beta^A[f]$ is quasidiagonal if and only if $f_* = 0$.*

Proof. By virtue of Theorem 2.2 it suffices to determine when the class $\beta^A[f]$ is in $Z_1(SA, B)$. The Bott map is a homeomorphism, by (1.2), and hence it is equivalent to ask when the class $[f] \in KK_0(A, B)$ lies in the subgroup $Z_0(A, B)$. By Theorem 2.3 it suffices to show that

$$[f] \in \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0$$

if and only if conditions (i) and (ii) hold. Condition (i) is necessary and sufficient for

$$[f] \in \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$$

by the UCT, and condition (ii) picks out those elements of $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$ which lie in $\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$. If $K_*(A)$ is torsionfree then

$$\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \;=\; \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$$

and the corollary follows. ∎

Suppose given an extension of C^*-algebras

$$0 \to B \xrightarrow{i} E \xrightarrow{p} A \to 0$$

which is split by a $*$-homomorphism $s : A \to E$. Then there is a unique class $\pi_s \in KK_0(E, B)$ called the *splitting morphism* with the property that

$$i_*(\pi_s) = [1] - [sp] \in KK_0(E, E).$$

Note that π_s induces a homomorphism $\gamma(\pi_s) : K_*(E) \to K_*(B)$ but this homomorphism generally does *not* arise from a $*$-homomorphism $E \to B \otimes \mathcal{K}$.

Every KK-class may be represented as the KK-product of a class induced by a homomorphism and by a splitting morphism ([1], Section 17.1.2 and Section 17.8.3). So we wish to know when a splitting morphism corresponds to a quasidiagonal class. Here is the answer.

PROPOSITION 3.2. *Suppose that A and $B \in \widetilde{\mathcal{N}}$. Further, suppose given an extension of C^*-algebras*

$$0 \to B \xrightarrow{i} E \xrightarrow{p} A \to 0$$

which is split by a $*$-homomorphism $s : A \to E$. Let $\pi_s \in KK_0(E, B)$ be the associated splitting morphism. Consider the class

$$\beta^E(\pi_s) \in KK_1(SE, B).$$

Then the following are equivalent:

(i) *$\beta^E(\pi_s)$ is a quasidiagonal class;*

(ii) $K_*(B) = 0$;
(iii) $KK_*(B,B) = 0$;
(iv) $\pi_{\mathrm{s}} = 0$.

Proof. The implication (iv) $\Rightarrow$ (i) is obvious. We shall show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Suppose first that $\beta^E(\pi_{\mathrm{s}})$ is a quasidiagonal class. Then $\gamma(\beta^E(\pi_{\mathrm{s}})) = 0$ by Theorem 2.3, which implies that $\gamma(\pi_{\mathrm{s}}) = 0$. Then

$$1 - s_* p_* \;=\; \gamma([1] - [sp]) \;=\; \gamma(i_*(\pi_{\mathrm{s}})) \;=\; i_*(\gamma(\pi_{\mathrm{s}})) \;=\; 0$$

so that $p_* : K_*(E) \to K_*(A)$ is an isomorphism and thus $K_*(B) = 0$ as required. Thus (i) $\Rightarrow$ (ii).

If $K_*(B) = 0$ then $KK_*(B,B) = 0$ by the UCT. Thus (ii) $\to$ (iii).

If $KK_*(B,B) = 0$ then the class $[1_B] \in KK_0(B,B)$ of the identity map $1_B : B \to B$ is trivial. However, $KK_*(E,B)$ is a right module over $KK_*(B,B)$ and so $\pi_{\mathrm{s}} = \pi_{\mathrm{s}} \otimes_B [1_B] = 0$ as required. Thus (iii) $\to$ (iv). ∎

Proposition 3.1 tells when a class $\beta^A[f]$ is quasidiagonal. Proposition 3.2 tells when the class $\beta^A(\pi_{\mathrm{s}})$ is quasidiagonal. The following theorem describes when a class with factorization

$$x = [f] \otimes_D \pi_{\mathrm{s}} = f^*(\pi_{\mathrm{s}})$$

corresponds to a quasidiagonal class $\beta^A(x)$. Every KK-class has this form. Thus this theorem gives a complete solution to the quasidiagonality of the associated class $\beta^A(x) \in KK_1(SA,B)$. We state the theorem for the class x for simplicity, remembering that $\beta^A(x)$ is a quasidiagonal class if and only if $x \in Z_0(A,B) \cong \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A),K_*(B))_1$.

THEOREM 3.3. *Suppose that $A \in \widetilde{\mathcal{N}}$. Let $x \in KK_0(A,B)$, with factorization $x = [f] \otimes_D \pi_{\mathrm{s}} = f^*(\pi_{\mathrm{s}})$, with respect to the map $f : A \to D$, and extension*

$$0 \to B \otimes \mathcal{K} \to D \xrightarrow{p} A \to 0,$$

with splitting $s : A \to D$, and splitting morphism $\pi_{\mathrm{s}} \in KK_0(D,B)$.

Then:

(a) *The following conditions are equivalent:*

$$\gamma(x) = 0 \in \mathrm{Hom}_{\mathbb{Z}}(K_*(A),K_*(B))_0, \tag{3.1}$$

$$\begin{aligned} \mathrm{Im}(f_* : K_*(A) \to K_*(D)) &\subseteq \mathrm{Ker}(\gamma(\pi_{\mathrm{s}}) : K_*(D) \to K_*(B)) \\ &= \mathrm{Im}(s_* : K_*(A) \to K_*(D)). \end{aligned} \tag{3.2}$$

(b) *Suppose that $\gamma(x) = 0$, so that $x \in \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A),K_*(B))_1 \subseteq KK_0(A,B)$. Then*

$$x = z \otimes_D \pi_{\mathrm{s}} \tag{3.3}$$

for some $z \in \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(D))_1 \subseteq KK_0(A, D)$. *The element* z *is unique modulo the subgroup*

$$s_*(\operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(A)))_1$$

and if desired we may take $z = i_*x$. *Conversely, any element* x *of form* (3.3) *is in the group* $\operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_1$.

(c) *Suppose that* $x \in \operatorname{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_1 \subseteq KK_0(A, B)$. *Then*

$$x = z \otimes_D \pi_s \tag{3.4}$$

for some $z \in \operatorname{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(D))_1 \subseteq KK_0(A, D)$. *The element* z *is unique modulo the subgroup*

$$s_*(\operatorname{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(A)))_1$$

and if desired we may take $z = i_*x$. *Conversely, any element of form* (3.4) *is in the group* $\operatorname{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_1$.

Proof. For Part (a) we compute:

$$\gamma(x) = \gamma(\,[f] \otimes_D \pi_s\,) = \gamma(\pi_s) f_*$$

and hence $\gamma(x) = 0$ if and only if $\operatorname{Im}(f_*) \subseteq \operatorname{Ker}(\,\gamma(\pi_s)\,)$. The identification $\operatorname{Ker}(\,\gamma(\pi_s)\,) = \operatorname{Im}(s_*)$ is immediate from the definition of $\gamma(\pi_s)$. This proves Part (a). Parts (b) and (c) follow from the decomposition of $KK_*(A, D)$ into components $KK_*(A, A)$ and $KK_*(A, B)$ which results from the splitting and the UCT. ∎

We note in passing the following naturality property of our decomposition of the UCT. It uses the notation and assumptions of Theorem 3.3.

PROPOSITION 3.4. *The diagram*

$$\begin{tikzcd}
\operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(D))_{*-1} \arrow[r, "\delta_D"] \arrow[d, "\gamma(\pi_s)_*"] & KK_*(A, D) \arrow[d, "(-)\otimes_D \pi_s"] \\
\operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1} \arrow[r, "\delta_B"] & KK_*(A, B)
\end{tikzcd}$$

commutes, where the maps δ *are the inclusion maps from the* UCT.

Proof. This does not follow immediately from the naturality of the UCT, since the map $(-) \otimes_D \pi_s$ is not induced by a map of C^*-algebras. We argue as follows. Expand the diagram to the diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(A))_{*-1} & \xrightarrow{\delta_A} & KK_*(A, A) \\ \downarrow{\scriptstyle s_*} & & \downarrow{\scriptstyle s_*} \\ \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(D))_{*-1} & \xrightarrow{\delta_D} & KK_*(A, D) \\ \downarrow{\scriptstyle \gamma(\pi_s)_*} & & \downarrow{\scriptstyle (-)\otimes_D \pi_s} \\ \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_{*-1} & \xrightarrow{\delta_B} & KK_*(A, B) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \quad .$$

Each column is split exact, since s is a splitting, and the upper square commutes by the naturality of the UCT (since the map s_* *is* induced by the map $s : A \to D$) and it is easy to see that the map δ_B is the quotient map, making the lower square commute. ∎

Here is our solution to the relative quasidiagonality problem, applied to a concrete extension.

THEOREM 3.5. *Suppose that $A \in \widetilde{\mathcal{N}}$ is quasidiagonal relative to B. Suppose given an essential extension*

$$0 \to B \otimes \mathcal{K} \to E_\tau \to A \to 0$$

representing $\tau \in KK_1(A, B)$ with associated faithful representation $e_\tau : E_\tau \to \mathcal{L}(H_B)$. Then the representation e_τ is B-quasidiagonal if and only if both of the following conditions hold:

(i) $\gamma(\tau) = 0$, *or, equivalently, the boundary homomorphism $K_*(A) \to K_{*-1}(B)$ is trivial; and*

(ii) $\tau \in \bigcap_n n\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_{*-1}(B))_0$ *or, equivalently,*

$$\tau \in \mathrm{Ker}[\varphi : \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0{}^\wedge]$$

where $G^\wedge$ denotes the $\mathbb{Z}$-adic completion of G.

If $K_*(A)$ is torsionfree then condition (ii) *is satisfied automatically, so that e_τ is a B-quasidiagonal representation if and only if the boundary homomorphism $K_*(A) \to K_{*-1}(B)$ is trivial.*

Proof. Conditions (i) and (ii) are exactly the conditions that guarantee that τ lies in the subgroup $QD(A, B)$. ∎

4. PURITY AND QUASIDIAGONALITY

In this section we take advantage of standard results in infinite abelian groups to deduce results on quasidiagonality.

PROPOSITION 4.1. (a) *Suppose that H is a countable abelian group. Then the following are equivalent:*

(i) $\mathrm{Pext}^1_{\mathbb{Z}}(G, H) = 0$ *for all countable abelian groups G;*
(ii) $\mathrm{Pext}^1_{\mathbb{Z}}(G, H) = 0$ *for the groups $G = \mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$;*
(iii) *H is algebraically compact.*

(b) *Suppose that G is a countable abelian group. Then the following are equivalent:*

(i) $\mathrm{Pext}^1_{\mathbb{Z}}(G, H) = 0$ *for all countable abelian groups H;*
(ii) $\mathrm{Pext}^1_{\mathbb{Z}}(G, H) = 0$ *for all countable direct sums of cyclic groups H;*
(iii) *G is the direct sum of cyclic groups.*

Proof. First concentrate on Part (a). Of course (a)(i) implies (a)(ii). The implication (a)(iii) implies (a)(i) is immediate from 53.4 of [11]. The implication (a)(ii) implies (a)(iii) is the least obvious, but it is also found in page 232 of [11].

Turning to part (b), the implication (b)(i) implies (b)(ii) is trivial, and the implication (b)(iii) implies (b)(i) follows from 53.4 of [11] as well. For the following argument that (b)(ii) implies (b)(iii) I am indebted to John Irwin. Suppose that G satisfies condition (b)(ii). Let $\widetilde{G}$ be the free abelian group on the (countable) set

$$\{[g] : g \in G\}$$

modulo the relations given by

$$n[g] = 0$$

if $g \in G$ has order n. There is an obvious surjection $\widetilde{G} \to G$ and hence a short exact sequence

$$\Theta : 0 \to K \to \widetilde{G} \to G \to 0.$$

This sequence is pure, since every torsion element of G lifts to a torsion element of $\widetilde{G}$ of the same order. The group $\widetilde{G}$ is countable by construction, hence K is countable. Further, $\widetilde{G}$ is a direct sum of cyclic groups. The group K is thus a subgroup of a direct sum of cyclic groups and by Kulikov's theorem (cf. 20.1 of [11]) K itself is a direct sum of cyclic groups. Thus

$$\Theta \in \mathrm{Pext}^1_{\mathbb{Z}}(G, K) = 0$$

and hence the extension Θ must be split. This implies that G is isomorphic to a subgroup of a direct sum of cyclic groups and hence (by Kulikov's theorem) is itself a direct sum of cyclic groups. ∎

The following theorem is the KK-version of the preceding, purely algebraic results. First a bit of notation. For G any countable abelian group, let C_G be a separable commutative C^*-algebra such that for each $j = 0, 1$,

$$K_j(C_G) = G.$$

Note that C_G exists by geometric realization ([17]) and is unique up to KK-equivalence by the UCT. Each C_G is quasidiagonal, since it is commutative, and hence C_G is B-quasidiagonal for all separable B. If desired we may choose

$$C_G = C_G^0 \oplus C_G^1$$

where $K_i(C_G^j) = G$ if $i = j$ and $K_i(C_G^j) = 0$ if $i \neq j$. Then we could use the C_G^j separately in parts (ii) and (iii) of Theorem 4.2.

THEOREM 4.2. (a) *Suppose that B is a separable C^*-algebra. Then the following are equivalent:*

(i) *for each $A \in \widetilde{\mathcal{N}}$ with A quasidiagonal relative to B, $QD(A, B) = 0$;*

(ii) *for $G = \mathbb{Q}$ and $G = \mathbb{Q}/\mathbb{Z}$, $QD(C_G, B) = 0$;*

(iii) *$K_*(B)$ is algebraically compact.*

(b) *Suppose given a quasidiagonal C^*-algebra $A \in \widetilde{\mathcal{N}}$. Then the following are equivalent:*

(i) *for each separable C^*-algebra B, $QD(A, B) = 0$;*

(ii) *for H any direct sum of cyclic groups, $QD(A, C_H) = 0$;*

(iii) *$K_*(A)$ is the direct sum of cyclic groups.*

Proof. First consider (a). The implication (a)(i) implies (a)(ii) is immediate. The implication (a)(iii) implies (a)(i) is elementary, since if $K_*(B)$ is algebraically compact then

$$\operatorname{Pext}^1_{\mathbb{Z}}(G, K_*(B)) = 0 \tag{4.1}$$

for all groups G, by Theorem 4.1(i). If $A \in \widetilde{\mathcal{N}}$ with A quasidiagonal relative to B then

$$QD(A, B) \cong \operatorname{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$$

by Theorem 2.3, and

$$\operatorname{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) = 0$$

by (4.1), completing the argument.

Next we show that (a)(ii) implies (a)(iii). The condition (a)(ii) implies that

$$\operatorname{Pext}^1_{\mathbb{Z}}(G, K_*(B)) = 0$$

for $G = \mathbb{Q}$ and $G = \mathbb{Q}/\mathbb{Z}$ and then Theorem 4.1 implies that $K_*(B)$ is algebraically compact. This completes the proof of part (a).

The proof of part (b) is quite similar, and we comment only on the deep implication (b)(ii) implies (b)(iii). Condition (b)(ii) together with Theorem 1.2 imply that

$$\operatorname{Pext}^1_{\mathbb{Z}}(K_*(A), H) = 0$$

whenever H is a direct sum of cyclic groups, and then Theorem 4.1(i) implies that $K_*(A)$ is itself a direct sum of cyclic groups. This completes the proof of Theorem 4.2. ∎

5. A PROBLEM OF L.G. BROWN

Let $\theta : K_*(A)_{\mathrm{t}} \to K_*(A)$ be the canonical inclusion of the torsion subgroup of $K_*(A)$. L.G. Brown page 63 in [4], showed that (with $B = \mathcal{K}$) there is a relation between quasidiagonality and the kernel of the induced map

$$\theta^* : \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A)_{\mathrm{t}}, K_*(B)).$$

We generalize his result as follows. Let G_{t} denote the torsion subgroup of a group G and $G_f = G/G_{\mathrm{t}}$ denote the maximal torsionfree quotient of G. Recall ([20]) that given A, there is an associated extension of C^*-algebras

$$0 \to A \otimes \mathcal{K} \to A_f \to SA_{\mathrm{t}} \to 0$$

whose K-theory long exact sequence degenerates to the pure short exact sequence

$$0 \to K_*(A)_{\mathrm{t}} \xrightarrow{\theta} K_*(A) \to K_*(A)_f \to 0. \tag{5.1}$$

In particular,

$$K_*(A_{\mathrm{t}}) \cong K_*(A)_{\mathrm{t}} \quad \text{and} \quad K_*(A_f) \cong K_*(A)_f.$$

Further, if $A \in \widetilde{\mathcal{N}}$ then so are both A_{t} and A_f. We established this in Theorem 1.1 of [20], for $A \in \mathcal{N}$ but once again it is clear by inspection of that proof that the statement holds for $A \in \widetilde{\mathcal{N}}$.

THEOREM 5.1. *Suppose that $A \in \widetilde{\mathcal{N}}$ and A is quasidiagonal relative to B. Then:*
(i) *There is a natural commutative diagram with exact columns:*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Ker}(Q\theta^*) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ker}(\theta^*) \arrow[d] \\
QD(A,B) \arrow[r, "\cong"] \arrow[d, "Q\theta^*"] & \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0 \arrow[d, "\theta^*"] \\
QD(A_{\mathrm{t}},B) \arrow[r, "\cong"] \arrow[d] & \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A)_{\mathrm{t}}, K_*(B))_0 \arrow[d] \\
0 & 0
\end{tikzcd}$$

(ii) *There is a natural exact sequence*

$$\mathrm{Hom}_{\mathbb{Z}}(K_*(A)_{\mathrm{t}}, K_*(B))_0 \xrightarrow{\delta'} QD(A_f, B) \to \mathrm{Ker}(\theta^*) \to 0,$$

where δ' *is the boundary map in the* Hom-Pext *long exact sequence associated to the pure short exact sequence* (5.1).

(iii) *If* $\mathrm{Im}(\delta') = 0$ *(this condition usually holds: for instance, it holds if* $K_*(A)_{\mathrm{t}}$ *is a direct summand of* $K_*(A)$ *or if* $K_*(B)$ *is torsion free, and, of course, it holds whenever* $\mathrm{Hom}_{\mathbb{Z}}(K_*(A)_{\mathrm{t}}, K_*(B)) = 0$*), then there is a natural isomorphism*

$$QD(A_f, B) \cong \mathrm{Ker}(\theta^*).$$

Proof. The short exact sequence (5.1) is pure exact and thus produces a six term Hom-Pext sequence in the first variable. Identifying entries using Theorem 2.3, one obtains the left column below, which is exact. The six term Hom-Ext exact sequence contributes the right column below, and this column is also exact.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(K_*(A)_{\mathrm{t}}, K_*(B))_0 \arrow[r, "\cong"] \arrow[d, "\delta'"] & \mathrm{Hom}_{\mathbb{Z}}(K_*(A)_{\mathrm{t}}, K_*(B))_0 \arrow[d, "\delta"] \\
QD(A_f, B) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A)_f, K_*(B))_0 \arrow[d] \\
QD(A, B) \arrow[r] \arrow[d, "Q\theta^*"] & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))_0 \arrow[d, "\theta^*"] \\
QD(A_{\mathrm{t}}, B) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A)_{\mathrm{t}}, K_*(B))_0 \arrow[d] \\
0 & 0
\end{tikzcd}.$$

The diagram commutes, the horizontal maps are injections, and the map

$$QD(A_f, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A_f), K_*(B))_0$$

is an isomorphism since $K_*(A_f)$ is torsionfree. An easy diagram chase shows that $\mathrm{Ker}(Q\theta^*) \cong \mathrm{Ker}(\theta^*)$ from which (i) is immediate. Part (ii) follows from expressing $\mathrm{Ker}(Q\theta^*)$ as the quotient of $QD(A_f, B)$ modulo the group $\mathrm{Im}(\delta')$. ∎

6. QUASIDIAGONALITY AND TORSION

One of the early applications of the Brown-Douglas-Fillmore theory was contained in work of L.G. Brown ([4]). He exhibited an example of a bounded operator T which was not quasidiagonal but such that $T \oplus T$ was quasidiagonal. In fact Brown showed that $T \oplus T$ generated a trivial extension.

Here is an analysis of such behavior from our perspective.

THEOREM 6.1. *Suppose that $A \in \widetilde{\mathcal{N}}$, A is quasidiagonal relative to B, and $x \in KK_1(A,B)$. Then:*

(i) *If $\gamma(x) \neq 0$ then x is not a quasidiagonal class.*

(ii) *If $\gamma(x)$ has infinite order in the group $\operatorname{Hom}_{\mathbb{Z}}(K_*(A), K_*(B))$ then no multiple of x is a quasidiagonal class.*

(iii) *If $\gamma(x) = 0$ then x is a quasidiagonal class if and only if it is in the kernel of the natural map*

$$\varphi : \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))^{\frown}.$$

(iv) *If $K_*(A)$ is torsionfree then x is a quasidiagonal class if and only if $\gamma(x) = 0$.*

Proof. This is all immediate from Theorem 2.3. ∎

We apply Theorem 6.1 to the setting of $x \in KK_1(A, \mathbb{C})$.

THEOREM 6.2. *Suppose that $A \in \widetilde{\mathcal{N}}$ and A is a quasidiagonal C^*-algebra. Suppose given an essential extension*

$$\tau : 0 \to \mathcal{K} \to E_\tau \xrightarrow{p} A \to 0$$

representing $\tau \in KK_1(A, \mathbb{C})$ so that $\gamma(\tau) : K_1(A) \to K_0(\mathcal{K}) \cong \mathbb{Z}$. Then:

(i) *If $\gamma(\tau) \neq 0$ then τ is not a quasidiagonal extension.*

(ii) *If $\gamma(\tau) = 0$ then τ is a quasidiagonal extension if and only if the short exact sequence*

$$0 \to \mathbb{Z} \to K_0(E_\tau) \xrightarrow{p_*} K_0(A) \to 0$$

is a pure short exact sequence.

(iii) *If $K_0(A)$ is torsionfree then τ is quasidiagonal if and only if $\gamma(\tau) = 0$.*

(iv) *If $K_0(A)$ is a direct sum of (finite and/or infinite) cyclic groups then τ is a quasidiagonal extension if and only if it is a trivial extension.*

Proof. The group $\operatorname{Hom}_{\mathbb{Z}}(K_1(A), \mathbb{Z})$ is torsionfree since $\mathbb{Z}$ is torsionfree. Thus Theorem 6.1(i) and (ii) imply (i). Part (ii) is a restatement of Theorem 6.1(iv). Part (iii) follows from Theorem 6.1(iv). Part (iv) holds since $\operatorname{Pext}^1_{\mathbb{Z}}(G, H) = 0$ whenever G is a direct sum of cyclic groups. ∎

REMARK 6.3. There are examples ([19]) where the surjection

$$\varphi : \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))^{\frown}$$

is not a *split* surjection. In fact, using the example of Christensen-Strickland cited there, it is possible to produce an extension $\tau \in KK_1(A, B)$ which satisfies all of the following conditions:

(1) $\gamma(\tau) = 0$;
(2) $\varphi(\tau) \neq 0$ so that τ is not a quasidiagonal class; and
(3) for some $k \in \mathbb{N}$, $k\tau$ is a quasidiagonal class but is not trivial.

This sort of phenomenon can occur only when both $K_*(A)$ and $K_*(B)$ have p-torsion for some fixed prime p and then only rarely.

Theorem 6.1 implies an early result ([6]) of Brown, Douglas, and Fillmore (BDF) on quasidiagonality. Recall that a bounded operator T on a separable Hilbert space H is *essentially normal* if $T^*T - TT^*$ is compact. The *essential spectrum* of T is the spectrum of $\pi T \in \mathcal{Q}(H)$. We let $\mathrm{ind}(T)$ denote the Fredholm index of the operator T.

THEOREM 6.4 (BDF). *Suppose that $T \in \mathcal{L}(H)$ is an essentially normal operator. Let X denote the essential spectrum of T. Then the following are equivalent:*
(i) *T is a quasidiagonal operator;*
(ii) *for each $\lambda \in \mathbb{C} - X$,* $\mathrm{ind}(T - \lambda I) = 0$;
(iii) *T is of the form (normal) + (compact).*

Proof. There is a natural extension

$$0 \to \mathcal{K} \to C^*\{T, \mathcal{K}\} \to C(X) \to 0$$

which gives rise to an element

$$[T] \in KK_1(C(X), \mathbb{C}).$$

The operator T is quasidiagonal if and only if $[T]$ is a quasidiagonal class, by 2.2 and the remark preceding it. The fact that $X \subset \mathbb{C}$ implies that $K_*(C(X) \cong K^*(X)$ is torsionfree. Hence $[T]$ is a quasidiagonal class if and only if $\gamma([T]) = 0$, by Theorem 6.1. So it suffices to compute $\gamma([T])$. BDF ([6]) show that there is a natural isomorphism

$$KK_1(C(X), \mathbb{C}) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(K^1(X), \mathbb{Z}) \cong \widetilde{H}^0(\mathbb{C} - X)$$

and this map takes $[T]$ to the function

$$\lambda \longmapsto \mathrm{ind}(T - \lambda I)$$

so that $[T] = 0$ if and only if $\mathrm{ind}(T - \lambda I) = 0$ where defined. This completes the proof. ∎

7. LIFTING QUASIDIAGONALITY

In this section we present a converse to the following theorem of Davidson, Herrero, and Salinas.

THEOREM 7.1. ([10]) *Suppose that*

$$\tau : 0 \to \mathcal{K} \to E_\tau \to A \to 0 \tag{7.1}$$

is an essential extension with associated faithful representation

$$e_\tau : E_\tau \to \mathcal{L}(H)$$

and suppose further that E_τ is separable and nuclear. If e_τ is a quasidiagonal representation then A is a quasidiagonal C^*-algebra.

There is an obvious obstruction to a converse to this theorem, known already to Halmos ([12]). Suppose that S is the unilateral shift. Then there is a canonical associated extension

$$\tau_S : 0 \to \mathcal{K} \to C^*\{S, \mathcal{K}\} \to A \to 0$$

and the quotient $A \cong C(S^1)$ is commutative, hence quasidiagonal. However the C^*-algebra $C^*\{S, \mathcal{K}\}$ itself is not quasidiagonal since it contains the unilateral shift S, which is not quasidiagonal. Halmos demonstrates this by observing that S has non-trivial Fredholm index, whereas any Fredholm quasidiagonal operator must have trivial Fredholm index. In modern jargon, the index map

$$\gamma : KK_1(C(S^1), \mathcal{K}) \to \mathrm{Hom}_{\mathbb{Z}}(K^1(S^1), \mathbb{Z}) \cong \mathbb{Z}$$

satisfies

$$\gamma(\tau_S)(z) = -1 \neq 0$$

which is an obstruction to quasidiagonality.

Here is the complete story, at least within the category $\widetilde{\mathcal{N}}$.

THEOREM 7.2. *Suppose given the essential extension* (7.1) *with* $A \in \widetilde{\mathcal{N}}$ *and quasidiagonal. Then the representation* $e_\tau : E_\tau \to \mathcal{L}(H)$ *is a quasidiagonal representation if and only if the following two conditions hold:*

(i) $\gamma(\tau) = 0$*; and*

(ii) *the resulting K-theory short exact sequence*

$$0 \to \mathbb{Z} \to K_0(E_\tau) \to K_0(A) \to 0$$

is a pure exact sequence.

If in addition $K_0(A)$ *is torsionfree then condition* (ii) *is automatically satisfied, so that* e_τ *is a quasidiagonal representation if and only if* $\gamma(\tau) = 0$.

Proof. Theorem 2.3 reduces in this case to the identification

$$QD(A, \mathcal{K}) \cong \mathrm{Pext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}).$$

Condition (i) is equivalent to $\tau \in \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z})$ and condition (ii) is simply a statement that $\tau \in \mathrm{Pext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z})$. If in addition $K_0(A)$ is torsionfree then

$$\mathrm{Pext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}).$$

This completes the proof. ∎

REMARK 7.3. It is instructive to compare our Theorem 7.2 with a related result due to N. Brown and M. Dădârlat ([5]) where they overlap. Consider the essential extension

$$\tau : 0 \to \mathcal{K} \to E_\tau \to A \to 0$$

with associated faithful representation

$$e_\tau : E_\tau \to \mathcal{L}(H)$$

and suppose that $A \in \widetilde{\mathcal{N}}$ with A quasidiagonal. Brown and Dădârlat show ([5], Theorem 3.4) specialized to this case) that if $\gamma(\tau) = 0$ then E_τ is also quasidiagonal. We show that if $\gamma(\tau) = 0$ *and* the resulting K-theory short exact sequence is *pure* then the representation e_τ is a quasidiagonal representation. If $\gamma(\tau) = 0$ but the K-theory short exact sequence is *not* pure then we conclude that even though E_τ *is* quasidiagonal, the representation e_τ is *not* a quasidiagonal representation.

Here is an example. Let G be any countable torsion group. Then

$$\mathrm{Ext}^1_{\mathbb{Z}}(G, \mathbb{Z}) \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z}) = \mathcal{P}(G)$$

the Pontrjagin dual group of G. For example, if G is finite then $\mathcal{P}(G) = G$. If $G = \bigoplus\limits_1^\infty \mathbb{Z}/p$, the sum of countably many copies of the group $\mathbb{Z}/p$, then $\mathcal{P}(G) = \prod\limits_1^\infty \mathbb{Z}/p$ which is, of course, uncountable. Note that $\mathcal{P}(\mathcal{P}(G)) = G$ by the Pontrjagin duality theorem, and hence if $G \neq 0$ then $\mathrm{Ext}^1_{\mathbb{Z}}(G, \mathbb{Z}) \neq 0$.

Choose a commutative C^*-algebra A with $K_0(A) = G$ and $K_1(A) = 0$. This is always possible, and A is unique up to KK-equivalence. Then

$$\mathrm{Hom}_{\mathbb{Z}}(K_*(A), \mathbb{Z}) = 0$$

and hence the index map γ is identically zero. Thus there is a natural isomorphism

$$KK_1(A, \mathcal{K}) \;\cong\; \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \;\cong\; \mathrm{Ext}^1_{\mathbb{Z}}(G, \mathbb{Z}) \;\cong\; \mathcal{P}(G).$$

Using Brown and Dădârlat's result, we conclude that if

$$0 \to \mathcal{K} \to E_\tau \to A \to 0$$

is any essential extension then the C^*-algebra E_τ is quasidiagonal. On the other hand,

$$QD(A, \mathcal{K}) \cong \mathrm{Pext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) = \mathrm{Pext}^1_{\mathbb{Z}}(G, \mathbb{Z}) = 0$$

since G is a torsion group and $\mathbb{Z}$ is torsionfree, by Theorem 9.1 of [23]. Thus among all of the various $\tau \in KK_1(A, \mathcal{K}) \cong \mathcal{P}(G)$ and associated representations

$$e_\tau : E_\tau \to \mathcal{L}(H),$$

the only representation e_τ that is quasidiagonal is the one corresponding to the trivial extension, where

$$[\tau] = 0 \in KK_1(A, \mathcal{K})$$

This also illustrates the phenomenon discovered by L.G. Brown and discussed in Section 6, since any non-trivial extension $\tau \in KK_1(A, \mathcal{K})$ will have the property that it itself is not quasidiagonal, but when added to itself enough times it becomes quasidiagonal and trivial.

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CLAUDE L. SCHOCHET, MATHEMATICS DEPARTMENT, WAYNE STATE UNIVERSITY, DETROIT, MI 48202, USA

E-mail address: claude@math.wayne.edu

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