J. OPERATOR THEORY
53:1(2005), 169–184

HILBERT-SCHMIDT SUBMODULES AND ISSUES OF UNITARY EQUIVALENCE

RONGWEI YANG

Communicated by William B. Arveson

ABSTRACT. The first half of this paper studies the M_q*-type* submodules over the bidisk. Because of their structural simplicity, M_q-type submodules are used to address several issues regarding the unitary equivalence of submodules. M_q-type submodules lie inside a much bigger class — the class of *Hilbert-Schmidt submodules* which we will define in the second half of the paper. Several facts are put in place to raise two conjectures about Hilbert-Schmidt submodules. The Hilbert-Schmidt submodule possesses a numerical invariant which is a natural analogue of Arveson's curvature invariant over the unit ball.

KEYWORDS: *Hilbert-Schmidt submodules.*

MSC (2000): Primary 47A13; Secondary 46E20.

0. INTRODUCTION

The Hardy space over the torus $H^2(\Gamma^2)$, under action defined by multiplication of functions, is a module over the polynomial ring $\mathbb{C}[z_1, z_2]$. Submodules of $H^2(\Gamma^2)$ have very complicated yet intriguing structure which has attracted a continuing effort in search of an elucidation. One prevalent idea in recent approaches is to define equivalence relations among submodules and study equivalence classes. Various kinds of questions regarding unitary equivalence thus arise. This paper has two objectives, one is to single out and study what we call M_q-type submodules of the form

$$M = q_1(z_1)H^2(\Gamma^2) + q_2(z_2)H^2(\Gamma^2),$$

where q_1 and q_2 are nontrivial inner functions. M_q-type submodules enable us to answer a few questions (cf. [10], [14]) regarding unitary equivalence. The other objective is to define the *Hilbert-Schmidt submodule*. This definition is motivated by the fact that, on the one hand, it is broad enough to include almost all known examples of submodules, and on the other hand, this class of submodules is good enough so that some fine analysis tools apply. Much work could be done for the Hilbert-Schmidt submodules, but the purpose in this paper is to raise two conjectures and relate a natural numerical invariant for the Hilbert-Schmidt submodules to Arveson's notion of curvature invariant over the unit ball B_n (cf. [3]).

We begin our discussion by fixing some notation. With every inner function $\theta(w)$ in the Hardy space $H^2(\Gamma)$ over the unit circle $\Gamma \subset \mathbb{C}$, there is an associated contraction $S(\theta)$ on $H^2(\Gamma) \ominus \theta H^2(\Gamma)$ defined by

$$S(\theta)f = P_\theta w f, \quad f \in H^2(\Gamma) \ominus \theta H^2(\Gamma),$$

where P_θ is the orthogonal projection from $H^2(\Gamma)$ onto $H^2(\Gamma) \ominus \theta H^2(\Gamma)$. The operator $S(\theta)$ is called a Jordan block, and its properties have been very well studied (cf. [4], [12]). On the Hardy space $H^2(\Gamma^2)$ with coordinate functions z_1 and z_2, the Toeplitz operators T_{z_1} and T_{z_2} are unilateral shifts of infinite multiplicity. One sees that a closed subspace $M \subset H^2(\Gamma^2)$ is a submodule if and only if M is invariant for both T_{z_1} and T_{z_2}. In the setting of $H^2(\Gamma^2)$, it is necessary to distinguish between the Hardy space $H^2(\Gamma)$ in the variable z_1 and that in the variable z_2, for which we denote by H_1 and H_2, respectively. Two evaluation operators $L(0)$ and $R(0)$ are defined by

$$L(0)f = f(0, z_2), \quad R(0)f = f(z_1, 0), \quad f \in H^2(\Gamma^2).$$

Two essential associates of a submodule $M \subset H^2(\Gamma^2)$ are the pairs (S_1, S_2) and (R_1, R_2) defined by

$$S_i f = (I - p) z_i f, \quad R_i g = z_i g, \quad i = 1, 2,$$

where $f \in H^2(\Gamma^2) \ominus M, g \in M$, and p is the orthogonal projection from $H^2(\Gamma^2)$ onto M. One verifies that (S_1, S_2) is a pair of commuting contractions on $H^2(\Gamma^2) \ominus M$ and (R_1, R_2) is a pair of commuting isometries acting on M. These two pairs of operators capture every piece of information about M and are subjects of many recent studies. Three results are important for the study in this paper (cf. [11], [8] and [14], respectively). As usual, $[A, B]$ means $AB - BA$.

THEOREM 0.1. *For a submodule M, $[S_1^*, S_2] = 0$ if and only if*

$$H^2(\Gamma^2) \ominus M = (H_1 \ominus q_1 H_1) \otimes (H_2 \ominus q_2 H_2),$$

where $q_1(z_1)$ and $q_2(z_2)$ are either one variable inner functions or the constant 0.

One observes that relative to the tensor product in Theorem 0.1, $S_1 = S(q_1) \otimes I$ and $S_2 = I \otimes S(q_2)$. It is also not hard to check that

$$(H_1 \ominus q_1 H_1) \otimes (H_2 \ominus q_2 H_2) = H^2(\Gamma^2) \ominus (q_1 H^2(\Gamma^2) + q_2 H^2(\Gamma^2)).$$

For convenience, submodules like $q_1 H^2(\Gamma^2) + q_2 H^2(\Gamma^2)$ are said to be of M_q-type in this paper.

THEOREM 0.2. *For a submodule M, $[R_1^*, R_2] = 0$ if and only if*

$$M = \psi H^2(\Gamma^2),$$

for some inner function $\psi(z_1, z_2)$.

THEOREM 0.3. *For a submodule M, if the unit disk $\mathbb{D}$ is not a subset of $\sigma_c(S_1) \cap \sigma_c(S_2)$, then $[R_1^*, R_2]$ and $[R_1^*, R_1][R_2^*, R_2]$ are both Hilbert-Schmidt.*

Here $\sigma_c(A)$ means the continuous spectrum of A. For a Hilbert-Schmidt operator A, we let $\|A\|_{\mathrm{HS}}$ denote its Hilbert-Schmidt norm. The condition in Theorem 0.3 is mild enough to include almost all known examples of submodules, and this fact, together with some other observations, motivates the definition of the *Hilbert-Schmidt submodule* in Section 5.

1. $[S_1^*, S_2]$ VERSUS $[R_1^*, R_2]$

In this section we study how the equation $[R_1^*, R_2] = 0$ affects the commutator $[S_1^*, S_2]$ and likewise how the equation $[S_1^*, S_2] = 0$ affects $[R_1^*, R_2]$. The study is based on Theorem 0.1 and Theorem 0.2. One verifies first (cf. [7]) that for every submodule M,

$$S_{z_1}^* S_{z_2} f - S_{z_2} S_{z_1}^* f = (I - p)\overline{z}_1 p z_2 f, \quad f \in H^2(\Gamma^2) \ominus M. \tag{1.1}$$

It is not hard to check that $p z_2 f \in M \ominus z_2 M$ for every $f \in H^2(\Gamma^2) \ominus M$. When $[R_1^*, R_2] = 0$, i.e., $M = \psi H^2(\Gamma^2)$ for some inner function $\psi(z_1, z_2)$, $\{\psi z_1^j : j \geqslant 0\}$ is an orthonormal basis for $M \ominus z_2 M$. Therefore

$$\begin{aligned} S_{z_1}^* S_{z_2} f - S_{z_2} S_{z_1}^* f &= (I - p)\overline{z}_1 p z_2 f \\ &= (I - p)\overline{z}_1 \Big(\sum_{j=0}^{\infty} \langle z_2 f, \psi z_1^j \rangle \psi z_1^j \Big) \\ &= \sum_{j=0}^{\infty} \langle z_2 f, \psi z_1^j \rangle (I - p)\overline{z}_1 (\psi z_1^j) \\ &= \langle z_2 f, \psi \rangle (I - p)\overline{z}_1 \psi \\ &= \langle f, \overline{z}_2(\psi - R(0)\psi) \rangle \overline{z}_1 (\psi - L(0)\psi), \end{aligned}$$

which shows that $S_{z_1}^* S_{z_2} - S_{z_2} S_{z_1}^*$ is a rank one operator, and moreover, that

$$\|[S_{z_1}^*, S_{z_2}]\|_{\mathrm{HS}}^2 = \|\psi - R(0)\psi\|^2 \|\psi - L(0)\psi\|^2. \tag{1.2}$$

Next we study $[R_1^*, R_2]$ on M_q-type submodules. To this end, let us first consider a backward shift invariant subspace $H^2(\Gamma) \ominus \theta H^2(\Gamma)$ and the map $D(\theta) : H^2(\Gamma) \ominus \theta H^2(\Gamma) \to \theta H^2(\Gamma)$ defined by

$$D(\theta) f(w) = \langle w f, \theta \rangle \theta(w).$$

$D^*(\theta)$ is evidently a rank 1 operator and one checks that

$$D^*(\theta) g = \langle g, \theta \rangle \overline{w}(\theta - \theta(0)).$$

Moreover,

$$\|D(\theta)\|_{\mathrm{HS}}^2 = \|D^*(\theta)\theta\|^2 = 1 - |\theta(0)|^2.$$

Given a pair of inner functions $q_1(z_1)$ and $q_2(z_2)$, we can decompose $H^2(\Gamma^2)$ as

$$\begin{aligned}\big((H_1 \ominus q_1 H_1) \oplus q_1 H_1\big) &\otimes \big((H_2 \ominus q_2 H_2) \oplus q_2 H_2\big)\\ &= (H_1 \ominus q_1 H_1) \otimes (H_2 \ominus q_2 H_2) \oplus (H_1 \ominus q_1 H_1) \otimes q_2 H_2 \oplus q_1 H_1 \otimes\\ &\qquad (H_2 \ominus q_2 H_2) \oplus q_1 H_1 \otimes q_2 H_2.\end{aligned}$$

It is not hard to see that

$$(H_1 \ominus q_1 H_1) \otimes (H_2 \ominus q_2 H_2) = H^2(\Gamma^2) \ominus (q_1 H^2(\Gamma^2) + q_2 H^2(\Gamma^2)).$$

Relative to the decomposition

$$\begin{aligned}&q_1 H^2(\Gamma^2) + q_2 H^2(\Gamma^2)\\ &\quad = (H_1 \ominus q_1 H_1) \otimes q_2 H_2 \oplus q_1 H_1 \otimes (H_2 \ominus q_2 H_2) \oplus q_1 H_1 \otimes q_2 H_2,\end{aligned} \tag{1.3}$$

R_1 and R_2 have the following matrix forms:

$$R_1 = \begin{pmatrix} S(q_1) \otimes I & 0 & 0 \\ 0 & S \otimes I & 0 \\ D(q_1) & 0 & S \otimes I \end{pmatrix}, \quad R_2 = \begin{pmatrix} I \otimes S & 0 & 0 \\ 0 & I \otimes S(q_2) & 0 \\ 0 & I \otimes D(q_2) & I \otimes S \end{pmatrix}.$$

One then calculates that

$$R_1^* R_2 = \begin{pmatrix} S^*(q_1) \otimes S & D^*(q_1) \otimes D(q_2) & D^*(q_1) \otimes S \\ 0 & S^* \otimes S(q_2) & 0 \\ 0 & S^* \otimes S(q_2) & S^* \otimes S \end{pmatrix},$$

and

$$R_2 R_1^* = \begin{pmatrix} S^*(q_1) \otimes S & 0 & D^*(q_1) \otimes S \\ 0 & S^* \otimes S(q_2) & 0 \\ 0 & S^* \otimes S(q_2) & S^* \otimes S \end{pmatrix},$$

from which it follows that

$$R_1^* R_2 - R_2 R_1^* = \begin{pmatrix} 0 & D^*(q_1) \otimes D(q_2) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that $R_1^* R_2 - R_2 R_1^*$ is in fact a rank 1 operator that maps $q_1 H_1 \otimes (H_2 \ominus q_2 H_2)$ into $(H_1 \ominus q_1 H_1) \otimes q_2 H_2$, and $\|[R_1^*, R_2]\|_{\mathrm{HS}}^2 = \|D^*(q_1)\|_{\mathrm{HS}}^2 \|D(q_2)\|_{\mathrm{HS}}^2 = (1 - |q_1(0)|^2)(1 - |q_2(0)|^2)$. We summarize these observations in

COROLLARY 1.1. *For a submodule M,*
(i) *If $[R_1^*, R_2] = 0$ then $[S_1^*, S_2]$ is of at most rank* 1*, and*

$$\|[S_{z_1}^*, S_{z_2}]\|_{\mathrm{HS}}^2 = \|\psi - R(0)\psi\|^2 \|\psi - L(0)\psi\|^2,$$

where ψ is the inner function such that $M = \psi H^2(\Gamma^2)$.

(ii) *If* $[S_1^*, S_2] = 0$ *then* $[R_1^*, R_2]$ *is of at most rank* 1, *and*

$$\|[R_1^*, R_2]\|_{\mathrm{HS}}^2 = (1 - |q_1(0)|^2)(1 - |q_2(0)|^2),$$

where q_1 *and* q_2 *are the one variable inner functions such that* $M = M_q$.

It is then not hard to see that $[R_1^*, R_2]$ and $[S_1^*, S_2]$ are both 0 if and only if $M = gH^2(\Gamma^2)$ for some one variable inner function g.

2. UNITARY EQUIVALENCE M_q-TYPE SUBMODULE

Two submodules are said to be *unitarily equivalent* if there is a unitary module map between them. The unitary equivalence of M_q-type submodules is not difficult to determine. The following lemma holds the key.

LEMMA 2.1. *If* M *is a submodule that contains two nontrivial one variable functions* $f_1(z_1)$ *and* $f_2(z_2)$, *then a submodule* N *is unitarily equivalent to* M *if and only if*

$$N = \phi M$$

for some inner function $\phi(z_1, z_2)$.

Proof. By a result in [1], if M is unitarily equivalent to N then there is a $\phi(z_1, z_2) \in L^\infty(\Gamma^2)$ with $|\phi(z_1, z_2)| = 1$ almost everywhere on Γ^2 such that

$$N = \phi M.$$

This in particular means $\phi(z_1, z_2) f_1(z_1) \in H^2(\Gamma^2)$, which implies that $\phi(z_1, z_2)$ is analytic in z_2. Similarly, $\phi(z_1, z_2) f_2(z_2) \in H^2(\Gamma^2)$ implies $\phi(z_1, z_2)$ is analytic in z_1. This shows that ϕ is an inner function in two variables. ∎

It is easy to see that if M is a submodule with finite codimension, then M satisfies the condition for Lemma 2.1. For example, if $g_1(z_1)$ and $g_2(z_2)$ are the characteristic polynomials of S_1 and S_2 (for S_1 and S_2 are finite matrices in this case), respectively, then

$$(I - p)(g_i) = g_i(S_i)(1 - p1) = 0, \quad i = 1, 2,$$

i.e., g_1 and g_2 are in M. The following result in [1] is now a direct consequence of Lemma 2.1.

COROLLARY 2.2. *If* M *is a submodule with finite codimension and if* N *is unitarily equivalent to* M, *then*

$$N = \phi M,$$

for some inner function $\phi(z_1, z_2)$.

COROLLARY 2.3. *If* $p_i, q_i, i = 1, 2$ *are one variable inner functions, and if*

$$M_p = p_1(z_1)H^2(\Gamma^2) + p_2(z_2)H^2(\Gamma^2), \quad M_q = q_1(z_1)H^2(\Gamma^2) + q_2(z_2)H^2(\Gamma^2),$$

then M_p *is unitarily equivalent to* M_q *only if* $M_p = M_q$.

Proof. By Lemma 2.1, there is an inner function ϕ such that

$$M_q = \phi M_p.$$

Since it implies $M_p = \overline{\phi} M_q$, $\overline{\phi}$ is also inner, which is possible only if ϕ is a constant. ∎

When $p_i, q_i, i = 1, 2$ are all finite Blaschke products, Corollary 2.3 is also a consequence of the Rigidity Theorems in [6]. The M_q-type submodule can be used to settle some questions regarding the unitary equivalence of the *fringe operator* and the *core operator*, both of which are very useful associates of submodules.

3. THE FRINGE OPERATOR AND UNITARY EQUIVALENCE

For a submodule M, the fringe operator F is defined on $M \ominus z_1 M$ by

$$Ff = [R_1^*, R_1] z_2 f.$$

The fringe operator has a very close connection with the pair (R_1, R_2) (cf. [5], [14], [16]) and was used in [14] to deduce a trace-index relation for (R_1, R_2). It is known that the unitary equivalence of submodules implies the equivalence of their respective fringe operators. But it is a question to decide how faithful the fringe operator represents a submodule. The answer to this question, as manifested by the following two examples, depends on the type of the fringe operator as well as the type of submodule on which it is defined.

EXAMPLE 3.1. For a submodule M, its fringe operator F is unitarily equivalent to the unilateral shift (denoted by S) if and only if M is unitarily equivalent to $H^2(\Gamma^2)$. The sufficiency follows from a result in [1] which asserts that M is unitarily equivalent to $H^2(\Gamma^2)$ if and only if $M = \psi H^2(\Gamma^2)$ for some inner function $\psi(z_1, z_2)$. Therefore, $M \ominus z_1 M = \psi H_2$, and it is easy to see that F is multiplication by z_1, i.e., the unilateral shift. On the other hand, if F is unitarily equivalent to S then the kernel of F^* is one dimensional. Pick $\psi \in \ker F^*$ with $\|\psi\| = 1$. Since F is an isometry, $Ff = z_2 f$ for every $f \in M \ominus z_2 M$, and it follows that

$$\langle z_1^m z_2^n \psi, z_1^i z_2^j \psi \rangle = \langle z_1^m F^n \psi, z_1^i F^j \psi \rangle = 0$$

for non-negative integers m, n, i, j with $(m, n) \neq (i, j)$. This implies

$$\int\limits_{\Gamma^2} |\psi(z_1, z_2)|^2 z_1^l z_2^s \mathrm{d}m(z) = 0$$

for all integers l, s not both 0, and hence $|\psi(z_1, z_2)|$ is the constant 1 almost everywhere on Γ^2. The fact that in this case $M = \psi H^2(\Gamma^2)$ is easy to verify.

In this example, the fringe operator is a faithful representation of the submodule because of its particular operator theoretical properties.

The M_q type submodules are good examples of a different situation. The following lemma is useful to this end.

LEMMA 3.2. $M \ominus z_1 M = q_1(z_1)(H_2 \ominus q_2(z_2)H_2) \oplus q_2(z_2)H_2$.

Proof. Based on (1.3), we can write M_q as

$$M_q = q_1 H_1 \otimes (H_2 \ominus q_2 H_2) \oplus H_1 \otimes q_2 H_2,$$

and the lemma follows easily. ∎

EXAMPLE 3.3. Under the decomposition in Lemma 3.2, one readily checks that for $f \in H_2 \ominus q_2 H_2, g \in q_2 H_2$,

$$F(q_1 f + g) = q_1 S(q_2) f + q_1(0) D(q_2) f + z_2 g.$$

Pick any inner function $\theta(z_1) \in H_1$ with $\theta(0) = q_1(0)$, let $M' = \theta H^2(\Gamma^2) + q_2 H^2(\Gamma^2)$ and consider the map $U : M \ominus z_1 M \to M' \ominus z_1 M'$ defined by

$$U(q_1 f + g) = \theta f + g.$$

It is easy to see that U is a unitary operator. Moreover, if F' denotes the fringe operator for M', then

$$\begin{aligned} F' U(q_1 f + g) &= F'(\theta f + g) \\ &= \theta S(q_2) f + \theta(0) D(q_2) f + z_2 g \\ &= \theta S(q_2) f + q_1(0) D(q_2) f + z_2 g \\ &= U F(q_1 f + g), \end{aligned}$$

which shows that F and F' are unitarily equivalent. But by Corollary 2.3, M' is not equivalent to M when θ is not a scalar multiple of q_1. Apparently, the fringe operator in this example only represents "a half" of the submodule.

4. THE CORE OPERATOR AND UNITARY EQUIVALENCE

If $K^M(\lambda, z)$, $\lambda, z \in D^2$ is the reproducing kernel for a submodule M, then the *core function* $G^M(\lambda, z)$ for M is

$$G^M(\lambda, z) := (1 - \overline{\lambda}_1 z_1)(1 - \overline{\lambda}_2 z_2) K^M(\lambda, z),$$

and the *core operator* is defined on $H^2(\Gamma^2)$ as

$$C_M(f)(z) := \int_{\Gamma^2} G^M(\lambda, z) f(\lambda) \mathrm{d}m(\lambda), \quad z \in D^2,$$

where $\mathrm{d}m(\lambda)$ is the normalized Lebesgue measure on Γ^2. For simplicity, we suppress the "M" in our writing of G^M and C_M when no confusion may result. It is shown in [10] that on every submodule M, C is a bounded self-adjoint operator with $\|C\| = 1$, and moreover,

$$C = 1 - R_1R_1^* - R_2R_2^* + R_1R_2R_1^*R_2^*. \tag{4.1}$$

EXAMPLE 4.1. When $M = H^2(\Gamma^2)$, $G(\lambda, z) = 1$, and hence C is the rank 1 operator which evaluates $f \in H^2(\Gamma^2)$ at $(0, 0)$.

The core operator has other essential connections with the pair (R_1, R_2) as well, in particular, with $[R_1^*, R_2]$ and $[R_1^*, R_1][R_2^*, R_2]$. For simplicity, we let

$$\Sigma_0 = \|[R_1^*, R_1][R_2^*, R_2]\|_{\mathrm{HS}}^2, \quad \Sigma_1 = \|[R_1^*, R_2]\|_{\mathrm{HS}}^2.$$

The next theorem from [10] is useful in the sequel.

THEOREM 4.2. *For a submodule M,*
(i) $\|C\|_{\mathrm{HS}}^2 = \Sigma_0 + \Sigma_1$, *and when C is Hilbert-Schmidt,* $\Sigma_0 - \Sigma_1 = 1$;
(ii) $\mathrm{tr}C = 1$ *when C is trace class.*

Since $G(\lambda, z)$ is the integral kernel of C, C is Hilbert-Schmidt if and only if $G \in L^2(\Gamma^2 \times \Gamma^2)$ and moreover $\|C\|_{\mathrm{HS}} = \|G\|$ (cf. [9]). For an eigenvalue of C, say μ, we let E_μ denote the corresponding eigenspace. The following lemma is also from [10].

LEMMA 4.3. *For every submodule M, $E_1 = M \ominus (z_1M + z_2M)$.*

In this section, we make a detailed study of the core operator for M_q-type submodules. First of all, from (1.3) M_q can be decomposed as

$$M_q = q_1(z_1)H^2(\Gamma^2) \oplus q_2(z_2)(H^2(\Gamma^2) \ominus q_1(z_1)H^2(\Gamma^2)).$$

This decomposition leads to an explicit expression of its reproducing kernel $K^{M_q}(\lambda, z)$, namely,

$$K^{M_q}(\lambda, z) = \frac{\overline{q_1(\lambda_1)}q_1(z_1) + \overline{q_2(\lambda_2)}q_2(z_2) - \overline{q_1(\lambda_1)q_2(\lambda_2)}q_1(z_1)q_2(z_2)}{(1 - \overline{\lambda}_1z_1)(1 - \overline{\lambda}_2z_2)},$$

and consequently,

$$G(\lambda, z) = \overline{q_1(\lambda_1)}q_1(z_1) + \overline{q_2(\lambda_2)}q_2(z_2) - \overline{q_1(\lambda_1)q_2(\lambda_2)}q_1(z_1)q_2(z_2). \tag{4.2}$$

This expression, in particular, shows that C is of rank 3 in this case. For handy reference, we write down the expression

$$\begin{aligned} |G(\lambda, z)|^2 = {} & |\overline{q_1(\lambda_1)}q_1(z_1)|^2 + |\overline{q_2(\lambda_2)}q_2(z_2)|^2 + |\overline{q_1(\lambda_1)}q_1(z_1)|^2|\overline{q_2(\lambda_2)}q_2(z_2)|^2 \\ & + \overline{q_1(\lambda_1)}q_1(z_1)q_2(\lambda_2)\overline{q_2(z_2)} + q_1(\lambda_1)\overline{q_1(z_1)q_2(\lambda_2)}q_2(z_2) \\ & - |\overline{q_1(\lambda_1)}q_1(z_1)|^2(q_2(\lambda_2)\overline{q_2(z_2)} + \overline{q_2(\lambda_2)}q_2(z_2)) \\ & - |\overline{q_2(\lambda_2)}q_2(z_2)|^2(q_1(\lambda_1)\overline{q_1(z_1)} + \overline{q_1(\lambda_1)}q_1(z_1)). \end{aligned} \tag{4.3}$$

Taking into account the fact $q_i,\ i = 1, 2$ are inner, one obtains that

$$\|C\|^2_{\mathrm{HS}} = \int\limits_{\Gamma^2\times\Gamma^2} |G(\lambda, z)|^2 \mathrm{d}m(\lambda)\mathrm{d}m(z) = 1 + 2(1 - |q_1(0)|^2)(1 - |q_2(0)|^2).$$

As one consequence, one sees that $\|C\|^2_{\mathrm{HS}}$, depending only on $q_1(0)$ and $q_2(0)$, can take on any value in $(1, 3]$. By Lemma 4.3, 1 is an eigenvalue of C, so we assume η_1 and η_2 are the other two nonzero eigenvalues of C (for C has rank 3!). So

$$\|C\|^2_{\mathrm{HS}} = 1 + \eta_1^2 + \eta_2^2.$$

By Theorem 4.2(ii), $1 + \eta_1 + \eta_2 = 1$, and hence

$$\eta_1 := (1 - |q_1(0)|^2)^{1/2}(1 - |q_2(0)|^2)^{1/2}, \quad \eta_2 = -\eta_1. \tag{4.4}$$

Since C is self-adjoint, when its kernel is neglected, C is unitarily equivalent to the diagonal 3×3 matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & -\eta_1 \end{pmatrix}.$$

The above arguments also show that a rank 3 core operator is determined by its Hilbert-Schmidt norm. Another interesting consequence of Lemma 4.2(ii) and Theorem 4.3 is the fact that there is no rank 2 core operator. This fact, as well as some other examples, make the following question seem interesting.

QUESTION 4.4. Is there a submodule for which $\mathrm{rank}(C) = 4$?

In fact, we suspect that if C is of finite rank the number of its positive eigenvalues is the number of its negative eigenvalues plus 1. For a submodule not of M_q-type, the eigenvalues of its core operator may be very difficult to compute, and one reason is that it is not clear what analytic properties they reflect.

It was shown in [10] that if M and N are unitarily equivalent submodules, then C_M and C_N are unitarily equivalent operators. Since the core operator C for $H^2(\Gamma^2)$ is the rank 1 projection, we readily have the following

COROLLARY 4.5. *M_q-type submodules are not unitarily equivalent to $H^2(\Gamma^2)$.*

However, the unitary equivalence of the core operators does not imply the unitary equivalence of the submodules.

EXAMPLE 4.6. It is clear from the calculations above (cf. (4.4)) that if $M = q_1(z_1)H^2(\Gamma^2) + q_2(z_2)H^2(\Gamma^2)$ and $M' = q_2(z_1)H^2(\Gamma^2) + q_1(z_2)H^2(\Gamma^2)$, then the core operators on M and M' are unitarily equivalent. However, if $q_1(w)$ is not a scalar multiple of $q_2(w)$, M and M' are not unitarily equivalent submodules by Corollary 2.3. On another matter, it was shown in [10] that if M is unitarily equivalent to a submodule of finite codimension, then the core operator for M has finite rank. It is natural to ask whether the converse is true. A counterexample is now easy to come by. For example, by Lemma 2.1 a submodule equivalent to M_q

must be of the form ϕM_q for some inner function ϕ. But ϕM_q does not have finite codimension unless $\phi = 1$. So if q_1 is singular, then M_q is not equivalent to any submodule of finite codimension.

5. HILBERT-SCHMIDT SUBMODULE AND TWO CONJECTURES

Submodules in $H^2(\Gamma^2)$ can have extremely complicated structure. In fact, it is known that for every strict contraction, say A, there are two submodules M and N with $N \subset M$ such that A is unitarily equivalent to the compression of T_{z_1} to $M \ominus N$. This means that the structure of submodules is at least as complicated as that of a general bounded linear operator on Hilbert space. So seeking out a manageable class of submodules is very important for our study. The following definition attempts to identify a candidate.

DEFINITION 5.1. A submodule $M \subset H^2(\Gamma^2)$ is said to be *Hilbert-Schmidt* if its core operator C is Hilbert-Schmidt, or equivalently, its core function $G(\lambda, z)$ is in $L^2(\Gamma^2 \times \Gamma^2)$. In this case, we set

$$\tau(M) := \int_{\Gamma^2 \times \Gamma^2} |G(\lambda, z)|^2 \mathrm{d}m(\lambda) \mathrm{d}m(z).$$

One nice feature of this definition is that given a Hilbert-Schmidt submodule M, one has naturally associates to it a self-adjoint Hilbert-Schmidt operator C and a representing L^2 function $G(\lambda, z)$, which is directly connected to the reproducing kernel $K^M(\lambda, z)$. The prominence of being Hilbert-Schmidt (instead of being trace class or in some Schatten-p class, for $p \neq 2$) in this definition is suggested by several observations. For example, it is known through Theorem 0.3 and Theorem 4.2 that if a submodule M is such that $\mathbb{D}$ is not a subset of $\sigma_c(S_1) \cap \sigma_c(S_2)$, then M is Hilbert-Schmidt. Almost all known examples of submodules, including some seemingly pathological ones (cf. [14]), satisfy this condition, and hence are Hilbert-Schmidt. While on the other hand the core operator fails to be trace class very easily.

EXAMPLE 5.2. If $M = [z_1 - z_2]$ is the submodule generated by function $z_1 - z_2$, then one can verify (with some calculations!) that its core operator has eigenvalues

$$1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \ldots,$$

and hence is not trace class.

Clearly, if a submodule M is Hilbert-Schmidt, then by Lemma 4.3, $\dim(M \ominus (z_1 M + z_2 M)) < \infty$. We have several good reasons to believe the converse is also true.

CONJECTURE 5.3. A submodule M is Hilbert-Schmidt if and only if $\dim(M \ominus (z_1M + z_2M)) < \infty$.

A quantitative property of (S_1, S_2) can be deduced for Hilbert-Schmidt submodules.

THEOREM 5.4. *If M is a Hilbert-Schmidt submodule, then $[S_1^*, S_2]$ is Hilbert-Schmidt with*

$$\|[S_1^*, S_2]\|_{\mathrm{HS}}^2 \leqslant \frac{\tau(M)+1}{2}.$$

Proof. By (1.1), $[S_1^*, S_2] = (I-p)\bar{z}_1 p z_2$. It is not hard to check that pz_2 maps $H^2(\Gamma^2) \ominus M$ into $M \ominus z_2M$ and $(I-p)\bar{z}_1 = 0$ on z_1M. Therefore, $pz_2 = [R_2^*, R_2]pz_2$ and $(I-p)\bar{z}_1 = (I-p)\bar{z}_1[R_1^*, R_1]$, and hence

$$\begin{aligned}[S_1^*, S_2] &= (I-p)\bar{z}_1 p z_2 \\ &= (I-p)\bar{z}_1[R_1^*, R_1][R_2^*, R_2]pz_2.\end{aligned}$$

This implies

$$\|[S_1^*, S_2]\|_{\mathrm{HS}}^2 \leqslant \|[R_1^*, R_1][R_2^*, R_2]\|_{\mathrm{HS}}^2,$$

and hence by Theorem 4.2,

$$\|[S_1^*, S_2]\|_{\mathrm{HS}}^2 \leqslant \frac{\tau(M)+1}{2}. \quad \blacksquare$$

The equality in Theorem 5.3 is attained for some submodules, for example, for $M = z_1z_2H^2(\Gamma^2)$ $\|[S_1^*, S_2]\|_{\mathrm{HS}}^2 = \frac{\tau(M)+1}{2} = 1$.

Although many unitary invariants can be defined for submodules, it seems difficult to find a complete one. Since submodule M is completely determined by its core function $G(\lambda, z)$, it is natural to attempt to extract a complete unitary invariant from $G(\lambda, z)$. Many examples have led us to believe that the function $|G(\lambda, z)|$ on $\Gamma^2 \times \Gamma^2$ is likely to be a candidate.

CONJECTURE 5.5. Two submodules M and N are unitarily equivalent if and only if $|G^M(\lambda, z)| = |G^N(\lambda, z)|$ almost everywhere on $\Gamma^2 \times \Gamma^2$.

If M and N are unitarily equivalent then there exists a $\phi(z) \in L^\infty(\Gamma^2)$ with $|\phi(z)| = 1$ almost everywhere on Γ^2 such that $N = \phi M$, and it then follows that $G^N(\lambda, z) = \overline{\phi(\lambda)}G^M(\lambda, z)\phi(z)$ almost everywhere on $\Gamma^2 \times \Gamma^2$. So the necessity of the conjecture is obvious. Here are two examples to support the sufficiency of this conjecture.

EXAMPLE 5.6. It is obvious that the core function for $H^2(\Gamma^2)$ is 1. In this example we show that $|G^M(\lambda, z)| = 1$ implies M is equivalent to $H^2(\Gamma^2)$. To see this fact, we recall that $E_1 = M \ominus (z_1M + z_2M) \neq \emptyset$, so

$$\tau = \|G\|^2 = 1$$

implies that C_M has only one non-zero eigenvalue, namely 1, and its multiplicity is 1. C_M is therefore a rank 1 operator, for which we can write $C_M = \psi \otimes \psi$, where

$\psi \in E_1$ with $\|\psi\| = 1$. It follows then $G^M(\lambda, z) = \overline{\psi(\lambda)}\psi(z)$. Since $|G^M(\lambda, z)| = 1$ a.e. on $\Gamma^2 \times \Gamma^2$, ψ is an inner function and one checks that $M = \psi H^2(\Gamma^2)$.

EXAMPLE 5.7. For a M_q-type submodule, we let $G_q(\lambda, z)$ denote its core function. In this example, we show that if $|G_p| = |G_q|$ on Γ^2 then $M_p = M_q$. We fix λ_1, λ_2 and z_2 in Γ^2 such that $\overline{p_2(\lambda_2)}p_2(z_2) \neq 1, \overline{q_2(\lambda_2)}q_2(z_2) \neq 1$, and for simplicity denote $\overline{p_1(\lambda_1)}$ by α_1, $\overline{q_1(\lambda_1)}$ by β_1, $\overline{p_2(\lambda_2)}p_2(z_2)$ by α_2 and $\overline{q_2(\lambda_2)}q_2(z_2)$ by β_2. So by (4.3),

$$|(\alpha_1 - \alpha_1\alpha_2)p_1(z_1) + \alpha_2| = |(\beta_1 - \beta_1\beta_2)q_1(z_1) + \beta_2|.$$

Squaring both sides, we have

$$\begin{aligned}|\alpha_1 - \alpha_1\alpha_2|^2 + |\alpha_2|^2 &+ 2\mathrm{Re}((\alpha_1 - \alpha_1\alpha_2)p_1(z_1)) \\ &= |\beta_1 - \beta_1\beta_2|^2 + |\beta_2|^2 + 2\mathrm{Re}((\beta_1 - \beta_1\beta_2)q_1(z_1)),\end{aligned}$$

which implies $\mathrm{Re}((\alpha_1 - \alpha_1\alpha_2)p_1(z_1) - (\beta_1 - \beta_1\beta_2)q_1(z_1))$ is a constant. For simplicity, we let $f(z_1) = (\alpha_1 - \alpha_1\alpha_2)p_1(z_1) - (\beta_1 - \beta_1\beta_2)q_1(z_1)$. Since $f \in H_1$, $f + \overline{f}$ being a constant certainly implies f is a constant. Since p_1 and q_1 both map Γ into Γ, it is not hard to check that this constant must be 0, which means p_1 and q_1 differ by a scalar multiple. Similarly, p_2 and q_2 differ by a scalar multiple, which shows that $M_p = M_q$.

6. ARVESON CURVATURE AND POISSON FLOW

The idea of core function and core operator applies to any other reproducing kernel Hilbert space. For example, a study was made in [18] for the Bergman space. In [2] and [3], Arveson studied the analytic function space H_n^2 on the unit ball $B_n \subset \mathbb{C}^n$ with the reproducing kernel

$$K(\lambda, z) = \frac{1}{1 - \overline{\lambda}_1 z_1 - \overline{\lambda}_2 z_2 - \cdots - \overline{\lambda}_n z_n}, \quad \lambda, z \in B_n.$$

H_n^2 is also a module over the polynomial ring $C[z]$ with module action defined by multiplication of functions. Likewise, the core function for a submodule $M \subset H_n^2$ is

$$G(\lambda, z) = (1 - \overline{\lambda}_1 z_1 - \cdots - \overline{\lambda}_n z_n)K^M(\lambda, z),$$

and the core operator is defined by

$$C(f)(z) = \langle f, G(z, \cdot)\rangle,$$

where $\langle -, - \rangle$ is the inner product in H_n^2. If T_i stands for multiplication by z_i on M, then it is not difficult to verify that

$$C = I - T_1T_1^* - T_2T_2^* - \cdots - T_nT_n^*,$$

which is in fact the square of the *defect operator* (denoted by Δ) for the *n-contraction* $(T_1, T_2, \dots, T_n)$. Let

$$T(z) = \bar{z}_1 T_1 + \bar{z}_2 T_2 + \cdots + \bar{z}_n T_n, \quad z \in B_n,$$

and

$$\begin{aligned} F(z) =& (1 - |z|^2)\Delta(I - T(z)^*)^{-1}(I - T(z))^{-1}\Delta \\ =& K^{-1}(z,z)\Delta K^*(z,T)K(z,T)\Delta. \end{aligned} \tag{6.1}$$

When Δ is trace class, Arveson shows that for almost every $\zeta \in \partial B_n$

$$K_0(\zeta) := \lim_{r \to 1^{-1}} \mathrm{tr} F(r\zeta)$$

exists, and he defines the *curvature* by

$$K(M) = \int_{\partial B_n} K_0(\zeta) \mathrm{d}m(\zeta),$$

where $\mathrm{d}m(\zeta)$ is the Lebesgue measure. The curvature invariant is the primary focus in [3].

This final section aims to identify a bidisk analogue of Averson's curvature invariant. Much of the study here can be generalized to the Hardy spaces over other type of domains. We let $k_\lambda(z)$ denote the normalized Szegö kernel for $H^2(\Gamma^2)$, i.e.,

$$k_\lambda(z) = \frac{\sqrt{(1 - |\lambda_1|^2)(1 - |\lambda_1|^2)}}{(1 - \bar{\lambda}_1 z_1)(1 - \bar{\lambda}_2 z_2)}.$$

So $|k_\lambda(z)|^2$ is in fact the Poisson-Szegö kernel. Here we let T_1 and T_2 stand for the Toeplitz operators T_{z_1} and T_{z_2}, respectively. So

$$k_\lambda(T) = \sqrt{(1 - |\lambda_1|^2)(1 - |\lambda_1|^2)}(1 - \bar{\lambda}_1 T_1)^{-1}(1 - \bar{\lambda}_2 T_2)^{-1}.$$

One sees that for every $f \in H^2(\Gamma^2)$, $k_\lambda(T)f(z) = k_\lambda(z)f(z)$ and hence

$$\|k_\lambda(T)f\|^2 = \int_{\Gamma^2} |k_\lambda(z)|^2 |f(z)|^2 \mathrm{d}m(z),$$

so by the property of the Poisson-Szegö kernel, for almost every $z \in \Gamma^2$ and λ converging nontangentially to z,

$$\lim_{\lambda \to z} \|k_\lambda(T)f\|^2 = |f(z)|^2. \tag{6.2}$$

In the setting of $H^2(\Gamma^2)$, the core operator is in general not a positive operator and hence does not has a square root, nor is it trace class in general (cf. Example 5.2). But for a Hilbert-Schmidt submodule M, there is a good substitute of (6.1), namely $Ck^*_\lambda(T)k_\lambda(T)C$, and we define

$$H(\lambda) = \mathrm{tr} C k^*_\lambda(T) k_\lambda(T) C = \|k_\lambda(T)C\|^2_{\mathrm{HS}}, \quad \lambda \in \mathbb{D}^2.$$

THEOREM 6.1. *For every Hilbert-Schmidt submodule, $H(z)$ is a 2-harmonic function with nontangential L^1 boundary value on Γ^2, and for almost every $z \in \Gamma^2$,*

$$H(z) = \int_{\Gamma^2} |G(\lambda, z)|^2 \mathrm{d}m(\lambda).$$

Proof. If $\{\eta_j, f_j(z) : \geqslant 0\}$ is the sequence of eigenvalues (counting multiplicity) and corresponding eigenfunctions of C such that $\{f_j(z) : \geqslant 0\}$ form an orthonormal basis for the range of C, then for every $\lambda \in \mathbb{D}^2$

$$\begin{aligned} H(\lambda) &= \mathrm{tr} C k_\lambda^*(T) k_\lambda(T) C \\ &= \sum_{j \geqslant 0} \langle C k_\lambda^*(T) k_\lambda(T) C f_j, f_j \rangle \\ &= \sum_{j \geqslant 0} \eta_j^2 \| k_\lambda(T) f_j \|^2. \end{aligned}$$

By (6.2), H has nontangential boundary value at almost every $z \in \Gamma^2$ (note here we regard $+\infty$ as a meaningful boundary value), and in fact, $H(z) = \sum_{j \geqslant 0} \eta_j^2 |f_j(z)|^2$ a.e. on Γ^2. Since G is the integral kernel of C,

$$G(\lambda, z) = \sum_{j \geqslant 0} \eta_j \overline{f_j(\lambda)} f_j(z),$$

therefore,

$$H(z) = \int_{\Gamma^2} |G(\lambda, z)|^2 \mathrm{d}m(\lambda),$$

for almost every $z \in \Gamma^2$. Since $|k_\lambda(z)|^2$ is the Poisson-Szegö kernel, H is 2-harmonic (i.e., harmonic in both z_1 and z_2!) and is in fact the least harmonic majorant of the function

$$\sum_{j \geqslant 0} \eta_j^2 |f_j(z)|^2 = \int_{\Gamma^2} |G(\lambda, z)|^2 \mathrm{d}m(\lambda), \quad z \in \mathbb{D}^2. \quad \blacksquare$$

Therefore, we have

$$H(0) = \int_{\Gamma^2} H(z) \mathrm{d}m(z) = \tau(M), \tag{6.3}$$

which means that, in the setting of $H^2(\Gamma^2)$, a meaningful analogue of Arveson's curvature invariant is indeed $\tau(M)$! However, algebraic properties of $\tau(M)$ are mostly unknown at this time. The proof of Theorem 6.1 provides an upper bound for the eigenfunctions of C.

COROLLARY 6.2. *If M is a Hilbert-Schmidt submodule and η_j, f_j are as in the proof of Theorem* 6.1, *then*

$$\sum_{j\geqslant 0} \eta_j^2 |f_j(z)|^2 \leqslant H(z), \quad z \in \mathbb{D}^2.$$

So in particular, when G is bounded, $\|f_j\|_\infty \leqslant |\eta_j|^{-1}\|G\|_\infty$ for every j.

The 2-harmonic function $H(z)$ is easy to calculate for some submodules.

EXAMPLE 6.3. If $M = \psi H^2(\Gamma^2)$ for some inner function ψ, then $H = 1$. If $M = M_p$, then on Γ^2,

$$\begin{aligned} H(z) &= \int_{\Gamma^2} |G(\lambda, z)|^2 \mathrm{d}m(\lambda) \\ &= 3 + 2\mathrm{Re}\big(\overline{p_1(0)}p_2(0)p_1(z_1)\overline{p_2(z_2)} - p_1(0)\overline{p_1(z_1)} - p_2(0)\overline{p_2(z_2)}\big). \end{aligned}$$

In this case, $H's$ harmonic extention into $\mathbb{D}^2$ bears the same expression, and, not surprisingly, $H(0) = 1 + 2(1 - |p_1(0)|^2)(1 - |p_2(0)|^2)$. So for two submodules M_p and M_q, $H_p = H_q$ implies that C_p and C_q are unitarily equivalent. It is not clear if this is a general fact. However, it is easy to see that the equation $H_p = H_q$ does not imply the unitary equivalence of M_p with M_q.

Acknowledgements. Part of the work in this paper was finished while the author was visiting the Niigata University, Japan. The hospitality of its Department of Mathematics is very appreciated. In particular, the author thanks Professor Izuchi for many valuable discussions.

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RONGWEI YANG, DEPARTMENT OF MATHEMATICS AND STATISTICS, SUNY AT ALBANY, ALBANY, NY 12222, USA
E-mail address: ryang@math.albany.edu

Received February 14, 2003; revised September 8, 2003.