J. OPERATOR THEORY
53:1(2005), 185–195

ON THE THOMPSON GROUP FACTOR

JAESEONG HEO

Communicated by Şerban Strătilă

ABSTRACT. In this article, we will study the structure of the von Neumann algebra $W^*(F, P)$ generated by the Thompson group von Neumann algebra $L(F)$ and a projection P on $l^2(F)$. We show that the algebra (not necessarily $*$) algebraically generated by two generating unitaries of the Thompson group factor $L(F)$ and the commutant $L(F)'$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$ and that $L^*_{x_0}$ is contained in the strong-operator closure of the algebra (not $*$) generated by L_{x_0} and the commutant $L(F)'$ where x_0 is one of generators in F.

KEYWORDS: *Thompson group, normal form, transitive algebra, approximately inner automorphism.*

MSC (2000): Primary 46L10, 46L35; Secondary 47A15.

1. INTRODUCTION AND PRELIMINARIES

The Thompson group F was introduced in the 1960s by Richard J. Thompson in connection with studies in logic. It was used to construct finitely presented groups with unsolvable word problems. The Thompson group is known as a very interesting group and it appears in a variety of mathematical areas: word problems, dynamical system, homotopy theory, group cohomology and analysis.

The Thompson group F can be realized as the group of piecewise linear homeomorphisms of $[0, 1]$ which, except at finitely many dyadic rational numbers, are differentiable with derivatives equal to powers of 2. Furthermore, it has the presentation

$$F = \langle x_0, x_1, \dots \mid x_i^{-1} x_n x_i = x_{n+1},\ 0 \leqslant i < n \rangle. \tag{1.1}$$

From this relation $x_i^{-1} x_n x_i = x_{n+1}$, $0 \leqslant i < n$, we have $x_{n+1} = x_0^{-n} x_1 x_0^n$ for $n \geqslant 1$, so that F is generated by x_0 and x_1.

If F_1 is a group defined by $\langle A, B \mid [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2] \rangle$, then there exists a group isomorphism from F_1 to F which maps A to x_0 and B to x_1. Hence the Thompson group F is the finite presentation with two generators and two relators. In the geometric realization of F, the corresponding homeomorphisms x_n are defined by

$$x_n(t) = \begin{cases} t & \text{if } 0 \leqslant t \leqslant 1 - 2^{-n}, \\ \frac{t}{2} + \frac{1}{2}(1 - 2^{-n}) & \text{if } 1 - 2^{-n} \leqslant t \leqslant 1 - 2^{-n-1}, \\ t - 2^{-n-2} & \text{if } 1 - 2^{-n-1} \leqslant t \leqslant 1 - 2^{-n-2}, \\ 2t - 1 & \text{if } 1 - 2^{-n-2} \leqslant t \leqslant 1. \end{cases}$$

See the expository note [3] for a good introduction, more details and historical remarks to the Thompson group.

Geoghegan conjectured that the Thompson group F does not contain a non-abelian free subgroup and that F is non-amenable. It was proved by Brin and Squier ([2]) that F has no free non-abelian subgroups and it is proved in [3] that F is not elementary amenable. In analysis, the existence of the Thompson group describes that either there is a non-amenable finitely presented group without a free subgroup on two generators or there is a finitely presented amenable group that is not elementary amenable.

Many questions about the Thompson group are still open, in particular it is unknown *whether or not F is amenable.* This question is of considerable interest since F is expected to be a counterexample to the von Neumann's conjecture for finitely presented groups. The operator algebra analogue of von Neumann's conjecture on embeddings of non-abelian free groups into non-amenable groups is: *Does any non-hyperfinite* II_1*-factor contain a copy of a free group factor* $L(\mathbb{F}_2)$ *on two generators?* This is still open and the Thompson group F is also expected to be a counterexample of this question.

If $\mathcal{B}$ is a subalgebra (not necessarily $*$-subalgebra) of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}$ has no non-trivial common invariant subspace in $\mathcal{H}$, then is $\mathcal{B}$ strong-operator dense in $\mathcal{B}(\mathcal{H})$? This question is well-known as the transitive algebra question. When Kadison posed the transitive algebra question, he thought that some self-adjoint maximal abelian subalgebra of $\mathcal{B}(\mathcal{H})$ together with some elements not in the subalgebra might generate a non-trivial (strong-operator closed) transitive algebra. However, Arveson proved in [1] that Kadison's original idea does not work. That is, if $\mathcal{A}$ is a transitive subalgebra of $\mathcal{B}(\mathcal{H})$ which contains a self-adjoint maximal abelian subalgebra, then $\mathcal{A}$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$. Even though this question has been considered by many people, it is still open. See the monograph [8] for a general discussion of the transitive algebra question and related topics.

To find non-trivial transitive algebras, we considered some factor $\mathcal{M}$ of type II_1 and some elements from its commutant $\mathcal{M}'$. We proved that two unitaries in the hyperfinite II_1 factor $\mathcal{R}$ with the irrational rotation relation and the commutant $\mathcal{R}'$ generate $\mathcal{B}(\mathcal{H})$ and that the algebra generated by only two generators of $L(\mathbb{F}_\infty)$ and the commutant $L(\mathbb{F}_\infty)'$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$ where $\mathbb{F}_\infty$ is the free group with countably infinite generators. See [6] and [7] for more details.

In this paper, we show that $W^*(F, P_x)$ is of type II or of type I where $W^*(F, P_x)$ is the von Neumann algebra generated by $L(F)$ and an orthogonal projection P_x of $l^2(F)$ onto $l^2(F(x))$ where $F(x)$ is the set of elements in F with their normal forms starting with x. It is proved that the algebra algebraically (not necessarily $*$) generated by two unitaries $\{L_{x_0}, L_{x_1}\}$ and the commutant $L(F)'$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$. Furthermore, we can see that each $L_{x_i}^* = L_{x_i}^{-1}$ is contained in the strong-operator closure of the algebra (not $*$) generated by L_{x_i} and the commutant $L(F)'$. Finally, we introduce an order two automorphism α of $L(F)$ which is not inner and ask a question to see if the Thompson group F is non-amenable. Is α not approximately inner?

2. SOME FACTORS GENERATED BY A C^*-ALGEBRA AND A PROJECTION

Let G be a discrete group with the identity e and $\mathcal{H}$ the Hilbert space $l^2(G)$ with the usual inner product. We shall assume that G is countable, so that $\mathcal{H}$ is separable. For each $g \in G$, let L_g denote the left translation of functions in $\mathcal{H}$ by g^{-1}. Then the map $g \mapsto L_g$ is a faithful unitary representation of G on the Hilbert space $\mathcal{H}$. That is, $L : G \to \mathcal{B}(\mathcal{H})$ is the left regular representation of G defined by $L_g(\chi_h) = \chi_{gh}$ for all $g, h \in G$ where $\{\chi_h : h \in G\}$ is the function defined by $\chi_h(h) = 1$ and $\chi_h(h') = 0$ for all $h' \in G \setminus \{h\}$.

Let $L(G)$ be the von Neumann algebra generated by $\{L_g : g \in G\}$. Similarly, let R_g be the right translation by g on $\mathcal{H}$ and $R(G)$ the von Neumann algebra generated by $\{R_g : g \in G\}$. Then $L(G)' = R(G)$ and $R(G)' = L(G)$. The function χ_g that is 1 at g and 0 elsewhere is a cyclic trace vector for $L(G)$ (and $R(G)$). In general, $L(G)$ and $R(G)$ are finite von Neumann algebras. They are factors (of type II_1) precisely when each conjugacy class in G (other than that of e) is infinite. In this case we say that G is an infinite conjugacy class (i.c.c.) group.

As a vector, each operator in $L(G)$ can be expressed as an l^2 sequence, that is, if $T \in L(G)$, then $T = \sum\limits_{g \in G} \lambda_g L_g$ with $\sum\limits_{g \in G} |\lambda_g|^2 < \infty$. The subset $\{g \in G : \lambda_g \neq 0\}$ of G is called the *support* of T and denoted by $\operatorname{supp} T$. If $\mathcal{S}$ is a subset of G, we define $T_{\mathcal{S}}$ as the element $\sum\limits_{g \in \mathcal{S}} \lambda_g L_g$ which has the support in $\mathcal{S}$. In convention, we will also denote L_g by g for each element $g \in G$.

Let $\Gamma = G_1 * \cdots * G_n$ with $n \in \{2, 3, \ldots\} \cup \{\infty\}$ be a free product of at least two but at most countably many cyclic groups. In [9], the weak closure $W^*(\Gamma, P_\Lambda)$ of $C^*(\Gamma, P_\Lambda)$ is a type II_∞-factor or a type I_∞-factor where $C^*(\Gamma, P_\Lambda)$ is a C^*-algebra generated by the reduced group C^*-algebra $C_{\mathrm{r}}(\Gamma)$ and a collection P_Λ of projections onto the l^2-spaces over certain subsets of Γ.

Note that every non-trivial element x of the Thompson group F can be expressed in a unique normal form

$$x = x_{i_1} \cdots x_{i_m} x_{j_k}^{-1} \cdots x_{j_1}^{-1}$$

where $0 \leqslant i_1 \leqslant \cdots \leqslant i_m$, $j_k \geqslant \cdots \geqslant j_1 \geqslant 0$, $i_m \neq j_k$ and if x_i and x_i^{-1} appear in the decomposition of x, then so does x_{i+1} or x_{i+1}^{-1} ([2]). Throughout this paper, F denotes a Thompson group with the presentation (1.1) unless is specified otherwise.

For $x \in F \setminus \{e\}$, let $F(x)$ be the set of all normal forms in F whose initial segments coincide with x, that is, the set of all normal forms in F starting with x. Let P_x denote the orthogonal projection from $l^2(F)$ onto $l^2(F(x))$. Let $C^*(F, P_x)$ be the C^*-subalgebra of $\mathcal{B}(l^2(F))$ generated by the reduced group C^*-algebra $C^*_{\mathrm{r}}(F)$ and the projection P_x and let $W^*(F, P_x)$ be the weak closure of $C^*(F, P_x)$ in $\mathcal{B}(l^2(F))$, that is, the von Neumann algebra generated by the group von Neumann algebra $L(F)$ and P_x.

THEOREM 2.1. *Let $F = \langle x_0, x_1, \ldots \mid x_i^{-1} x_n x_i = x_{n+1},\ 0 \leqslant i < n\rangle$ be a Thompson group. Then we have*

$$W^*(F, P_x) \cong \begin{cases} \mathcal{B}(\mathcal{H}) & \text{if the normal form of } x \text{ contains some } x_i^{-1}, \\ \text{type II-factor} & \text{otherwise.} \end{cases}$$

Proof. In convention, we will denote L_g by g for each element $g \in G$ if no confusion. For any $x \in F \setminus \{e\}$, let $\mathfrak{M}_x$ be the von Neumann algebra $W^*(F, P_x)$ generated by $L(F)$ and the projection P_x in $\mathcal{B}(l^2(F))$. Hence $\mathfrak{M}_x$ contains $L(F)$, so that $\mathfrak{M}_x' \subset R(F) = L(F)'$. It is not hard to see that $P_x \notin L(F)$. Indeed, if $P_x \in L(F)$, we have that

$$P_x R_h = R_h P_x \quad \text{for all } h \in F.$$

In particular, we have that $P_x R_{x^{-1}} = R_{x^{-1}} P_x$. However, $x = P_x R_{x^{-1}}(e) = R_{x^{-1}} P_x(e)$ $= 0$, which is absurd.

First we consider the case where the normal form of x does not contain x_i^{-1}, $i = 0, 1, \ldots$. We put $x = x_{p_1} \cdots x_{p_n}$, $n \geqslant 1$. Take any element $\xi \in \mathfrak{M}_x'$, so that $\xi^* \in \mathfrak{M}_x'$. Then we have that $P_x \xi^*(h) = \xi^* P_x(h)$ for all $h \in F$. If h has the normal form beginning with x, then

$$P_x(h\xi) = P_x \xi^*(h) = \xi^* P_x(h) = \xi^*(h) = h\xi. \tag{2.1}$$

Otherwise, that is, if h does not have a normal form beginning with x, then we have that

$$P_x(h\xi) = P_x \xi^*(h) = \xi^* P_x(h) = 0. \tag{2.2}$$

Note that ξ can be expressed as an l^2 sequence since $\xi \in \mathfrak{M}_x' \subset R(F)$. Consider $g \in \operatorname{supp}\xi$. If g has the normal form whose initial segment coincides with x, then by taking $h = e$, we obtain that $P_x(h\xi) = P_x(\xi) \neq 0$. This contradicts (2.2).

If g has the normal form not beginning with x but containing x as some segment, then we can write g as $g_1 x \cdots$ where g_1 does not contain x as a segment of the normal form. By taking $h = g_1^{-1}$, we have that $P_x(h\xi) \neq 0$, which also contradicts to (2.2).

If g has the normal form not containing x but containing x^{-1} as some segment, that is, $g = g_2 x^{-1} \cdots$ where g_2 does not contain x^{-1}, then by taking $h = x g_2^{-1}$ we have $P_x(h\xi) \neq h\xi$, which contradicts to (2.1).

Suppose that g has the normal form not containing x and x^{-1} as a segment. We put $g = x_{i_1} \cdots x_{i_m} x_{j_k}^{-1} \cdots x_{j_1}^{-1}$. If $i_1 < p_n$ for some $p_n \in \mathbb{N}$, by taking $h = x$, we get $P_x(h\xi) \neq h\xi$, which contradicts to (1.1). Thus we have that $i_1 \geqslant p_n$, and by considering $\operatorname{supp}\xi^*$, we also see that $j_1 \geqslant p_n$. If $i_1 = p_n$ for some $p_n \in \mathbb{N}$, take $h = x_{p_1} \cdots x_{p_{n-1}}$ where $p_1 \leqslant \cdots \leqslant p_n$. Then we have that $P_x(h\xi) \neq 0$, which contradicts to (2.2). Similarly, we also get a contradiction if $j_1 = p_n$. From above argument, we can see that $\mathfrak{M}_x' = R_{F_{p_n+1}}$, where F_{p_n+1} is the subgroup in F generated by $x_{p_n+1}, x_{p_n+2}, \ldots$. Therefore, $\mathfrak{M}_x$ is of type II.

Now we consider the case where the normal form of x contains x_i^{-1} for some $i \in \mathbb{N} \cup \{0\}$. That is, suppose that x can be expressed as following:

$$x = x_{p_1} \cdots x_{p_t} x_{q_s}^{-1} \cdots x_{q_1}^{-1} \quad \text{or} \quad x_{q_s}^{-1} \cdots x_{q_1}^{-1}, \quad s \geqslant 1.$$

Let $g \in F$ be in $\operatorname{supp}\xi$. If g has the normal form containing x as a segment, then we get a contradiction because of the same reason as the above. Suppose that g has the normal form not containing x. If g has a positive part, that is, $g = x_{i_1} \cdots x_{i_m} x_{j_k}^{-1} \cdots x_{j_1}^{-1}$, $m \geqslant 1$, then by taking $h = x$, we obtain that $P_x(h\xi) \neq h\xi$, which contradicts to (2.1). By considering $\operatorname{supp}\xi^*$, we can see that g can not have a negative part. Hence $\mathfrak{M}_x' = \mathbb{C} \cdot I$, so that $\mathfrak{M}_x \cong \mathcal{B}(\mathcal{H})$ is of type I. ∎

3. INVARIANT SUBSPACE AND TRANSITIVE ALGEBRA QUESTION

DEFINITION 3.1. We call a subset (or a subalgebra) $\mathcal{X}$ of a II_1-factor $\mathcal{M}$ *transitive* with respect to $\mathcal{M}$ if $\mathcal{X}$ has no non-trivial invariant projections in $\mathcal{M}$. Simply, we say that $\mathcal{X}$ is transitive in $\mathcal{M}$.

This definition is similar to the original definition of transitivity (in the factor of type I_∞). Similar definitions can be carried over to factors of type II_∞ or III. In a factor of type II_1, there are many non-trivial strong-operator closed transitive subalgebras. Furthermore, the transitive algebra question could also be considered for algebras generated by special kinds of operators. If $\mathfrak{A}$ is a transitive algebra generated by self-adjoint operators, then $\mathfrak{A}$ is a von Neumann algebra and must be equal to $\mathcal{B}(\mathcal{H})$. What is the situation if $\mathfrak{A}$ is generated by isometries or normal operators? In spite of a great deal of interest in this question, no transitive algebras other than $\mathcal{B}(\mathcal{H})$ are yet known.

In this section we are concerned if a transitive subset in a type II_1-factor $\mathcal{M}$ together with its commutant $\mathcal{M}'$ (we always assume that $\mathcal{H} = L^2(\mathcal{M}, \tau)$ even though most of our results and definitions do not depend on the choices of representations of $\mathcal{M}$) generates a *non-trivial* strong-operator closed transitive algebra in $\mathcal{B}(\mathcal{H})$.

THEOREM 3.2. *Let x_0 and x_1 be generators of the Thompson group F. Then the (non-selfadjoint) algebra generated by $\{L_{x_0}, L_{x_1}\}$ together with the commutant $L(F)'$ is strong-operator dense in $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space $l^2(F)$.*

Proof. If $P \in L(F)$ is a projection invariant under L_{x_j} for $j = 0, 1$, that is, $PL_{x_j}P = L_{x_j}P$, then P commutes with L_{x_j} since

$$\begin{aligned}\|L_{x_j}P - PL_{x_j}\|_2^2 &= \tau((L_{x_j}P - PL_{x_j})^*(L_{x_j}P - PL_{x_j})) \\ &= \tau(P - L_{x_j}^*PL_{x_j}P - PL_{x_j}^*PL_{x_j} + L_{x_j}^*PL_{x_j}) \\ &= 0\end{aligned}$$

where the third equality follows from the property of the trace τ and the equation $PL_{x_j}P = L_{x_j}P$. Since L_{x_0} and L_{x_1} generate $L(F)$ as a von Neumann algebra, L_{x_0} and L_{x_1} have no common non-trivial invariant projection in $L(F)$. Hence the set $\{L_{x_0}, L_{x_1}\}$ is transitive in $L(F)$. For each $n \in \mathbb{N}$, we define T_n by

$$T_n = \frac{1}{n}\sum_{j=1}^{n} L_{x_0^j x_1^j} R_{x_0^j x_1^j}.$$

Note that a strong-operator closed transitive algebra containing one dimensional projection must be $\mathcal{B}(\mathcal{H})$ (see [8]). To prove that the set $\{L_{x_0}, L_{x_1}\}$ together with $L(F)'$ generates $\mathcal{B}(\mathcal{H})$, we only have to show that T_n strongly tends to the one dimensional projection onto the unit vector χ_e as n goes to infinity. Since each $L_{x_0^j x_1^j} R_{x_0^j x_1^j}$ is a unitary operator and T_n is the convex combination of n unitary operators, each T_n is uniformly bounded. It is easy to see that $T_n\chi_e = \chi_e$.

Now we will show that $T_n(\chi_g)$ tends to zero when n tends to infinity for every $g \in F \setminus \{e\}$. In this case, we have that for $g \neq e$

$$T_n\chi_g = \frac{1}{n}\sum_{j=1}^{n} L_{x_0^j x_1^j} R_{x_0^j x_1^j}\chi_g = \frac{1}{n}\sum_{j=1}^{n} \chi_{x_0^j x_1^j g x_1^{-j} x_0^{-j}}.$$

Assume that the following relation holds:

$$x_0^j x_1^j g x_1^{-j} x_0^{-j} = x_0^k x_1^k g x_1^{-k} x_0^{-k} \quad \text{for } j < k,$$

where $1 \leqslant j, k \leqslant n$. Then we obtain that $g = x_1^{-j} x_0^{k-j} x_1^k g x_1^{-k} x_0^{j-k} x_1^j$, that is, g commutes with $x_1^{-j} x_0^{k-j} x_1^k$. Since $x_1^{-j} x_0^{k-j} x_1^k = x_0^{k-j} x_1^k x_{2k-j+1}^{-j}$, g commutes with

$$x_0^{k-j} x_1^k x_{2k-j+1}^{-j}, \quad j < k.$$

Expressing $g = x_{i_1} \cdots x_{i_m} x_{j_l}^{-1} \cdots x_{j_1}^{-1}$ as a normal form, we have that

$$(x_0^{k-j} x_1^k x_{2k-j+1}^{-j})(x_{i_1} \cdots x_{i_m} x_{j_l}^{-1} \cdots x_{j_1}^{-1}) = (x_{i_1} \cdots x_{i_m} x_{j_l}^{-1} \cdots x_{j_1}^{-1})(x_0^{k-j} x_1^k x_{2k-j+1}^{-j}).$$

By comparing both sides after changing into a normal form, the above equality holds only if g is of the form $(x_0^{k-j} x_1^k x_{2k-j+1}^{-j})^d$ for some integer d. If $x_0^s x_1^s g x_1^{-s} x_0^{-s} =$

$x_0^t x_1^t g x_1^{-t} x_0^{-t}$ for some $1 \leqslant s, t \leqslant n$, we obtain that $k = t$ and $j = s$. Indeed, if $t > s$ and if

$$(x_0^{t-s} x_1^t x_{2t-s+1}^{-s})(x_0^{k-j} x_1^k x_{2k-j+1}^{-j})^d = (x_0^{k-j} x_1^k x_{2k-j+1}^{-j})^d (x_0^{t-s} x_1^t x_{2t-s+1}^{-s}),$$

then we get that $k = t$ and $j = s$ by comparing normal forms of both sides.

Therefore, we have that

$$\|T_n \chi_g\|^2 = \Big\| \frac{1}{n} \sum_{j=1}^{n} \chi_{x_0^j x_1^j g x_1^{-j} x_0^{-j}} \Big\|^2 \leqslant (n-2)\frac{1}{n^2} + \Big(\frac{2}{n}\Big)^2 = \frac{n+2}{n^2},$$

so that the strong-operator limit of T_n is one dimensional projection onto the subspace generated by the unit vector χ_e. This completes the proof. ∎

PROPOSITION 3.3. *Let x_0 be one of generators in F. We see that $L_{x_0^{-1}}$ is in the strong operator closure of the algebra generated by L_{x_0} and the commutant $L(F)'$.*

Proof. To show this, we have only to show that for $g_1, \dots, g_k \in F$ and any $\varepsilon > 0$, there is an operator T in the algebra algebraically generated by L_{x_0} and $R(F)$ such that

$$\|T\chi_{g_p} - L_{x_0^{-1}}\chi_{g_p}\| < \varepsilon, \quad p = 1, \dots, k.$$

Choosing sufficiently large $m \in \mathbb{N}$ such that $(k+20)/\varepsilon^2 < m$, we define an operator T by

$$T = \frac{1}{m}\sum_{j=1}^{m} L_{x_0^j}(R_{g_1^{-1}x_0^{j+1}g_1} + \cdots + R_{g_k^{-1}x_0^{j+1}g_k}).$$

Then T satisfies the above property. Indeed, we have that for each $1 \leqslant p \leqslant k$

$$\begin{aligned}\|Tg_p - L_{x_i^{-1}}g_p\|^2 &= \Big\| \frac{1}{m}\sum_{j=1}^{m}\sum_{q\neq l} x_0^j g_p g_q^{-1} x_i^{-(j+1)} g_q \Big\|^2 \\ &= \Big\langle \frac{1}{m}\sum_{j=1}^{m}\sum_{q\neq l} x_0^j g_p g_q^{-1} x_0^{-(j+1)} g_q, \frac{1}{m}\sum_{j'=1}^{m}\sum_{q'\neq l} x_0^{j'} g_p g_{q'}^{-1} x_0^{-(j'+1)} g_{q'} \Big\rangle.\end{aligned}$$

Assume that

$$x_0^j g_p g_q^{-1} x_0^{-(j+1)} g_q = x_0^{j'} g_p g_{q'}^{-1} x_0^{-(j'+1)} g_{q'} \tag{3.1}$$

for some j, q, j', q'. If $j = j'$, then the equation (3.1) becomes $g_q^{-1} x_0^{-(j+1)} g_q = g_{q'}^{-1} x_0^{-(j+1)} g_{q'}$. Thus we get $g_{q'} g_q^{-1} x_0^{-(j+1)} (g_{q'} g_q^{-1})^{-1} = x_0^{-(j+1)}$, so that $x_0^{-(j+1)}$ commutes with $g_{q'} g_q^{-1}$. By considering the normal form of $g_{q'} g_q^{-1}$, we see that $g_{q'} g_q^{-1}$ is of the form x_0^n for some integer n. If $q = q'$, then

$$x_0^j g_p g_q^{-1} x_0^{-(j+1)} = x_0^{j'} g_p g_q^{-1} x_0^{-(j'+1)}.$$

It is easy to see that $x_0^{j-j'} g_p g_q^{-1} x_0^{-j+j'} = g_p g_q^{-1}$. That is, $g_p g_q^{-1}$ commutes with $x_0^{j-j'}$, so that $g_p g_q^{-1}$ must be of the form x_0^n for some integer n. Assume that $j \neq j'$

and $q \neq q'$. Then we have $x_0^j g_p g_q^{-1} x_0^{-(j+1)} g_q = x_0^{j'} g_p g_{q'}^{-1} x_0^{-(j'+1)} g_{q'}$, so that

$$x_0^{j-j'} g_p g_q^{-1} x_0^{-(j+1)} g_q = g_p g_{q'}^{-1} x_0^{-(j'+1)} g_{q'}. \tag{3.2}$$

By comparing the order of x_0 and x_0^{-1} in the normal forms, we see that the equality (3.2) cannot hold.

It follows from the argument in preceding paragraph that

$$\begin{aligned}\|Tg_p - L_{x_i^{-1}} g_p\|^2 &= \Big\| \frac{1}{m} \sum_{j=1}^{m} \sum_{q \neq l} x_0^j g_p g_q^{-1} x_i^{-(j+1)} g_q \Big\|^2 \\ &\leqslant \frac{1}{m^2}((k-5)m + 16m) < \varepsilon^2,\end{aligned}$$

which completes the proof. ∎

REMARK 3.4. In [6], we proved that if x is a generator of a non-abelian free group $\mathbb{F}_n$, $n = 2, 3, \ldots, \infty$, then $L_{x^{-1}}$ is in the strong operator closure of the (non-selfadjoint) algebra generated by L_x and the commutant $L(\mathbb{F}_n)'$. We also got a similar result for the hyperfinite II_1-factor $\mathcal{R}$ ([7]). Hence we can ask *if, for any unitary element U in a type II_1-factor $\mathcal{M}$, U^* is always in the strong operator closure of the (non-selfadjoint) algebra generated by U and the commutant $\mathcal{M}'$.*

By Haagerup's result ([5]), $L_{x_0} + L_{x_1}$ has a non-trivial invariant projection in $L(F)$. Indeed, the Fuglede-Kadison determinant of $L_{x_0} + L_{x_1}$ is 1 since

$$\log \triangle(L_{x_0} + L_{x_1}) = \log \triangle(I + L_{x_1 x_0^{-1}}) = \frac{1}{2\pi} \int_0^{2\pi} \log(1 + \mathrm{e}^{\mathrm{i}\theta}) \mathrm{d}\theta = 0.$$

Hence $L_{x_0} + L_{x_1}$ has a non-trivial invariant subspace in $L(F)$.

Furthermore, it is known that $L(F)$ is a McDuff factor of type II_1, so that $L(F)$ is singly generated. We suspect if $L_{x_0} + L_{x_1}$ is a single generator of $L(F)$, so that we investigate what is the relative commutant of $L_{x_0} + L_{x_1}$ in $L(F)$. Suppose that T is a self-adjoint element in $L(F)$ such that $T(L_{x_0} + L_{x_1}) = (L_{x_0} + L_{x_1})T$. We have $TL_{x_0} - L_{x_1}T = L_{x_0}T - TL_{x_1}$, so that from taking adjoints we get

$$L_{x_0^{-1}} T - T L_{x_1^{-1}} = T L_{x_0^{-1}} - L_{x_1^{-1}} T.$$

Hence

$$\begin{aligned} TL_{x_0} - L_{x_1}T = L_{x_0}T - TL_{x_1} &= L_{x_0}(TL_{x_1^{-1}} - L_{x_0^{-1}}T)L_{x_1} \\ &= -L_{x_0}(TL_{x_0^{-1}} - L_{x_1^{-1}}T)L_{x_1} = -L_{x_0}L_{x_1^{-1}}(L_{x_1}T - TL_{x_0})L_{x_0^{-1}}L_{x_1} \\ &= L_{x_0}L_{x_1^{-1}}(TL_{x_0} - L_{x_1}T)L_{x_0^{-1}}L_{x_1}, \end{aligned}$$

so that

$$\begin{aligned} L_{x_1^{-1}}TL_{x_0} - T &= L_{x_1^{-1}}L_{x_0}L_{x_1^{-1}}(TL_{x_0} - L_{x_1}T)L_{x_0^{-1}}L_{x_1} \\ &= L_{x_1^{-1}}L_{x_0}(L_{x_1^{-1}}TL_{x_0} - T)L_{x_0^{-1}}L_{x_1}. \end{aligned}$$

Therefore, $L_{x_1^{-1}}TL_{x_0} - T$ commutes with $L_{x_0^{-1}}L_{x_1}$ and

$$L_{x_0^{-1}}L_{x_1}(L_{x_1^{-1}}TL_{x_0} - T) = L_{x_0^{-1}}TL_{x_0} - L_{x_0^{-1}}L_{x_1}T$$

also commutes with $L_{x_0^{-1}}L_{x_1}$. Moreover, we have

$$L_{x_0^{-1}}TL_{x_0} - L_{x_0^{-1}}L_{x_1}T - (L_{x_0^{-1}}TL_{x_0} - L_{x_0^{-1}}L_{x_1}T)^* = TL_{x_1^{-1}}L_{x_0} - L_{x_0^{-1}}L_{x_1}T$$

commutes with $L_{x_0^{-1}}L_{x_1}$. Thus $T - L_{x_0^{-1}}L_{x_1}TL_{x_0^{-1}}L_{x_1}$ commutes with $L_{x_0^{-1}}L_{x_1}$, so that so does

$$(L_{x_0^{-1}}L_{x_1})^k(T - L_{x_0^{-1}}L_{x_1}TL_{x_0^{-1}}L_{x_1})(L_{x_0^{-1}}L_{x_1})^k.$$

From the above argument, we obtain that

$$\begin{aligned}\sum_{k=0}^{n}&(L_{x_0^{-1}}L_{x_1})^k(T - L_{x_0^{-1}}L_{x_1}TL_{x_0^{-1}}L_{x_1})(L_{x_0^{-1}}L_{x_1})^k \\ &= T - (L_{x_0^{-1}}L_{x_1})^{n+1}L_{x_0^{-1}}L_{x_1}TL_{x_0^{-1}}L_{x_1}(L_{x_0^{-1}}L_{x_1})^{n+1}\end{aligned}$$

commutes with $L_{x_0^{-1}}L_{x_1}$. Since the weak operator limit of

$$T - (L_{x_0^{-1}}L_{x_1})^{n+1}L_{x_0^{-1}}L_{x_1}TL_{x_0^{-1}}L_{x_1}(L_{x_0^{-1}}L_{x_1})^{n+1}$$

is T, T commutes with $L_{x_0^{-1}}L_{x_1}$. Hence $T(L_{x_0} + L_{x_1}) = (L_{x_0} + L_{x_1})T = L_{x_0}(I + L_{x_0^{-1}}L_{x_1})T = L_{x_0}TL_{x_0^{-1}}(L_{x_0} + L_{x_1})$. Note that if $A(L_{x_0} + L_{x_1}) = 0$ for some $A \in L(F)$, then $A = 0$. We get $T = L_{x_0}TL_{x_0^{-1}}$, which implies that T commutes with L_{x_0}. So T also commutes with L_{x_1}. Therefore, T is a multiple of scalar. Thus we suspect that $L_{x_0} + L_{x_1}$ is a single generator of $L(F)$, that is, $L_{x_0} + L_{x_1}$ and $L_{x_0^{-1}} + L_{x_1^{-1}}$ generate $L(F)$ as a von Neumann algebra.

4. ORDER TWO AUTOMORPHISM ON THE THOMPSON GROUP

Let $\mathcal{M}$ be a factor of type II_1. We denote by $\mathrm{Aut}(\mathcal{M})$ (respectively, $\mathrm{Inn}(\mathcal{M})$) the automorphism (respectively, the inner automorphism) group of $\mathcal{M}$ with the topology of strong pointwise convergence in $\mathcal{M}$. Let $\overline{\mathrm{Inn}(\mathcal{M})}$ be the closure of $\mathrm{Inn}(\mathcal{M})$ with respect to the strong pointwise convergence topology, which is called the *approximately inner* automorphism group. The following theorem is the fundamental theorem proved by Connes.

THEOREM 4.1. ([4]) *Let $\mathcal{M}$ be a II_1-factor.*

(i) *$\mathcal{M}$ is hyperfinite if and only if the symmetry $\alpha : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$, $x \otimes y \mapsto y \otimes x$, is in $\overline{\mathrm{Inn}(\mathcal{M} \otimes \mathcal{M})}$.*

(ii) *A discrete i.c.c. group G is amenable if and only if $L(G)$ is hyperfinite.*

We consider an automorphism of the Thompson group factor $L(F)$. In particular, we will consider an order two automorphism $\alpha : L(F) \to L(F)$ since a symmetry is an automorphism of order two. In order to see if F is amenable or non-amenable, we investigate if α is approximately inner. This is motivated by Connes' remarkable fundamental theorem.

To get such an automorphism, we first consider the geometric realization of the Thompson group F. Take an order two automorphism $\alpha : F \to F$ which rotates each element in F by 180° with a center $(1/2, 1/2)$. Then α is the automorphism of F given by $\alpha(x)(t) = 1 - x(1-t)$. We still denote by the same notation α the automorphism on $L(F)$ induced by α. Then we can see that

$$\alpha(x_0) = x_0^{-1} \quad \text{and} \quad \alpha(x_1) = x_0 x_1 x_0^{-2}.$$

Inductively, we can find that $\alpha(x_n) = x_0^n x_1 x_0^{-n-1}$ for $n \geqslant 1$. Then such α extends to an automorphism of $L(F)$

We claim that α is an outer automorphism of $L(F)$. Indeed, if $\alpha \in \mathrm{Inn}(L(F))$, then there is a unitary $u \in L(F)$ such that $\alpha(x) = u^* x u$ for all $x \in L(F)$. We can write u as the sum of normal forms, that is,

$$u = \sum \lambda_{i_1 \cdots i_m j_k \cdots j_1} x_{i_1} \cdots x_{i_m} x_{j_k}^{-1} \cdots x_{j_1}^{-1}$$

where $\sum |\lambda_{i_1 \cdots i_m j_k \cdots j_1}|^2 = 1$. Since $\alpha(x_0^n) = u^* x_0^n u$, we have $u x_0^{-n} = x_0^n u$ for all integer n. Since we have this equality $\sum |\lambda_{i_1 \cdots i_m j_k \cdots j_1}|^2 = 1$, for any $\varepsilon > 0$ we can take a finite set $\mathcal{C}$ of coefficients such that $\sum\limits_{\mathcal{C}} |\lambda_{i_1 \cdots i_m j_k \cdots j_1}|^2 > 1 - \varepsilon$. Take a sufficiently large integer N such that

$$N > 2 \max\{|x| : x \text{ is a summand of } u \text{ and the coefficient of } x \text{ is in } \mathcal{C}\}.$$

Then we have $\alpha(x_0^N) = u^* x_0^N u$, so that $x_0^N u = u x_0^{-N}$. This implies that the length of each summand of u is larger than $N/2$, which contradicts the choice of the set $\mathcal{C}$. Therefore, we have that $\alpha \notin \mathrm{Inn}(L(F))$.

However, we do not know whether or not α is approximately inner. Since the Thompson group is expected to be non-amenable by many people, we can ask a question *"is α approximately inner or not?"* in order to see if the Thompson group factor F is non-amenable. If we could prove that α is not approximately inner, then it follows from Theorem 4.1 that the Thompson group factor $L(F)$ will be not hyperfinite, so that F will be not amenable.

Acknowledgements. This work was supported by a grant No. R01-2001-000-00001-0 and R14-2003-006-01000-0 from the Korea Science and Engineering Foundation (KOSEF).

REFERENCES

[1] W. ARVESON, A density theorem for operator algebras, *Duke Math. J.* **23**(1967), 635–648.

[2] M. BRIN, C. SQUIER, Groups of piecewise linear homeomorphisms of the real line, *Invent. Math.* **79**(1985), 485–498.

[3] J. CANNON, W. FLOYD, W. PARRY, Introductory notes on Richard Thompson's groups, *Enseign. Math. (2)* **42**(1996), 215–256.

[4] A. CONNES, Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$, *Ann. of Math.* **104**(1976), 73–115.

[5] U. HAAGERUP, Spectral decomposition of all operators in a II_1-factor, which is embeddable in $\mathcal{R}^\omega$, *MSRI*, preprint, 2001.

[6] J. HEO, On transitive operator algebras containing the free group von Neumann algebras, preprint, 2001.

[7] J. HEO, Invariant subspace question relative to the hyperfinite II_1-factors, preprint, 2002.

[8] H. RADJAVI, P. ROSENTHAL, *Invariant Subspaces*, Springer-Verlag, Berlin 1973.

[9] W. SZYMAŃSKI, S. ZHANG, Type II_∞ factors generated by purely infinite simple C^*-algebras associated with free groups, *Proc. Amer. Math. Soc.* **128**(2000), 813–818.

JAESEONG HEO, DEPARTMENT OF MATHEMATICS, CHUNGNAM NATIONAL UNIVERSITY, TAEJON 305-764, KOREA
E-mail address: hjs@math.cnu.ac.kr

Received February 17, 2003.