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POLYNOMIAL CONDITIONS ON OPERATOR SEMIGROUPS

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ABSTRACT. It is known that if $AB - BA$ is quasinilpotent for every A and B in a multiplicative semigroup $\mathcal{S}$ of compact operators on a complex Banach space, then $\mathcal{S}$ is triangularizable. Possible extensions of this result are examined when $AB - BA$ is replaced with a general noncommutative polynomial in A and B. Easily checkable conditions on polynomials are found which enable us to reduce the problem to the case of finite groups acting on finite-dimensional spaces. In particular, all homogeneous noncommutative polynomials f in two variables with the following property are determined: if $f(A, B)$ is quasinilpotent for all A and B in $\mathcal{S}$, then $\mathcal{S}$ has a chain of invariant subspaces such that every induced semigroup on a "gap" of the chain is a matrix group that is finite modulo its centre. A triangularizability theorem which is a direct generalization of the known result on $AB - BA$ mentioned above, is obtained by replacing the polynomial $xy - yx$ with suitable polynomials of the form $f(xy, yx)$.

KEYWORDS: *Operator semigroups, invariant subspaces, triangularizability.*

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0. INTRODUCTION

Let $f(x, y)$ be a noncommutative polynomial in two variables with complex coefficients. Let $\mathcal{S}$ be a (multiplicative) semigroup of bounded linear operators on a Banach space. We say that *f is zero on $\mathcal{S}$* if $f(A, B) = 0$ for all pairs A and B in $\mathcal{S}$. More generally, we say that *f is quasinilpotent on $\mathcal{S}$* if $f(A, B)$ is quasinilpotent for all such pairs. (If the Banach space is finite-dimensional, we just say "nilpotent".) We are interested in connections between these conditions and reducibility of semigroups, i.e., existence of common invariant subspaces for all members of the semigroup.

The simplest and most familiar example is $f(x, y) = xy - yx$. If this polynomial is zero on a semigroup $\mathcal{S}$ of compact operators (in particular, operators acting on a finite-dimensional space), then $\mathcal{S}$ is (simultaneously) triangularizable (i.e., its lattice of invariant subspaces contains a maximal subspace chain), by a consequence of Lomonosov's results ([5]). A more recent result ([9]) asserts that it is sufficient to assume merely that $xy - yx$ is quasinilpotent. The proof of this theorem makes use of Turovskii's globalization ([14]) of Lomonosov's result; its finite-dimensional antecedent was proved much earlier by Guralnick ([3]). The more general case in which the polynomial f is of degree one in one of the variables was treated in [7].

We shall consider polynomials that are homogeneous in each of two variables. Note that if such a polynomial f is quasinilpotent on a semigroup $\mathcal{S}$ of compact operators, then f is quasinilpotent on

$$\mathbb{C}\mathcal{S} = \{cS : c \in \mathbb{C}, S \in \mathcal{S}\}$$

and also on its norm closure $\overline{\mathbb{C}\mathcal{S}}$ by continuity of spectrum on compact operators. One could of course consider polynomials in more than two variables, but in all instances of interest in this paper this question reduces to the case of two variables. It is well known (see, for example, pp. 156–158 in [8]) that if the coefficients of such a polynomial add up to zero, then it is necessarily quasinilpotent on any triangularizable semigroup of compact operators.

If algebras of operators are considered as opposed to semigroups, then satisfactory answers to some reducibility questions on compact operators are readily obtainable from the classical theory of polynomial-identity rings. The celebrated Amitsur-Levitzki Theorem ([1]), for example, states that the algebra $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ matrices satisfies

$$s_{2n}(A_1, A_2, \ldots, A_{2n}) = 0$$

for all choices of A_i, where s_m denotes the so-called standard polynomial of degree m in m noncommutative variables,

$$s_m(x_1, \ldots, x_m) = \Sigma(\text{sign } \tau)x_{\tau(1)}x_{\tau(2)} \cdots x_{\tau(m)},$$

with τ ranging over all permutations in m symbols. Replacing x_i by $x^{i-1}y$ for all i one obtains a nontrivial polynomial in two variables that is zero on $\mathcal{M}_n(\mathbb{C})$. There are in fact polynomials $f(x,y)$ of degree n in y to serve the purpose. One such polynomial is

$$f_n(x,y) = s_{n+1}(y, xy, x^2y, \ldots, x^ny)y^{-1},$$

and it is known that no polynomial of degree less than n in either variable would do for $\mathcal{M}_n$. On the other hand, $f_{n-1}(x,y)$ can be shown not to be nilpotent on $\mathcal{M}_n$, but $(f_{n-1}(x,y))^2$ is certainly zero on a subalgebra $\mathfrak{A}$ of $\mathcal{M}_n$ if $\mathfrak{A}$ has a nontrivial invariant subspace. This gives a criterion for reducibility of an algebra $\mathfrak{A}$ of operators on an n-dimensional space. For these and many related results see the brief and elegant monograph by Formanek ([2]). A standard reference is [12].

The preceding paragraph yields an easy consequence for algebras $\mathfrak{A}$ of compact operators on an infinite-dimensional space. Recall that if such an $\mathfrak{A}$ has no nontrivial invariant subspaces, then its uniform closure $\overline{\mathfrak{A}}$ contains all finite-rank operators. (This is a corollary of Lomonosov's Lemma; see, e.g. [8]). In particular, it has copies of $\mathcal{M}_n$ in it for all n; thus no polynomial can be zero, or quasinilpotent, on it. Observe that if a polynomial f is zero on an algebra $\mathfrak{A}$, then it is also zero on its norm closure $\overline{\mathfrak{A}}$. It follows that if f is a polynomial of degree n in x (or in y) and if $\mathfrak{A}$ is an algebra of compact operators on which f is quasinilpotent, then $\mathfrak{A}$ has invariant subspaces. In fact, $\mathfrak{A}$ has a block triangularization with "diagonal blocks" of size at most n, i.e., its lattice of invariant subspaces contains a chain with the property that if $\mathcal{M}$ and $\mathcal{N}$ are in the chain with $\mathcal{M} \subset \mathcal{N}$ and no other member between $\mathcal{M}$ and $\mathcal{N}$, then $\mathcal{M}$ has codimension at most n in $\mathcal{N}$. These and other related results are discussed in [4].

Returning now to our main theme of polynomials on semigroups, we observe that it is of course much easier for a polynomial f to be zero on an irreducible semigroup — as opposed to an irreducible algebra. The following example is typical, as we shall see later in our discussions of operator groups.

EXAMPLE 0.1. Let p and q be primes, not necessarily distinct. Let A be a nonscalar diagonal $p \times p$ matrix satisfying $A^q = I$ and let B be the cyclic $p \times p$ matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

It is easy to see that the group $\mathcal{G}(p,q,A)$ generated by A and B is irreducible. Furthermore, for every member T of the group, T^p is diagonal and T^{pq} is scalar. This gives rise to many polynomials that are zero on $\mathcal{G}(p,q,A)$. Two examples are $x^p y^p - y^p x^p$ and $\sum_{j=0}^{k} a_j y^j x^{pq} y^{k-j}$ with $\sum_{j=0}^{k} a_j = 0$ for any k.

Abstract groups or semigroups satisfying "semigroup identities" have been studied extensively. These correspond to the special case of binomials $f(x,y) = w_1(x,y) - w_2(x,y)$, where w_1 and w_2 are words in x and y. By definition, a free semigroup satisfies no such identity; neither does any faithful representation of it. But every (faithful representation of a) group on $\mathbb{C}^n$ satisfies all the identities of $\mathcal{M}_n(\mathbb{C})$, including $f_n(x,y)$, as seen above. For a recent treatment of topics in semigroup identities see Chapter 5 of [6].

Our main results will be on compact operators (including those acting on finite-dimensional spaces). Imposing simple conditions on the coefficients of f, we shall prove that any irreducible semigroup of compact operators on which f is quasinilpotent is necessarily a finite group modulo scalars. This will yield reducibility results, e.g., for those semigroups on infinite-dimensional spaces which contain a nonzero compact operator.

It should be noted here that for reducibility questions, the only homogeneous polynomials f of interest are those whose coefficients add up to zero. Otherwise, $f(S,S) = cS^n$ for some fixed n, where $c \neq 0$; thus the condition that f

is quasinilpotent on $\mathcal{S}$ implies that every member of $\mathcal{S}$ is quasinilpotent. This entails triangularizabilty if $\mathcal{S}$ consists of compact operators ([14]).

We should also mention that without some compactness assumptions one cannot go far. For Hilbert spaces, for example, it is not known whether the simplest polynomial $xy - yx$ can be zero on an irreducible semigroup of operators (even if the semigroup is singly generated: the invariant subspace problem). For ℓ^1, on the other hand, there is a singly generated, irreducible semigroup of quasinilpotent operators ([10]).

1. A PRELIMINARY LEMMA AND ITS CONSEQUENCES

Let $\mathcal{S}$ be an irreducible semigroup of compact operators that either acts on an infinite-dimensional space or has a nonzero member of rank less than n and acts on $\mathbb{C}^n$. If a homogeneous polynomial f is quasinilpotent (in particular, if f is zero) on $\mathcal{S}$, then it turns out to be quasinilpotent (respectively zero) on an irreducible semigroup of singular 2×2 matrices. To prove this we need the following lemma, which is convenient to state and prove for finite dimensions first. It is easy to find all minimal irreducible semigroups of rank $\leqslant 1$ on $\mathbb{C}^2$; the point of the lemma is that such a "small" semigroup is embedded naturally and spatially in the semigroups under consideration.

In what follows, $\mathbb{R}^+$ denotes the set of positive numbers and $\overline{\mathbb{R}^+\mathcal{S}}$ is the norm closure of the set $\{\alpha S : \alpha \in \mathbb{R}^+, S \in \mathcal{S}\}$. A family of operators on $\mathcal{X}$ is called *irreducible* if its members do not share a closed invariant subspace other than $\{0\}$ and $\mathcal{X}$. We treat $\mathbb{C}^n$ as an inner-product space when convenient.

LEMMA 1.1. *Let $\mathcal{S}$ be an irreducible semigroup in $\mathcal{M}_n(\mathbb{C})$ with $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$ and $n \geqslant 2$. Assume that $\mathcal{S}$ contains a nonzero operator of rank less than n. Then $\mathcal{S}$ contains a subsemigroup $\mathcal{S}_0$ with the following property: $\mathcal{S}_0$ has invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ with $\mathcal{M}_1$ a subspace of codimension two in $\mathcal{M}_2$ such that the semigroup induced by $\mathcal{S}_0$ on $\mathcal{M}_2/\mathcal{M}_1$ is, up to simultaneous similarity, generated by one of the three pairs (A, B) below:*

(i) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$;

(ii) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$;

(iii) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} t & t \\ 1-t & 1-t \end{pmatrix}$ *for some t satisfying $t(1-t) \neq 0$.*

REMARK 1.2. (i) If $\mathcal{S}$ contains operators of rank one, this lemma can be proved very easily; the general case requires more work. We note here that the pairs in all the three cases above can be put together in one form, e.g.,

$$A' = \begin{pmatrix} 0 & 1 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} \gamma & 0 \\ 1 & 0 \end{pmatrix},$$

from which we obtain the case (a) if $\beta = \gamma = 0$, (b) if $\beta = 0$, $\gamma = 1$, and a form simultaneously similar to (c) if $0 \neq \beta\gamma \neq 1$. The forms given above are preferred not just because they represent structurally different semigroups, but in order to simplify the calculations that are needed later to test a given polynomial.

(ii) Note that (a) and (b) yield 5-element semigroups, but the semigroup in (c) is finite if and only if t is a root of unity (in which case the number of its elements is $4m$, where m is the order of t).

(iii) Zassenhaus ([15]) presents analogous forms for minimal nontriangular semigroups over finite fields, but they are not directly applicable to our questions.

Proof of Lemma 1.1. Let r be the minimal positive rank present in $\mathcal{S}$, so that $1 \leqslant r \leqslant n-1$. The ideal $\mathcal{J}$ of $\mathcal{S}$ consisting of all members of rank r or 0 is irreducible. (See, e.g., Lemma 2.1.10 of [8], p. 29]). Thus we can assume with no loss of generality that $\mathcal{S} = \mathcal{J}$.

(i) Assume first that $\mathcal{S}$ contains nonzero idempotents P and Q with $PQ = QP = 0$. Letting $\mathcal{R}_1$ and $\mathcal{R}_2$ denote the ranges of P and Q respectively, we express the matrices in $\mathcal{S}$ relative to the decomposition $\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_3$ (with $\mathcal{R}_3$ a complement of $\mathcal{R}_1 + \mathcal{R}_2$ in $\mathbb{C}^n$). Thus

$$P = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $P\mathcal{S}Q \neq \{0\}$ by irreducibility, there is a nonzero member

$$N_1 = \begin{pmatrix} 0 & M & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of $P\mathcal{S}Q$, so that M is invertible by the minimality of r. By passing to $T\mathcal{S}T^{-1}$, where $T = I \oplus M \oplus I$, we can assume $M = I$. Again, $Q\mathcal{S}P \neq \{0\}$, so we pick a member N_2 of $Q\mathcal{S}P$ whose rank is r. Since N_1N_2 is a nonzero member of $P\mathcal{S}P$, its restriction to $\mathcal{R}_1$ is an invertible matrix X, and $X^{-1} \in P\mathcal{S}P|\mathcal{R}_1$, since $P\mathcal{S}P|\mathcal{R}_1 \setminus \{0\}$ is a group. (For a proof, see Lemma 3.1.6 in [8], p. 48.) So $\mathcal{S}$ contains the two matrices

$$N_2 = \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N_3 = \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the restriction to $\mathcal{R}_1 \oplus \mathcal{R}_2$ of the two matrices N_1 and N_3 are

$$\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

Pick one-dimensional subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively and consider the restrictions of these two operators to $\mathcal{V}_1 \oplus \mathcal{V}_2$. It follows that the semigroup $\mathcal{S}_0$ generated by N_1 and N_3 has a two-dimensional invariant subspace on which it restricts to the semigroup of basic matrices together with the zero matrix.

Hence the lemma is proved in this case (with $\mathcal{M}_1 = \{0\}$ and $\mathcal{M}_2 = \mathcal{V}_1 \oplus \mathcal{V}_2$) and the generators of the restriction are

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(ii) Now assume that

(a) for no idempotents $P \neq 0$ and $Q \neq 0$ in $\mathcal{S}$, the equations $PQ = QP = 0$ hold; but

(b) there is a nonzero nilpotent N in $\mathcal{S}$.

We can assume, by taking powers of N, that $N^2 = 0$. Now $N\mathcal{S}$ cannot consist entirely of nilpotents, because otherwise, $\operatorname{tr}(NS) = 0$ for all $S \in \mathcal{S}$ implies that $\mathcal{S}$ is reducible. (See, e.g., Corollary 2.1.6 of [8], p. 28].) Thus $N\mathcal{S}$ contains a nonzero idempotent P, whose rank is necessarily r. (This can be proved, e.g., by applying Lemma 3.4.2 of [8], p. 62, to the semigroup generated by a nonnilpotent member of $N\mathcal{S}$.) Similarly, $\mathcal{S}N$ has an idempotent member $Q \neq 0$. Note that $QP = 0$, so that $PQ \neq 0$ by the assumption (a) above. Denoting the range and nullspace of an operator T by $\mathcal{R}_T$ and $\mathcal{N}_T$ respectively, we observe that $\mathcal{R}_P + \mathcal{R}_Q$ is $2r$-dimensional, because $QP = 0$, so that the subspace $\mathcal{M} = \mathcal{N}_P \cap (\mathcal{R}_P + \mathcal{R}_Q)$ has dimension r. Since $\mathcal{M} \cap \mathcal{R}_P = 0$, it follows that

$$\mathcal{R}_P \oplus \mathcal{M} = \mathcal{R}_P + \mathcal{R}_Q.$$

Since $\mathcal{V} = \mathcal{N}_P \cap \mathcal{N}_Q$ is a complement of $\mathcal{R}_P + \mathcal{R}_Q$, we have a decomposition $\mathbb{C}^n = \mathcal{R}_P \oplus \mathcal{M} \oplus \mathcal{V}$ relative to which

$$P = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & M & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $PQ \neq 0$, it has rank r, and so does M. Thus M is invertible. As in the paragraph (i) above, we can assume by a simultaneous similarity that $M = I$. Thus we also get

$$N = PQ = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}.$$

Now $Q\mathcal{S}P \neq \{0\}$ by irreducibility. Hence there is a nonzero operator of the form

$$\begin{pmatrix} T & 0 & 0 \\ T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in $\mathcal{S}$. Since $P\mathcal{S}P|\mathcal{R}_P \setminus \{0\}$ is a group (as in the proof of (i)) containing T, we have $T^{-1} \in P\mathcal{S}P|\mathcal{R}_P$. It follows that

$$R = \begin{pmatrix} T & 0 & 0 \\ T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}.$$

Next let $\mathcal{S}_0$ be the subsemigroup generated by N and R (which includes $P = NR$, $Q = RN$, and 0). Now pick nonzero x in $\mathcal{R}_P$ and let $y = Rx - x$. The two-dimensional span of x and y is invariant under N and R, and the corresponding restrictions are

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus the assertion of the lemma is proved with $\mathcal{M}_1 = 0$ and $\mathcal{M}_2$ the span of x and y.

(iii) For the last case, we assume that $\mathcal{S}$ contains no nilpotents other than zero. This implies that there are no pairs of nontrivial idempotents P and Q with $PQ = 0$, because otherwise, $Q\mathcal{S}P$ consists of nilpotents; this yields $Q\mathcal{S}P = \{0\}$, which contradicts irreducibility.

Pick an idempotent P in $\mathcal{S}$ and express the matrices of $\mathcal{S}$ relative to $\mathcal{R}_P \oplus \mathcal{N}_P$. By irreducibility again, $P\mathcal{S}(1-P) \neq \{0\}$. Choose T in $\mathcal{S}$ with $T = PT$ and $PT(1-P) \neq 0$. Now $PT(1-P)\mathcal{S}P$ does not consist of nilpotents (because $\operatorname{tr} PT(1-P)\mathcal{S} \neq 0$ by irreducibility); so we can pick R with $R = RP$ and $PT(1-P)RP \neq 0$. Thus we have

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} T_0 & T_1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad R = \begin{pmatrix} R_0 & 0 \\ R_1 & 0 \end{pmatrix},$$

where T_1R_1 is not nilpotent by construction. Now T_0 and R_0 are both nonzero, because $\mathcal{S}$ contains no nilpotents other than zero; since they both belong to $P\mathcal{S}P|\mathcal{R}_P$, they are invertible and their inverses are in the group $P\mathcal{S}P|\mathcal{R}_P$, as in the preceding proofs. Now

$$\begin{pmatrix} I & T_0^{-1}T_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_0 & T_1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} I & 0 \\ R_1R_0^{-1} & 0 \end{pmatrix} = \begin{pmatrix} R_0 & 0 \\ R_1 & 0 \end{pmatrix}\begin{pmatrix} R_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

belong to $\mathcal{S}$. Since $T_0^{-1}T_1R_1R_0^{-1} \neq 0$, we can rename T_1 and R_1, so that the two idempotents

$$T = \begin{pmatrix} I & T_1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} I & 0 \\ R_1 & 0 \end{pmatrix}$$

are in $\mathcal{S}$ with $T_1R_1 \neq 0$.

We claim that T and R are not simultaneously triangularizable. If they were, then $RT - TR$ and thus $(RT-TR)^2$ would be nilpotent, but

$$(RT - TR)^2 = \begin{pmatrix} -T_1R_1 & T_1 \\ R_1 & R_1T_1 \end{pmatrix}^2 = \begin{pmatrix} T_1R_1(1+T_1R_1) & 0 \\ 0 & R_1T_1(1+R_1T_1) \end{pmatrix},$$

which would imply that $T_1R_1(1+T_1R_1)$ is nilpotent. Now if this is the case, then $1 + T_1R_1$ is not invertible, because T_1R_1 is not nilpotent; since $1 + T_1R_1$ is in $P\mathcal{S}P|\mathcal{R}_P$, and since $P\mathcal{S}P|\mathcal{R}_P \setminus \{0\}$ is a group, we conclude that $1 + T_1R_1 = 0$. This means that $TR = 0$, which was ruled out at the beginning of the proof of (iii).

To complete the proof, we use the well-known (and easily verified) fact that any two idempotents on $\mathbb{C}^n$ can be simultaneously block-triangularized so that every diagonal block is either 1×1 or 2×2. This together with the preceding paragraph shows that there is at least one irreducible 2×2 diagonal block in this block form of the pair (R, T). Thus the pair has invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ with $\mathcal{M}_2 \ominus \mathcal{M}_1$ two-dimensional such that relative to the decomposition

$$\mathcal{M}_1 \oplus (\mathcal{M}_2 \ominus \mathcal{M}_1) \oplus (\mathbb{C}^n \ominus \mathcal{M}_2)$$

the matrices of T and R have the forms

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{pmatrix},$$

where (T_{22}, R_{22}) is an irreducible pair of 2×2 idempotents. Letting $A = T_{22}$ and $B = R_{22}$ and observing that both A and B have to have rank one by irreducibility, we apply a similarity to assume, with no loss, that

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Now $B^2 = B$ implies $\alpha + \delta = 1$, $\beta\gamma = \alpha(1 - \alpha)$. Also, $\beta\gamma \neq 0$ by irreducibility. By applying another simultaneous diagonal similarity, we can assume $\beta = \alpha$, so that

$$B = \begin{pmatrix} \alpha & \alpha \\ 1 - \alpha & 1 - \alpha \end{pmatrix}$$

with $\alpha(1 - \alpha) \neq 0$. ∎

The proof above yields a little more than the asserted claim. We record the following obvious consequence of the proof for use in the last section of the paper.

LEMMA 1.3. *Let $\mathcal{S}$ be as in the preceding lemma. Then the subsemigroup $\mathcal{S}_0$ of the lemma has a restriction of*

type (i) *if $\mathcal{S}$ contains nonzero idempotents P and Q with $PQ = QP = 0$,*

type (ii) *if $\mathcal{S}$ contains no such pair of idempotents but has a nonzero nilpotent member,*

type (iii) *otherwise.*

COROLLARY 1.4. *Let $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$ be as in Lemma* 1.1 *and let r denote the minimal positive rank in $\mathcal{S}$. If $\mathcal{S}$ does not contain nonzero nilpotents (in particular, if $2r > n$), then $\mathcal{S}$ contains a subsemigroup that induces an $\mathcal{S}_0$ of type* (iii).

Proof. The semigroup $\mathcal{S}_0$ of type (iii) was obtained assuming no nonzero nilpotents. In particular, assume that $2r > n$, and let N be a nilpotent member of $\mathcal{S}$. If N were nonzero, we could assume with no loss of generality that $N^2 = 0$. But then the kernel of N, would have dimension at least r, contradicting the inequality $2r > n$. ∎

It is now easy to prove the appropriate version of Lemma 1.1 for compact operators.

COROLLARY 1.5. *Let $\mathcal{S}$ be an irreducible semigroup of compact operators on an infinite-dimensional Banach space. Then $\overline{\mathbb{R}^+\mathcal{S}}$ contains a subsemigroup $\mathcal{S}_0$ as described in Lemma* 1.1.

Proof. Assume with no loss of generality that $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$. Since $\mathcal{S}$ is irreducible, Turovskii's Theorem ([14], or see p. 198 in [8]) implies that $\mathcal{S}$ contains nonquasinilpotent operators. Thus it also contains nonzero finite-rank operators. (See, e.g., Lemma 7.4.5 of [8], p. 169.) Denoting by r the minimal nonzero rank of operators in $\mathcal{S}$ we observe that operators of rank r or zero in $\mathcal{S}$ form an ideal, which is irreducible. (See, e.g., p. 200 in [8].) The infinite-dimensionality of $\mathcal{X}$ now allows us to pick a pair $\{A, B\}$ in $\mathcal{S}$ of rank r whose ranges are not the same.

Let $\mathcal{M}$ be the span of ranges of A and B and note that $A\mathcal{S} \cup B\mathcal{S}$ is a subsemigroup (in fact a right ideal) of $\mathcal{S}$ that leaves $\mathcal{M}$ invariant. Furthermore, it is easy to deduce from the irreducibility of $\mathcal{S}$ that the restriction $\mathcal{S}_1$ of $A\mathcal{S} \cup B\mathcal{S}$ to $\mathcal{M}$ is also irreducible. (Just note that the restriction $A\mathcal{S}|\mathcal{M}$ must contain a basis for all linear transformations from $\mathcal{M}$ to the range of A; a similar assertion holds for $B\mathcal{S}|\mathcal{M}$.) Now the dimension of $\mathcal{M}$ is greater than r, because $A\mathcal{X} \neq B\mathcal{X}$. Thus Lemma 1.1 is applicable to the semigroup $\mathcal{S}_1$. ∎

2. THE SIGNIFICANT CASE OF DIMENSION TWO

The results of Section 1 reduce the question of which homogeneous polynomials f are zero on an irreducible semigroup with at least one nonzero, noninvertible member to the special case of semigroups acting on $\mathbb{C}^2$. To see this, just note that $f(R, S) = 0$ for a pair of operators R, S implies $f(R_0, S_0) = 0$ whenever $\mathcal{M}_1$ and $\mathcal{M}_2$ are simultaneous invariant subspaces for the pair (R, S) with $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and where (R_0, S_0) is the induced pair of operators on $\mathcal{M}_2/\mathcal{M}_1$. Further economy can be achieved in determining which polynomials f are quasinilpotent on such an $\mathcal{S}$: we shall need only check whether the square of f is zero for one of the three types of 2×2 matrix semigroups in Lemma 1.1.

For a first reading, the present section can be omitted by taking the view that it is elementary to verify whether a given polynomial (or its square) vanishes on some irreducible semigroup of singular 2×2 matrices. Then one can proceed to Section 3 and rephrase Definition 3.1 accordingly.

The first lemma of this section gives necessary and sufficient conditions for f to be zero on each of the three possible cases when $\mathcal{S} \subseteq \mathcal{M}_2(\mathbb{C})$. They are, of necessity, longer to state than apply to a given polynomial. They become much shorter in the second lemma, where f is checked for nilpotence (the criterion that is of greater use to us).

All but one of the conditions to be checked that are given in the following lemma amount to the assertion that certain coefficients of f sum to zero. The exception occurs in checking for type (iii) semigroups, and it asserts that a certain system of polynomial equations in one variable (obtained from f) has a suitable solution.

To shorten the statements of the lemmas somewhat we adopt some simple notation.

NOTATION 2.1. Let f be any noncommutative polynomial in two variables x and y.

(i) f_{ij} will denote the four unique summands of f in the expression

$$f = f_{11} + f_{12} + f_{21} + f_{22},$$

where f_{11} is the sum of all monomial terms in f starting with x and ending with x; f_{12} the sum of those starting with x and ending with y, and so on.

(ii) $f^{(1,1)}$ denotes the polynomial obtained from f when x^k is replaced with x and y^k with y for all $k \geqslant 2$ (i.e., f is reduced by $x^2 = x$ and $y^2 = y$). If x^k and y^k are replaced with x and 0 for all $k \geqslant 2$, the resulting polynomial is denoted by $f^{(1,0)}$ (reducing f by $x^2 = x$ and $y^2 = 0$). Similarly, we define $f^{(0,1)}$ and $f^{(0,0)}$ reducing f by $\{x^2 = 0, y^2 = y\}$ and $\{x^2 = 0, y^2 = 0\}$ respectively.

Observe that

$$(f_{ij})^{(u,v)} = (f^{(u,v)})_{ij}$$

for all u and v in $\{0, 1\}$ and all i and j in $\{1, 2\}$, so we shall omit these parentheses.

LEMMA 2.2. *Let f be a noncommutative polynomial of homogeneous degree r in x and homogeneous degree s in y, with $r \geqslant 1$ and $s \geqslant 1$.*

(i) *f is zero on the semigroup generated by*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

if and only if:

(a) $f_{i,j}^{(0,0)}(1,1) = 0$ *for* $i, j = 1, 2$ *(or, equivalently: the coefficients of the two monomials $xyxy\cdots$ and $yxyx\cdots$ are zero); and*

(b) $f(1,1) = 0$*; furthermore the coefficients of $x^r y^s$ and $y^s x^r$ are zero if* $\min(r, s) = 1$.

(ii) *f is zero on the semigroup generated by*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

if and only if:

(a) $f_{ij}^{(0,1)}(1,1) = 0$ *for* $i, j = 1, 2$, *and* $f_{ij}^{(1,0)}(1,1) = 0$ *for* $i, j = 1, 2$; *and*

(b) *the coefficients of $x^r y^s$ and $y^s x^r$ are zero, and* $f_{1i}(1,1) + f_{2i}(1,1) = 0$ *for* $i = 1, 2$, *and* $f_{i1}(1,1) + f_{i2}(1,1) = 0$ *for* $i = 1, 2$.

(iii) *f is zero on the semigroup generated by*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} t & t \\ 1-t & 1-t \end{pmatrix}$$

for some t with $t \neq t^2$ if and only if:

(a) *the system of eight equations*

$$f_{ij}^{(1,1)}(\xi, 1) = 0, \quad f_{ij}^{(1,1)}\Big(\frac{1}{\xi}, 1\Big) = 0$$

is satisfied for some $\xi = t \neq t^2$; and

(b) $f_{1i}(1,1) + f_{2i}(1,1) = 0$ *for* $i = 1, 2$, *and* $f_{i1}(1,1) + f(_{i2}(1,1) = 0$ *for* $i = 1, 2$.

Proof. (i) If f is zero on $\mathcal{S}$, then $f(A, B) = 0$ implies, together with $A^2 = B^2 = 0$ and the fact that AB and BA are linearly independent, that the coefficients of the two monomials $ABAB\cdots$ and $BABA\cdots$ are zero, which is the same as (a). Clearly, $f(1,1) = 0$ yields $f(A, A) = 0$. Now if $\min(r, s) \geqslant 2$, then (b) is vacuously satisfied. Assume $r = 1$. The equation $f(A, AB) = 0$, together with $A^2 = 0$ and $(AB)^2 = AB$, implies that the coefficient of $y^s x$ is zero. Similarly, $f(A, BA) = 0$ implies, since $A^2 = 0$ and $(BA)^2 = BA$, that the coefficient of xy^s is zero.

If $s = 1$, we interchange x and y in the argument just given to verify that the coefficients of yx^r and $x^r y$ are zero, proving (b).

For the converse, assume (a) and (b) hold. The equation $f(1,1) = 0$ implies $f(S, S) = 0$ for every S in the semigroup

$$\mathcal{S} = \{A, B, AB = AB = (AB)^2, \quad BA = (BA)^2, \quad 0\}$$

generated by A and B. We must show that $f(S, T) = 0$ for every nonzero and distinct S and T in $\mathcal{S}$. If S and T are both nilpotent, then (a) implies the desired equation, because every monomial in f contains either x^m or y^m with $m \geqslant 2$. If S and T are both idempotents, then $ST = TS = 0$ implies $f(S, T) = 0$. It remains to verify the case of one nilpotent and one idempotent. By interchanging x and y we can assume $S^2 = 0$. By a simultaneous similarity, we can also assume $S = A$, reducing the required checking to $f(A, AB)$ and $f(A, BA)$. In these instances, either $ST = 0$ or $TS = 0$. But (b) quarantees that every monomial in f with nonzero coefficient contains both subwords xy and yx, yielding $f(S, T) = 0$.

(ii) If f is zero on $\mathcal{S}$, then the equations in (a) are easily verified by considering $f(A, B) = 0$ and $f(B, A) = 0$ together with the linear independence of the nonzero members A, B, AB and BA of $\mathcal{S}$. Applying f to the idempotents

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

in $\mathcal{S}$, and observing that $(BA)(AB) = 0$, we conclude that the coefficient of $x^r y^s$ is zero. Similarly, $f(BA, AB) = 0$ implies that the coefficient of $y^s x^r$ is zero.

Applying f to (B, AB) and observing that $B(AB) = B$ and $(AB)B = AB$, together with the independence of B and AB, yields

$$f_{11}(1,1) + f_{12}(1,1) = 0 \quad \text{and} \quad f_{21}(1,1) + f_{22}(1,1) = 0.$$

Similarly, the two idempotents B and BA satisfy $B(BA) = BA$ and $(BA)B = B$, yielding

$$f_{11}(1,1) + f_{21}(1,1) = 0 \quad \text{and} \quad f_{12}(1,1) + f_{22}(1,1) = 0.$$

Thus (b) is satisfied.

To prove the converse, assume (a) and (b). We must prove that $f(R,S) = 0$ for all pairs from the set. Now as seen above, $f(A,B) = f(B,A) = 0$ by (a). The preceding calculations also show that $f(R,S) = 0$ whenever (R,S) satisfies any one of the following conditions:

(1) $RS = 0$;
(2) $SR = 0$;
(3) $R^2 = R$, $S^2 = S$, $RS = S$; and
(4) $R^2 = R$, $S^2 = S$, $RS = R$.

It is easy to see that this exhausts all the remaining pairs (R,S) with $R \neq S$. But the equation $f(S,S) = 0$ for all S in $\mathcal{S}$ follows from $f(1,1) = 0$, which is a direct consequence of (b).

(iii) Before the proof in this case, we make a few observations on the semigroup $\mathcal{S}_0$ generated by the two given idempotents A and B. The semigroup $\mathbb{C}\mathcal{S}_0$ contains the idempotents $C = AB/t$ and $D = BA/t$. It is easy to see that the only irreducible ordered pairs of idempotents in $\mathbb{C}\mathcal{S}_0$ are $(A,B), (B,A), (C,D)$ and (D,C). Of these, the first two are (simultaneously) similar: just note that the invertible matrix

$$T = \begin{pmatrix} t & t \\ 1-t & -t \end{pmatrix}$$

satisfies $AT = TB$ and $BT = TA$. The second two pairs are also similar with the same T. But (C,D) is similar to (A,B'), where B' is obtained from B by replacing t with $1/t$ with

$$R = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

we obtain $AR = RC$ and

$$B'R = \begin{pmatrix} \frac{1}{t} & \frac{1}{t} \\ 1-\frac{1}{t} & 1-\frac{1}{t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1-t}{t} & 0 \end{pmatrix} = RD.$$

Thus unless $t = -1$, there are two distinct irreducible ordered pairs of idempotents up to similarity, one corresponding to t and another to $1/t$. In particular, f is zero on the semigroup generated by (A,B) if and only if it is zero on the semigroup generated by (A,B'). (We have of course used the fact that a homogeneous polynomial is zero on $\mathcal{S}$ if and only if it is zero on $\mathbb{C}\mathcal{S}$.)

Now assume f is zero on $\mathcal{S}$. It follows from $f(A,B) = 0$ and the linear independence of A, B, AB, and BA that $f_{ij}^{(1,1)}(A,B) = 0$ for $i,j = 1,2$. Note that $ABA = tA$. Considering $f_{11}^{(1,1)}$ first, we observe that for some scalars α_i (which are sums of coefficients from the original polynomial f),

$$\begin{aligned} f_{11}^{(1,1)}(A,B) &= \alpha_1 ABA + \alpha_2 ABABA + \cdots = (\alpha_1 t + \alpha_2 t^2 + \cdots)A \\ &= \frac{(\alpha_1 t^2 + \alpha_2 t^3 + \cdots)A}{t} = \frac{f_{11}^{1,1}(t,1)A}{t} = 0. \end{aligned}$$

Thus $f_{11}^{(1,1)}(t,1) = 0$. It follows from the preceding paragraph, using (A,B') instead of (A,B), that $f_{11}^{(1,1)}(1/t,1) = 0$ as well. Next we use $(AB)^2 = tAB$ to get, for some β_i,

$$f_{12}^{(1,1)}(A,B) = \beta_1 AB + \beta_2 ABABA + \cdots = \frac{(\beta_1 t + \beta_2 t^2 \cdots)AB}{t} = \frac{f_{12}^{(1,1)}(t,1)AB}{t} = 0,$$

which yields $f_{12}^{(1,1)}(t,1) = 0$. Similarly, we obtain $f_{12}^{(1,1)}(1/t,1) = 0$.

The proofs for the remaining equations are similar. Thus (a) is proved. To show (b), note that $f(A,C) = 0$. Now A and C are idempotents with $AC = C$ and $CA = A$. As we saw in the proof of (ii) above, this by itself yields

$$f_{11}(1,1) + f_{21}(1,1) = 0 \quad \text{and} \quad f_{12}(1,1) + f_{22}(1,1) = 0.$$

Similarly, applying f to the idempotents A and $D = BA/t$ (which satisfy the relations $AD = A$ and $DA = D$) we obtain the other equations in (b).

To verify the converse, assume (a) and (b). The semigroup $\mathcal{S}$ is contained in scalar multiples of the four matrices $A, B, C = AB/t$, and $D = BA/t$. Now if R and S are (not necessarily distinct) idempotents in $\{A, B, C, D\}$, then it is easily seen that every pair (R,S) other than

$$(A,B), (B,A), (C,D), \quad \text{and} \quad (D,C),$$

satisfies either $\{RS = S,\ SR = R\}$ or $\{RS = R,\ SR = S\}$. In all these cases (b) implies $f(R,S) = 0$ as in the proof of (ii) above. Letting B' be as defined above, we deduce from (a) that $f(A,B) = f(A,B') = 0$. But we have shown that each of the remaining (irreducible) pairs is similar either to (A,B) or to (A,B'). ∎

Note that the condition (a) in (i) automatically holds whenever $|r-s| \geqslant 2$.

The reason for splitting the conditions in each of the three cases into (a) and (b) in the statement of the preceding lemma is that we only need (a) in determining which polynomials f with $f(1,1) = 0$ are nilpotent on a semigroup, as in the next lemma.

LEMMA 2.3. *Let g be a noncommutative polynomial of homogeneous degrees r and s in x and y respectively, with $g(1,1) = 0$, and let $f(x,y) = (g(x,y))^2$. Let $\mathcal{S}$ be the semigroup generated by one of the pairs in* (i), (ii), *or* (iii) *of Lemma* 2.2. *Then g is nilpotent on $\mathcal{S}$ if and only if the corresponding condition* (a) *of that lemma holds for f in each case.*

Proof. If g is nilpotent on $\mathcal{S}$, then f is obviously zero on $\mathcal{S}$, because $\mathcal{S} \subseteq \mathcal{M}_2(\mathbb{C})$; this implies (a) for f in each case. (It also implies (b).)

For the converse, assume (a). It follows, as in the proof of the preceding lemma that

$$f(A,B) = f(B,A) = 0$$

in each case. In case (iii), it also follows that $f(AB, BA) = f(BA, AB) = 0$ as in that lemma. Now it is easy to see that all the remaining pairs (R,S) from $\{A, B, AB, BA\}$ in all the three cases are reducible, i.e., triangularizable. Since $g(1,1) = 0$, this means that the diagonal entries of $g(R,S)$ in the triangularization are zero for all these pairs, and hence $(g(R,S))^2 = 0$. ∎

COROLLARY 2.4. *Let f be a polynomial homogeneous in x and in y, whose square does not satisfy* (a) *in any of the conditions* (i), (ii), *or* (iii) *of Lemma* 2.2. *If f is nilpotent on a semigroup $\mathcal{S}$ of* 2×2 *matrices of rank* $\leqslant 1$, *then $\mathcal{S}$ is reducible.*

Proof. This is just Lemma 2.3 together with the observation that if $\mathcal{S}$ is irreducible, then Lemma 1.1 is applicable to $\overline{\mathbb{R}^+\mathcal{S}}$ and the semigroup $\mathcal{S}_0$ of that lemma coincides with one of the minimal doubly generated, semigroups satisfying (i), (ii), or (iii). Note that the case in which the coefficients of f (and thus those of f^2) do not add up to zero is trivial as mentioned in the introduction: in that case every member of $\mathcal{S}$ is nilpotent, implying that $\mathcal{S}$ is reducible without appeal to the remaining hypotheses. ∎

3. THE GENERAL CASE OF COMPACT OPERATORS

In view of reductions carried out above, it is convenient to introduce an adjective for polynomials that are never nilpotent on an irreducible semigroup of singular 2×2 matrices.

DEFINITION 3.1. Let f be homogeneous in each of its two variables. We say that f is *rigid* if its square does not satisfy the condition (a) in any of the parts (i), (ii) and (iii) of Lemma 2.2.

It is not hard to check polynomials for rigidity. For example, $(xy)^m - (yx)^m$ is rigid for every positive integer m; so is $(xy)^m + yg(x,y)x$ for any polynomial g that is homogeneous of degree $m-1$ in each variable. None of the polynomials $f_n(x,y)$ obtained from standard polynomials in Section 0 is rigid for $n \geqslant 2$. A sufficient condition for rigidity of f is that its coefficients do not add up to zero; for if they do, then f can only be nilpotent on a semigroup when every member of the semigroup is nilpotent. A necessary condition for rigidity is that f be of the same degree in x as in y. (Otherwise f^2 satisfies (a) in part (i) of Lemma 2.2. We shall come back to this later.)

We start with a simple and immediate consequence of the results above. Recall that a family of operators is called *triangularizable* if the lattice of its invariant subspaces contains a maximal subspace chain. (If the chain has members $\mathcal{M}$ and

$\mathcal{N}$ with $\mathcal{M} \subseteq \mathcal{N}$ and no member between the two, then $\mathcal{M}$ has codimension at most one in $\mathcal{N}$.)

PROPOSITION 3.2. *Let $\mathcal{S}$ be a semigroup of operators of rank at most one on a Banach space. If a rigid polynomial is nilpotent (equivalently quasinilpotent) on $\mathcal{S}$, then $\mathcal{S}$ is triangularizable.*

Proof. If $\mathcal{M}$ and $\mathcal{N}$ are invariant subspaces of $\mathcal{S}$ with $\mathcal{M} \subseteq \mathcal{N}$, then the quotient semigroup induced on $\mathcal{N}/\mathcal{M}$ consists of operators of rank at most one, and any polynomial that is nilpotent on $\mathcal{S}$ is also nilpotent on this quotient. Thus we need only show that $\mathcal{S}$ is reducible. (See the Triangularization Lemma in [8], p. 155.) But this follows from Corollary 1.5 together with the rigidity hypothesis. ∎

It is easy to see that for any given nonrigid polynomial f there is an irreducible semigroup of rank-one operators, acting on any given Banach space, on which f is nilpotent. In the extreme cases of nonrigidity, a polynomial can be nilpotent on every pair of operators of rank one on a Banach space:

EXAMPLE 3.3. Let f be any polynomial with $f(1,1) = 0$ and let f^2 satisfy all the three conditions (a) in parts (i), (ii), and (iii) of Lemma 2.2. (The polynomials $x^m y^n x - y^n x^{m+1}$ for positive integers m and n are easily checked samples.) Then f is nilpotent on the entire semigroup of operators of rank at most one. The polynomials $x^{m+1} y^n x - x y^n x^{m+1}$ are identically zero on this semigroup.

Proof. Let S and T be any operators of rank one. Since $f(1,1) = 0$, we see that $f(S,T)$ is nilpotent if S and T are simultaneously triangularizable. If they are not, then let $\mathcal{M}$ be the two-dimensional span of their ranges and $\mathcal{M}'$ a complement of $\mathcal{M}$ in the underlying Banach space $\mathcal{X}$. Observe that relative to this decomposition, S and T have matrices of the form

$$\begin{pmatrix} S_0 & * \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} T_0 & * \\ 0 & 0 \end{pmatrix}$$

and that the pair (S_0, T_0) is irreducible. Then it is easily verified that, after multiplication by scalars, S_0 and T_0 have the forms

$$\begin{pmatrix} 0 & 1 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \gamma & 0 \\ 1 & 0 \end{pmatrix},$$

which makes (S_0, T_0) simultaneously similar to the generators (A, B) in one of the case (i), (ii), or (iii) of Lemma 1.1. (See the remarks preceding the proof of that lemma.) Hence $f(S_0, T_0)$ is nilpotent by hypothesis, and so is

$$f(S,T) = \begin{pmatrix} f(S_0, T_0) & * \\ 0 & 0 \end{pmatrix}. \quad ∎$$

In the general case (i.e., when the minimal positive rank present in a semigroup is greater than one, the hypothesis of rigidity does not necessarily yield triangularizability, but an appropriate generalization. Recall that if $\mathcal{C}$ is a maximal chain of invariant subspaces for a family $\mathcal{F}$ of operators, then $\mathcal{C}$ gives rise to a *block triangularization of* $\mathcal{F}$ whose *diagonal blocks* are irreducible; these are quotients of $\mathcal{F}$ induced on $\mathcal{N}/\mathcal{M}$, where $\mathcal{M}$ and $\mathcal{N}$ are in $\mathcal{C}$ with $\mathcal{M} \subset \mathcal{N}$ and no member of $\mathcal{C}$ is properly between $\mathcal{M}$ and $\mathcal{N}$. (If there are no such adjacent members in $\mathcal{C}$, i.e, if $\mathcal{C}$ is a continuous chain, then there are no diagonal blocks, of course.)

THEOREM 3.4. *Let $\mathcal{S}$ be a semigroup of compact operators on a Banach space. If a rigid polynomial is quasinilpotent on $\mathcal{S}$, then $\mathcal{S}$ has a block triangularization in which every diagonal block is contained in some $\mathbb{C}\mathcal{G}$, where $\mathcal{G}$ is a finite group acting on a finite-dimensional space.*

Proof. By Corollary 1.4, together with the rigidity assumption, $\mathcal{S}$ is reducible if the underlying Banach space is infinite-dimensional. Apply Zorn's Lemma to obtain a maximal chain $\mathcal{C}$ of invariant subspaces for $\mathcal{S}$. Since quotient algebras induced on $\mathcal{N}/\mathcal{M}$ satisfy the hypothesis for all $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{C}$ with $\mathcal{M} \subset \mathcal{N}$, it follows that if $\mathcal{C}$ does not contain a member between $\mathcal{M}$ and $\mathcal{N}$, then $\mathcal{N}/\mathcal{M}$ is finite-dimensional. For the rest of the proof, we can assume with no loss of generality that $\mathcal{S}$ acts on $\mathcal{N}/\mathcal{M}$.

We now have a rigid polynomial f that is nilpotent on an irreducible $\mathcal{S} \subseteq \mathcal{M}_n(\mathbb{C})$. Not only is $g = f^n$ zero on $\overline{\mathbb{C}\mathcal{S}}$, but it turns out to be zero on a larger semigroup if $\mathcal{S}$ is not essentially finite, as we now proceed to show.

Let Φ be a ring automorphism of matrices induced by a (not necessarily continuous) field automorphism ϕ of $\mathbb{C}$, i.e., $\Phi(M)$ is obtained from M by applying ϕ to M entrywise. Since the coefficients of the polynomial g may change under ϕ, we cannot conclude that g is zero on $\Phi(\mathcal{S})$, but it is certainly zero on the semigroup $\Phi^{-1}(\overline{\mathbb{C}\Phi(\mathcal{S})})$, which contains $\mathcal{S}$. It then follows from the Finiteness Lemma in [8], p. 75, that $\mathcal{S}$ is contained in a semigroup $\widehat{\mathcal{S}} \subseteq \mathcal{M}_n(\mathbb{C})$ on which g is zero, and there is a nonzero idempotent in $\widehat{\mathcal{S}}$ such that the restriction of $E\widehat{\mathcal{S}}E$ to the range of E is, up to simultaneous similarity, of the form $\mathbb{C}\mathcal{G}$ with $\mathcal{G}$ a finite unitary group.

All we have to do now is to show that E is the identity matrix. But otherwise, Lemma 1.1 would be applicable to $\widehat{\mathcal{S}}$ implying that g is zero on an irreducible semigroup of 2×2 matrices of rank at most one, which contradicts the rigidity of f. ∎

Note that the finite group $\mathcal{G}$ in the theorem depends on the diagonal block. A specialization to finite dimensions follows.

COROLLARY 3.5. *Let $\mathcal{S}$ be a semigroup of operators on a finite-dimensional space, and assume that a rigid polynomial is nilpotent on $\mathcal{S}$. Then either $\mathcal{S}$ has an invariant subspace or $\mathcal{S} \subseteq \mathbb{C}\mathcal{G}$, where $\mathcal{G}$ is similar to a finite unitary group.*

The following result is just a rephrasing of Theorem 3.4.

COROLLARY 3.6. *If an abstract semigroup Σ has a faithful, irreducible representation as compact operators on which a rigid polynomial is quasinilpotent, then there is a group G that is finite modulo its centre such that $\Sigma \subset G$ (if Σ does not have a zero) or $\Sigma \subseteq G \cup \{0\}$ (if it does).*

Before giving a further reduction of the finite-group case, we give two more corollaries of Theorem 3.4.

COROLLARY 3.7. *Let $\mathcal{S}$ be a semigroup of (not necessarily compact) operators on a Hilbert space on which a rigid polynomial is quasinilpotent. Assume that $\mathcal{S}$ contains a cyclic, diagonalizable, compact operator K whose eigenvalues either:*

(a) *have distinct moduli; or*

(b) *form an algebraically independent set (over $\mathbb{Q}$).*

Then $\mathcal{S}$ is triangularizable. In particular, $\mathcal{S}$ is commutative if it is self-adjoint.

Proof. Let $\mathcal{S}_0$ be the ideal of $\mathcal{S}$ consisting of compact operators. We first show that $\mathcal{S}_0$ is triangularizable. Since $\mathcal{S}_0$ has a block triangularization by Theorem 3.4, we must verify that every diagonal block $\mathcal{S}_1$ given by that theorem acts on a one-dimensional space. Observe that $\mathcal{S}_1$ contains a quotient K_1 of the compact operator K, and that K_1 also satisfies the hypotheses listed for K. Hence $\mathbb{C}\mathcal{G}$ cannot contain K_1 if $\mathcal{G}$ is a finite group, unless its underlying space is one-dimensional. Since $\mathcal{S}_0$ is an ideal of $\mathcal{S}$, we have shown that $\mathcal{S}$ is reducible, but since every quotient of $\mathcal{S}$ contains an operator of type K, the triangularizability of $\mathcal{S}$ follows (e.g., by the Triangularization Lemma in [8], p. 155).

If $\mathcal{S} = \mathcal{S}^*$, then $SK - KS$ and $(SK - KS)^*$ are compact and triangularizable together with all of $\mathcal{S}$. Note that the diagonal blocks of SK and KS (acting on one-dimensional spaces) are the same; so are those of S^*K^* and K^*S^*. Thus the diagonal blocks of $(SK - KS)(SK - KS)^*$ are all zero, implying that this operator is quasinilpotent by Ringrose's theorem ([11] or [8], p. 156). But it is also Hermitian. Hence $SK = KS$ for all $S \in \mathcal{S}$. Since K is cyclic, this shows that $\mathcal{S}$ is commutative (and is contained in the commutant of K). ∎

The following result generalizes Theorem 3.2 in finite dimensions. It has an obvious adaptation to the infinite-dimensional case.

COROLLARY 3.8. *Let $\mathcal{S}$ be a semigroup of operators on an n-dimensional space $\mathcal{X}$ on which a rigid polynomial is nilpotent. If k is the maximal rank of members of $\mathcal{S}$, then $\mathcal{S}$ has a chain*

$$\{0\} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_m = \mathcal{X}$$

of distinct invariant subspaces, where $m \geqslant n/k$.

Proof. If $\mathcal{C}$ is a maximal chain of invariant subspaces for $\mathcal{S}$, each irreducible quotient semigroup induced by $\mathcal{S}$ relative to $\mathcal{C}$ that acts on $\mathbb{C}^j$ with $j \geqslant 2$ contains invertible operators by Theorem 3.4 and thus acts on a space of dimension at most k. ∎

4. FURTHER REDUCTION OF THE GROUP CASE WITH APPLICATIONS

The main theorem of Section 3 has reduced our general triangularizability question to this: given a homogeneous polynomial, can we determine whether or not it is zero (or nilpotent) on some finite irreducible group of $n \times n$ matrices (with $n \geqslant 2$)? The following result shows that we need only check this for very special finite groups.

LEMMA 4.1. *A homogeneous polynomial is zero (resp. nilpotent) on some finite irreducible unitary group $\mathcal{F} \subseteq \mathcal{M}_n(\mathbb{C})$ with $n \geqslant 2$ if and only if it is zero (respectively nilpotent) on a group of the form $\mathcal{G}(p,q,A)$ as described in Example* 0.1.

Proof. Let $\mathcal{G}$ be a minimal nonabelian subgroup of $\mathcal{F}$. Then it follows from O.J. Schmidt's theorem ([13]) that $\mathcal{G}$ is a solvable group, and thus it contains a normal (and abelian) subgroup of some prime index p. It is easy to verify that $\mathcal{G}$ has an invariant subspace $\mathcal{M}$ such that $\mathcal{G}|\mathcal{M}$ is generated by a diagonal operator and a cyclic operator, i.e., αA and βB, where A and B are as in Example 0.1, and α and β are scalars. (A proof of this is given in [8], p. 85.) We conclude that f^m is zero on $\mathcal{F}$ if and only if it is zero on some $\mathcal{G}(p,q,A)$. ∎

Unlike the criteria for rigidity of a polynomial, the above condition is not very easy to check for a general polynomial (in terms of its coefficients, say). For one thing, the relation between p and q enters the picture: If $p = q$, for example, it is not hard to see that $\mathcal{G}(p,q,A)$ contains $\mathcal{G}(p,p,A')$, where A' has the pleasant feature of a geometric progression, i.e., $A' = \mathrm{diag}\ (1, \omega, \ldots, \omega^{p-1})$ with ω a primitive root of 1; checking a polynomial against this smaller group is then not too complicated. If p divides $q-1$, so that the finite field with q elements has a nontrivial p-th root λ of unity, again a reduction in checking is possible to a subgroup $\mathcal{G}(p,q,A')$, where

$$A' = \mathrm{diag}\ (w, w^{\lambda}, w^{\lambda^2}, \ldots, w^{\lambda^{p-1}}).$$

In the case of an arbitrary pair (p,q), criteria for a general polynomial are quite cumbersome to state. However, sufficient conditions are possible to obtain in special cases, resulting in reducibility and triangularizability theorems. The case of degree one, where the only rigid polynomials are those of the form $axy + byx$ (and where $xy - yx$ is the only interesting subcase) is known to give a triangularizability theorem ([3],[9]) as mentioned above. We now present a direct generalization of this result. For an easily applicable special case see Corollary 4.4 below.

THEOREM 4.2. *Let f be a rigid polynomial of the form $f(x,y) = g(xy,yx)$ and assume that $g(t,1)$ is not divisible by $t^p - 1$ for any prime p. If f is quasinilpotent on a semigroup of compact operators, then $\mathcal{S}$ is triangularizable.*

Proof. We must only show that f cannot be nilpotent on any of the finite groups $\mathcal{G}(p,q,A)$. Note, incidentally, that we shall not need the rigidity hypothesis for this part of the proof. Suppose f is nilpotent on some such group and let A and B be the generators as in Example 0.1. Observe that for any diagonal member T of the group, the matrix

$$f(B^{-1}T, B) = g(B^{-1}TB, T)$$

is nilpotent. Since $B^{-1}TB$ and T are commuting diagonal matrices, this implies that $g(B^{-1}TB, T) = 0$. Now by taking $T = B^{-m}AB^m$ for an appropriate integer m, and multiplying by a scalar, we can assume that the $(1,1)$ and $(2,2)$ entries of the matrix T are 1 and θ respectively, where θ is a primitive q-th root of unity. The $(1,1)$ entry of $g(B^{-1}TB, T)$ is then $g(\theta, 1)$. We have shown that $g(\theta, 1) = 0$. Replacing T by T^k, we also deduce that $g(B^{-1}T^kB, T) = 0$, which implies

$$g(\theta^k, 1) = 0 \quad \text{for all integers } k.$$

This means that the (commutative) polynomial $g(x, 1)$ is divisible by $x^q - 1$, which contradicts the hypothesis. ∎

The next example concerns a quadratic case of the polynomials discussed above.

EXAMPLE 4.3. Let $f(x, y) = (xy)^2 + axy^2x + byx^2y$ with a and b in $\mathbb{C}$. If f is quasinilpotent on a semigroup of compact operators, then $\mathcal{S}$ is triangularizable.

Proof. As before, we need only consider the case $1 + a + b = 0$. Now let $g(x, y) = x^2 + axy + byx$, so that $f(x, y) = g(xy, yx)$. Since $g(x, 1) = x^2 - x$ is not divisible by $x^p - 1$ for any prime p, we shall be done if we prove that f is rigid.

Let $h = f^2$. Since the coefficient of $(xy)^4$ in h is one, h does not satisfy (a) of (i) in Lemma 2.2. Also, an easy calculation shows that

$$h_{11}^{(0,1)}(1, 1) = a \quad \text{and} \quad h_{22}^{(1,0)}(1, 1) = b.$$

Since a and b are not both zero, h fails to satisfy (a) of (ii) in Lemma 2.2. Another calculation yields

$$h_{12}^{(1,1)}(\xi, 1) = \xi^3(\xi + ab - 1) \quad \text{and} \quad h_{21}^{(1,1)}(\xi, 1) = ab\xi^3.$$

Now if $ab \neq 0$, then $h_{21}^{(1,1)}(\xi, 1) = 0$ implies $\xi = 0$. If ab is zero, then $h_{12}^{(1,1)}(\xi, 1) = 0$ yields $\xi(\xi - 1) = 0$. Thus in either case, the system of equations in Lemma 2.2 (iii) has no solution other than 0 and 1. We have shown that f is rigid. ∎

The next special case of Theorem 4.2 concerns a family of polynomials that are automatically rigid. Note that if $g(\xi, \eta)$ is homogeneous (not necessarily separately in ξ and η) then $g(xy, yx)$ is homogeneous of the same degree r in each of the variables x and y.

COROLLARY 4.4. *Let* $g(\xi, \eta) = \sum_{j=0}^{k} a_j \xi^j \eta^{k-j}$ *with* $a_0 a_k \neq 0$, *and assume that* $g(\xi, 1)$ *is not divisible by* $\xi^p - 1$ *for any prime* p. *If* $g(xy, yx)$ *is quasinilpotent on a semigroup* $\mathcal{S}$ *of compact operators, then* $\mathcal{S}$ *is triangularizable.*

Proof. We must only verify that $f(x,y) = g(xy,yx)$ is rigid. But if $h = f^2$, it is easily seen that h does not satisfy (i)(a) of Lemma 2.2, because the coefficient of $(xy)^{2k}$ is nonzero. Also, since

$$h_{22}(x,y) = a_0 a_k (yx)^k (xy)^k,$$

we see that $h_{22}^{(1,0)}(1,1) = a_0 a_k \neq 0$, so (ii)(a) of Lemma 2.2 is not satisfied. Finally, $f_{22}^{(1,1)}(t,1) = a_0 a_k t^{2k-1}$, which shows that the system in (iii)(a) of the lemma cannot have a nonzero solution. ∎

The nondivisibility hypothesis above is necessary.

EXAMPLE 4.5. Let $g(\xi,\eta) = \sum_{j=0}^{k} a_j \xi^j \eta^{k-j}$ and assume $\xi^p - 1$ divides $g(\xi,1)$ for a prime p. Then $g(xy,yx)$ is zero on $\mathcal{G}(p,p,A)$ with

$$A = \text{diag}\ (1,\omega,\ldots,\omega^{p-1})$$

and ω a primitive root of unity.

Proof. A straightforward calculation shows that for any pair S and T in this group, there exists some power $\theta = \omega^m$ of w such that $ST = \theta TS$. Thus

$$g(ST,TS) = \Sigma a_j (ST)^j (TS)^{k-j} = \Sigma a_j \theta^j (TS)^k = g(\theta,1)(TS)^k = 0$$

by hypothesis. ∎

We conclude this section with another immediate consequence of Theorem 4.2.

COROLLARY 4.6. *No semigroup has a nontrivial irreducible representation in $\mathcal{M}_n(\mathbb{C})$ satisfying the identity*

$$(g(xy,yx))^n = 0$$

if $g(xy,yx)$ is a rigid polynomial in x and y and $g(t,1)$ is not divisible by $t^p - 1$ for any prime p.

5. ON THE CASE OF NONRIGID POLYNOMIALS

There are nonrigid polynomials (e.g., those of the form $g(xy,yx)$ discussed in the proof of Theorem 4.2) that cannot be nilpotent on any of the minimal finite groups $\mathcal{G}(p,q,A)$. For the results in this section the full force of rigidity is not needed.

PROPOSITION 5.1. *Let f be any polynomial, homogeneous (of possibly different degrees) in x and in y. Assume that f is not nilpotent on any group $\mathcal{G}(p,q,A)$. If $\mathcal{S}$ is a maximal irreducible semigroup of compact operators on which f is quasinilpotent, then $\mathcal{S}$ contains matrices of rank one.*

Proof. Suppose $\mathcal{S}$ did not contain an operator of rank one. By [14], $\mathcal{S}$ does not consist entirely of quasinilpotent operators. Since $\mathcal{S} = \overline{\mathbb{C}\mathcal{S}}$ by maximality, it contains nonzero idempotents. Let E be a minimal nonzero idempotent in $\mathcal{S}$. Then $E\mathcal{S}E$, when restricted to the range of E, is of the form $\mathbb{C}\mathcal{G}$, where $\mathcal{G}$ is a finite group. (This can be proved by an argument similar to the one given in the proof of Theorem 3.4, making use of the Finiteness Lemma ([8], p. 75); note that rigidity is not needed for this application.) Now since E has rank greater than one, and since f is nilpotent on $\mathcal{G}$, we obtain a contradiction by applying Lemma 4.1. ∎

COROLLARY 5.2. *Let $f(x,y)$ be a homogeneous polynomial of the form $g(xy, yx)$ such that $g(t,1)$ is not divisible by $t^p - 1$ for any prime p. Then any maximal irreducible semigroup of compact operators on which f is quasinilpotent contains rank-one matrices.*

Proof. Such a polynomial cannot be nilpotent on any $\mathcal{G}(p,q,A)$ by the argument given in the proof of Theorem 4.2, which did not use the rigidity hypothesis. ∎

The rigidity conditions can be substantially relaxed when we apply the results above to groups. We need the following lemma. Its proof uses the same ideas as in that of the Finiteness Lemma ([8], p. 75), but since we require a distinct conclusion, not obtainable directly from that lemma, we include a proof. Note, also, that the group in the following result is not required to be irreducible.

LEMMA 5.3. *Let $\mathcal{G}$ be any group in $\mathcal{M}_k(\mathbb{C})$ with $k \geqslant 2$ on which a homogeneous polynomial f is nilpotent. If $\mathcal{G}$ is not contained in $\mathbb{C}\mathcal{F}$ for some finite group $\mathcal{F}$, then $\mathcal{G}$ can be extended to a semigroup $\mathcal{S}$ on which f is nilpotent such that $\mathcal{S}$ contains either a nonzero nilpotent or two nonzero idempotents P and Q with $PQ = QP = 0$. In particular, the latter occurs if $\mathcal{G}$ is a unitary group.*

Proof. By hypothesis, the group

$$\mathcal{G}_0 = \{A \in \mathbb{C}\mathcal{G} : \det A = 1\}$$

is not finite. We can assume, with no loss of generality that $\mathcal{G} = \mathcal{G}_0 = \overline{\mathcal{G}}_0$.

(a) First assume that $\mathcal{G}$ is not bounded. Choose $S_n \in \mathcal{G}$ with $\|S_n\| \to \infty$. Pass to a subsequence to assume $S_n/\|S_n\|$ is convergent to T, and note that $\|T\| = 1$ and $\det T = 0$. If T is nilpotent, we are done since $T \in \overline{\mathbb{C}\mathcal{G}}$. Otherwise, for large enough n, the matrix $A = S_n$ is, after a similarity, of the form

$$\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

with $\rho(B) > \rho(C)$, where ρ denotes the spectral radius. Thus the matrices $A_1 = A/\rho(B)$ and $A_2 = A^{-1}/\rho(C^{-1})$ in $\mathbb{C}\mathcal{G}$ both have spectral radius 1. Since we can assume with no loss of generality that $\overline{\mathbb{C}\mathcal{G}}$ does not contain nonzero nilpotents, it follows that $\{A_1^n\}$ and $\{A_2^n\}$ have subsequences converging to nonzero idempotents P and Q respectively. (See, e.g., Lemma 3.4.2 of [8], p. 62.) Since $(C/\rho(B))^n$ and $(B^{-1}/\rho(C^{-1}))^n$ both converge to zero, we get $PQ = QP = 0$.

(b) To complete the proof of the lemma we now assume that $\mathcal{G}$ is compact, and thus, without loss of generality, a unitary group. Since $\mathcal{G}$ is not finite, it does not have a finite exponent. In particular, the set

$$\Omega = \left\{ \frac{\lambda}{\mu} : \lambda \in \sigma(A), \mu \in \sigma(A), A \in \mathcal{G} \right\}$$

is infinite. Thus the set $\{\omega^k : \omega \in \Omega\}$ is dense in $\{z : |z| = 1\}$. It follows that $\overline{\mathcal{G}}$ has a member S with eigenvalues λ and μ such that λ/μ is transcendental. The operator $A = S/\mu$ then has 1 and $\alpha = \lambda/\mu$ in its spectrum.

Fix a transcendental number β with $|\beta| < 1$. There exists a field automorphism ϕ of $\mathbb{C}$ such that $\phi(\alpha) = \beta$ and $\phi(\beta) = \alpha$. The induced ring automorphism Φ on $\mathcal{M}_n(\mathbb{C})$ then takes $\overline{\mathbb{C}\mathcal{G}}$ to $\mathcal{G}_1 \cup \{0\}$, where $\mathcal{G}_1$ is a group. If the minimal and characteristic polynomials of $T \in \overline{\mathbb{C}\mathcal{G}}$ are $f_1 = \Sigma a_j x^j$ and $f_2 = \Sigma b_j x^j$ respectively, then those of $\Phi(T)$ are $\phi(f_1) = \Sigma\phi(a_j)x^j$ and $\phi(f_2) = \Sigma\phi(b_j)x^j$. In particular, $\Phi(T)$ is diagonalizable for all T in $\overline{\mathbb{C}\mathcal{G}}$. Now $B = \Phi(A)$ has eigenvalues $\phi(1) = 1$ and $\phi(\alpha) = \beta$. Thus powers of $B/\rho(B)$ and $B^{-1}/\rho(B^{-1})$ have subsequences approaching idempotents P and Q in the closure of $\Phi(\overline{\mathbb{C}\mathcal{G}})$ with $PQ = QP = 0$. We deduce, finally, that the semigroup

$$\mathcal{S} = \Phi^{-1}(\overline{\Phi(\overline{\mathbb{C}\mathcal{G}})})$$

which is an extension of $\mathcal{G}$, contains nonzero idempotents $E = \Phi^{-1}(P)$ and $F = \Phi^{-1}(Q)$ with $EF = FE = 0$. Since f is nilpotent on $\overline{\mathbb{C}\mathcal{G}}$, $\phi(f)$ is nilpotent on $\Phi(\overline{\mathbb{C}\mathcal{G}})$ and thus on its closure. Hence $\phi^{-1}(\phi(f))$ is nilpotent on $\mathcal{S}$. ∎

COROLLARY 5.4. *If in Lemma* 5.3, $\mathcal{G}$ *is also assumed irreducible, then the extension* $\mathcal{S}$ *contains a nonzero nilpotent.*

Proof. We need only observe that if P and Q are idempotents obtained in the lemma, then the set $P\mathcal{S}Q$ consists of nilpotents and is different from $\{0\}$ by irreducibility. ∎

In the following result f is only assumed to satisfy the two easily verifiable conditions of rigidity.

COROLLARY 5.5. *Let f be a homogeneous polynomial of the form $g(xy, yx)$ with $g(t, 1)$ not divisible by $t^p - 1$ for any prime p, and assume that f^2 does not satisfy conditions* (i)(a) *or* (ii)(a) *of Lemma* 2.2. *If $\mathcal{G}$ is a group in $\mathcal{M}_n(\mathbb{C})$ on which f is nilpotent, then $\mathcal{G}$ is triangularizable.*

Proof. Since quotients induced by $\mathcal{G}$ on a chain of invariant subspaces are also groups satisfying the hypothesis, we need only show that $\mathcal{G}$ is reducible if $n \geqslant 2$. Assume not. By Lemma 4.1 and the proof of Theorem 4.2, $\mathbb{C}\mathcal{G}$ is not of the form $\mathbb{C}\mathcal{F}$ with $\mathcal{F}$ a finite group. Hence it is not contained in such a $\mathbb{C}\mathcal{F}$ either. Then by the preceding corollary, $\mathcal{G}$ can be extended to a semigroup $\mathcal{S}$ containing a nonzero nilpotent and with f nilpotent on $\mathcal{S}$.

It follows from Lemma 1.3 that $\overline{\mathbb{C}\mathcal{S}}$ gives rise to one of the types (i) and (ii) on which f is nilpotent, which is a contradiction. ∎

For compact groups, only a vestige of rigidity conditions needs to be checked as the next result indicates. If $g(\xi, \eta)$ is any (jointly) homogeneous polynomial of degree r in noncommutative variables ξ and η, then g has two "leading coefficients" $g(1,0)$ and $g(0,1)$, i.e., the coefficients of the terms ξ^r and η^r. Now it is easily seen that the "one-third" of rigidity corresponding to type (i) in Lemma 2.3, i.e., the condition that $f(x,y) = g(xy, yx)$ is not nilpotent on that type, is equivalent to the requirement that $g(1,0)$ and $g(0,1)$ are not both zero.

COROLLARY 5.6. *Let g be any jointly homogeneous polynomial in two variables such that:*

(i) $g(1,0)$ *and* $g(0,1)$ *are not both zero; and*

(ii) $g(t,1)$ *is not divisible by* $t^p - 1$ *for any prime* p.

If a compact group $\mathcal{G}$ has a faithful representation in $\mathbb{C}^n$ on which $g(xy, yx)$ is nilpotent, then $\mathcal{G}$ is abelian.

Proof. Assume with no loss that $\mathcal{G}$ is a group of unitaries contained in $\mathcal{M}_n(\mathbb{C})$. To prove that $\mathcal{G}$ is (simultaneously) diagonalizable, it suffices to show triangularizability. Since the hypotheses are satisfied by quotients as before, we need only show that $\mathcal{G}$ is reducible if $n \geqslant 2$. But assuming otherwise, we deduce from Lemma 5.3 that cG has an extension semigroup containing nonzero idempotents P and Q with $PQ = QP = 0$. This leads, via Lemma 1.3, to the conclusion that g is nilpotent on a semigroup of type (i) in Lemma 1.1. This in turn contradicts the hypothesis that $g(1,0)$ and $g(0,1)$ are not both zero. ∎

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