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VON NEUMANN MODULES, INTERTWINERS AND SELF-DUALITY

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ABSTRACT. Von Neumann modules are self-dual ([9]). We apply the ideas of Muhly, Skeide and Solel [4] of considering von Neumann $\mathcal{B}$-modules as intertwiner spaces for representations of $\mathcal{B}'$ to obtain a new, simple and self-contained proof for self-duality of von Neumann modules. This simplifies also the approach of [4].

KEYWORDS: *C^*-modules, self-duality, von Neumann modules, commutants.*

MSC (2000): 46L08, 46L10, 46H25, 46H15, 46L53, 46L55.

1. PRELIMINARIES

Let E be a Hilbert module over a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ acting (non-degenerately) on the Hilbert space G. We define the Hilbert space $H = E \odot G$ as the interior *tensor product over* $\mathcal{B}$ of the right $\mathcal{B}$-module E and the $\mathcal{B}$-$\mathbb{C}$-module G with inner product $\langle x_1 \odot g_1, x_2 \odot g_2 \rangle = \langle g_1, \langle x_1, x_2 \rangle g_2 \rangle$. Every $x \in E$ gives rise to a mapping $L_x \colon g \mapsto x \odot g$ in $\mathcal{B}(G, H)$ and it is easy to verify that $L_{xb} = L_x b$ and $L_x^* L_y = \langle x, y \rangle$.

We, therefore, may and will identify every Hilbert $\mathcal{B}$-module over a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ as a concrete submodule $E \subset \mathcal{B}(G, H)$ of operators, where $H = E \odot G$. Following Skeide [9], [10] we say E is a *von Neumann $\mathcal{B}$-module*, if E is strongly closed in $\mathcal{B}(G, H)$.

On H we define a (normal unital) representation $\rho' \colon \mathcal{B}' \to \mathcal{B}(H)$ of the commutant $\mathcal{B}'$ of $\mathcal{B}$ by $\rho'(b') = \mathrm{id}_E \odot b'$. (This is well-defined, because b' is a bilinear mapping on the $\mathcal{B}$-$\mathbb{C}$-module G, and also checking normality is routine.) In the special case when E is the GNS-module of a completely positive mapping with values in $\mathcal{B}$ (see Paschke [5]), ρ' is known as *commutant lifting* ([1]).

Following Skeide [8], [9], the *$\mathcal{B}'$-center* of the $\mathcal{B}'$-$\mathcal{B}'$-module $\mathcal{B}(G, H)$ is defined as

$$C_{\mathcal{B}'}(\mathcal{B}(G, H)) = \{x \in \mathcal{B}(G, H) \colon \rho'(b')x = xb', b' \in \mathcal{B}'\}.$$

As observed, for instance, by Goswami [2], it is easy to check that $C_{\mathcal{B}'}(\mathcal{B}(G,H))$ is itself a von Neumann $\mathcal{B}$-module.

Clearly, E is contained in $C_{\mathcal{B}'}(\mathcal{B}(G,H))$.

It is the starting point in [4] to show that E is all of $C_{\mathcal{B}'}(\mathcal{B}(G,H))$. Once known that von Neumann modules are *self dual*, i.e. every bounded right linear mapping $\Phi\colon E \to \mathcal{B}$ (so-called $\mathcal{B}$-*functionals*) has the form $\langle x, \bullet\rangle$ for a (unique) $x \in E$ (see [9] and [10] for a proof using *complete quasi orthonormal systems*, a suitable generalization of orthonormal bases in Hilbert spaces), this task is easy: Like for Hilbert spaces a strongly closed (and, therefore, self-dual) submodule with zero-complement is all. And since EG is total in H the complement of E in $C_{\mathcal{B}'}(\mathcal{B}(G,H))$ is, indeed, $\{0\}$.

REMARK 1.1. The other important observation in [4] is that for an arbitrary (normal unital) representation ρ' of $\mathcal{B}'$ on a Hilbert space, $C_{\mathcal{B}'}(\mathcal{B}(G,H))$ is a von Neumann $\mathcal{B}$-module acting totally on G, which gives a one-to-one correspondence between von Neumann $\mathcal{B}$-modules contained in $\mathcal{B}(G,H)$ and representations ρ' of $\mathcal{B}'$ on H.

This approach becomes particularly fruitful when the von Neumann modules are two-sided so that there is around another (normal unital) representation ρ on H of a second von Neumann algebra $\mathcal{A}$. Switching the roles of $\mathcal{B}$ and $\mathcal{A}'$, the result is a duality between $\mathcal{A}$-$\mathcal{B}$-modules and $\mathcal{B}'$-$\mathcal{A}'$-modules generalizing the duality between a von Neumann algebra and its commutant. One application is a complete theory of normal representations of the adjointable operators on a von Neumann $\mathcal{B}$-module on a von Neumann $\mathcal{A}$-module (this can be, e.g., a Hilbert space).

In this short note we show directly that $E = C_{\mathcal{B}'}(\mathcal{B}(G,H))$ (Section 2) and we show that $C_{\mathcal{B}'}(\mathcal{B}(G,H))$ is self-dual (Section 3), thus showing that E is self-dual (Section 4). The only prerequisite for the first statement is von Neumann's *double commutant theorem*, the only prerequisite for the second statement is a technical lemma (Lemma 3.1) which asserts that every $\mathcal{B}$-functional Φ is represented by an operator in $\mathcal{B}(H,G)$. The simplicity of the proofs, in particular in Section 2, improves also accessibility of [4] (showing equality of E and $C_{\mathcal{B}'}(\mathcal{B}(G,H))$, i.e. one half of the one-to-one correspondence mentioned in Remark 1.1, without making use of self-duality) and, therefore, of the whole theory of von Neumann modules.

2. $E = C_{\mathcal{B}'}(\mathcal{B}(G,H))$

Every $a \in \mathcal{B}^a(E)$ (the *algebra of adjointable operators* on E) gives rise to a bounded operator $x \odot g \mapsto ax \odot g$ on H. In that way, we identify $\mathcal{B}^a(E)$ as a $*$-subalgebra of $\mathcal{B}(H)$. It is easy to see that $\mathcal{B}^a(E)$ is a von Neumann subalgebra of $\mathcal{B}(H)$.

It follows that the *matrix $*$-algebra*

$$\mathcal{M} = \begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix}$$

with the obvious operations is a von Neumann algebra on $G \oplus H$. Let us compute its commutant.

PROPOSITION 2.1. *The commutant of $\mathcal{M}$ is $\mathcal{M}' = \left\{ \begin{pmatrix} b' & 0 \\ 0 & \rho'(b') \end{pmatrix} : b' \in \mathcal{B}' \right\}$.*

Proof. Suppose $\begin{pmatrix} b' & y'^* \\ x' & a' \end{pmatrix} \in \mathcal{B}(G \oplus H)$ is an element in $\mathcal{M}'$. As it must commute with $\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, b \in \mathcal{B}$, we find

$$\begin{pmatrix} b'b & 0 \\ x'b & 0 \end{pmatrix} = \begin{pmatrix} bb' & by'^* \\ 0 & 0 \end{pmatrix}.$$

As this must hold for all $b \in \mathcal{B}$ (in particular also for $b = \mathbf{1}$), we find $x' = y' = 0$ and $b' \in \mathcal{B}'$. The remaining part $\begin{pmatrix} b' & 0 \\ 0 & a' \end{pmatrix}$ must commute with $\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, x \in E$. Therefore,

$$\begin{pmatrix} 0 & a'x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xb' \\ 0 & 0 \end{pmatrix}.$$

We find $a'(x \odot g) = a'xg = xb'g = \rho'(b')(x \odot g)$ for all $x \in E$, $g \in G$ and, therefore, $a' = \rho'(b')$. ∎

The commutant of $\mathcal{M}'$ is, clearly,

$$\mathcal{M}'' = \begin{pmatrix} \mathcal{B} & C_{\mathcal{B}'}(\mathcal{B}(H,G)) \\ C_{\mathcal{B}'}(\mathcal{B}(G,H)) & \rho'(\mathcal{B}')' \end{pmatrix}.$$

By the *double commutant theorem* $\mathcal{M}'' = \mathcal{M}$. Therefore, we do not only show the statement of this section's headline, but, as an additional benefit, we identify also $\mathcal{B}^a(E)$ as the commutant of the image of $\mathcal{B}'$ under ρ'. (This can also be done by using *Morita equivalence* for von Neumann algebras; see [7].)

PROPOSITION 2.2. *$E = C_{\mathcal{B}'}(\mathcal{B}(G,H))$ and $\mathcal{B}^a(E) = \rho'(\mathcal{B}')'$.*

This is a key result in [4], now proved without making use of self-duality.

3. $C_{\mathcal{B}'}(\mathcal{B}(G,H))$ IS SELF-DUAL

A $\mathcal{B}$-functional $\Phi \in \mathcal{B}^r\big(C_{\mathcal{B}'}(\mathcal{B}(G,H)), \mathcal{B}\big)$ gives rise to a linear mapping

$$L_\Phi \colon \operatorname{span} C_{\mathcal{B}'}(\mathcal{B}(G,H))G \to G, \quad L_\Phi(x \odot g) = (\Phi x)g.$$

The following lemma is a special case of the more general Lemma 3.9 of [9] (or, still more general, Lemma 2.3.7 of [10]). The essential part of the proof consists in showing that for computing the operator norm of L_Φ it is sufficient to take the supremum only over elementary tensors $x \odot g$ with $\|x\| \leqslant 1, \|g\| \leqslant 1$. As in our case the proof simplifies considerably, we include it.

LEMMA 3.1. $\|L_\Phi\| = \|\Phi\|$. *Therefore, L_Φ extends to a bounded operator in $\mathcal{B}(H, G)$, identified with Φ.*

Proof. Suppose that there is a cyclic vector $g_0 \in G$, i.e. $\mathcal{B}g_0$ is dense in G. (Otherwise, use a decomposition of G into subspaces G_α cyclic for $\mathcal{B}$ and take into account the facts, firstly, that also H decomposes accordingly into cyclic subspaces H_α and, secondly, that the norm of an element in a direct sum of operator spaces $\mathcal{B}(G_\alpha, H_\alpha)$ is just the supremum over the single norms.) Then every element in H can be approximated by those of the form $h = x \odot g_0$. Use polar decomposition $x = x_0 |x|$ of x and put $g = |x| g_0$. Then $\|h\| = \|g\|$ because $g \in |x| G$. In particular, every unit vector in H can be approximated by $x \odot g$ where x is a partial isometry in E and g is a unit vector in G. ∎

PROPOSITION 3.2. *$C_{\mathcal{B}'}(\mathcal{B}(G, H))$ is self-dual.*

Proof. From $\Phi\rho'(b')(x \odot g) = \Phi(x \odot b'g) = \Phi x b' g = b'\Phi x g = b'\Phi(x \odot g)$ we see that Φ intertwines b' and $\rho'(b')$. Therefore the adjoint $y = \Phi^*$ of Φ is an element in $C_{\mathcal{B}'}(\mathcal{B}(G, H))$ such that $\Phi x = \langle y, x \rangle$ for all $x \in C_{\mathcal{B}'}(\mathcal{B}(G, H))$. ∎

REMARK 3.3. Well-definedness of the mapping L_Φ follows from the not completely elementary standard result Proposition 4.5 in [3] (or Proposition 4.2.22 of [10]) that the vector subspace $\operatorname{span}\{x \odot g \colon x \in E, g \in G\}$ of H is, actually, the algebraic tensor product over $\mathcal{B}$ of E and G. The mapping $L_\Phi = \Phi \odot \mathrm{id}_G$ is, then, well-defined. A more elementary alternative, as worked out in [9] and [10], consists in passing through the proof of Lemma 3.1 for the well-defined mapping $x \otimes g \mapsto (\Phi x)g$ on the seminormed space $E \otimes G$. As this mapping is bounded, length-zero elements of $E \otimes G$ may be divided out and the quotient is exactly the tensor product.

4. SYNTHESIS

THEOREM 4.1. *Every von Neumann $\mathcal{B}$-module is self-dual.*

Proof. $E = C_{\mathcal{B}'}(\mathcal{B}(G, H))$ and $C_{\mathcal{B}'}(\mathcal{B}(G, H))$ is self-dual. ∎

REMARK 4.2. It seems that Lemma 3.1 forms always an essential part of the proofs of self-duality, which cannot be replaced by simpler arguments.

REMARK 4.3. To be honest, we should mention that (unlike the approach by complete quasi orthonormal systems) the preceding arguments cannot be used to show the *Riesz representation theorem* (Hilbert spaces are self-dual), but, actually, reduce the statement about von Neumann modules to that about Hilbert spaces.

An equivalent form of the *Riesz representation theorem* is that all bounded operators between Hilbert spaces have an adjoint. Without this, in the proof of Proposition 3.2 it was not possible to pass from Φ to Φ^*. One may see the failure of the argument clearly, by taking $\mathcal{B} = \mathbb{C}$ and $G = \mathbb{C}$ and for H only a pre-Hilbert space. This is, actually, the only place in these notes, where we are not able to write down an adjoint explicitly on the algebraic domain $\operatorname{span} EG$. (The adjoint of $x\colon g \mapsto x \odot g$ is, of course, $x^*\colon y \odot g \mapsto \langle x, y\rangle g$.)

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We wish to thank Paul Muhly for pointing out that the statement of Section 3 is contained in Rieffel ([7], Proposition 6.10). (This follows by observing that a *generator of $Normod_N$* is a faithful normal representation of the W^*-algebra N, in our case the identity representation of the von Neumann algebra $N = \mathcal{B}$.) Rieffel's proof, however, refers to an older result ([6], Theorem 1.4) about tensor products of Banach modules whose proof, in turn, refers to literature on tensor products of Banach spaces. It is considerably less elementary than Lemma 3.1.

We would also like to mention that the intention of Proposition 6.10 in [7] is to reprove and specify better Paschke's result [5] that every Hilbert module over a W^*-algebra admits a self-dual extension. Our intention is rather to reprove the result from [9] which characterizes self-dual Hilbert modules over von Neumann algebras as those which are von Neumann modules, that is as those which are strongly closed subspaces of a canonically associated operator space. The idea is to underline that the definition of von Neumann module from [9] and its consequent use allows to arrive by rather elementary means at many results which in the abstract frame of W^*-modules require more sophisticated tools.

REFERENCES

[1] W.B. ARVESON, Subalgebras of C^*-algebras, *Acta Math.* **123**(1969), 141–224.

[2] D. GOSWAMI, Quantum stochastic dilation of completely positive semigroups and flows, Ph.D. Dissertation, ISI Delhi, Delhi 1999.

[3] E.C. LANCE, *Hilbert C^*-modules*, Cambridge Univ. Press, Cambridge 1995.

[4] P.S. MUHLY, M. SKEIDE, B. SOLEL, On product systems of W^*-modules and their commutants, preprint, Campobasso 2003.

[5] W.L. PASCHKE, Inner product modules over B^*-algebras, *Trans. Amer. Math. Soc.* **182**(1973), 443–468.

[6] M.A. RIEFFEL, Multipliers and tensor products of L^p-spaces of locally compact groups, *Studia Math.* **33**(1969), 71–82.

[7] M.A. RIEFFEL, Morita equivalence for C^*-algebras and W^*-algebras, *J. Pure Appl. Algebra* **5**(1974), 51–96.

[8] M. SKEIDE, Hilbert modules in quantum electro dynamics and quantum probability, *Comm. Math. Phys.* **192**(1998), 569–604.

[9] M. SKEIDE, Generalized matrix C^*-algebras and representations of Hilbert modules, *Math. Proc. Roy. Irish Acad.* **100A**(2000), 11–38.

[10] M. SKEIDE, Hilbert modules and applications in quantum probability, Habilitationsschrift, Cottbus, 2001, `http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html`.

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