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HYPERCYCLIC OPERATORS, MIXING OPERATORS, AND THE BOUNDED STEPS PROBLEM

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ABSTRACT. The main topic of this paper is the Hypercyclicity Criterion. We construct mixing operators which fail Kitai's Criterion, thus answering a question of Shapiro. We show that the Bounded Steps Problem introduced by Bès and Peris is equivalent to the Hypercyclicity Criterion. Then we show that hypercyclic operators which satisfy an additional regularity assumption satisfy the Hypercyclicity Criterion: if T is hypercyclic and upper-triangular, or if T has a dense set of vectors with bounded orbit, or if $T \oplus T$ is cyclic, then $T \oplus T$ is hypercyclic. We give similar results for cyclic operators.

KEYWORDS: *Hypercyclic operators, Hypercyclicity Criterion, mixing operators, Kitai's Criterion, direct sums of cyclic and hypercyclic operators.*

MSC (2000): 47A16.

1. INTRODUCTION

Let X be a real or complex separable Banach space, and let $\mathcal{B}(X)$ denote the algebra of bounded operators on X. We first give the definition of a hypercyclic sequence of operators [19], which are also called *universal* sequences:

DEFINITION 1.1. Let $(T_n)_{n\geqslant 0}$ be a sequence of bounded operators on X. The sequence $(T_n)_{n\geqslant 0}$ is *hypercyclic* when there exists a vector x of X such that the set $\{T_n x : n \geqslant 0\}$, is dense in X. Such a vector x is called a hypercyclic vector for $(T_n)_{n\geqslant 0}$. If $T \in \mathcal{B}(X)$, the operator T on X is hypercyclic when the sequence $(T^n)_{n\geqslant 0}$ of powers of T is hypercyclic.

The study of hypercyclic operators on a Banach space was initiated in 1969 when Rolewicz proved that any multiple ωB, $|\omega| > 1$, of the standard backward shift B on ℓ_p, $1 \leqslant p < \infty$, is hypercyclic, and it really began with Kitai's work [24]. She stated in particular a simple criterion implying that T is hypercyclic (in Kitai's original statement, it was additionally required that the two dense sets V and W in the criterion below coincide):

KITAI'S CRITERION. *Suppose that T is a bounded operator on X such that there exist two dense subsets V and W of X and a map $S : W \longrightarrow W$ with the properties that:*
(1) *for every $v \in V$, the sequence $(T^n v)_{n\geqslant 0}$ tends to 0;*
(2) *for every $w \in W$, the sequence $(S^n w)_{n\geqslant 0}$ tends to 0;*
(3) *for every w in W, $TSw = w$.*
Then T is hypercyclic.

This does not look very engaging at first sight, but it can be reformulated in a more intuitive way by saying that an operator with a dense set of orbits going to zero and a dense set of backward orbits going to zero is hypercyclic. This criterion is widely used to obtain hypercyclic operators, for instance the Godefroy-Shapiro Criterion [19] is a consequence of Kitai's Criterion.

Kitai's Criterion was independently rediscovered by Gethner and Shapiro in [18], and it was noted by L. Drewnowski that the assumptions could be significantly weakened so as to give a much more general criterion, known today as the Hypercyclicity Criterion:

HYPERCYCLICITY CRITERION. *Suppose that T is a bounded operator on X such that there exist a strictly increasing sequence (n_k) of positive integers, two dense subsets V and W of X and a sequence (S_{n_k}) of maps (not necessarily linear nor continuous) $S_{n_k} : W \longrightarrow X$ such that:*
(1) *for every $v \in V$, the sequence $(T^{n_k} v)_{k\geqslant 0}$ tends to 0;*
(2) *for every $w \in W$, the sequence $(S_{n_k} w)_{k\geqslant 0}$ tends to 0;*
(3) *for every w in W, the sequence $(T^{n_k} S_{n_k} w)_{k\geqslant 0}$ tends to w.*
Then T is hypercyclic.

It is an open problem [25], [10] to know whether every hypercyclic operator satisfies the Hypercyclicity Criterion with respect to some sequence (n_k). This question is deeper than one may think, and the Hypercyclicity Criterion is not only a practical tool to check hypercyclicity. Bès and Peris [10] proved that an operator T satisfies the Hypercyclicity Criterion if and only if $T \oplus T$ is hypercyclic. Thus the Hypercyclicity Criterion Problem boils down to the following question of Herrero:

QUESTION 1.2. [22]. If T is a hypercyclic operator on X, is it true that $T \oplus T$ is hypercyclic on $X \oplus X$?

There is a similar question involving Kitai's Criterion: an operator satisfying Kitai's Criterion is not only hypercyclic (for every pair (U, V) of nonempty open subsets of X there exists an integer n such that $T^n(U) \cap V \neq \emptyset$), but *mixing*: for every pair (U, V) of nonempty open subsets of X there exists an integer N such that for every $n \geqslant N$, $T^n(U) \cap V \neq \emptyset$. It is a question of Shapiro [33] to know whether the converse is true:

QUESTION 1.3. [33]. Does every mixing operator obey the hypotheses of Kitai's Criterion?

Another problem in hypercyclicity is the "Bounded Steps Problem", which was motivated by the paper [10]:

QUESTION 1.4. Let T be a hypercyclic operator and let (n_k) be a sequence such that $\sup(n_{k+1} - n_k) < +\infty$. Is the sequence (T^{n_k}) hypercyclic?

This problem is related to a result of Ansari [1], namely that T^N is hypercyclic for every $N \geqslant 2$ when T is hypercyclic: in fact T^N and T even have the same hypercyclic vectors. It is stated in [10] that whenever $\sup(n_k - n_{k+1}) < +\infty$, every hypercyclic vector for T is also hypercyclic for (T^{n_k}). But it is not so, as is shown in [28]: if x is any hypercyclic vector for an operator T, there exists a sequence (n_k) with $\sup(n_{k+1} - n_k) \leqslant 2$ such that x is not a hypercyclic vector for the sequence (T^{n_k}).

In this paper, we address Questions 1.2, 1.3 and 1.4. In the first part of this paper we answer Shapiro's question 1.3 in the negative, and exhibit mixing operators which fail Kitai's Criterion, using Salas's construction of hypercyclic compact perturbations of the identity [32] and results of Atzmon [3], [4] and Esterle and Zarrabi [15]. We show that every separable space of infinite dimension supports a mixing operator which fails Kitai's Criterion (Theorem 2.6). We also give a simple category proof of Salas's result [32] that the spaces ℓ_p, $1 \leqslant p < +\infty$, or c_0 support a hypercyclic operator which is not mixing (Proposition 2.8). The second part is devoted to the "Bounded Steps Problem": we show that it is in fact equivalent to the Hypercyclicity Criterion (Theorem 3.1). This last result has been obtained independently by Peris and Saldivia in [30]. We also give several equivalent formulations of the Hypercyclicity Criterion, which are used in the next section to give a positive answer to Question 1.2 when T is a hypercyclic operator satisfying some additional regularity assumption: if T is a hypercyclic upper-triangular operator, then T satisfies the Hypercyclicity Criterion (Theorem 4.3). If T is a hypercyclic operator which has a dense set of vectors with bounded orbit, then T satisfies the Hypercyclicity Criterion (Theorem 4.4). The last section gathers miscellaneous results, in particular, remarking that there is a similar "Cyclicity Criterion" for cyclic operators, we give a necessary and sufficient condition for an upper-triangular operator to satisfy $T \oplus T$ cyclic (Corollary 5.2). We also show (Theorem 6.1) that whenever T is a hypercyclic operator, $T \oplus T$ is norm-weak topologically transitive.

2. MIXING OPERATORS WHICH FAIL KITAI'S CRITERION

Let us first recall the notion of a hereditarily hypercyclic operator [10]:

DEFINITION 2.1. Let T be an operator on X and (n_k) a strictly increasing sequence of positive integers. Then T is *hereditarily hypercyclic* with respect to the sequence (n_k) if for every subsequence (n_{k_j}) of (n_k), the sequence $(T^{n_{k_j}})$ is hypercyclic.

It is proved in [10] that T satisfies the Hypercyclicity Criterion if and only if T is hereditarily hypercyclic with respect to some sequence (n_k). There is also a characterization of mixing operators in terms of hereditarily hypercyclic sequences. The following lemma is well known, but we include the short proof for completeness's sake. Remark that what is called "hereditarily hypercylic" in Ansari's paper [1] corresponds to "hereditarily hypercyclic with respect to the whole sequence (n) of integers" in our terminology.

LEMMA 2.2. *The operator T is mixing if and only if it is hereditarily hypercyclic with respect to the whole sequence (n) of integers.*

Proof. The operator T is not mixing if and only if there exists a pair (U, V) of non empty open subsets of X and an infinite sequence (n_k) of integers such that $T^{n_k}(U) \cap V$ is always empty: this means exactly that the sequence (T^{n_k}) is not hypercyclic. ∎

The operators we consider here are perturbations of the identity operator by a backward shift. These operators were introduced by Salas in [32]: let $(e_n)_{n\geqslant 1}$ be the canonical basis of one of the spaces ℓ_p, $1 \leqslant p < +\infty$, or c_0, and let $(w_n)_{n\geqslant 1}$ be a bounded sequence of positive numbers. The backward shift with weights w_n is defined by $Be_1 = 0$ and $Be_n = w_n e_{n-1}$ for $n \geqslant 2$. Then the operator $I + B$ is hypercyclic ([32], Theorem 3.3). The argument of the proof of this is of a surprisingly elementary nature, and does not use the Hypercyclicity Criterion. Shortly afterwards, Leon-Saavedra and Montes-Rodriguez proved in [25] that these operators did satisfy the Criterion. It turns out that they are even mixing:

LEMMA 2.3. *Let $(w_n)_{n\geqslant 1}$ be any sequence of positive numbers. Let B be the backward shift with weights w_n. Then $I + B$ is mixing.*

Proof. It suffices to go back to the proof of Theorem 3.3 of [32]: if (p_j) is any sequence of integers, our goal is to construct a hypercyclic vector for the sequence (T^{p_j}). If $(z_k)_{k\geqslant 1}$ is a dense sequence of vectors of ℓ_p, $1 \leqslant p < +\infty$, or c_0 with finite support such that for every $k \geqslant 1$, $\max \operatorname{supp}(z_k) \leqslant k$, a hypercyclic vector for $I + B$ is obtained by constructing inductively a fast increasing sequence $(n_k)_{k\geqslant 1}$ of integers and a sequence $(y_k)_{k\geqslant 1}$ of finitely supported vectors such that: for every $k \geqslant 2$, $\max \operatorname{supp}(y_{k-1}) < \min \operatorname{supp}(y_k)$, and for every $k \geqslant 1$,

$$\|y_k\| \leqslant 2^{-(k+1)}(1 + \|B\|)^{-(n_k+1)} \quad \text{and} \quad \Big\|(I+B)^{n_k}\Big(\sum_{j=1}^{k} y_j\Big) - z_k\Big\| \leqslant 2^{-k}.$$

Then the vector $y = \sum_{k=1}^{+\infty} y_k$ is a hypercyclic vector for $I + B$, and it is in fact hypercyclic for the sequence $((I + B)^{n_k})$. Now the integers (n_j) are obtained through

Lemma 3.2 of [32], which says that a certain $2^k \times 2^k$ matrix C_n with combinatory numbers as entries can be inverted so as to give a solution X_n of a certain system $C_n X_n = B_n$ with the entries of X_n small enough. This solution exists provided n is large enough, and such large n's are choosen as n_j's. Since the only requirement here is that n be large enough, it can be chosen so as to belong to the sequence (p_k), and thus the construction yields a $((I+B)^{n_k})$-hypercyclic vector. ∎

If the weights (w_n) are small enough, it is possible to describe completely the orbits of $I+B$ which have polynomial growth:

PROPOSITION 2.4. *Suppose that the sequence of weights (w_n) is decreasing and that*

$$n(w_1 w_2 \cdots w_n)^{1/n}$$

tends to zero as n tends to infinity. Let x be any non-zero vector of ℓ_p, $1 \leqslant p < +\infty$, or c_0, and consider the orbit $\{(I+B)^n x : n \geqslant 0\}$ of x. For any positive integer k,

$$\|(I+B)^n x\| = \mathrm{o}(n^k)$$

if and only if x is a linear combination of the first k basis vectors $e_1, \ldots, e_k$.

The proof of Proposition 2.4 relies on the work of Atzmon [3], [4] and Esterle and Zarrabi [15] regarding the local properties of powers of operators. When an orbit has polynomial growth, a strong dichotomy applies: let X be any separable Banach space and T any bounded operator on X. If x is any vector of X and $\|T^n x\| = o(n^k)$ for some $k \geqslant 1$, then we have either $\limsup n\|(I-T)^n x\| > 0$ or $(T-I)^k x = 0$. Results of this kind were used in [4] in order to obtain hyperinvariant subspaces for certain operators T with $\sigma(T) = \{1\}$. A short proof relying on a Phragmen-Lindelöf argument is given in [15] under weaker hypotheses.

Proof. We apply the result quoted in the preceding paragraf with $T = I+B$: $(I-T)^n = (-B)^n$ and since the sequence (w_n) decreases, $\|B^n\| = w_1 w_2 \cdots w_n$. Thus our hypotheses imply that $n\|(I-T)^n\|$ tends to zero, and if x satisfies $\|T^n x\| = o(n^k)$ for some $k \geqslant 1$, then $(T-I)^k x = 0$, i.e. $B^k x = 0$. Now $B^k x = 0$ if and only if x is supported by the first k basis vectors $e_1, \ldots, e_k$. The converse is clear: if $x = \sum\limits_{i=1}^{k} x_i e_i$,

$$(I+B)^n x = \sum_{i=1}^{k} x_i \sum_{j=0}^{n} \binom{n}{j} B^j e_i = \sum_{i=1}^{k} x_i \sum_{j=0}^{k-1} \binom{n}{j} B^j e_i.$$

Hence $\|(I+B)^n x\| = O(\binom{n}{k-1}) = O(n^{k-1}) = o(n^k)$. ∎

As a direct consequence we obtain:

THEOREM 2.5. *If (w_n) is a decreasing sequence of positive weights such that*

$$n(w_1 w_2 \cdots w_n)^{1/n}$$

tends to zero as n tends to infinity and B is the backward shift on ℓ_p, $1 \leqslant p < +\infty$, or c_0 with weights (w_n), $I + B$ is a mixing operator which does not obey the hypotheses of Kitai's Criterion.

Proof. If $\|(I+B)^n x\|$ tends to zero, then $\|(I+B)^n x\| = \mathrm{o}(n)$ so by Proposition 2.4, x is proportional to e_1, and since $(I+B)e_1 = e_1$, this forces x to be equal to 0. ∎

For instance if $w_n = 2^{-n}$ for every $n \geqslant 1$, the assumptions of Theorem 2.5 are fulfilled. Theorem 2.5 may look somewhat surprising because it is when the weights are very small that the operator $I + B$ behaves in the wildest way in the sense that no vector with an infinite support has an orbit with polynomial growth. But this is not so surprising as it may seem at first sight: suppose for instance that all the weights w_n are bounded below by some positive constant $\delta > 0$. Then every complex number λ with $|1 - \lambda| < \delta$ is an eigenvalue of $I + B$ and an associated eigenvector is

$$x_\lambda = \sum_{n=0}^{+\infty} \Big(\frac{\lambda - 1}{\delta}\Big)^{n-1} e_n.$$

Then if $|\lambda| < 1$, $(I+B)^n x_\lambda$ tends to zero. Such operators satisfy Kitai's Criterion and in fact much more is true: if $H_+\,(I+B)$ denotes the linear span of the kernels $\ker\,(I + B - \lambda I)$ for $|\lambda| > 1$ and $H_-\,(I+B)$ the linear span of the kernels $\ker\,(I + B - \lambda I)$ for $|\lambda| < 1$, then $H_+\,(I+B)$ and $H_-\,(I+B)$ are dense in ℓ_p, $1 \leqslant p < +\infty$, or c_0 ([20], Remark 1) and $I + B$ satisfies the assumptions of the Godefroy-Shapiro Criterion ([19]; see also Proposition 1 of [20]).

The constructions of Proposition 2.4 and Theorem 2.5 have been carried out in the setting of ℓ_p-spaces, but they work as well for any separable Banach space:

THEOREM 2.6. *Let X be any separable Banach space. Then X supports a mixing operator which fails Kitai's Criterion.*

Proof. Ansari's construction [2] of hypercyclic operators on any separable Banach space relies on the result of Salas presented above: let (x_i, x_i^*) be a bounded biorthogonal system for X and (w_i) be positive weights such that $\sum w_i \|x_i\| \|x_i^*\| < +\infty$. The operator defined on X by $Bx = \sum_{i=1}^{+\infty} w_i x_{i+1}^*(x) x_i$ is nuclear and $I + \widetilde{B}$ is hypercyclic. This follows from the fact that $I + \widetilde{B}$ is a quasi-extension in the sense of Definition 5 in [2], of $I + B$ on the space ℓ_1, where B is the backward shift on ℓ_1 with weights (w_i): if $X_1 = \Big\{ \sum_{i=1}^{\infty} a_i x_i : a_i \in \mathbb{C}, \sum_{i=1}^{\infty} |a_i| < +\infty \Big\}$, the norm on X_1 is defined by

$$\Big\| \sum_{i=1}^{\infty} a_i x_i \Big\| = \sum_{i=1}^{\infty} |a_i|,$$

and $J : \ell_1 \to X_1$ which maps e_i onto x_i is an isomorphism. Then $\widetilde{B}$ is the restriction of JBJ^{-1} to X_1. By Lemma 1 of [2], a subset of X_1 is dense in X whenever it is dense for the norm $\|\cdot\|_1$. Here for every infinite sequence (n_k) there exists a vector x in ℓ_1 such that $\{(I+B)^{n_k}x\}$ is dense. Then $\{(I+\widetilde{B})^{n_k}Jx\}$ is dense in X, so the sequence $((I+\widetilde{B})^{n_k})$ is hypercyclic. Thus $I+\widetilde{B}$ is mixing, and if the weights (w_i) are small enough, the same argument as in Proposition 2.4 shows that $I+\widetilde{B}$ does not satisfy Kitai's Criterion. ∎

It was pointed out to me by Alfredo Peris that the same argument as in Theorem 2.5 implies that some spaces do not support any operator satisfying Kitai's Criterion. More precisely:

PROPOSITION 2.7. *Let X be a complex separable hereditarily indecomposable Banach space X whose dual is also hereditarily indecomposable. No operator on X satisfies Kitai's Criterion.*

Proof. Let T be a bounded operator on X. Then T can be written as $T = \lambda I + S$, where $|\lambda| = 1$ and S is quasinilpotent (see for instance [20]). Any rotation of a hypercyclic operator being hypercyclic [26], $\frac{1}{\lambda}T = I + \frac{1}{\lambda}S$ is hypercyclic. Write $B = \frac{1}{\lambda}S$. Since B is quasinilpotent, $n\|B^n x\|$ tends to zero for every x. Using the results of Atzmon and Esterle-Zarrabi quoted above, we obtain as in Theorem 2.5 that if x has bounded orbit under the action of $\frac{1}{\lambda}T$, x belongs to $\ker B$. Since $B \neq 0$, the set of such vectors cannot be dense: $\frac{1}{\lambda}T$ does not satisfy Kitai's Criterion, hence T does not either. ∎

Of course there exist hypercyclic operators which are not mixing: such operators have been constructed on the spaces ℓ_p, $1 \leqslant p < +\infty$, or c_0 by Salas [32]. We give here a very simple proof of this fact which relies on a category argument:

PROPOSITION 2.8. *Let X be one of the spaces ℓ_p, $1 \leqslant p < +\infty$, or c_0. The set of mixing operators on X is not a G_δ subset of $\mathcal{B}(X)$ for the Strong Operator Topology* (SOT)*, while the set of hypercyclic operators is one. Thus such spaces support hypercyclic operators which are not mixing.*

The proof of this proposition uses also the bilateral shifts involved in Salas' proof, but avoids any explicit characterization of the hypercyclic weighted shifts in terms of the weights.

Proof. We identify ℓ_p with $\ell_p(\mathbb{Z})$ with canonical basis $(e_n)_{n\in\mathbb{Z}}$. Let $\alpha = (\alpha_n)_{n\in\mathbb{Z}}$ be a sequence of 0's and 1's (i.e. an element of 2^ω). Consider the operator T_α defined on $\ell_p(\mathbb{Z})$ by $T_\alpha e_k = \omega_k e_{k-1}$ with

$$\begin{cases} \omega_k = \dfrac{1}{2} & \text{if } k \leqslant 0, \\ \omega_k = 2 & \text{if } k > 0 \text{ and } \alpha_k = 0, \\ \omega_k = \dfrac{1}{k!} & \text{if } k > 0 \text{ and } \alpha_k = 1. \end{cases}$$

Then T_α is a bounded operator on $\ell_p(\mathbb{Z})$ of norm 2, and for every k, n in $\mathbb{Z}$,

$$T_\alpha^n e_k = \omega_k \omega_{k-1} \cdots \omega_{k-n+1} e_{k-n}.$$

It is easy to check that for every k, $T_\alpha^n e_k \to 0$ as $n \to +\infty$. Thus for every finitely supported vector x, $T_\alpha^n x \to 0$. On the other hand, set $S_\alpha e_{k-1} = {\omega_k}^{-1} e_k$ for every $k \in \mathbb{Z}$ and extend it by linearity to the set of finitely supported vectors. Then

$$S_\alpha^n e_k = (\omega_{k+1} \cdots \omega_{k+n})^{-1} e_{k+n}.$$

If the sequence α belongs to $2^{<\omega}$, i.e. $\alpha_n = 0$ except possibly for finitely many n's, then there is an N such that for every $n \geqslant N$, $\omega_n = 2$, and hence $S_\alpha^n e_k \to 0$ for every basis vector e_k. This implies that for every x with finite support, $S_\alpha^n x \to 0$. Now for every such vector $T_\alpha S_\alpha x = x$, thus the hypotheses of Kitai's Criterion are satisfied and T_α is mixing.

If now α does not belong to $2^{<\omega}$, let (n_k) be the strictly increasing sequence of integers n such that $\alpha_n = 1$. For every k, $(T_\alpha^*)^{n_k} e_1 = \omega_1 \cdots \omega_{n_k} e_{n_{k+1}}$, hence

$$\|(T_\alpha^*)^{n_k} e_1\| \leqslant \frac{2^{n_k - 1}}{n_k!} \to 0.$$

This implies that for every x in $\ell_p(\mathbb{Z})$, $\langle T_\alpha^{n_k} x, e_1 \rangle \to 0$. Thus the vector x cannot be hypercyclic, $(T_\alpha^{n_k})$ is not hypercyclic, and T_α is not mixing. We have proved that T_α is mixing if and only if α belongs to $2^{<\omega}$.

Consider now the map Φ which maps any sequence α of 2^ω to the operator T_α. This map is continuous from 2^ω endowed with the product topology to $\mathcal{B}(\ell_p(\mathbb{Z})) \cap \overline{B}(0,2)$ with the Strong Operator Topology (SOT). Suppose indeed that α_0 is any sequence in 2^ω and F is a finite dimensional space with normalized basis $(f_1, \ldots, f_r)$, and let C be a positive constant such that for every r-tuple of scalars $(\lambda_1, \ldots, \lambda_r)$,

$$\sum_{i=1}^{r} |\lambda_i| \leqslant C \Big\| \sum_{i=1}^{r} \lambda_i f_i \Big\|.$$

For any $\varepsilon > 0$, let $x_1, \ldots, x_r$ be vectors with support in $[-N, N]$ such that for every $i = 1, \ldots, r$, $\|x_i - f_i\| < \frac{\varepsilon}{4C}$, and α be such that $\alpha_n = (\alpha_0)_n$ for $n = 1, \ldots, N$. Let $y = \sum_{i=1}^{r} \lambda_i f_i$ be any vector of F and $x = \sum_{i=1}^{r} \lambda_i x_i$. Then the triangular inequality and the fact that $T_\alpha e_n = T_{\alpha_0} e_n$ for $n \leqslant N$ imply that

$$\|T_\alpha y - T_{\alpha_0} y\| \leqslant 4\|x - y\| = 4 \Big\| \sum_{i=1}^{r} \lambda_i (f_i - x_i) \Big\| \leqslant \varepsilon \|y\|.$$

Hence the norm of the restriction of $T_\alpha - T_{\alpha_0}$ to F is less than ε, which proves that Φ is continuous. If we denote by $HC(\ell_p(\mathbb{Z}))$ the set of hypercyclic operators on $\ell_p(\mathbb{Z})$ and by $MX(\ell_p(\mathbb{Z}))$ the set of mixing operators on $\ell_p(\mathbb{Z})$, then

$$\Phi^{-1}(MX(\ell_p(\mathbb{Z})) \cap \overline{B}(0,2)) = 2^{<\omega}$$

which is not a G_δ subset of 2^ω. Hence $MX(\ell_p(\mathbb{Z})) \cap \overline{B}(0,2)$ is not a G_δ subset for the SOT-topology of $\mathcal{B}(\ell_p(\mathbb{Z})) \cap \overline{B}(0,2)$. But $HC(\ell_p(\mathbb{Z})) \cap \overline{B}(0,2)$ is a SOT-G_δ subset of $\mathcal{B}(\ell_p(\mathbb{Z})) \cap \overline{B}(0,2)$:

$$HC(\ell_p(\mathbb{Z})) = \bigcap_{p,q} \bigcup_{n} \{T \in \mathcal{B}(X) : T^n(U_p) \cap U_q \neq \emptyset\}$$

where (U_k) is a basis of the topology of X. It is easy to check that each one of the sets

$$\Omega_{n,p,q} = \{T \in \mathcal{B}(X) : T^n(U_p) \cap U_q \neq \emptyset\}$$

is open for the SOT-topology. Thus it is impossible that $MX(\ell_p(\mathbb{Z})) \cap \overline{B}(0,2)$ and $HC(\ell_p(\mathbb{Z})) \cap \overline{B}(0,2)$ coincide, and there exists an operator T_α which is hypercyclic on $\ell_p(\mathbb{Z})$ without being mixing. ∎

This "category" proof of Salas's result was motivated by the following question, which seems to be open:

QUESTION 2.9. Does every separable Banach space of infinite dimension support a hypercyclic operator which is not mixing?

3. THE BOUNDED STEPS PROBLEM

3.1. MAIN RESULT. The Bounded Steps Problem arises from the paper [10] by Bès and Peris, where the authors considered sequences (n_k) such that $(n_{k+1} - n_k)$ is bounded above by some positive integer. It is stated there that every hypercyclic vector for T is also hypercyclic for (T^{n_k}). But it is not so, as was pointed out by Montes-Rodriguez and Salas in [28]. We reproduce the argument here for the reader's convenience: suppose that x is a hypercyclic vector for T. Then $Tx \neq x$ and there exists an open set U containing x such that U and $T(U)$ are disjoint. Consider for the sequence (n_k) the set of integers n such that $T^n x$ does not lie in U. If $n \notin \{n_k\}$ then $T^n x \in U$ and hence $T^{n+1}x \in T(U)$ which implies that $T^{n+1}x \notin U$: $n+1 \in \{n_k\}$. This proves that $n_{k+1} - n_k$ is smaller than 2 for every k, but the sequence $(T^{n_k}x)$ is not dense in X because it avoids the open set U. This observation should be compared with Ansari's result [1], which states that if T is hypercyclic and $N \geqslant 2$, T^N and T have the same hypercyclic vectors. Here we generalize the argument above and show the following:

THEOREM 3.1. *Let T be any operator on X. The following assertions are equivalent:*

(i) *T satisfies the Hypercyclicity Criterion;*

(ii) *for every integer $N \geqslant 2$ and every sequence (n_k) such that $n_{k+1} - n_k \leqslant N$ for every k, the sequence (T^{n_k}) is hypercyclic;*

(iii) *for every sequence (n_k) such that $n_{k+1} - n_k \leqslant 2$ for every k, the sequence (T^{n_k}) is hypercyclic.*

The main step in the proof of Theorem 3.1 is Theorem 3.2 below.

3.2. ANOTHER FORMULATION OF THE HYPERCYCLICITY CRITERION. Recall that T is hypercyclic if and only if T is topologically transitive. Hence $T \oplus T$ is hypercyclic if and only if for any two pairs (U_1, V_1) and (U_2, V_2) of non-empty open subsets of X there exists an integer n such that $T^n(U_1) \cap V_1$ and $T^n(U_2) \cap V_2$ are simultaneously non-empty. We show that this can be much weakened:

THEOREM 3.2. *Let T be any operator on X. The following assertions are equivalent:*

(i) *$T \oplus T$ is hypercyclic;*

(ii) *for every pair (U, V) of non empty open subsets of X, there exists an integer n such that $T^n(U) \cap V \neq \emptyset$ and $T^{n+1}(U) \cap V \neq \emptyset$;*

(iii) *there exists a positive integer p such that for every pair (U, V) of non empty open subsets of X, there exists an integer n such that $T^n(U) \cap V \neq \emptyset$ and $T^{n+p}(U) \cap V \neq \emptyset$.*

One interest of this result is to show that the Hypercyclicity Criterion Problem boils down to an interversion of two quantifiers: if T is hypercyclic and U and V are two non-empty open subsets of X, there exist integers n and p such that $T^n(U) \cap V$ and $T^{n+p}(U) \cap V$ are non-empty. But if the integer p can be chosen to be independent of U and V, then $T \oplus T$ is hypercyclic.

Proof. That (i) implies (ii) is easy: just pick an n such that both sets $T^n(U) \cap V$ and $T^n(U) \cap T^{-1}(V)$ are non-empty. It is obvious that (ii) implies (iii). In order to prove that (iii) implies (i), consider U_1, V_1, U_2, V_2 four non empty open subsets of X. Our aim is to find an integer n such that $T^n(U_1) \cap V_1 \neq \emptyset$ and $T^n(U_2) \cap V_2 \neq \emptyset$. Let v_1 be a T-hypercyclic vector belonging to V_1. There exists an integer r_1 such that $T^{r_1} v_1 = u_1$ is in U_1. Now T^{r_1} has dense range (T^* has no eigenvalue) and there exists u_2 in U_2 of the form $u_2 = T^{r_1} w_2$ for some vector w_2. Let v_2 be any element of V_2 and $\delta > 0$ such that the open ball $B(v_2, \delta)$ with center v_2 and radius δ is contained in V_2 and $B(u_2, \delta)$ is contained in U_2. Recall now the following result, due to Bourdon [11] in the complex case and to Bès [9] in the real case: if x is a hypercyclic vector for T and p is any non zero polynomial, the vector $p(T)x$ is also hypercyclic for T. Hence $(T^p - I)v_1$ is hypercyclic: there exists an integer q_1 such that

$$\|T^{q_1}(T^p - I)v_1 - (v_2 - w_2)\| < \frac{\delta}{2\|T\|^{r_1}}$$

and an integer p_1 such that

$$\|T^{p_1} v_1 - (v_2 - T^{q_1} v_1)\| < \frac{\delta}{2\|T\|^{r_1}}.$$

Consider the vector $z_2 = T^{p_1}u_1 + T^{q_1+p}u_1$: z_2 belongs to U_2. Indeed,

$$\begin{aligned}\|z_2 - u_2\| &= \|T^{r_1}(T^{p_1}v_1 + T^{q_1+p}v_1 - w_2)\| \\ &\leqslant \|T^{r_1}\|\|T^{p_1}v_1 - (v_2 - T^{q_1}v_1) + v_2 - T^{q_1}v_1 + T^{q_1+p}v_1 - w_2\| \\ &< \delta.\end{aligned}$$

Since $B(u_2,\delta) \subseteq U_2$, $z_2 \in U_2$. In the same way, $y_2 = T^{p_1}v_1 + T^{q_1}v_1$ is in V_2. We now apply the second condition of Theorem 3.2 to the pairs of open sets (U_k, V_k) with $U_k = B(u_1, 2^{-k})$ and $V_k = B(v_1, 2^{-k})$: there exist two sequences (u_k) and (u'_k) converging to u_1 and a sequence (n_k) of integers such that the sequences $(T^{n_k}u_k)$ and $(T^{n_k+p}u'_k)$ both converge to the limit v_1. Hence $T^{n_k}(T^{p_1}u_k + T^{q_1+p}u'_k)$ converges to y_2 and there exists a k_0 such that for every $k \geqslant k_0$, $T^{n_k}(T^{p_1}u_k + T^{q_1+p}u'_k)$ is in V_2. But now $T^{p_1}u_k + T^{q_1+p}u'_k$ converges to z_2, which is in U_2 and there exists a $k_1 \geqslant k_0$ such that for every $k \geqslant k_1$, $T^{n_k}(T^{p_1}u_k + T^{q_1+p}u'_k)$ is in V_2. Thus for $k \geqslant k_1$, $T^{n_k}(U_2) \cap V_2 \neq \emptyset$. Moreover, since (u_k) converges to u_1 and $(T^{n_k}u_k)$ converges to v_1, $T^{n_k}u_k$ is in $T^{n_k}(U_1) \cap V_1$ for k large enough. This yields that if k is large enough, $T^{n_k}(U_1) \cap V_1$ and $T^{n_k}(U_2) \cap V_2$ are both non empty, and this proves our claim. ∎

3.3. PROOF OF THEOREM 3.1. We will use repeatedly the following characterization of hypercyclicity, which appears in [19]:

LEMMA 3.3. *Let T belong to $\mathcal{B}(X)$. The following assertions are equivalent:*
(i) *the sequence (T^{n_k}) is hypercyclic;*
(ii) *for every pair (U, V) of non empty open subsets of X there exists an integer r such that $T^{n_r}(U) \cap V \neq \emptyset$.*

Let us first prove that (i) implies (ii) in Theorem 3.1: let (n_k) be a sequence such that for every k, $n_{k+1} - n_k \leqslant N$ with $N \geqslant 2$. The operator T satisfies the Hypercyclicity Criterion, hence [10] there exists a sequence (p_k) such that T is hereditarily hypercyclic with respect to (p_k). This implies that the direct sum $T \oplus T \oplus \cdots \oplus T$ of N copies of T is also hypercyclic: let indeed (U_i, V_i), $i = 1, \ldots, N$, be N pairs of non empty open subsets of X. There exists a subsequence $(p_{k_{j_1}})$ of p_k such that $T^{p_{k_{j_1}}}(U_1) \cap V_1 \neq \emptyset$ for every j_1. Now the sequence $(T^{p_{k_{j_1}}})$ is hypercyclic, so there exists a subsequence $(p_{k_{j_2}})$ of $(p_{k_{j_1}})$ such that $T^{p_{k_{j_2}}}(U_2) \cap V_2 \neq \emptyset$ for every j_2. Continuing in this way we obtain a sequence $(p_{k_{j_N}})$ such that for every j_N and $i = 1, \ldots, N$, $T^{p_{k_{j_N}}}(U_i) \cap V_i \neq \emptyset$, and hence $T \oplus T \oplus \cdots \oplus T$ is hypercyclic. Let now U, V be non empty open subsets of X: let $U_i = T^{-(i-1)}(U)$ for $i = 1, \ldots, N$. There exists an integer $k \geqslant i$ such that for $i = 1, \ldots, N$, $T^k(U_i) \cap V = \emptyset$, which exactly means that $T^{k-i+1}(U) \cap V = \emptyset$. Now $k - (N-1), k - (N-2), \ldots, k-1, k$ are N consecutive integers, so at least one of them is equal to some element n_{k_r} of the sequence (n_k), and $T^{n_{k_r}}(U) \cap V = \emptyset$. By Lemma 3.3, (T^{n_k}) is hypercyclic.

(ii) $\Rightarrow$ (iii) is obvious.

It remains to prove that (iii) implies (i). This follows from the following lemma:

LEMMA 3.4. *The following assertions are equivalent:*

(i) *for every sequence* (n_k) *such that* $n_{k+1} - n_k \leqslant 2$ *for every* k, *the sequence* (T^{n_k}) *is hypercyclic;*

(ii) *for every pair* (U, V) *of non empty open subsets of* X *there exists an integer* n *such that* $T^n(U) \cap V \neq \emptyset$ *and* $T^{n+1}(U) \cap V \neq \emptyset$.

Proof. (i) $\Rightarrow$ (ii): Suppose that (ii) is not true, and let U and V be such that $T^n(U) \cap V$ and $T^{n+1}(U) \cap V$ are never simultaneously non empty, i.e. $T^n(U) \cap V \neq \emptyset$ implies that $T^{n+1}(U) \cap V = \emptyset$. Let (n_k) be the sequence of integers n such that $T^n(U) \cap V = \emptyset$. If n does not belong to $\{n_k\}$, $T^n(U) \cap V \neq \emptyset$ and hence $T^{n+1}(U) \cap V = \emptyset$, i.e. $n+1$ belongs to $\{n_k\}$. This proves that $n_{k+1} - n_k \leqslant 2$ for every k. But $T^{n_k}(U) \cap V = \emptyset$ for every k and (T^{n_k}) cannot be hypercyclic, a contradiction to (i).

(ii) $\Rightarrow$ (i): If (i) is false, let (n_k) be such that $n_{k+1} - n_k \leqslant 2$ for every k and (T^{n_k}) is not hypercyclic. By Lemma 3.3 there exist U and V such that $T^{n_k}(U) \cap V = \emptyset$ for every k. Suppose now that $T^n(U) \cap V = \emptyset$: $n \notin \{n_k\}$, so $n+1 \in \{n_k\}$ and $T^{n+1}(U) \cap V = \emptyset$. Thus (ii) cannot be true. ∎

The conclusion then follows from Theorem 3.2.

3.4. STILL ANOTHER FORMULATION OF THE HYPERCYCLICITY CRITERION. We give here another formulation of the Hypercyclicity Criterion which will be constantly used in the next section:

PROPOSITION 3.5. *Let* T *be any operator on* X. *The following assertions are equivalent:*

(i) $T \oplus T$ *is hypercyclic;*

(ii) *for every pair* (U, V) *of non-empty open subsets of* X *and every neighbourhood* W *of* 0, *there exists an integer* n *such that* $T^n(U) \cap W \neq \emptyset$ *and* $T^n(W) \cap V \neq \emptyset$.

Condition (ii) was introduced by Godefroy and Shapiro in their paper [19], where it is proved that any operator satisfying condition (ii) is hypercyclic. That (i) implies (ii) is obvious, and the converse has been proved by Bernal-Gonzalez and Grosse-Erdmann in [8]. This converse is also a direct consequence of Theorem 3.2, so we give a proof here:

Proof. In order to show that $T \oplus T$ is hypercyclic, consider two non-empty open subsets U and V of X. Let W be any neighbourhood of 0: there exists an integer n such that $T^n(U) \cap (W \cap T^{-1}(W)) \neq \emptyset$ and $T^n(W \cap T^{-1}(W)) \cap V \neq \emptyset$. Hence $T^n(U) \cap W \neq \emptyset$, $T^{n+1}(U) \cap W \neq \emptyset$, $T^{n+1}(W) \cap V \neq \emptyset$ and $T^n(W) \cap V \neq \emptyset$. Thus for every u, v in X there exist sequences u_k, w_k, u'_k and w'_k with $u_k \to u$, $w_k \to 0$, $u'_k \to u$ and $w'_k \to 0$ and a sequence n_k of positive integers such that

$T^{n_k}u_k \to 0$, $T^{n_k}w_k \to v$, $T^{n_k+1}u'_k \to 0$, and $T^{n_k+1}w'_k \to v$. Hence $u_k + w_k \to u$ and $T^{n_k}(u_k + w_k) \to v$, $u'_k + w'_k \to u$ and $T^{n_k+1}(u'_k + w'_k) \to v$. If u and v belong to U and V respectively, $T^{n_k}(U) \cap V$ and $T^{n_k+1}(U) \cap V$ are both non-empty when k is large enough. This proves Proposition 3.5. ∎

4. SOME PARTIAL ANSWERS TO THE HYPERCYCLICITY CRITERION PROBLEM

The general flavor of the results to be given in this section is the following: if T is a hypercyclic operator which satisfies some regularity conditions, then $T \oplus T$ is hypercyclic. We begin with the following result, which gives a link between cyclicity and hypercyclicity:

PROPOSITION 4.1. *Let T be a hypercyclic operator on a separable space of infinite dimension. The following assertions are equivalent:*

(i) *$T \oplus T$ is hypercyclic;*

(ii) *$T \oplus T$ is cyclic;*

(iii) *for every non-empty open subsets U_1, V_1, U_2, V_2 of X, there exists a polynomial p such that $p(T)(U_1) \cap V_1$ and $p(T)(U_2) \cap V_2$ are non-empty;*

(iv) *for every pair (U, V) of non-empty open subsets of X and every neighbourhood W of 0, there exists a polynomial p such that $p(T)(U) \cap W$ and $p(T)(W) \cap V$ are both non-empty.*

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii): If T is hypercyclic and $T \oplus T$ is cyclic, the set of cyclic vectors for $T \oplus T$ form a dense subset of $X \oplus X$. Let indeed $x \oplus y$ be a cyclic vector for $T \oplus T$, and let p be any non-zero polynomial. Then $p(T)x \oplus p(T)y$ is cyclic for $T \oplus T$:

$$\mathbb{K}[T \oplus T](p(T)x \oplus p(T)y) = (p(T) \oplus p(T))\mathbb{K}[T \oplus T](x \oplus y).$$

Now $p(T) \oplus p(T)$ has dense range: since T is hypercyclic, $q(T)$ has dense range for any non-zero polynomial q [11], [9]. Thus $\mathbb{K}[T \oplus T](p(T)x \oplus p(T)y)$ is dense in $X \oplus X$ and $p(T)x \oplus p(T)y$ is cyclic. Thus for every non-empty open subsets U_1, V_1, U_2, V_2 of X there exists a polynomial p such that $p(T)(U_1) \cap V_1$ and $p(T)(U_2) \cap V_2$ are non-empty. This proves (iii).

(iii)⇒(iv) is obvious.

(iv)⇒(i): We are going to show that condition (ii) of Proposition 3.5 is fulfilled. Let (U, V) be non-empty open subsets of X and W a neighborhood of 0. Let p be a polynomial such that $p(T)(U) \cap W$ and $p(T)(W) \cap V$ are both non-empty, and let x be a hypercyclic vector for T belonging to U such that $p(T)x$ belongs to W. There exists an integer n such that $T^n x$ belongs to $W \cap p(T)^{-1}(V)$. Since x is in U, $T^n(U) \cap W$ is non empty. Moreover, $p(T)T^n x = T^n p(T)x$ is in V, while $p(T)x$ is in W: $T^n(W) \cap V$ is non empty, and this proves our claim. ∎

REMARK 4.2. The argument which was used in the proof of (iv)⇒(i) can also be applied in many other cases to obtain "cyclic formulations" of the Hypercyclicity Criterion: for instance if T is hypercyclic, $T \oplus T$ is hypercyclic if and only if for every pair (U, V) of non-empty open subsets of X there exists a polynomial p such that $p(T)(U) \cap V$ and $p(T)T(U) \cap V$ are non-empty. It may also be of interest to remark that the only argument which was used in the proof of (iv)⇒(i) is that $p(T)$ and T are commuting operators. Thus the same proof yields that if for every pair (U, V) of non-empty open subsets of X and every neighbourhood W of 0 there exists an operator A in the commutant of T such that $A(U) \cap W$ and $A(W) \cap V$ are non-empty, then $T \oplus T$ is hypercyclic.

Our first application of Proposition 4.1 is to show that hypercyclic upper-triangular operators satisfy the Hypercyclicity Criterion. If X is any separable space and T is any operator on X, T is said to be *upper-triangular* if it admits an increasing sequence $(E_n)_{n\geqslant 1}$ of invariant subspaces such that the dimension of E_n is equal to n for every $n \geqslant 1$ and X is the closed linear span of the finite-dimensional spaces E_n, $n \geqslant 1$. If X is a Hilbert space, T is upper-triangular if and only if T has an upper-triangular matrix with respect to some orthonormal basis of X. Theorem 4.3 was suggested to us by a result of Ansari ([1], Theorem 4).

THEOREM 4.3. *Let T be a hypercyclic operator such that*

$$\bigcup_{p\in\mathbb{K}[\xi]\setminus\{0\}} \ker p(T)$$

is dense in X. Then T satisfies the Hypercyclicity Criterion. In particular:

(i) *if T is upper-triangular and hypercyclic, T satisfies the Hypercyclicity Criterion;*

(ii) *if the linear span of the eigenvectors of T is dense and T is hypercyclic, T satisfies the Hypercyclicity Criterion.*

Theorem 4.3 generalizes Theorem 2.3 of [6] and applies to all the backward shifts and all the perturbations of the identity operator constructed by Salas in [32], as well as to all the operators considered in [21], [23], [2], ... It applies also to these operators which have a dense set of eigenvectors associated to eigenvalues of modulus equal to 1 [16].

Proof. Let (U, V) be non-empty open subsets of X and W an open neighbourhood of 0. There exists a vector u in U and a non-zero polynomial p such that $p(T)u = 0$. In particular for every $r > 0$, $rp(T)(U) \cap W \neq \varnothing$. Now since T is hypercyclic, $p(T)$ has dense range [11], [9], and there exists an $r > 0$ such that $p(T)(rW)$ intersects V. For this particular r, $rp(T)(U) \cap W$ and $rp(T)(W) \cap V$ are non-empty. By Proposition 4.1, $T \oplus T$ is hypercyclic. ∎

Another result along the same lines is:

THEOREM 4.4. *Let T be a hypercyclic operator which has a dense set of vectors whose orbit is bounded. Then T satisfies the Hypercyclicity Criterion.*

Theorem 4.4 generalizes some results of [10] (every chaotic operator satisfies the Hypercyclicity Criterion) and [6] and [8] (a hypercyclic operator which has a dense set of vectors with precompact orbit satisfies the Hypercyclicity Criterion).

Proof. Let (U,V) be non-empty open subsets of X and W an open neighbourhood of 0. There exists a vector u in U which has a bounded orbit: there is a positive constant M such that for every n, $T^n(U) \cap MW$ is non-empty. Then for every n, $M^{-1}T^n(U) \cap W$ is non-empty. Now, T being hypercyclic, there exists an n such that $T^n(M^{-1}W) \cap V$ is non-empty. Taking $p(t) = M^{-1}t^n$ for this n, $p(T)(U) \cap W$ and $p(T)(W) \cap V$ are both non-empty and $T \oplus T$ is hypercyclic. ∎

REMARK 4.5. If T is hypercyclic and A is the set of vectors with bounded orbit, then A is either dense or rare: this follows directly from the fact that, A being invariant under the action of T, the interior of the norm-closure of A is an open T-invariant subset of X. Hence it is either empty or equal to the whole space. Thus Theorem 4.4 implies the following: if T is a hypercyclic operator such that the set of vectors with bounded orbit is not rare, then T satisfies the Hypercyclicity Criterion.

REMARK 4.6. It is also possible to consider backward orbits instead of (forward) orbits in Theorem 4.4 (this also generalizes a result of [6]): a sequence (x_n) is a backward orbit for the vector x if $x_0 = x$ and $T^n x_n = x$ for every $n \geqslant 1$. Suppose that T is hypercyclic and has a dense set of vectors which admit a bounded backward orbit. Let (U,V) be non-empty open subsets of X and W an open neighbourhood of 0. There exists a vector v in V with a bounded backward orbit (v_n). Let M be such that for every n, v_n belongs to MW. Then $T^n(MW)$ intersects V for all n. Now there exists an n such that $T^n(MU) \cap W$ is non-empty. Taking $p(t) = Mt^n$ and applying Proposition 4.1 yields the result: a hypercyclic operator with a dense set of vectors with bounded backward orbit (or such that the set of vectors with bounded backward orbits is not rare) satisfies the Hypercyclicity Criterion. It is also interesting to note that both conditions on orbits and backward orbits can be mixed to get the same result: if T is a hypercyclic operator such that the union of the set of vectors with bounded orbit and the set of vectors with bounded backward orbit is dense in X, then T satisfies the Hypercyclicity Criterion.

The idea which lies at the core of these last two results can be summarized as follows: if T is hypercyclic and has sufficiently many points with a "regular" orbit, then T is "strongly hypercyclic" in the sense that $T \oplus T$ is hypercyclic. This should be put in regard to what happens when considering continuous chaotic maps $\phi : (E,d) \to (E,d)$, on a metric space (E,d): ϕ is chaotic in the sense of Devaney if it is topologically transitive, has a dense set of periodic points, and has sensitive dependence on initial conditions: there exists a positive real number δ such that for every x in X and every neighborhood V of X, there exists a vector y in V and a nonnegative integer n such that $\phi^n x$ and $\phi^n y$ are more than the distance δ apart. It is in fact proved in [5] that if ϕ is topologically transitive and has dense periodic points, then, automatically, ϕ has sensitive dependence on initial conditions: what seems to be the most remarkable feature of chaotic maps follows from topological transitivity and regularity. The situation is the same in Theorems 4.3 and 4.4.

5. A CYCLICITY CRITERION

Consider a cyclic operator T: it is well-known that $T \oplus T$ can be non-cyclic: if the range of T has codimension 1, as is the case for instance for the forward shift on the Hilbert space, then $T \oplus T$ cannot be cyclic. Moreover the set of cyclic vectors for T spans the space, but is not necessarily dense in X. We nonetheless have the following criterion, which is the exact analogue of the Hypercyclicity Criterion for cyclic operators:

CYCLICITY CRITERION. *Let T be any operator on X. Suppose that there exist two dense subsets V and W of X, a sequence (p_k) of polynomials, and a sequence (S_k) of maps (not necessarily linear nor continuous) $S_k : W \to X$ such that:*
(i) *for every $x \in V$, $p_k(T)x \to 0$;*
(ii) *for every $x \in W$, $S_k x \to 0$;*
(iii) *for every $x \in W$, $p_k(T)S_k x \to x$.*
Then $T \oplus T$ is cyclic.

Proof. Let U_1, V_1, U_2, V_2 be four non empty open subsets of X. Our aim is to find a polynomial p such that $p(T)(U_1) \cap V_1 \neq \emptyset$ and $p(T)(U_2) \cap V_2 \neq \emptyset$. Let u_1, v_1, u_2, v_2 belong to $U_1 \cap V$, $V_1 \cap W$, $U_2 \cap V$ and $V_2 \cap W$ respectively. Then $u_1 + S_k v_1 \to u_1$, $p_k(T)(u_1 + S_k v_1) \to v_1$, $u_2 + S_k v_2 \to u_2$, $p_k(T)(u_2 + S_k v_2) \to v_2$, thus $p_k(T)(U_1) \cap V_1$ and $p_k(T)(U_2) \cap V_2$ are non-empty for k large enough. The sequence $(p_k(T) \oplus p_k(T))$ is hypercyclic, in particular $T \oplus T$ is cyclic and has a dense set of cyclic vectors. Remark also that the same proof shows that whenever T satisfies the Cyclicity Criterion, any direct sum $T \oplus T \oplus \cdots \oplus T$ of N copies of T is cyclic on $X \oplus X \oplus \cdots \oplus X$. ∎

REMARK 5.1. Suppose that X is a complex space. In order that $T \oplus T$ be cyclic, it is necessary that the point spectrum $\sigma_p(T^*)$ of the adjoint T^* be empty. If not, there exists an α such that the range of $T - \alpha I \oplus T - \alpha I$ has codimension at least 2, hence $T - \alpha I \oplus T - \alpha I$ cannot be cyclic. So $T \oplus T$ itself cannot be cyclic.

The following corollary is a slight improvement of Theorem 4 of [1]:

COROLLARY 5.2. *Let T be an upper-triangular operator on a complex space X. Then $T \oplus T$ is cyclic if and only if the point spectrum $\sigma_p(T^*)$ of the adjoint is empty.*

Proof. Take V to be the union of the kernels of the operators $p(T)$ with p a non-zero polynomial with rational coefficients. We enumerate these polynomials as (q_k). For every v in V, $q_k(T)v = 0$. Now since $\sigma_p(T^*)$ is empty, $q_k(T)$ has dense range. Let (w_n) be any dense sequence of X, and let $R_k w_n$ be such that $\|q_k(T)(R_k w_n) - w_n\| < 2^{-k}$ for $n \leqslant k$ with $R_k w_n \neq 0$. Consider now $p_k(X) = 2^k \max(\|R_k w_n\|, n \leqslant k) q_k(X)$ and S_k defined by

$$S_k w_n = \frac{R_k w_n}{2^k \max(\|R_k w_n\|, n \leqslant k)}$$

for $n \leqslant k$ and $S_k w_n = 0$ for $n > k$. Then $p_k(T)v = 0$ for every v in V, $S_k w_n$ tends to zero for every n, and $\|p_k(T)(S_k w_n) - w_n\| < 2^{-k}$ for $k \geqslant n$, which proves that for every n, $p_k(T)(S_k w_n) \to w_n$ as $k \to +\infty$. Hence T satisfies the Cyclicity Criterion and $T \oplus T$ is cyclic. ∎

The following statement is a reformulation of Remark 2.6 of [10] in a special case, so we omit the proof and refer the reader to [10]:

PROPOSITION 5.3. *Let T be any operator such that $\sigma_p(T^*) = \emptyset$. The following assertions are equivalent:*

(i) *T satisfies the Cyclicity Criterion;*

(ii) *there exists a sequence $(p_k)_{k\geqslant 1}$ of polynomials such that the sequence of operators $(p_k(T) \oplus p_k(T))_{k\geqslant 1}$ is hypercyclic;*

(iii) *$T \oplus T$ is cyclic.*

The condition $\sigma_p(T^*) = \emptyset$ is of course fulfilled when dealing with hypercyclic operators. If the polynomials p_k in the Cyclicity Criterion can be chosen to be multiples of monomials $p_k(X) = \lambda_k X^{n_k}$, then $T \oplus T$ is supercyclic: this is the Supercyclicity Criterion as presented in [27]. It is not known whether a supercyclic operator such that $\sigma_p(T^*)$ is empty satisfies the Supercyclicity Criterion [27]. On the other hand, it was pointed out to me by Aharon Atzmon that a cyclic operator with $\sigma_p(T^*) = \emptyset$ does not necessarily satisfy that $T \oplus T$ is cyclic: if N is a normal operator on a Hilbert space, $N \oplus N$ is not cyclic. If N is the multiplication operator by the independent variable x on $L^2([0,1])$, N is a cyclic self-adjoint operator such that $\sigma_p(N) = \emptyset$, but $N \oplus N$ is not cyclic.

6. MISCELLANEA

6.1. NORM-WEAK TOPOLOGICAL TRANSITIVITY.

Our aim in this section is to prove the following:

THEOREM 6.1. *Let T be a hypercyclic operator on X. Let U_1 and U_2 be two non-empty open sets, and $\widetilde{V}_1$ and $\widetilde{V}_2$ be two non-empty weakly open sets. There exists an integer n such that $T^n(U_1) \cap \widetilde{V}_1$ and $T^n(U_2) \cap \widetilde{V}_2$ are non-empty.*

Proof. The proof of this uses the same arguments as the proofs of Theorem 3.2 and Proposition 3.5, except for the fact that it is not possible to use sequences any more. We begin as in the proof of Theorem 3.2 by considering a hypercyclic vector v_1 belonging to $\widetilde{V}_1$ and an element $u_1 = T^{r_1}v_1$ of U_1. There exist integers p_1 and q_1 (with $p_1 > q_1$) such that the vectors $z_2 = T^{p_1}u_1 + T^{q_1+1}u_1$ and $y_2 = T^{p_1}v_1 + T^{q_1}v_1$ belong to U_2 and $\widetilde{V}_2$ respectively. Now there exist $\varepsilon > 0$ and functionals $x_1^*, \ldots, x_r^*$ such that

$$\{x \in X : \forall\, i = 1, \ldots, r |\langle x_i^*, x - v_1\rangle| < \varepsilon\} \subseteq \widetilde{V}_1$$

and

$$\{x \in X : \forall\, i = 1, \ldots, r |\langle x_i^*, x - y_2\rangle| < \varepsilon\} \subseteq \widetilde{V}_2.$$

Consider for $N > r(p_1 + 1)$ the map

$$\begin{aligned} \Phi : \mathbb{K}_{N-1}[\xi] &\longrightarrow \quad \mathbb{K}^r \times \cdots\cdots\cdots\cdots \times \mathbb{K}^r \\ p \quad &\longmapsto (\langle x_i^*, p(T)u_1\rangle)_{i=1,\ldots,r} \times \cdots \times (\langle x_i^*, p(T)T^{p_1}u_1\rangle)_{i=1,\ldots,r}. \end{aligned}$$

This linear map between a space of dimension greater than $r(p_1 + 1)$ and a space of dimension $r(p_1 + 1)$ cannot be injective, so there exists a non-zero polynomial p such that $\langle x_i^*, p(T)q(T)u_1\rangle = 0$ for every $i = 1, \ldots, r$ and every polynomial q of degree less or equal to p_1. Moreover p has dense range, so if $W_k = B(0, 2^{-k}) \cap T^{-1}(B(0, 2^{-k}))$ and $V_k = T^{-1}(B(v_1, 2^{-k}))$, there exists an $r_k > 0$ such that $p(T)(r_k W_k) \cap V_k \neq \emptyset$. Setting $p_k(t) = r_k p(t)$, we obtain that there exist two sequences w_k and w'_k such that $w_k \to 0$, $w'_k \to 0$ and $p_k(T)w_k \to v_1$, $p_k(T)Tw'_k \to v_1$. Moreover $\langle x_i^*, p_k(T)q(T)u_1\rangle = 0$ for $i = 1, \ldots, r$ and $q \in \mathbb{K}_{p_1}[\xi]$. We are going to show that $p_k(U_1) \cap \widetilde{V}_1$ and $p_k(U_2) \cap \widetilde{V}_2$ are non-empty if k is large enough. Consider first $p_k(T)(u_1 + w_k)$: for every $i = 1, \ldots, r$,

$$\langle x_i^*, p_k(T)(u_1 + w_k) - v_1\rangle = \langle x_i^*, p_k(T)w_k - v_1\rangle$$

and this quantity goes to 0 as k tends to infinity. Thus $p_k(T)(u_1 + w_k)$ belongs to $\widetilde{V}_1$ if k is large enough. Since $u_1 + w_k \to u_1$, $p_k(U_1) \cap \widetilde{V}_1$ is non-empty. Consider now $p_k(T)(T^{p_1}(u_1 + w_k) + T^{q_1+1}(u_1 + w'_k))$:

$$\begin{aligned} &\langle x_i^*, p_k(T)(T^{p_1}(u_1 + w_k) + T^{q_1+1}(u_1 + w'_k)) - y_2\rangle \\ &\qquad\qquad = \langle x_i^*, p_k(T)(T^{p_1}w_k + T^{q_1+1}w'_k) - (T^{p_1}v_1 + T^{q_1}v_1)\rangle. \end{aligned}$$

Now $T^{p_1}(p_k(T)w_k - v_1) \to 0$ and $T^{q_1}(p_k(T)Tw'_k - v_1) \to 0$. Hence the quantity above goes to 0 and $p_k(T)(T^{p_1}(u_1 + w_k) + T^{q_1+1}(u_1 + w'_k))$ is in $\widetilde{V}_2$ for k large enough. Since $T^{p_1}(u_1 + w_k) + T^{q_1+1}(u_1 + w'_k) \to z_2$, $p_k(U_2) \cap \widetilde{V}_2$ is non-empty for k large enough. The proof of Proposition 4.1 then shows that there exists an n such that $T^n(U_1) \cap \widetilde{V}_1$ and $T^n(U_2) \cap \widetilde{V}_2$ are non-empty, which proves our claim. ∎

6.2. A CONNECTEDNESS RESULT. Ansari's striking result [1] that T and T^N, $N \geqslant 2$, have the same hypercyclic vectors relies in a crucial way on a connectedness argument. This is also the case for the results [13], [29] and [12], which yield Ansari's result as a corollary. Thus it seems plausible that an affirmative answer to Question 1.2 would involve a connectedness argument. The following proposition shows that connected sets appear in a natural way while studying hypercyclic operators:

PROPOSITION 6.2. *Let U and V be two non-empty connected open subsets of X, and let T be a cyclic operator on X with a dense set of cyclic vectors. Then*

$$\Omega = \bigcup_{p \in \mathbb{K}[\xi] \setminus \{0\}} p(T)(U) \cap V$$

is a connected dense subset of V.

Proof. Let $q_-(T)x_-$ and $q_+(T)x_+$ be two elements of Ω with x_- and x_+ in U and q_- and q_+ two non-zero polynomials. Let z be any cyclic vector for T belonging to U, and let p_- be a non-zero polynomial such that $p_-(T)z$ and $q_-(T)p_-(T)z$ are so close to x_- and $q_-(T)x_-$ respectively that for every $t \in [0,1]$, the vectors $tp_-(T)z + (1-t)x_-$ and $tq_-(T)p_-(T)z + (1-t)q_-(T)x_-$ lie in U and V respectively. We can also choose the degree of p_- to be greater than the degrees of q_- and q_+. In the sequel of the proof, when choosing a polynomial, we will always choose it so that its degree is greater than the degrees of all the previous polynomials. The path

$$\begin{aligned} \Phi : [0,1] &\longrightarrow X \\ t &\longmapsto q_-(T)(tp_-(T)z + (1-t)x_-) \end{aligned}$$

is a path in Ω such that $\Phi(0) = q_-(T)x_-$ and $\Phi(i) = q_-(T)p_-(T)z$. In the same way there is a polynomial p_+ of large enough degree such that $p_+(T)z$ is so close to x_+ and $q_+(T)p_+(T)z$ is so close to $q_+(T)x_+$ that $q_+(T)x_+$ and $q_+(T)p_+(T)z$ can be joined by a path in Ω. Thus it remains to connect $q_-(T)p_-(T)z$ and $q_+(T)p_+(T)z$. Since U is a connected open set in a normed space, there is a polygonal line

$$u_0 = p_-(T)z, u_1 = p_1(T)z, \ldots, u_{r-1} = p_{r-1}(T)z \text{ and } u_r = p_+(T)z$$

of vectors of U such that for every $i = 1, \ldots, r-1$ there exists a polynomial q_i with $q_i(T)p_i(T)z$ in V. We will now connect u_i and u_{i+1} by a path in Ω. Consider

$$U_i^\varepsilon = \{x \in U : d(x, [u_i, u_{i+1}]) < \varepsilon\}$$

the ε-dilation in U of the segment $[u_i, u_{i+1}]$: it is an open set containing u_i and u_{i+1}. If ε is small enough, U_i^ε is convex: for every $t \in [0,1]$ there exists an open ball of radius $\varepsilon_t > 0$ centered at $tu_i + (1-t)u_{i+1}$ contained in U. Since the segment $[u_i, u_{i+1}]$ is compact, finitely many such balls $\widetilde{B}_j, j = 1, \ldots, s$ suffice to cover this segment. Let ε be the distance between $[u_i, u_{i+1}]$ and the complement of $\bigcup_{j=1,\ldots,s} \widetilde{B}_j$: $\varepsilon > 0$ and if $d(x, [u_i, u_{i+1}]) < \varepsilon$ then x is in U. It follows that U_i^ε is indeed a convex subset of U. Let now $\alpha_i : [0,1] \to V$ be a continuous map such that $\alpha_i(0) = q_i(T)u_i$ and $\alpha_i(1) = q_{i+1}(T)u_{i+1}$. Since $\alpha_i([0,1])$ is compact, there exist finitely many balls $B_j, j = 1, \ldots, m$ contained in V which cover $\alpha_i([0,1])$ and satisfy $B_k \cap B_{k+1} \neq \emptyset$ for $k = 1, \ldots, m-1$, $u_i \in B_1$ and $u_{i+1} \in B_m$. Now let $w_k = \widetilde{p}_k(T)z$ be

$m-1$ points of U_i^ε such that $\widetilde{q}_k(T)w_k$ belongs to $B_k \cap B_{k+1}$ for some polynomial $\widetilde{q}_k$ of high degree. We first connect u_i to w_1 by the path

$$\Phi : t \longmapsto t\widetilde{q}_1(T)w_1 + (1-t)q_i(T)u_i = (t\widetilde{q}_1(T)\widetilde{p}_1(T) + (1-t)q_i(T)p_i(T))(z).$$

This path lies in V since $q_i(T)u_i$ and $\widetilde{q}_1(T)w_1$ are in B_1 which is convex, and

$$\Phi(t) = p_t(z)$$

where z is in U and p_t is non-zero (the degrees of the polynomials have been chosen to be strictly increasing). Hence this is a path in Ω which connects u_i and w_1. In the same way,

$$\Phi : t \longmapsto t\widetilde{q}_2(T)w_2 + (1-t)\widetilde{q}_1(T)w_1$$

is a path with range in $B_2 \cap \Omega$ which connects w_1 and w_2. Going on in this way, we finally connect w_{m-1} and u_{i+1}: we have connected u_i and u_{i+1} in Ω. This being true for every i, the conclusion follows. ∎

6.3. A LAST REMARK. We have seen that the Hypercyclicity Criterion Problem boils down to showing that $T \oplus T$ is cyclic whenever T is hypercyclic. But it is easy to show that $T \oplus (-T)$ is always cyclic in this case: if x is a hypercyclic vector for T, then $x \oplus x$ is cyclic for $T \oplus (-T)$. Let v_1 and v_2 be two vectors of X. Since T^2 is hypercyclic [1], there exist a sequence (p_k) of integers such that $T^{2p_k}x \to \frac{1}{2}(v_1+v_2)$ and also a sequence (q_k) of integers such that $T^{2q_k+1}x \to \frac{1}{2}(v_1-v_2)$. Set $p_k(X) = X^{2p_k} + X^{2q_k+1}$: $p_k(T)x \to v_1$ and $p_k(-T)x \to v_2$, which proves our claim. Moreover, the same proof shows that $x \oplus T^k x$ is cyclic for $T \oplus (-T)$ for any k, so the set of cyclic vectors for $T \oplus (-T)$ is dense in $X \oplus X$.

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