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INVARIANT SUBSPACES FOR COMMUTING PAIRS WITH NORMAL BOUNDARY DILATION AND DOMINATING TAYLOR SPECTRUM

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ABSTRACT. Let $T \in L(H)^n$ be a commuting tuple of continuous linear operators on a separable complex Hilbert space. In this article we show that interior points of the Fredholm spectrum of T can be made accessible to the Scott Brown technique by establishing factorizations of the corresponding point evaluations via the holomorphic functional calculus. This allows us to improve a series of known results in the context of the invariant-subspace and the reflexivity problem. In particular we deduce that each commuting pair $T = (T_1, T_2) \in L(H)^2$ possessing a ∂D-unitary dilation and dominating Taylor spectrum in a strictly pseudoconvex open subset $D \Subset \mathbb{C}^2$ has a non-trivial invariant subspace.

KEYWORDS: *H^∞-functional calculus, invariant subspaces, dominating Taylor spectrum, strictly pseudoconvex sets.*

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1. INTRODUCTION

A classical theorem of Brown, Chevreau and Pearcy from 1979 guarantees that a contraction T on a complex Hilbert space H possesses a non-trivial invariant subspace if the spectrum of T is dominating in the open unit disc $\mathbb{D}$ (see [2]).

In 1996, a corresponding multivariable result has been obtained by Eschmeier, who asked for joint invariant subspaces of spherical contractions, i.e. commuting n-tuples $T = (T_1, \ldots, T_n) \in L(H)^n$ satisfying $\sum_{i=1}^{n} \|T_i x\|^2 \leqslant \|x\|^2$ for every $x \in H$. More precisely, Theorem 3.3 in [6] states that each spherical contraction $T \in L(H)^n$ which possesses a spherical dilation and dominating Harte spectrum $\sigma^{\mathrm{H}}(T)$ in the open Euclidean unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ has a non-trivial joint invariant subspace. As carried out by Pott [11], an analogous result holds for commuting

n-tuples possessing a (holomorphic) normal boundary dilation and dominating Harte spectrum over a strictly pseudoconvex open subset $D \Subset \mathbb{C}^n$.

Since, for $n = 1$, the Harte spectrum and the (ordinary) spectrum of T coincide and, by a famous theorem of Sz.-Nagy, each single contraction possesses a unitary power dilation, the cited theorem of Eschmeier is in fact a generalization of the classical single-operator result. However, as remarked in [6], it is quite natural to ask whether the above invariant-subspace result for spherical contractions remains true if the Harte spectrum $\sigma^{\mathrm{H}}(T)$ is replaced by the Taylor spectrum $\sigma(T)$ which is in many respects a more natural choice of a joint spectrum for a commuting n-tuple $T \in L(H)^n$ (and always contains the Harte spectrum). It is the aim of the present paper to answer this question in the affirmative if $n = 2$ even in the more general strictly pseudoconvex case.

To be more specific, let $X \subset \mathbb{C}^n$ be a Stein submanifold of fixed dimension $1 \leqslant \dim_{\mathbb{C}}(X) \leqslant n$ and let $\emptyset \neq D \Subset X$ be a relatively compact, strictly pseudoconvex open subset which will be kept fixed throughout this article. In the following, we write $A(D)$ for the algebra of all continuous complex-valued functions on $\overline{D}$ which are holomorphic on D equipped with the supremum norm, and we denote by $H^\infty(D)$ the algebra of all bounded holomorphic functions on D equipped with its canonical dual algebra structure (see Section 2.1 in [4] for details).

For the remainder of this paper, H always denotes a separable complex Hilbert space. A commuting n-tuple $T \in L(H)^n$ possessing a contractive $A(D)$-functional calculus $\Phi_T : A(D) \to L(H)$ will be called a von Neumann n-tuple over D. As pointed out in Section 2.3 of [4], such a representation Φ_T is unique and necessarily extends Taylor's holomorphic functional calculus for T.

By a ∂D-unitary dilation of a von Neumann n-tuple $T \in L(H)^n$ over D we mean a commuting n-tuple $N \in L(K)^n$ of normal operators acting on some possibly larger Hilbert space $K \supset H$ with $\sigma(N) \subset \partial D$ and $\Phi_T(f) = P_H \Phi_N(f)|H$ for each $f \in A(D)$. Here P_H denotes the orthogonal projection from K onto H. Note that, according to the unitary dilation theorem of Sz.-Nagy and the von Neumann inequality, each single contraction $T \in L(H)$ fits into the context of von Neumann n-tuples possessing a ∂D-unitary dilation (with $n = 1$ and $D = \mathbb{D}$).

Our aim is to show that a von Neumann pair $T \in L(H)^2$ over a strictly pseudoconvex subset $D \Subset X$ of a Stein submanifold $X \subset \mathbb{C}^2$ has a non-trivial invariant subspace if it possesses a ∂D-unitary dilation and dominating Taylor spectrum $\sigma(T)$ in D (Corollary 4.7).

As in the single-variable theory it suffices to prove this invariant-subspace result under some additional continuity hypotheses on the representation Φ_T. To be more precise, recall that a representation $\Phi : A(D) \to L(H)$ is said to be of class $C_{0\cdot}$ if it is sequentially weak*-SOT continuous. In this case Φ extends in a unique way to a representation of $H^\infty(D)$ (usually again denoted by Φ) having the same continuity property. We call a von Neumann n-tuple $T \in L(H)^n$ over D of type $C_{0\cdot}$ if the corresponding representation Φ_T is of class $C_{0\cdot}$. If T^* rather than T itself satisfies the $C_{0\cdot}$-condition, then we say that T belongs to the class

$C_{\cdot 0}$. A well-known argumentation scheme can be used to show that if T is a von Neumann n-tuple over D being neither of class $C_{0\cdot}$ nor of class $C_{\cdot 0}$, then T has a non-trivial hyperinvariant subspace or the components of T are scalar multiples of the identity (cf. Section 3.1 in [4]). From this and a simple duality argument it follows that the proof of our invariant-subspace theorem reduces to the case where T is of class $C_{\cdot 0}$.

The richness condition on the Taylor spectrum of T guarantees that in this case the $H^\infty(D)$-functional calculus Φ_T (being surely weak* continuous by the $C_{\cdot 0}$-condition) is isometric. In general a von Neumann n-tuple $T \in L(H)^n$ over D is said to be of class $\mathbb{A}$ if it possesses a ∂D-unitary dilation and, in addition, an isometric and weak* continuous $H^\infty(D)$-functional calculus Φ_T (which then necessarily extends the $A(D)$-functional calculus of T). The solution of the desired invariant-subspace problem now relies on the structure theory of von Neumann n-tuples in the class $\mathbb{A} \cap C_{\cdot 0}$ as developed in [4].

To avoid technical difficulties, let us assume for a moment that $X = \mathbb{C}^n$. (Note that a restriction to the case $n = 2$ is not necessary at this stage.) We show first that each von Neumann n-tuple T of the class $\mathbb{A} \cap C_{\cdot 0}$ over D for which the set $\sigma_{\mathrm{e}}(T) \cup \operatorname{Int}(\sigma(T) \setminus \sigma_{\mathrm{e}}(T))$ is dominating in D is reflexive (Proposition 4.4). The proof of this statement is an application of the Scott Brown technique and depends on the fact that we are able to factorize point evaluations over the essential spectrum (Section 2) and the interior points of the Fredholm spectrum (Section 3).

The announced invariant-subspace result can now be obtained from the above reflexivity assertion by combining the standard reductions, namely $\sigma^{\mathrm{H}}(T) \subset \sigma_{\mathrm{e}}(T)$ and $T \in [C_{\cdot 0}]$, with the observation that in the case $n = 2$ the Taylor spectrum of T possesses the representation $\sigma(T) = \sigma^{\mathrm{H}}(T) \cup \sigma_{\mathrm{e}}(T) \cup \operatorname{Int}(\sigma(T) \setminus \sigma_{\mathrm{e}}(T))$.

In particular, applying Corollary 4.7 to the special case $X = \mathbb{C}^2$ and $D = \mathbb{B}_2$, one immediately obtains that each spherical contraction $T \in L(H)^2$ of length 2 with a spherical dilation and dominating Taylor spectrum in $\mathbb{B}_2$ possesses a non-trivial invariant subspace.

2. ESSENTIAL SPECTRUM AND HARTE SPECTRUM

Fix a Stein submanifold $X \subset \mathbb{C}^n$, a relatively compact strictly pseudoconvex open subset $\emptyset \neq D \Subset X$ and a separable complex Hilbert space H. Let $T \in L(H)^n$ denote a commuting n-tuple of operators possessing a weak* continuous functional calculus $\Phi : H^\infty(D) \to L(H)$ over the dual algebra $H^\infty(D)$.

For any natural number $M \geqslant 1$, Φ induces a representation $\Phi^{(M)}$ of $H^\infty(D)$ on the Hilbert space H^M given by the assignment $f \mapsto \Phi(f)^{(M)}$. The dual representation corresponding to Φ is the map $\Phi^* : H^\infty(D^*) \to L(H), f \mapsto \Phi(f_*)^*$, where D^* denotes the image of D under the complex conjugation $z \mapsto \overline{z}, z \in \mathbb{C}^n$, and $f_* \in H^\infty(D_*)$ is defined by $f_*(z) = \overline{f(\overline{z})}$ for $z \in D^*$ (see the remarks following Corollary 2.3.7 in [4] for a more detailed explanation).

Given vectors $x, y \in H$, Φ induces a weak* continuous vector functional

$$x \otimes y : H^\infty(D) \to \mathbb{C}, \quad f \mapsto \langle \Phi(f)x, y \rangle$$

on $H^\infty(D)$. To distinguish the vector functionals belonging to $\Phi, \Phi^{(M)}$ and Φ^* we denote them by $x \otimes y, x \otimes_M y$ and $x \otimes_* y$, respectively. The space of all weak* continuous linear forms on $H^\infty(D)$ will be abbreviated by $Q(D)$ in the sequel. The main aim of this paper is to represent point evaluations

$$\mathcal{E}_\lambda : H^\infty(D) \to \mathbb{C}, \quad f \mapsto f(\lambda) \quad \text{with } \lambda \in D,$$

as vector functionals via $\Phi^{(M)}$ with some $M \geqslant 1$, i.e. to solve the equation $\mathcal{E}_\lambda = x \otimes_M y$ with $x, y \in H^M$ for as many $\lambda \in D$ as possible.

Before considering this factorization problem in more detail we have to fix some notations concerning the Koszul complex and the Taylor spectrum. Let $p \in \{0, \dots, n\}$ be an integer. We write $\Lambda^p(n, H)$ for the complex vector space of all p-forms in n indeterminates $e_1, \dots, e_n$ with coefficients in H. Recall that $\Lambda^0(n, H) = H$ and that, for $1 \leqslant p \leqslant n$, each element $x \in \Lambda^p(n, H)$ has a unique representation

$$x = \sum_{1 \leqslant i_1 < \dots < i_p \leqslant n} x_{(i_1, \dots, i_p)} e_{i_1} \wedge \dots \wedge e_{i_p} = \sum_{|I| = p} x_I e_I \quad \text{with } x_I \in H.$$

Here, the symbol $\sum\limits_{|I|=p}$ has to be understood as an abbreviation of the sum in the middle term and, by definition, $e_I = e_{i_1} \wedge \dots \wedge e_{i_p}$, for $I = (i_1, \dots, i_p) \in \mathbb{N}^p$. If $|I| = 0$, then we set $e_I = 1 \in \mathbb{C}$. The formula $\Big\langle \sum\limits_{|I|=p} x_I e_I, \sum\limits_{|I|=p} y_I e_I \Big\rangle = \sum\limits_{|I|=p} \langle x_I, y_I \rangle$ defines a scalar product on the space $\Lambda^p(n, H)$ turning it into a Hilbert space which is isometrically isomorphic to $H^{\binom{n}{p}}$. Consequently, the direct sum

$$\Lambda(n, H) = \bigoplus_{p=0}^{n} \Lambda^p(n, H)$$

can be identified with H^{2^n} (as a Hilbert space).

Given a commuting n-tuple $T = (T_1, \dots, T_n) \in L(H)^n$, the Koszul complex $K_\bullet(T, H)$ induced by T is the finite complex

$$0 \longleftarrow K_0(T, H) \overset{\delta_1(T)}{\longleftarrow} K_1(T, H) \overset{\delta_2(T)}{\longleftarrow} \cdots \overset{\delta_n(T)}{\longleftarrow} K_n(T, H) \longleftarrow 0$$

consisting of the spaces $K_p(T, H) = \Lambda^p(n, H)$ and the unique linear mappings $\delta_p(T) : \Lambda^p(n, H) \to \Lambda^{p-1}(n, H)$ satisfying

$$\delta_p(T)(x e_I) = \sum_{j=1}^{p} (-1)^{j-1} (T_{i_j} x) e_{i_1} \wedge \dots \wedge \widehat{e}_{i_j} \wedge \dots \wedge e_{i_p} \quad \text{for } x \in H, |I| = p.$$

The cochain Koszul complex $K^\bullet(T, H)$ induced by T is given by

$$0 \longrightarrow K^0(T, H) \overset{\delta^0(T)}{\longrightarrow} K^1(T, H) \overset{\delta^1(T)}{\longrightarrow} \cdots \overset{\delta^{n-1}(T)}{\longrightarrow} K^n(T, H) \longrightarrow 0,$$

where $K^p(T,H) = \Lambda^p(n,H)$ and $\delta^p(T) : \Lambda^p(n,H) \to \Lambda^{p+1}(n,H)$ is the unique linear map with the property

$$\delta^p(T)(xe_I) = \sum_{j=1}^{n} (T_j x) e_j \wedge e_I \quad \text{for all } x \in H, |I| = p.$$

It is elementary to check that there exists a universal constant $C = C(n) > 0$ such that

$$\|\delta_p(T)\|, \|\delta^p(T)\| \leqslant C \max_{j=1,\dots,n} \|T_j\| \quad \text{for all } p = 0, \dots, n.$$

The following well-known algebraic relation between the boundary maps of the Koszul complex and the Koszul cochain complex plays a central role in the solution of the factorization problem for point evaluations.

LEMMA 2.1. *Let $S, T \in L(H)^n$ be two commuting n-tuples such that the components of T commute pairwise with the components of S. Then, for $p = 0, \dots, n$ and $M = \binom{n}{p}$, the identity*

$$\delta^{p-1}(T)\delta_p(S) + \delta_{p+1}(S)\delta^p(T) = \sum_{j=1}^{n} S_j^{(M)} T_j^{(M)}$$

holds on $H^M = \Lambda^p(n,H)$.

Before we can demonstrate how the lemma is actually used to factorize point evaluations, we have to fix some more notations. Recall that the Taylor spectrum $\sigma(T)$ of a commuting n-tuple $T \in L(H)^n$ is by definition the set

$$\sigma(T) = \{\lambda \in \mathbb{C}^n : K^\bullet(\lambda - T, H) \text{ is not exact}\}.$$

Here, $\lambda - T$ denotes the n-tuple $(\lambda_1 - T_1, \dots, \lambda_n - T_n)$. The non-exactness of $K^\bullet(\lambda - T, H)$ is equivalent to the condition that at least one of the corresponding cohomology groups

$$H^p(\lambda - T, H) = \ker \delta^p(\lambda - T) / \operatorname{ran} \delta^{p-1}(\lambda - T) \quad \text{with } 0 \leqslant p \leqslant n$$

is non-trivial. Since we are in the Hilbert space case, a point $\lambda \in \mathbb{C}^n$ belongs to the so called essential Taylor spectrum

$$\sigma_{\mathrm{e}}(T) = \{\lambda \in \mathbb{C}^n : \dim H^p(\lambda - T, H) = \infty \text{ for at least one } p \in \{0, \dots, n\}\}$$

if and only if there exist a natural number $p \in \{0, \dots, n\}$ as well as an orthonormal sequence

$$(x_k) \text{ in } \Lambda^p(n,H) \ominus \operatorname{ran} \delta^{p-1}(\lambda - T) \quad \text{satisfying} \quad \delta^p(\lambda - T)x_k \xrightarrow{k\to\infty} 0.$$

The Harte spectrum (essential Harte spectrum) of T can be defined as the union

$$\sigma^{\mathrm{H}}(T) = \sigma_{\mathrm{l}}(T) \cup \sigma_{\mathrm{r}}(T), \quad \sigma^{\mathrm{H}}_{\mathrm{e}}(T) = \sigma_{\mathrm{le}}(T) \cup \sigma_{\mathrm{re}}(T)$$

of the following subsets of the Taylor spectrum:

$$\begin{aligned}
\sigma_{\mathrm{le}}(T) &= \{\lambda \in \mathbb{C}^n : \dim H^0(\lambda - T, H) = \infty \text{ or } \delta^0(\lambda - T) \text{ has non-closed range}\},\\
\sigma_{\mathrm{l}}(T) &= \{\lambda \in \mathbb{C}^n : H^0(\lambda - T, H) \neq 0 \text{ or } \delta^0(\lambda - T) \text{ has non-closed range}\},\\
\sigma_{\mathrm{re}}(T) &= \{\lambda \in \mathbb{C}^n : \dim H^n(\lambda - T, H) = \infty\} = \sigma_{\mathrm{le}}(T^*)^*,\\
\sigma_{\mathrm{r}}(T) &= \{\lambda \in \mathbb{C}^n : H^n(\lambda - T, H) \neq 0\} = \sigma_{\mathrm{l}}(T^*)^*.
\end{aligned}$$

Note that a point $\lambda \in \mathbb{C}^n$ belongs to $\sigma_{\mathrm{le}}(T)$ if and only if, for each finite dimensional subspace $M \subset H = \Lambda^0(n, H)$, there exists an orthonormal sequence (x_k) in $H \ominus M$ satisfying $\delta^0(\lambda - T)x_k \overset{k\to\infty}{\longrightarrow} 0$. Moreover,

$$\bigcap_{j=1}^{n} \ker(\lambda_j - T_j) = H^0(\lambda - T, H) \neq \{0\} \quad \text{whenever } \lambda \in \sigma_{\mathrm{l}}(T) \setminus \sigma_{\mathrm{le}}(T).$$

The following observation allows us to factorize point evaluations in points belonging to the essential spectrum $\sigma_{\mathrm{e}}(T)$.

LEMMA 2.2. *Let $\lambda^1, \dots, \lambda^r \in D$ be arbitrary points, $r \in \mathbb{N}$. Then there exists a constant $c > 0$ (depending on $\lambda^1, \dots, \lambda^r$ and the representation Φ) with the following property:*

Let $c_1, \dots, c_r \geqslant 0$ be non-negative real numbers such that $\sum_{j=1}^{r} c_j \leqslant 1$. Suppose that, for some $p \in \{0, \dots, n\}$, there exists an orthonormal system $(x^j)_{j=1,\dots,r}$ in $\Lambda^p(n, H) \ominus \bigvee_{j=1}^{r} \operatorname{ran} \delta^{p-1}(\lambda^j - T)$. Then the vector

$$x = \sum_{j=1}^{r} \sqrt{c_j} x^j \in \Lambda^p(n, H) = H^{\binom{n}{p}}$$

satisfies the estimate $\left\| x \otimes_{\binom{n}{p}} x - \sum_{j=1}^{r} c_j \mathcal{E}_{\lambda^j} \right\| \leqslant c \sum_{j=1}^{r} \|\delta^p(\lambda^j - T)x^j\|$.

Proof. For abbreviation, we set $N = \binom{n}{p}$. Starting with an arbitrary $f \in H^\infty(D)$, $\|f\|_{\infty,D} \leqslant 1$, we consider the expression

$$\begin{aligned}
\Big(x \otimes_N x - \sum_{j=1}^{r} c_j \mathcal{E}_{\lambda^j}\Big)(f) &= \sum_{j=1}^{r} \sqrt{c_j} \langle \Phi^{(N)}(f) x^j, x\rangle - \sum_{j=1}^{r} \sqrt{c_j} \langle f(\lambda^j) x^j, x\rangle\\
&= \sum_{j=1}^{r} \sqrt{c_j} \langle \Phi^{(N)}(f - f(\lambda^j)) x^j, x\rangle.
\end{aligned}$$

For each $1 \leqslant j \leqslant r$, there exists an n-tuple $g^j = (g_1^j, \dots, g_n^j) \in H^\infty(D)^n$ of functions with norm $\|g_i^j\|_{\infty,D} \leqslant \gamma$, for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant r$, solving the division problem

$$f - f(\lambda^j) = \sum_{i=1}^{n} (\lambda_i^j - z_i) g_i^j$$

(see Proposition 2.1.6 (c) in [4] and note that the constant $\gamma > 0$ does not depend on f but may depend on $\lambda^1, \dots, \lambda^r$). Applying the functional calculus $\Phi^{(N)}$ to this equation and making use of Lemma 2.1, we obtain the identity

$$\begin{aligned}\Phi^{(N)}(f - f(\lambda^j)) &= \sum_{i=1}^{n} (\lambda_i^j - T_i^{(N)}) \Phi^{(N)}(g_i^j)\\ &= \delta^{p-1}(\lambda^j - T)\delta_p(\Phi(g^j)) + \delta_{p+1}(\Phi(g^j))\delta^p(\lambda^j - T)\end{aligned}$$

for $j = 1, \dots, r$. In view of the orthogonality relation $x \perp \bigvee_{j=1}^{r} \operatorname{ran} \delta^{p-1}(\lambda^j - T)$ we deduce that the following estimate holds uniformly for f in the unit ball of $H^\infty(D)$:

$$\begin{aligned}\Big|\Big(x \otimes_N x - \sum_{j=1}^{r} c_j \mathcal{E}_{\lambda^j}\Big)(f)\Big| &= \Big|\sum_{j=1}^{r} \sqrt{c_j} \langle \delta_{p+1}(\Phi(g^j))\delta^p(\lambda^j - T)x^j, x\rangle\Big|\\ &\leqslant C\|\Phi\|\gamma \sum_{j=1}^{r} \|\delta^p(\lambda^j - T)x^j\|.\end{aligned}$$

This observation completes the proof. ∎

As a canonical application (cf. Albrecht and Chevreau [1] and Eschmeier [5]) one obtains the following factorization results.

COROLLARY 2.3. (i) *For each* $\lambda \in \sigma_{\mathrm{e}}(T) \cap D$, *there exists an orthonormal sequence* (x_k) *in* $\Lambda(n, H) = H^N, N = 2^n$, *satisfying* $x_k \otimes_N x_k \xrightarrow{k\to\infty} \mathcal{E}_\lambda$.

(ii) *If* $\lambda_1, \dots, \lambda_r \in \sigma_{\mathrm{le}}(T) \cap D$, *then, for any choice of non-negative real numbers* $c_1, \dots, c_r \geqslant 0$ *with* $\sum_{j=1}^{r} c_j \leqslant 1$, *there exists an orthogonal sequence* (x_k) *of vectors in the unit ball of* H *such that* $x_k \otimes x_k \xrightarrow{k\to\infty} \sum_{j=1}^{r} c_j \mathcal{E}_{\lambda_j}$.

(iii) *The assertion of part* (ii) *remains true after replacing* $\sigma_{\mathrm{le}}(T)$ *by* $\sigma_{\mathrm{re}}(T)$.

Proof. A point $\lambda \in D$ belongs to $\sigma_{\mathrm{e}}(T)$ if and only if there exists an index $p \in \{0, \dots, n\}$ and an orthonormal sequence (x_k) in $\Lambda^p(n, H) \ominus \operatorname{ran} \delta^{p-1}(\lambda - T) \subset \Lambda(n, H)$ satisfying $\delta^p(\lambda - T)x_k \xrightarrow{k\to\infty} 0$. Applying the previous lemma, we conclude that (i) holds.

Using the remarks concerning the left essential spectrum $\sigma_{\mathrm{le}}(T)$ to be found in the preceding section, one easily checks that, for $\lambda_1, \dots, \lambda_r \in \sigma_{\mathrm{le}}(T) \cap D$, there is an orthonormal family

$$(x_k^j)_{k\geqslant 1}^{1\leqslant j\leqslant r} \quad \text{in } H \text{ such that} \quad \delta^0(\lambda_j - T)x_k^j \xrightarrow{k\to\infty} 0 \quad \text{for each } j = 1, \dots, r.$$

Once again by the previous lemma, the vectors $x_k = \sum_{j=1}^{r} \sqrt{c_j} x_k^j$, $k \geqslant 1$, have all the desired properties.

Part (iii) follows by a standard duality argument: Given $\lambda_1, \ldots, \lambda_r \in \sigma_{\mathrm{re}}(T) \cap D$, we apply part (ii) to obtain an orthogonal sequence (x_k) of vectors in the unit ball of H such that $x_k \otimes_* x_k \stackrel{k\to\infty}{\longrightarrow} \sum_{j=1}^{r} c_j \mathcal{E}_{\lambda_j^*}$. To conclude the proof, it suffices to check the elementary identity $\left\| x \otimes x - \sum_{j=1}^{r} c_j \mathcal{E}_{\lambda_j} \right\| = \left\| x \otimes_* x - \sum_{j=1}^{r} c_j \mathcal{E}_{\lambda_j^*} \right\|$ for every $x \in H$. ∎

As we just have seen, it is possible to factorize convex combinations of point evaluations over the left and the right essential spectrum. The following lemma allows us to treat the Fredholm part of the Harte spectrum.

LEMMA 2.4. *Suppose that $\lambda^1, \ldots, \lambda^r \in D$ are pairwise distinct and satisfy either $\lambda^1, \ldots, \lambda^r \in \sigma_{\mathrm{l}}(T) \setminus \sigma_{\mathrm{le}}(T)$ or $\lambda^1, \ldots, \lambda^r \in \sigma_{\mathrm{r}}(T) \setminus \sigma_{\mathrm{re}}(T)$. Then, for any choice of non-negative real numbers $c_1, \ldots, c_r \geqslant 0$ with $\sum_{j=1}^{r} c_j \leqslant 1$, there exist vectors $x, y \in H$ such that $\|x\|, \|y\| \leqslant 1$ and $x \otimes y = \sum_{j=1}^{r} c_j \mathcal{E}_{\lambda^j}$.*

Proof. A duality argument (cf. the proof of Corollary 2.3) reduces the assertion to the case where $\lambda^1, \ldots, \lambda^r \in (\sigma_{\mathrm{l}}(T) \setminus \sigma_{\mathrm{le}}(T)) \cap D$. In this situation, we find an (automatically linearly independent) system $(y_1, \ldots, y_r)$ of joint eigenvectors $0 \neq y_j \in \bigcap_{i=1}^{n} \ker(\lambda_i^j - T_i)$. Of course we are allowed to assume that $c = \sum_{j=1}^{r} c_j > 0$. By a result of Zenger (Proposition 2.6.20 in [10]), there exist a vector $x \in H$, $\|x\| = 1$, as well as complex numbers $\mu_1, \ldots, \mu_r \in \mathbb{C}$ such that $y = \sum_{j=1}^{r} \mu_j y_j$ satisfies $\|y\| = 1$, and $\langle \mu_j y_j, x \rangle = c_j / c$, $j = 1, \ldots, r$. To complete the proof, fix $f \in H^\infty(D)$ and choose $g_i^j \in H^\infty(D)$, $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant r$, with $f - f(\lambda^j) = \sum_{i=1}^{n} (\lambda_i^j - z_i) g_i^j$ to obtain

$$
\begin{aligned}
(cy \otimes x)(f) - \sum_{j=1}^{r} c_j f(\lambda^j) &= \sum_{j=1}^{r} \mu_j c \langle \Phi(f - f(\lambda^j)) y_j, x \rangle \\
&= \sum_{j=1}^{r} \mu_j c \Big\langle \sum_{i=1}^{n} \Phi(g_i^j)(\lambda_i^j - T_i) y_j, x \Big\rangle = 0. \quad ∎
\end{aligned}
$$

In connection with Corollary 2.3 this implies the possibility to factorize absolute convex combinations of point evaluations over the Harte spectrum.

COROLLARY 2.5. *Suppose that* $\lambda_1, \ldots, \lambda_r \in \sigma^{\mathrm{H}}(T) \cap D$ *are pairwise distinct. Then, given any complex numbers* $c_1, \ldots, c_r \in \mathbb{C}$ *with* $\sum_{j=1}^{r} |c_j| \leqslant 1$, *there exist sequences* $(x_k)_{k\geqslant 1}$, $(y_k)_{k\geqslant 1}$ *in* H^{16} *satisfying* $\|x_k\|, \|y_k\| \leqslant 4$, $k \geqslant 1$, *and* $x_k \otimes_{16} y_k \xrightarrow{k\to\infty} \sum_{j=1}^{r} c_j \mathcal{E}_{\lambda_j}$.

Proof. We first restrict ourselves to the case of convex combinations, that is, we assume that $c_1, \ldots, c_r \geqslant 0$ are non-negative real numbers.

We write $c_{\lambda_j} = c_j$ and $\Lambda = \{\lambda_j : j = 1, \ldots, r\} \subset \sigma_{\mathrm{l}}(T) \cup \sigma_{\mathrm{r}}(T)$, and decompose a given convex combination

$$L = \sum_{j=1}^{r} c_{\lambda_j} \mathcal{E}_{\lambda_j} = \sum_{i=1}^{4} L_i \quad \text{with } L_i = \sum_{\lambda \in J_i} c_\lambda \mathcal{E}_\lambda \text{ for } i = 1, \ldots, 4,$$

where $J_1 \subset \Lambda \cap \sigma_{\mathrm{le}}(T)$, $J_2 \subset \Lambda \cap (\sigma_{\mathrm{l}}(T) \setminus \sigma_{\mathrm{le}}(T))$, $J_3 \subset \Lambda \cap \sigma_{\mathrm{re}}(T)$ and $J_4 \subset \Lambda \cap (\sigma_{\mathrm{r}}(T) \setminus \sigma_{\mathrm{re}}(T))$ are chosen to be pairwise disjoint sets with $J_1 \cup \cdots \cup J_4 = \Lambda$. Corollary 2.3 and the preceding lemma guarantee the existence of sequences $(x_k^i)_{k\geqslant 0}$, $(y_k^i)_{k\geqslant 0}$, $i = 1, \ldots, 4$, in the closed unit ball of H such that $x_k^i \otimes y_k^i \xrightarrow{k\to\infty} L_i$, $i = 1, \ldots, 4$. Setting $x_k = (x_k^1, x_k^2, x_k^3, x_k^4) \in H^4$ and $y_k = (y_k^1, y_k^2, y_k^3, y_k^4) \in H^4$, $k \geqslant 1$, we obtain sequences in the closed ball of radius 2 centered at the origin in H^4 satisfying $x_k \otimes_4 y_k \to L$.

Finally, if L is actually an absolute convex combination, then a decomposition into real and imaginary parts and the corresponding positive and negative parts yields the desired sequences in H^{16}. ∎

3. INTERIOR POINTS OF THE FREDHOLM SPECTRUM

Let $K \subset \mathbb{C}^n$ be a compact set and let $\lambda \in K$ be an arbitrary point in K. In what follows, a subset $\omega \subset K$ will be said to be an analytically deformed disc inside K with center λ, if there exist a Stein open zero neighborhood $V \subset \mathbb{C}^n$ and a biholomorphic mapping $h : V \to h(V) \subset \mathbb{C}^n$ such that

$$h(V) \supset K, \quad h(0) = \lambda, \quad \text{and} \quad \omega = h(\{0\} \times D_\varepsilon(0)) \subset K \quad \text{for some } \varepsilon > 0.$$

Here $\{0\}$ stands for $\{(0, \ldots, 0)\} \subset \mathbb{C}^{n-1}$ and $D_\varepsilon(0) = \{z \in \mathbb{C} : |z| < \varepsilon\}$ denotes a planar open disc with radius ε centered at the origin.

If $T \in L(H)^n$ is a commuting n-tuple of operators, then we define

$$\sigma_0(T) = \left\{ \lambda \in \sigma(T) : \begin{array}{c} \text{There exists an analytically deformed disc } \omega \text{ inside} \\ \sigma(T) \text{ with center } \lambda \text{ such that } \omega \subset \sigma(T) \setminus \sigma_{\mathrm{e}}(T) \end{array} \right\}.$$

Note that, for trivial reasons, the inclusion

$$\mathrm{Int}(\sigma(T) \setminus \sigma_{\mathrm{e}}(T)) \subset \sigma_0(T)$$

holds. Moreover, if $\omega = h(\{0\} \times D_\varepsilon(0))$ is an analytically deformed disc inside $\sigma(T)$ with center $\lambda \in \sigma(T)$ and $\omega \subset \sigma(T) \setminus \sigma_{\mathrm{e}}(T)$, then obviously the set $\omega^* =$

$h_*(\{0\} \times D_\varepsilon(0)) \subset \sigma(T^*) \setminus \sigma_e(T^*)$ is an analytically deformed disc inside $\sigma(T^*)$ with center λ^*. Therefore we have

$$\sigma_0(T^*) = \sigma_0(T)^*.$$

The definition of $\sigma_0(T)$ is motivated by the following observation.

LEMMA 3.1. *Let $T \in L(H)^n$ be a commuting n-tuple and suppose that, for some $\varepsilon > 0$, the inclusion $\{0\} \times D_\varepsilon(0) \subset \sigma(T) \setminus \sigma_e(T)$ holds. Then there exists an integer $p \in \{0, \dots, n\}$ such that* $\dim H^p(f_k(T), H) < \infty, k \in \mathbb{N}$, *and*

$$\overline{\lim_{k\to\infty}} \dim H^p(f_k(T), H) = \infty,$$

where $f_k : \mathbb{C}^n \to \mathbb{C}^n$ is defined by $f_k(z) = (z_1, \dots, z_{n-1}, z_n^k)$ for each $k \in \mathbb{N}$.

Proof. Choose a decreasing sequence $(r_j)_{j\geqslant 1}$ in $(0, \varepsilon)$ such that $r_j \downarrow 0$. For abbreviation, we set $w_{k,j} = (0, \dots, 0, r_j^k)$ with $k, j \in \mathbb{N}$. For every choice of $k, j \in \mathbb{N}$, the equation

$$f_k(z) = w_{k,j}$$

has exactly k distinct solutions in $\mathbb{C}^n$, namely

$$z_{k,j}^{(m)} = \Big(0, \dots, 0, r_j \exp\Big(2\pi \mathrm{i}\frac{m-1}{k}\Big)\Big) \in \{0\} \times D_\varepsilon(0) \quad m = 1, \dots, k.$$

Therefore we have $f_k^{-1}(\{w_{k,j}\}) \subset \sigma(T) \setminus \sigma_e(T)$, implying

$$w_{k,j} \in \sigma(f_k(T)) \setminus \sigma_e(f_k(T)) \quad \text{for all } k, j \in \mathbb{N}$$

by the spectral mapping theorems for the Taylor and the essential Taylor spectrum. Similarly, $0 \in \sigma(f_k(T)) \setminus \sigma_e(f_k(T))$, for $k \in \mathbb{N}$, and hence the dimension $\dim H^p(f_k(T), H)$ is finite, $0 \leqslant p \leqslant n$.

Since, by definition, we have $w_{k,j} \stackrel{j\to\infty}{\longrightarrow} 0$ for any fixed $k \in \mathbb{N}$, and since the dimension of the cohomology groups of the Koszul complex is upper-semicontinuous (Proposition 9.4.5 in [9]), we can choose a strictly increasing sequence $(j(k))_{k\geqslant 1}$ of integers such that the corresponding sequence $\omega_k = w_{k,j(k)}$, $k \geqslant 1$, fulfills the estimate

$$\dim H^p(f_k(T), H) = \dim H^p(-f_k(T), H) \geqslant \dim H^p(\omega_k - f_k(T), H)$$

whenever $0 \leqslant p \leqslant n$. Using the pre-images $\zeta_k^{(m)} = z_{k,j(k)}^{(m)}$, $m = 1, \dots, k$, of ω_k under f_k, one has the identification

$$H^p(\omega_k - f_k(T), H) = \bigoplus_{m=1}^{k} H^p(\zeta_k^{(m)} - T, H) \quad \text{where } 0 \leqslant p \leqslant n$$

(see the proof of Theorem 10.3.13 in [9]). For $k \in \mathbb{N}$ and $0 \leqslant p \leqslant n$, consider the sets

$$N_k = \{\zeta_k^{(m)} : m = 1, \dots, k\} \quad \text{and} \quad N_k^p = \{z \in N_k : \dim H^p(z - T, H) \geqslant 1\},$$

and observe that $N_k \subset \sigma(T)$ is the union $N_k = \bigcup_{p=0}^{n} N_k^p$. Since, for each $k \in \mathbb{N}$, the set $N_{(n+1)k}$ has exactly $(n+1)k$ elements, there exists a sequence $(p_k)_{k\geqslant 1}$ in $\{0,\dots,n\}$ satisfying $\sharp N_{(n+1)k}^{p_k} \geqslant k$. Since $(p_k)_{k\geqslant 1}$ hits at least one $p \in \{0,\dots,n\}$ infinitely many times, we can find a strictly increasing sequence (k_j) in $\mathbb{N}$ satisfying $p_{k_j} = p, j \in \mathbb{N}$. Writing $\kappa(j) = (n+1)k_j$ we deduce the estimate

$$\begin{aligned}\infty > \dim H^p(f_{\kappa(j)}(T), H) &\geqslant \dim H^p(\omega_{\kappa(j)} - f_{\kappa(j)}(T), H)\\ &= \dim \bigoplus_{m=1}^{\kappa(j)} H^p(\zeta_{\kappa(j)}^{(m)} - T, H) \geqslant \sharp N_{\kappa(j)}^p.\end{aligned}$$

Because of $\sharp N_{\kappa(j)}^p \geqslant k_j \geqslant j$, $j \in \mathbb{N}$, the proof is complete. ∎

Replacing T by the n-tuple induced by T on the cohomology groups of $f_k(T)$, one is lead exactly to the situation described in the next lemma. Given an arbitrary subset $K \in \mathbb{C}^n$, we write $\mathcal{O}(K)$ to denote the set of all holomorphic functions $f : U \to \mathbb{C}$ defined on an open neighborhood U of K in $\mathbb{C}^n$.

LEMMA 3.2. *Let Z be a Hilbert space of dimension $d = \dim(Z) < \infty$ and let $S \in L(Z)^n$ be an n-tuple of the form $S = (0,\dots,0,N)$, where $N \in L(Z)$ is a nilpotent operator. Then we have $\sigma(S) = \{0\}$ and there exists an orthonormal basis $(x_1,\dots,x_d)$ of Z such that*

$$\langle f(S)x_i, x_i\rangle = f(0) \quad \text{for each } f \in \mathcal{O}(\{0\}), 1 \leqslant i \leqslant d.$$

Proof. By the projection property of the Taylor spectrum, we have $\sigma(S) = \{0\}$. For later use we remark that the continuous algebra homomorphism

$$\mathcal{O}(\{0\}) \to L(Z), \quad f \mapsto \tau_N(f(0,\dots,0,\cdot)),$$

where τ_N denotes the holomorphic functional calculus of N, coincides with the holomorphic functional calculus of S (use Lemma 5.1.1 (b) of [9]).

If $l \in \mathbb{N}$ denotes the smallest natural number with $N^l = 0$, then we have the orthogonal direct sum decomposition

$$Z = \ker N^l = \bigoplus_{j=1}^{l} (\ker N^j \ominus \ker N^{j-1})$$

containing no zero summands. We intend to show that, whenever $1 \leqslant j \leqslant l$ and $x \in \ker N^j \ominus \ker N^{j-1}$ satisfies $\|x\| = 1$, then x solves the equation

$$\langle f(S)x, x\rangle = f(0) \quad \text{for every } f \in \mathcal{O}(\{0\}).$$

This clearly suffices to finish the proof. To realize this claim, suppose that we are given an $\varepsilon > 0$ and a function $f \in \mathcal{O}(D_\varepsilon(0)^n)$. Then define $g = f(0,\dots,0,\bullet) \in$

$\mathcal{O}(D_\varepsilon(0))$ and let

$$q(z) = f(0) + \sum_{m=1}^{j-1} a_m z^m \quad \text{for } z \in D_\varepsilon(0)$$

be the Taylor polynomial of order $(j-1)$ of g at the origin. Since $g - q$ has a zero of order $\geqslant j$ at the origin, there exists a function $h \in \mathcal{O}(D_\varepsilon(0))$ satisfying $g(z) = q(z) + z^j h(z), z \in D_\varepsilon(0)$. Applying the holomorphic functional calculus leads to the formula

$$f(S) = g(N) = f(0) + \sum_{m=1}^{j-1} a_m N^m + h(N) N^j$$

which implies that $f(S)x - f(0)x \in \ker(N^{j-1})$ and hence completes the proof. ∎

LEMMA 3.3. *Let H be a separable Hilbert space and let $A \subset H$ be a subset which, for each natural number $k \geqslant 1$, contains an orthonormal system $S \subset A$ of length $\sharp S = k$. Then A contains a weak zero sequence of unit vectors.*

Proof. Choose an arbitrary dense sequence $(z_n)_{n \geqslant 1}$ in the closed unit sphere of H. Our aim is to construct a sequence $(e^{(j)})_j$ of unit vectors in A satisfying

$$|\langle e^{(j)}, z_i \rangle| \leqslant \frac{1}{\sqrt{j}} \quad \text{for all } i, j \in \mathbb{N} \text{ with } i \leqslant j.$$

The latter relation clearly implies that $(e^{(j)})_j$ tends to zero weakly.

Fix $j \geqslant 1$. To obtain $e^{(j)}$, we first choose an orthonormal system $(e_1, \ldots, e_{j^2})$ in H. Since we have $\sum_{k=1}^{j^2} |\langle e_k, z_i \rangle|^2 \leqslant 1$ for every $i \in \mathbb{N}$, each of the sets $N_i = \{k : |\langle e_k, z_i \rangle| > 1/\sqrt{j}\}$ satisfies $\sharp N_i < j$. Hence there exists an element $k \in \{1, \ldots, j^2\} \setminus \bigcup_{i=1}^{j} N_i$. The vector $e^{(j)} = e_k$ then clearly has the desired properties. ∎

Let $\Psi : \mathcal{O}(\sigma(T)) \to L(H)$ denote the holomorphic functional calculus of T, and let $Z = M \ominus N$ be a Ψ-semi-invariant subspace. Writing $P \in L(M)$ for the orthogonal projection of M onto Z, it is not hard to show that the induced map $\widehat{P} : M/N \to Z$ is a topological isomorphism making the diagram

$$\begin{array}{ccc} Z & \xrightarrow{T_Z} & Z \\ {\scriptstyle \widehat{P}}\uparrow & & \uparrow{\scriptstyle \widehat{P}} \\ M/N & \xrightarrow{(T|M)/N} & M/N \end{array}$$

commutative. Here, as usual, $T_Z = (PT_1|Z, \ldots, PT_n|Z) \in L(Z)^n$, denotes the compression of T onto Z.

Now suppose that the spectral inclusion $\sigma(T_Z) \subset \sigma(T)$ holds and that $U \subset \mathbb{C}^n$ is a Stein open neighborhood of $\sigma(T)$. Then the uniqueness property of Taylor's holomorphic functional calculus (see Lemma 5.1.1 (b) in [9]) guarantees that $\Psi_Z|\mathcal{O}(U)$ coincides with the holomorphic functional calculus of T_Z restricted to $\mathcal{O}(U)$; here $\Psi_Z(f) = Pf(T)|Z, f \in \mathcal{O}(\sigma(T))$ denotes the compression of Ψ onto Z.

After these preliminaries we are able to prove the following factorization result.

THEOREM 3.4. *Let $T \in L(H)^n$ be a commuting n-tuple of continuous linear operators on a separable complex Hilbert space. Given $\lambda \in \sigma_0(T)$, there exists a weak zero sequence (x_k) of unit vectors in H^N, $N = 2^n$ such that, for every Stein open neighborhood $U \subset \mathbb{C}^n$ of $\sigma(T)$, we have*

$$\langle f(T^{(N)})x_k, x_k\rangle = f(\lambda) \quad \textit{for all } f \in \mathcal{O}(U), k \in \mathbb{N}.$$

Proof. The proof is divided into two steps.

(1) Instead of $\lambda \in \sigma_0(T)$ we first make the stronger requirement that $\lambda = 0$ and $\{0\} \times D_\varepsilon(0) \subset \sigma(T) \setminus \sigma_e(T)$. According to Lemma 3.1 and Lemma 3.3 it suffices to check that, for each $k \in \mathbb{N}$, the set

$$A = \left\{ x \in H^N : \begin{array}{c} \langle f(T^{(N)})x, x\rangle = f(0) \text{ whenever } f \in \mathcal{O}(U) \text{ and} \\ U \subset \mathbb{C}^n \text{ is a Stein open neighborhood of } \sigma(T) \end{array} \right\}$$

contains an orthonormal system of length $d_k = \dim H^p(f_k(T), H)$, where $p \in \{0, \ldots, n\}$ and $f_k : \mathbb{C}^n \to \mathbb{C}^n$ are defined as in the statement of Lemma 3.1.

To check this, fix $k \in \mathbb{N}$ and set $F = f_k(T) = (T_1, \ldots, T_{n-1}, T_n^k) \in L(H)^n$. Since the components of $T^{(N)}$ commute with $\delta^j(F), j = 0, \ldots, n$, and since the holomorphic functional calculus preserves intertwining relations, the finite dimensional space

$$Z = \ker \delta^p(F) \ominus \operatorname{ran} \delta^{p-1}(F) \subset \Lambda(n, H) = H^N$$

is semi-invariant for the holomorphic functional calculus of T. (Note that the mapping $\delta^{p-1}(F) : \Lambda^{p-1}(n, H) \to \ker \delta^p(F)$ has finite codimensional and hence closed range, since by hypothesis $0 \in \sigma(T) \setminus \sigma_e(T)$.) If $P_Z \in L(\ker \delta^p(F))$ denotes the orthogonal projection onto Z, then, modulo the topological identification

$$H^p(F, H) = \ker \delta^p(F)/\operatorname{ran} \delta^{p-1}(F) \xrightarrow{\widehat{P}_Z} Z,$$

the n-tuple induced by T on the cohomology group $H^p(F, H)$ coincides with the compression $S \in L(Z)^n$ of $T^{(N)}$ onto Z. Since $F = (T_1, \ldots, T_{n-1}, T_n^k)$ induces the zero tuple on $H^p(F, H)$, the components of S satisfy $S_1 = \cdots = S_{n-1} = 0 = S_n^k$. Fix an orthonormal basis B of Z as in the statement of Lemma 3.2. Then, for any Stein open neighborhood $U \supset \sigma(T) \supset \{0\} = \sigma(S)$ and any $f \in \mathcal{O}(U)$, the remarks preceding the theorem imply

$$\langle f(T^{(N)})x, x\rangle = \langle f(S)x, x\rangle = f(0) \quad \text{for every } x \in B.$$

Hence A contains an orthonormal system of length d_k, as was to be shown.

(2) Given $\lambda \in \sigma_0(T)$, we fix a corresponding biholomorphic map $h : V \to h(V) \subset \mathbb{C}^n$ ($V \subset \mathbb{C}^n$ Stein open neighborhood of the origin) such that $h(V) \supset \sigma(T)$, $h(0) = \lambda$ and $h(\{0\} \times D_\varepsilon(0)) \subset \sigma(T) \setminus \sigma_e(T)$ for some $\varepsilon > 0$.

The n-tuple $W = h^{-1}(T)$ then satisfies

$$\sigma(W) \setminus \sigma_e(W) = h^{-1}(\sigma(T) \setminus \sigma_e(T)) \supset \{0\} \times D_\varepsilon(0).$$

By part **(1)** of the proof, there exists a weak zero sequence (x_k) of unit vectors in H^N such that

$$\langle g(W^{(N)})x_k, x_k\rangle = g(0)$$

holds whenever $\widetilde{U}$ is a Stein open neighborhood of $\sigma(W)$ and $g \in \mathcal{O}(\widetilde{U})$. Now suppose that $U \supset \sigma(T)$ is a Stein open set in $\mathbb{C}^n$. Then we may take $\widetilde{U} = h^{-1}(h(V) \cap U) \supset \sigma(W)$ to deduce that, for any $f \in \mathcal{O}(U)$, the desired identity

$$\langle f(T^{(N)})x_k, x_k\rangle = \langle f \circ h(W^{(N)})x_k, x_k\rangle = f \circ h(0) = f(\lambda)$$

holds. ∎

Note that up to now we did not make use of the weak* continuous $H^\infty(D)$-functional calculus $\Phi : H^\infty(D) \to L(H)$ for T. But since Φ necessarily extends the holomorphic functional calculus of T (see Lemma 2.3.4 in [4]), we finally obtain:

COROLLARY 3.5. *Let $T \in L(H)^n$ be a commuting n-tuple possessing a weak* continuous functional calculus $\Phi : H^\infty(D) \to L(H)$ over a relatively compact strictly pseudoconvex open subset $D \Subset X$ of a Stein submanifold $X \subset \mathbb{C}^n$. Then, for each $\lambda \in \sigma_0(T) \cap D$, there exists a weak zero sequence (x_k) of unit vectors in H^N, $N = 2^n$, satisfying $x_k \otimes_N x_k = \mathcal{E}_\lambda$, $k \in \mathbb{N}$.*

4. REFLEXIVITY AND INVARIANT SUBSPACES

In the present section we demonstrate how the factorization technique established above can be applied to attack the invariant-subspace and the reflexivity problem for special classes of von Neumann n-tuples over D.

Recall that a representation $\Phi : H^\infty(D) \to L(H)$ is of type $C_{0\cdot}$ if it is sequentially weak*-SOT continuous, whereas we call Φ of type $C_{\cdot 0}$ if the dual representation Φ^* satisfies the $C_{0\cdot}$-condition. If Φ belongs both to the class $C_{0\cdot}$ and the class $C_{\cdot 0}$, then Φ is said to be a C_{00}-representation.

Let $\Phi : H^\infty(D) \to L(H)$ be a weak* continuous representation. If, for each sequence $(L_k)_{k \geqslant 1}$ in $Q(D)$, there exist vectors $x \in H$ and $y_k \in H$, $k \geqslant 1$, such that

$$L_k = x \otimes y_k \quad k \geqslant 1,$$

then Φ is said to possess the factorization property $(\mathbb{A}_{1,\aleph_0})$. If, in this case, $\Phi = \Phi_T$ for a von Neumann n-tuple T of class $\mathbb{A}$ over D, then T is called to be of class $\mathbb{A}_{1,\aleph_0}$.

Given a real number $\theta \geqslant 0$ we write $\mathcal{L}(\theta) = \mathcal{L}^\Phi(\theta)$ to denote the set of all elements $L \in Q(D)$ with the following property: For each $\varepsilon > 0$ and any choice of vectors $a_1, \ldots, a_s, b_1, \ldots, b_s \in H$ ($s \in \mathbb{N}$ arbitrary), there exist vectors $x, y \in H$ satisfying:

(i) $\|x\| \leqslant 1, \|y\| \leqslant 1$;
(ii) $\|L - x \otimes y\| < \theta + \varepsilon$;
(iii) $\|x \otimes b_i\| < \varepsilon, \|a_i \otimes y\| < \varepsilon, i = 1, \ldots, s$.

It is well known (see Lemma 1.1 in [7]) that the sets $\mathcal{L}(\theta) \subset Q(D), \theta \geqslant 0$, are norm-closed and absolutely convex. If there are real numbers $\gamma > \theta \geqslant 0$ such that the inclusion

$$\{L \in Q(D) : \|L\| \leqslant \gamma\} \subset \mathcal{L}(\theta)$$

holds, then we say that Φ has property $(\Delta_{\theta,\gamma})$.

Using results of Bercovici, Foiaş and Pearcy one can show that if Φ has property $(\Delta_{0,1})$, then the operator algebra $\Phi(H^\infty(D))$ is super-reflexive (cf. Remark 2.5 of [7], and Proposition 0.1 of [1]). Moreover, in the case of a C_{00}-representation Φ, the validity of property $(\Delta_{\theta,\gamma})$, where $\gamma > \theta \geqslant 0$ are fixed numbers, implies that Φ has property $(\Delta_{0,1})$ (see Lemma 2.2 in [7]). If Φ is even a contractive C_{00}-representation and $\Phi^{(N)}$ satisfies property $(\Delta_{\theta,\gamma})$ for some numbers $\gamma > \theta \geqslant 0$, then Φ has property $(\Delta_{0,1})$ (see Proposition 2.3 of [7]).

The reader should recall the following fact which is quite useful in the above context (see Lemma 1.1 in [5]).

LEMMA 4.1. *Let $\Phi : H^\infty(D) \to L(H)$ be a weak* continuous representation and let (x_k) be a weak zero sequence in H. If Φ is of class $C_{0\cdot}$, then, for any vector $z \in H$, we have $z \otimes x_k \xrightarrow{k} 0$. If Φ is of class $C_{\cdot 0}$, then, for any vector $z \in H$, the analogous relation $x_k \otimes z \xrightarrow{k} 0$ holds.*

The factorization results obtained in the preceding section allow us to formulate a concrete richness condition on the spectrum of C_{00}-representations implying reflexivity. At least in the strictly pseudoconvex case, the following theorem is an improvement of Albrecht and Chevreau ([1], Theorem 2.1).

THEOREM 4.2. *Let $T \in L(H)^n$ be a von Neumann n-tuple of class C_{00} over D. If $\sigma_{\mathrm{e}}(T) \cup \sigma_0(T)$ is dominating in D, then Φ_T has property $(\Delta_{0,1})$.*

Proof. Set $N = 2^n$ and observe that, by Lemma 4.1 and the factorization results stated as Corollary 2.3 (i) and Corollary 3.5, the inclusion

$$\{\mathcal{E}_\lambda : \lambda \in (\sigma_{\mathrm{e}}(T) \cup \sigma_0(T)) \cap D\} \subset \mathcal{L}^{\Phi_T^{(N)}}(0)$$

holds. Since the right-hand side is norm-closed and absolutely convex, we may use the richness condition on the spectrum to deduce that $\Phi_T^{(N)}$ has property $(\Delta_{0,1})$. By the remarks preceding Lemma 4.1, this suffices to prove the theorem. ∎

Our next aim is to characterize the class $\mathbb{A}_{1,\aleph_0}$ by means of a spectral condition in the spirit of Eschmeier ([8], Corollary 3.7 (viii)).

PROPOSITION 4.3. *Given a von Neumann n-tuple $T \in L(H)^n$ of class $\mathbb{A}$ over D, the following assertions are equivalent:*

(i) *Φ_T has property $(\mathbb{A}_{1,\aleph_0})$;*

(ii) *there is a Φ_T-semi-invariant subspace $Z \subset H$ such that the compression T_Z of T onto Z is a von Neumann n-tuple of class $\mathbb{A}$ over D which is of type $C_{\cdot 0}$ and has dominating Harte spectrum $\sigma^{\mathrm{H}}(T_Z)$ in D.*

If one of these conditions is satisfied, then the dual operator algebra $\Phi_T(H^\infty(D))$ is super-reflexive.

Proof. That (i) implies (ii) is a consequence of Lemma 4.3.10 and Corollary 4.4.3 (g) in [4]. Now suppose that (ii) holds. According to Corollary 5.2.6 and Corollary 4.4.3 in [4], it suffices to check that, for some $N \geqslant 1$, the n-tuple $T_Z^{(N)}$ has the ρ-almost factorization property for some $\rho > 0$. We thus have to show that, for a suitable choice of $N \in \mathbb{N}$ and $\rho > 0$, the factorization problem

$$\|L - x \otimes_{T_Z^{(N)}} y\| < \varepsilon \quad \text{and} \quad \|x\|, \|y\| \leqslant \rho \|L\|^{1/2}$$

is solvable for every choice of $L \in Q(D)$ and $\varepsilon > 0$. Corollary 2.5 asserts that each functional belonging to the absolutely convex hull

$$\Gamma(\{\mathcal{E}_\mu : \mu \in \sigma^{\mathrm{H}}(T_Z) \cap D\})$$

can ρ-almost be factorized via $\Phi_{T_Z}^{(16)}$ with $\rho = 8$. The richness condition imposed on the Harte spectrum $\sigma^{\mathrm{H}}(T_Z)$ allows us to finish the proof of the equivalence assertion. The super-reflexivity statement follows from Corollary 4.4.3 in [4]. ∎

Note that for the above proof to work it is essential that we are able to factorize *absolute convex combinations* of point evaluations over the Harte spectrum. To make the spectra $\sigma_{\mathrm{e}}(T)$ and $\sigma_0(T)$ accessible to the Scott Brown technique, we use another approximation device.

Let $\theta \geqslant 0$ be a real number. We write $\mathcal{E}_\theta^r(T)$ for the set of all elements $L \in Q(D)$ for which there exist sequences (x_k), (y_k) in H satisfying the following requirements:

(i) $\|x_k\| \leqslant 1, \|y_k\| \leqslant 1, k \geqslant 1$;

(ii) $\overline{\lim\limits_{k\to\infty}} \|L - x_k \otimes y_k\| \leqslant \theta$;

(iii) $x_k \otimes z \xrightarrow{k} 0$ for each $z \in H$.

As pointed out in Remark 3.1 of [3], one can achieve that in addition the sequence (y_k) converges to zero weakly in H.

Given real numbers $0 \leqslant \theta < \gamma \leqslant 1$, we say that T has property $E_{\theta,\gamma}^r$ if the inclusion

$$\{L \in Q(D) : \|L\| \leqslant \gamma\} \subset \overline{\Gamma}(\mathcal{E}_\theta^r(T))$$

holds, where $\overline{\Gamma}(\cdots)$ denotes the closed absolutely convex hull. Corollary 4.4.3 in [4] asserts that if, for some von Neumann n-tuple T of class $\mathbb{A}$ over D and $N \in \mathbb{N}$, the n-tuple $T^{(N)}$ satisfies the condition $E^r_{\theta,\gamma}$ for some $0 \leqslant \theta < \gamma \leqslant 1$, then T is of class $\mathbb{A}_{1,\aleph_0}$ and $\Phi_T(H^\infty(D))$ is super-reflexive.

PROPOSITION 4.4. *Let $T \in L(H)^n$ be a von Neumann n-tuple of class $\mathbb{A}$ over D. Suppose that T is of type $C_{\cdot 0}$ and that $\sigma_{\mathrm{e}}(T) \cup \sigma_0(T)$ is dominating in D. Then T satisfies property $\mathbb{A}_{1,\aleph_0}$ and the dual operator algebra $\Phi_T(H^\infty(D))$ is super-reflexive.*

Proof. Set $N = 2^n$. Applying Lemma 4.1 and the factorization results obtained in the previous section (Corollary 2.3 (i) and Corollary 3.5) we deduce that the inclusion

$$\{\mathcal{E}_\lambda : \lambda \in (\sigma_{\mathrm{e}}(T) \cup \sigma_0(T)) \cap D\} \subset \mathcal{E}^r_0(T^{(N)})$$

holds. Forming the absolutely convex hull on both sides we infer that $T^{(N)}$ has property $E^r_{0,1}$. Now, a look at the above-mentioned Corollary 4.4.3 in [4] completes the proof. ∎

As a consequence of the above reflexivity statement, we are able to extend an invariant-subspace result for commuting n-tuples with normal boundary dilations due to Eschmeier (cf. Theorem 3.3 in [6]).

THEOREM 4.5. *Let $T \in L(H)^n$ be a von Neumann n-tuple over D possessing a ∂D-unitary dilation. If $\sigma^{\mathrm{H}}(T) \cup \sigma_{\mathrm{e}}(T) \cup \sigma_0(T)$ is dominating in D, then T has a non-trivial invariant subspace.*

Proof. If there exists an element $\lambda \in \sigma^{\mathrm{H}}(T) \setminus \sigma_{\mathrm{e}}(T)$, then one of the spaces $\bigcap\limits_{i=1}^{n} \ker(\lambda_i - T_i)$ or $\sum\limits_{i=1}^{n} (\lambda_i - T_i)H$ is a non-trivial hyperinvariant subspace for T. Without loss of generality, we therefore may assume that $\sigma_{\mathrm{e}}(T) \cup \sigma_0(T)$ is dominating in D. If T is neither of type $C_{0\cdot}$ nor of type $C_{\cdot 0}$, then T has a non-trivial invariant subspace by Theorem 3.1.1 in [4]. By a simple duality argument, we may restrict ourselves to the $C_{\cdot 0}$ case. But then an application of the preceding proposition suffices to finish the proof. ∎

The formulation of the above theorem becomes particularly simple in the case of commuting pairs of operators.

LEMMA 4.6. *For each commuting pair $T = (T_1, T_2) \in L(H)^2$, the identity*

$$\sigma(T) = \sigma^{\mathrm{H}}(T) \cup \sigma_{\mathrm{e}}(T) \cup \mathrm{Int}(\sigma(T) \setminus \sigma_{\mathrm{e}}(T)) = \sigma^{\mathrm{H}}(T) \cup \sigma_{\mathrm{e}}(T) \cup \sigma_0(T)$$

holds.

Proof. Suppose that $\lambda \in \sigma(T) \setminus \sigma^{\mathrm{H}}(T)$. Since in this case, the exterior cohomology groups of $\lambda - T$ are trivial in this case, this implies $H^1(\lambda - T, H) \neq 0$. But then either $\lambda \in \sigma_{\mathrm{e}}(T)$ or $\dim H^1(\lambda - T, H)$ is finite. In the latter case, $\lambda - T$ is a Fredholm pair and $\mathrm{ind}(\lambda - T) = \sum\limits_{i=0}^{2} (-1)^i \dim H^i(\lambda - T, H) = -\dim H^1(\lambda - T, H) < 0$, implying that the whole component of $\mathbb{C}^2 \setminus \sigma_{\mathrm{e}}(T)$ containing λ belongs to $\sigma(T) \setminus \sigma_{\mathrm{e}}(T)$, and hence $\lambda \in \mathrm{Int}(\sigma(T) \setminus \sigma_{\mathrm{e}}(T)) \subset \sigma_0(T)$. ∎

As an immediate consequence we obtain an invariant-subspace result based on a richness condition on the whole Taylor spectrum.

COROLLARY 4.7. *Let $D \Subset X$ be a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^2$ and let $T \in L(H)^2$ be a von Neumann pair over D possessing a ∂D-unitary dilation. If the Taylor spectrum $\sigma(T)$ is dominating in D, then T possesses a non-trivial invariant subspace.*

In particular, each spherical contraction of length 2 with spherical dilation and dominating Taylor spectrum in the open Euclidean unit ball $\mathbb{B}_2$ has a non-trivial invariant subspace. However, the question whether an analogous result holds for spherical contractions of length $n \geqslant 3$ still remains open.

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