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SOME CHARACTERIZATIONS OF WEAKLY COMPACT OPERATORS ON H^∞ AND ON THE DISK ALGEBRA. APPLICATION TO COMPOSITION OPERATORS

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ABSTRACT. We characterize weakly compact operators from H^∞ or from $A(\mathbb{D})$ in terms of absolutely continuous operators. From this, we easily obtain that weakly compact composition operators on H^∞ are compact. We prove the same result for the disk algebra.

KEYWORDS: *Composition operator, disk algebra, bounded analytic functions, weakly compact operator.*

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INTRODUCTION

The starting point of this paper was to investigate the weak compactness of composition operators on the space H^∞ of bounded analytic functions on the open unit disk $\mathbb{D}$ of the complex plane, and on the disk algebra $A(\mathbb{D})$. The composition operators were investigated so far in many ways. The composition operators are very often investigated on H^p spaces $(1 < p < \infty)$, where weak compactness is a trivial problem (because of reflexivity). A very good survey of the properties of such operators (mainly, on the Hilbert space H^2) is contained in the monograph [8].

In the following, the spaces $L^1/\overline{H}^1_0$ and H^∞ are in duality via the bracket

$$\langle h, f\rangle = h * f(0), \quad \text{where } h \in L^1/\overline{H}^1_0 \text{ and } f \in H^\infty.$$

The technique that we use to solve the problem for H^∞ is based on a characterization of relatively weakly compact subsets of L^1/H^1 due to J. Chaumat [2] (see also [3]). Actually the situation is rather general and motivated the first section, where we characterize dual weakly compact operators in terms of absolutely continuous operator. We easily obtain a similar characterization for weakly compact operators on the disk algebra (even if we shall not use this in the sequel).

As a corollary, we obtain both for H^∞ and the disk algebra that the membership of the classical operator ideals is equivalent for composition operators.

Actually, after this work was completed, F. Bayart and D. Li told us that the results concerning the composition operators were not new: this is due to R. Aron, P. Galindo and M. Lindström [1] (see also Ülger [9]) for H^∞, and to P. Galindo and M. Lindström [5] for the disk algebra. Nevertheless, our approach is different and seems to be far more elementary: for instance, we do not use Carleson's corona theorem. Moreover, we obtain these results as consequences of some new (as far as we know) general results on weakly compact operators on H^∞ and on the disk algebra. Note that the works of Aron, Galindo, Lindström and Ülger concern homomorphisms between algebras of analytic functions.

Now, we precise some definitions of classical classes of operators ideals. We refer to [4] for general studies of these classes.

DEFINITION 0.1. A bounded operator T from a Banach space X to a Banach space Y is *p-summing* if there is a constant $C > 0$ such that, for any finite family of vectors (x_n) in X,

$$\Big(\sum_n \|T(x_n)\|^p\Big)^{1/p} \leqslant C \sup_{\substack{\chi\in X^* \\ \|\chi\|=1}} \Big(\sum_n |\chi(x_n)|^p\Big)^{1/p}.$$

We denote $\pi_p(T)$ the smallest constant C satisfying the previous property.

DEFINITION 0.2. A bounded operator T from a Banach space X to a Banach space Y is *γ-summing* if there is a constant $C > 0$ such that, for any finite family of vectors (x_n) in X,

$$\mathbb{E}\Big\| \sum_n g_n T(x_n)\Big\| \leqslant C \sup_{\substack{\chi\in X^* \\ \|\chi\|=1}} \Big(\sum_n |\chi(x_n)|^2\Big)^{1/2}.$$

We denote $\pi_\gamma(T)$ the smallest constant C satisfying the previous property.

We obtain an equivalent notion (*as-summing operator*) if we replace the family of (normalized) Gaussian independent variables $(g_n)_n$ by a family of Bernoulli independent variables.

DEFINITION 0.3. A bounded operator T from a Banach space X to a Banach space Y is *nuclear* if there exist some vectors $y_j \in Y$ and some functionals $\chi_j \in X^*$ such that $\sum\limits_j \|\chi_j\| \cdot \|y_j\| < \infty$ and for every $x \in X$,

$$T(x) = \sum_j \chi_j(x) y_j.$$

DEFINITION 0.4. A bounded operator T from a Banach space X to a Banach space Y is *absolutely continuous* if one can find $p \in [1, \infty[$, a Banach space Z and a p-summing operator $\widetilde{T}$ from X to Z such that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$

verifying for every $x \in X$

$$\|T(x)\| \leqslant C_\varepsilon \|\widetilde{T}(x)\| + \varepsilon \|x\|.$$

On p. 311 of [4] this class of operators is introduced as class $\mathcal{H}(X,Y)$.

1. ABSOLUTELY CONTINUOUS OPERATORS

The purpose of this section is the study of general weakly compact operators both on H^∞ and the disk algebra. Actually, it turns out that a part of the characterization is not proper to the Hardy spaces. We recall that weak-star–weak continuity means continuity from a dual space equipped with the weak-star topology, to a space equipped with the weak topology.

THEOREM 1.1. *Let X be a separable Banach space, Y be a Banach space and $T : X^* \to Y$ be a bounded operator. We suppose that there exists an operator j from X^* to some space Z, which is weak-star–weak continuous with the following property: for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for every $f \in X^*$,*

$$\|T(f)\|_Y \leqslant C_\varepsilon \|j(f)\|_Z + \varepsilon \|f\|_{X^*}.$$

Then T is a weak-star–weak continuous (in particular T is a weakly compact operator).

Proof. It is sufficient to prove that for every ξ in the unit ball of Y^*, the restriction to the unit ball of X of the functional $\xi \circ T$ is weak-star continuous at 0. The weak-star topology on this unit ball of X^* is metrizable (by separability of X). Now, let $(f_k)_{k\in\mathbb{N}}$ be a sequence in the unit ball of X^* weak-star convergent to 0.

Now, we suppose that for some ξ in the unit ball of Y^*, the sequence $\xi \circ T(f_k)$ does not tend to 0. Then, there is a $\delta_0 > 0$, a subsequence $(f_{j_k})_{k\in\mathbb{N}}$ and a sequence of modulus one complex scalars $(\alpha_k)_{k\in\mathbb{N}}$ such that for every k, $\xi \circ T(\alpha_k f_{j_k}) \geqslant \delta_0$. Of course, $\alpha_k f_{j_k}$ is still star-weakly null in X^*. Hence $j(\alpha_k f_{j_k})$ is weakly null in Z. By the classical Mazur theorem, this implies that some convex combination of the $\alpha_k j(f_{j_k})$ converges in norm to 0. By hypothesis, we deduce that the corresponding convex combination of the $T(\alpha_k f_{j_k})$ converges in norm to 0. More precisely, there exist sequences p_n and q_n of integers, with $p_n \leqslant q_n$, and positive coefficients $c_{n,k}$, with $\sum\limits_{p_n \leqslant k \leqslant q_n} c_{n,k} = 1$, such that $\lim\limits_{n\to+\infty} \Big\| \sum\limits_{p_n \leqslant k \leqslant q_n} c_{n,k}\alpha_k j(f_{j_k}) \Big\|_Z = 0$.

Now fix $\varepsilon = \delta_0/4$. There exists a corresponding C_ε by hypothesis. For N_0 large enough and every $n \geqslant N_0$: $\Big\| \sum\limits_{p_n \leqslant k \leqslant q_n} c_{n,k}\alpha_k j(f_{j_k}) \Big\|_Z \leqslant \varepsilon C_\varepsilon^{-1}$. We then have

$$\Big\| \sum_{k=p_n}^{q_n} c_{n,k}\alpha_k T(f_{j_k}) \Big\|_Y \leqslant C_\varepsilon \Big\| \sum_{k=p_n}^{q_n} c_{n,k}\alpha_k j(f_{j_k}) \Big\|_Z + \varepsilon \Big\| \sum_{k=p_n}^{q_n} c_{n,k}\alpha_k f_{j_k} \Big\|_X$$

$$\leqslant \varepsilon + \varepsilon \sum_{k=p_n}^{q_n} c_{n,k} \| f_{j_k} \|_X \leqslant 2\varepsilon.$$

So that,

$$\delta_0 \leqslant \sum_{k=p_n}^{q_n} c_{n,k}\alpha_k \xi \circ T(f_{j_k}) \leqslant \Big\| \sum_{k=p_n}^{q_n} c_{n,k}\alpha_k T(f_{j_k}) \Big\|_Y \leqslant \frac{\delta_0}{2}.$$

This gives a contradiction. ∎

THEOREM 1.2. *Let Y be a Banach space and $T : H^\infty \to Y^*$ be a bounded operator. The following assertions are equivalent:*

(i) *T is weakly compact and weak-star continuous.*

(ii) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $f \in H^\infty$ with $\|f\|_\infty \leqslant 1$ and $\|f\|_1 \leqslant \delta$, we have $\|T(f)\| \leqslant \varepsilon$.*

(iii) *For every $p \geqslant 1$, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for every $f \in H^\infty$, $\|T(f)\| \leqslant C_\varepsilon \|f\|_p + \varepsilon \|f\|_\infty$.*

(iv) *There exists some $p \geqslant 1$ such that: for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for every $f \in H^\infty$, $\|T(f)\| \leqslant C_\varepsilon \|f\|_p + \varepsilon \|f\|_\infty$.*

Proof. (ii)⇔(iii)⇔(iv). These implications are easy.

(iv)⇒(i). This is given by the previous theorem: let us precise that $L^1/\overline{H}^1_0$ is separable and here the operator j is nothing but the formal identity from H^∞ to H^p. It is weak-star–weak continuous because $H^{p'} \subset H^1$, where p' is the conjugate exponent of p.

(i)⇒(ii). Note that $T = S^*$, where $S : Y \to L^1/\overline{H}^1_0$. Moreover, by the Gantmacher theorem S is also weakly compact. Hence the range K of the unit ball of Y by S is a relatively weakly compact subset of $L^1/\overline{H}^1_0$. Now, we use the characterization of relatively weakly compact subsets of L^1/H^1, due to J. Chaumat [2] (the same obviously holds for $L^1/\overline{H}^1_0$ via the conjugate): for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every f in the unit ball of H^∞, with $\|f\|_1 \leqslant \delta$, we have $\sup_{h\in K} |\langle h, f\rangle| \leqslant \varepsilon$. Replacing K by its definition, we compute

$$\sup_{h\in K} |\langle h, f\rangle| = \sup_{\chi\in B_Y} |\langle S(\chi), f\rangle| = \sup_{\chi\in B_Y} |\langle \chi, T(f)\rangle| = \|T(f)\|_\infty.$$

The conclusion follows. ∎

Our proof relies on the theorem of Chaumat. Actually, this is equivalent: indeed, if we suppose that every (bounded) operator $T : H^\infty \to Y^*$, weakly compact and weak-star continuous, verifies the conclusion (ii) of the previous theorem, then we have the result of Chaumat. The key point is to consider the operator T from H^∞ to ℓ^∞ given by $T(h) = (h * f_j(0))_{j\in\mathbb{N}}$, where f_j is weakly convergent in $L^1/\overline{H}^1_0$.

Now, we give a similar characterization of weakly compact operators on the disk algebra. Actually, this is an easy consequence of the proof of a similar result for $C(K)$ spaces due to Niculescu (see the proof of Theorem 15.2, p. 309 in [4], which is due to H. Jarchow and A. Pełczyński; see also [6] for generalization to C^*-algebras) and on the result of J. Chaumat on lifting of weakly compact subsets of L^1/H^1_0. First, we give the following general result.

PROPOSITION 1.3. *Let X be subspace of $C(K)$ such that every weakly compact subset of $X^* = M(K)/X^\perp$ is the range by the canonical surjection of a weakly compact subset of $M(K)$. Then every weakly compact operator T from X to some Banach space Y has the following property: there exists some probability measure μ subject to the property that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for every $x \in X$,*

$$\|T(x)\| \leqslant C_\varepsilon \|x\|_{L^1(\mu)} + \varepsilon \|x\|_\infty.$$

Proof. We mimic the proof of Theorem 15.2, p. 309 in [4]. Nevertheless, for sake of completeness, we give the details: $W = \overline{T^*(B_{Y^*})}$ is a weakly compact subset of $M(K)/X^\perp$. Hence, by hypothesis, there exists a weakly compact subset $\widetilde{W}$ of $M(K)$ such that $\pi(\widetilde{W}) = W$, where π is the canonical surjection from $M(K)$ onto $M(K)/X^\perp$. By Theorem 15.1, p. 309 in [4], there exists a positive measure μ in $M(K)$ such that every $\nu \in \widetilde{W}$ is absolutely continuous with respect to μ and such that the set of all the corresponding densities is a relatively weakly compact subset of $L^1(K,\mu)$.

Now, let us suppose that T does not verify the conclusion of the proposition. We then have a sequence (f_n) of norm one functions of X and an $\varepsilon > 0$ such that, for every n,

$$\|T(f_n)\| > n\|f_n\|_{L^1(K,\mu)} + \varepsilon.$$

We obviously have that $\|f_n\|_{L^1(K,\mu)}$ tends to 0, since $\|T(f_n)\| \leqslant \|T\|$.

As $\|T(f_n)\| > \varepsilon$, there exists y^*_n in the unit ball of Y^* such that $y^*_n(T(f_n)) > \varepsilon$. Now, we consider $T^*(y^*_n) \in W = \pi(\widetilde{W})$. We may write $T^*(y^*_n) = \pi(g_n)$ for some $g_n \in L^1(K,\mu)$ (identifying an absolutely continuous measure and the density lying in $L^1(K,\mu)$). Moreover, the set of the g_n is a relatively weakly subset of $L^1(K,\mu)$ and $\|g_n\|_{L^1(K,\mu)} \leqslant M$ for some $M > 0$. We compute

$$\begin{aligned}\int_K f_n g_n \mathrm{d}\mu &= \langle f_n, g_n\rangle_{C(K),M(K)} = \langle f_n, \pi(g_n)\rangle_{X,X^*} \\ &= \langle f_n, T^*(y^*_n)\rangle_{X,X^*} = \langle T(f_n), y^*_n\rangle_{Y,Y^*} > \varepsilon.\end{aligned}$$

Now, since the set of g_n is uniformly integrable (as seen by weak compactness in $L^1(K,\mu)$), we have $\lim\limits_{\mu(B)\to 0} \sup\limits_n \int_B |g_n| \mathrm{d}\mu = 0$. We fix $\delta > 0$ verifying $\delta < \varepsilon/4M$ and the condition: $\mu(B) \leqslant \delta \Rightarrow \int_B |g_m| \mathrm{d}\mu \leqslant \varepsilon/4$, for every m.

We fix n such that $\|f_n\|_{L^1(K,\mu)} \leqslant \delta^2$. Let $B_n = \{x \in K : |f_n(x)| > \delta\}$, we then have $\mu(B_n) \leqslant \delta$. We obtain

$$\varepsilon < \int_K f_n g_n \mathrm{d}\mu \leqslant \int_{B_n} |g_n| \mathrm{d}\mu + \delta \int_{B_n^c} |g_n| \mathrm{d}\mu \leqslant \frac{\varepsilon}{4} + \delta M \leqslant \frac{\varepsilon}{2}.$$

This gives a contradiction. ∎

We deduce

PROPOSITION 1.4. *Let $T : A(\mathbb{D}) \to Y$ be a bounded operator. Then T is weakly compact if and only if there exists some probability measure μ on the torus, and for every $\varepsilon > 0$, some $C_\varepsilon > 0$ such that for every $x \in X$,*

$$\|T(x)\| \leqslant C_\varepsilon \|x\|_{L^1(\mu)} + \varepsilon \|x\|_\infty.$$

Proof. It is sufficient to prove that the disk algebra shares the lifting property of the previous proposition. Actually, this comes from the Chaumat's theorem and the decomposition

$$A(\mathbb{D})^* = L^1/H^1 \oplus_1 M_{\text{sing}}(\mathbb{T})$$

where $M_{\text{sing}}(\mathbb{T})$ is the space of singular measure, with respect to the Haar measures on the torus. ∎

2. WEAKLY COMPACT COMPOSITION OPERATORS

DEFINITION 2.1. Let φ be a map from $\mathbb{D}$ to $\mathbb{D}$. We denote by H_φ the composition operator defined on H^∞ by $H_\varphi(f) = f \circ \varphi$, for every $f \in H^\infty$. It is clear that H_φ maps H^∞ to H^∞ if and only if $\varphi \in H^\infty$.

The following lemma belongs to the folklore. Nevertheless, we prove it for the sake of completeness.

LEMMA 2.2. *On the unit ball of H^∞, the weak-star topology $\sigma(H^\infty, L^1/\overline{H}_0^1)$ is the topology of uniform convergence on every compact subset of $\mathbb{D}$.*

Proof. First we notice that the topologies are metrizable. Indeed $L^1/\overline{H}_0^1$ is separable, so that the weak star topology is metrizable on the unit ball of the dual. The fact that the topology of convergence on every compact subset of an $\Omega \subset \mathbb{C}$ is metrizable, is standard. Now, it is sufficient to prove that the convergent sequences in both topologies are the same.

Let $(f_k)_{k\in\mathbb{N}}$ in the unit ball of H^∞, weak-star convergent to $f \in H^\infty$. Let us fix a compact subset K of $\mathbb{D}$, we may suppose that K is the closed ball, with center 0 and radius $r < 1$. First, testing the weak-star convergence with the characters, we have that for every $n \in \mathbb{N}$, $\widehat{f_k}(n)$ tends to $\widehat{f}(n)$. We have

$$\sup_{|z|\leqslant r} |f_k(z) - f(z)| = \sup_{|z|=r} |f_k(z) - f(z)| \leqslant \sum_{n\in\mathbb{N}} r^n |\widehat{f_k}(n) - \widehat{f}(n)|.$$

The last term obviously tends to zero when k tends to the infinity. The result follows.

Now, let $(f_k)_{k\in\mathbb{N}}$ in the unit ball of H^∞, converging to $f \in H^\infty$ on every compact subset of $\mathbb{D}$. We fix $h \in L^1$ (so that its class belongs to $L^1/\overline{H}_0^1$) and $\varepsilon > 0$. There exists some $r < 1$ such that $\|P_r * h - h\|_1 \leqslant \varepsilon/4$, where P_r is the Poisson kernel with parameter r. Then

$$\begin{aligned} |h * (f_k - f)(0)| &\leqslant \frac{\varepsilon}{4}\|f_k - f\|_\infty + |P_r * h * (f_k - f)(0)| \\ &\leqslant \frac{\varepsilon}{2} + \|h\|_1 \cdot \sup_{|z|=r} |f_k(z) - f(z)|. \end{aligned}$$

Now, by uniform convergence on the closed ball, with center 0 and radius r, there exists k_ε such that for every integer k greater than k_ε we have

$$\|h\|_1 \cdot \sup_{|z|=r} |f_k(z) - f(z)| \leqslant \frac{\varepsilon}{2}. \quad ∎$$

LEMMA 2.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$, an analytic map. Then H_φ is the dual operator of some operator $h_\varphi : L^1/\overline{H}_0^1 \to L^1/\overline{H}_0^1$.*

Proof. It is sufficient to notice that H_φ is continuous from H^∞, with the weak-star topology, to H^∞, with the weak-star topology. This point is trivial with the previous lemma. ∎

Now, we characterize the weakly compact composition operators on the space of bounded analytic functions on $\mathbb{D}$.

THEOREM 2.4. *If H_φ is weakly compact then $\|\varphi\|_\infty < 1$.*

Proof. By the previous lemma H_φ is weak-star continuous and we can use Theorem 1.2.

Now we suppose that $\|\varphi\|_\infty = 1$, so that there exists a sequence $z_n \in \mathbb{D}$ such that $|\varphi(z_n)|$ converges to 1. Extracting a subsequence if necessary, we may suppose that $\varphi(z_n)$ converges to some a, on the torus. Now, we fix $\varepsilon_0 \in (0, 1)$, this gives δ by the property in Theorem 1.2. We consider f in the disk algebra such that $\|f\|_\infty \leqslant 1$, $\|f\|_1 \leqslant \delta$ and $f(a) = 1$ (a "peak function": for instance, take, with N sufficiently large, $f(z) = 2^{-N}(\bar{a}z + 1)^N$). Clearly, we have by continuity of f:

$$\|f \circ \varphi\|_\infty \geqslant \lim_{n\to\infty} |f \circ \varphi(z_n)| = |f(a)| = 1.$$

This contradicts $\|f \circ \varphi\|_\infty \leqslant \varepsilon_0$. ∎

REMARK 2.5. As the compactness is well known (and is easy), one could try to use the Dunford-Pettis property of H^∞ to conclude. More precisely, if H_φ

is weakly compact then its square is compact by the Dunford-Pettis property for H^∞. We obtain that $\|\varphi \circ \varphi\|_\infty < 1$, but this does not imply that $\|\varphi\|_\infty < 1$ in general.

Indeed a simple example is given by $\varphi(z) = -(1/2)(z+1)$.

We have the following consequence.

THEOREM 2.6. *Let $\varphi : \mathbb{D} \to \mathbb{D}$, an analytic map. The following assertions are equivalent:*

(i) $\|\varphi\|_\infty < 1$.
(ii) *H_φ factorizes through the identity map from H^∞ to H^1.*
(iii) *H_φ is 1-summing.*
(iv) *H_φ is p-summing for some finite $p \geqslant 1$.*
(v) *H_φ is γ-summing.*
(vi) *H_φ is nuclear.*
(vii) *H_φ is compact.*
(viii) *H_φ is weakly compact.*

Proof. The implications (ii)⇒(iii)⇒(iv)⇒(v)⇒(viii), (vi)⇒(iii) and (vi)⇒(vii) ⇒(viii) are obvious or consequences of the general theory of operator ideals. (viii)⇒(i) is proved by the previous theorem.

(i)⇒(vi). We may suppose that the range of φ is included in some ball, with center 0 and radius $r < 1$. Let $\chi_n \in (H^\infty)^*$ be defined by $\chi_n(h) = \widehat{h}(n)$ (for $n \in \mathbb{N}$). Then, we have the nuclear decomposition

$$H_\varphi = \sum_{n\in\mathbb{N}} \chi_n \varphi^n, \quad \text{where } \sum_{n\in\mathbb{N}} \|\chi_n\|_{(H^\infty)^*} \|\varphi^n\|_\infty \leqslant \sum_{n\in\mathbb{N}} r^n < \infty.$$

(i)⇒(ii). We may suppose that the range of φ is included in some ball, with center 0 and radius $r < 1$. By the Cauchy formula, for every $f \in H^\infty$, we have $\|f \circ \varphi\|_\infty \leqslant (1-r)^{-1}\|f\|_1$. By density of H^∞ in H^1, we have the conclusion. ∎

Now, we treat the case of the disk algebra. The results are similar.

DEFINITION 2.7. Let φ be a map from $\mathbb{D}$ to $\mathbb{D}$. We denote by A_φ the composition operator defined on $A(\mathbb{D})$ by $A_\varphi(f) = f \circ \varphi$, for every $f \in A(\mathbb{D})$. It is clear that A_φ maps $A(\mathbb{D})$ to $A(\mathbb{D})$ if and only if $\varphi \in A(\mathbb{D})$.

Note that we could have defined A_φ for any $\varphi \in A(\mathbb{D})$ such that $\|\varphi\|_\infty \leqslant 1$. Nevertheless, the only case that we “forgot" is when $|\varphi(z)| = 1$ for some $z \in \mathbb{D}$. In such a case, φ is constant by the maximum modulus principle and A_φ is trivial.

THEOREM 2.8. *Let $\varphi \in A(\mathbb{D})$, with $\varphi(\mathbb{D}) \subset \mathbb{D}$. The following assertions are equivalent:*

(i) $\|\varphi\|_\infty < 1$.
(ii) *A_φ factorizes through the formal identity from H^∞ to H^1.*
(iii) *A_φ is 1-summing.*

(iv) A_φ *is p-summing for some finite* $p \geqslant 1$.
(v) A_φ *is* γ*-summing.*
(vi) A_φ *is nuclear.*
(vii) A_φ *is compact.*
(viii) A_φ *is weakly compact.*

Proof. This is the same proof as for H^∞, except for the implication (viii)⇒(i). We suppose that some $a \in \mathbb{T}$ belongs to $\varphi(\mathbb{T})$. We consider $(f_k)_{k\in\mathbb{N}^*}$ the sequence in the unit sphere of the disk algebra defined by

$$f_k(z) = \frac{1}{k-(k-1)\bar{a}z}.$$

As A_φ is weakly compact, there exists $f \in A(\mathbb{D})$ and a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ such that $(f_{n_j} \circ \varphi)_{j\in\mathbb{N}}$ weakly converges to f: for every $z \in \mathbb{T}$, $f_{n_j}(\varphi(z))$ tends to $f(z)$. Introducing the set $E = \{z \in \mathbb{T} : \varphi(z) = a\}$, we have: for every $z \in E$, $f(z) = 1$ and for every $z \notin E$, $f(z) = 0$. As the torus is connected and E is non empty by hypothesis, we obtain that $E = \mathbb{T}$. Hence $\varphi = a$. This contradicts $\varphi(\mathbb{D}) \subset \mathbb{D}$. We conclude that $\varphi(\mathbb{T}) \subset \mathbb{D}$. ∎

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