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OPERATORS WITH COMMON HYPERCYCLIC SUBSPACES

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ABSTRACT. We provide a reasonable sufficient condition for a countable family of operators to have a common hypercyclic subspace. We also extend a result of the third author and A. Montes [22], thereby obtaining a common hypercyclic subspace for certain countable families of compact perturbations of operators of norm no larger than one.

KEYWORDS: *Hypercyclic vectors, subspaces, and operators; universal families.*

MSC (2000): 47A16.

1. INTRODUCTION

It is known that for any separable infinite dimensional Banach space X, there is a continuous linear operator $T : X \to X$ which is hypercyclic; that is, there is a vector x such that the set $\{x, Tx, \ldots, T^n x, \ldots\}$ is norm dense in X [3], [5]. Moreover, a simple Baire category argument shows that the set $HC(T)$ of such so-called *hypercyclic vectors* x is a dense G_δ in X [21], and its linear structure is well understood: While $HC(T)$ must always contain a dense subspace [9], [20], it not always contains a *closed* infinite dimensional one; see [16] for a complete characterization of when this occurs. (Throughout, when we say that $HC(T)$ contains a vector space V we mean of course that every $x \in V$ *except* $x = 0$ is hypercyclic for T.) Thus, for example it was shown that for the simplest example of a hypercyclic operator on a Banach space, namely the Rolewicz operator

$$B_2 : \ell_2 \to \ell_2,\ B_2(x_1, x_2, \ldots) = 2(x_2, x_3, \ldots),$$

$HC(B_2)$ contains an infinite dimensional vector space but that this vector space cannot be closed ([25], Theorem 3.4).

In recent years, an increasing amount of attention has been paid to the set $\bigcap\limits_{T\in\mathcal{F}} HC(T)$ of common hypercyclic vectors of a given family $\mathcal{F}$ of hypercyclic operators acting on the same Banach space X. Again, by a Baire category argument $\bigcap\limits_{T\in\mathcal{F}} HC(T)$ is a dense subset of X whenever $\mathcal{F}$ is countable. Moreover, L. Bernal and C. Moreno [6] showed this set contains a dense vector space if we ask in addition that the members be hereditarily hypercyclic. Finally S. Grivaux proved that this additional hypothesis can be suppressed ([17], Proposition 4.3).

Other important recent work is by E. Abakumov and J. Gordon [1], who showed that

$$\bigcap_{\{\lambda\in\mathbb{C}:|\lambda|>1\}} HC(B_\lambda) \neq \emptyset,$$

where B_λ is the Rolewicz operator with 2 replaced by λ. In fact it is simple to derive from this that the above intersection contains a dense subspace of ℓ_2. On the other hand, in [4], F. Bayart showed that under the assumption of a strong form of the hypercyclicity condition, uncountable collections of hypercyclic operators can indeed contain an infinite dimensional *closed* subspace of common hypercyclic vectors. Similar results were obtained by G. Costakis and M. Sambarino [13], who also provided a criterion for the existence of common hypercyclic vectors.

Our interest here will be in the following problem:

PROBLEM 1. Let $\mathcal{F}$ be a countable family of operators acting on a Banach space X. When does $\bigcap_{T\in\mathcal{F}} HC(T)$ contain a closed infinite dimensional subspace?

In Section 2 we show that a family of operators acting on a common Banach space may fail to support a common hypercyclic subspace, even if each operator in the family has a hypercyclic subspace (Example 2.1). Moreover, if the family is uncountable it may even fail to have single common hypercyclic *vector* (Example 2.2). In Section 3 we extend a result of A. Montes ([25], Theorem 2.1) by providing a reasonable sufficient condition on a countable family of hypercyclic operators acting on a Banach space to have a common infinite dimensional hypercyclic subspace (Corollary 3.5). We then apply this to extend a result of the third author and A. Montes [22], thereby obtaining a common hypercyclic subspace for certain countable families of operators of the form $T = U + K$ where $\|U\| \leqslant 1$ and K is compact.

2. TWO EXAMPLES

Example 2.1 was provided to us by an anonymous referee. An operator T is said to be *hereditarily hypercyclic* with respect to a given increasing sequence of positive integers (n_k) provided $\{T^{n_k}\}_{k\in\mathbb{N}}$ is hereditarily universal (cf. Section 3).

EXAMPLE 2.1. Consider the operators $T_1 := (I + B_{\rm w}) \oplus B_2$ and $T_2 := B_2 \oplus (I + B_{\rm w})$ acting on $\ell_2 \oplus \ell_2$, where B_2 and I are the Rolewicz' and the identity operator on ℓ_2, respectively, and B_{w} is the compact shift on ℓ_2 defined by

$$B_{\mathrm{w}} e_n := \begin{cases} \frac{1}{n} e_{n-1} & \text{if } n \geqslant 2, \\ 0 & \text{if } n = 1. \end{cases} \tag{2.1}$$

We show next that

(i) Each of T_1, T_2 has a hypercyclic subspace, and

(ii) T_1 and T_2 do not support a common hypercyclic subspace.

To see (i), notice that B_2 is hereditarily hypercyclic with respect to the entire sequence (n), and $I + B_{\mathrm{w}}$ is hereditarily hypercyclic with respect to some sequence (n_k) ([22], Lemma 4.5). Hence T_1 and T_2 are hereditarily hypercyclic with respect to some sequence (n_k) and by Theorem 2.1 of [23] it suffices to verify that the essential spectrum of T_i intersects the closed unit disk $(i = 1, 2)$. Now, the sequence $(e_n \oplus 0)$ is orthonormal in $\ell_2 \oplus \ell_2$. Also, $(T_1 - I)(e_n \oplus 0) = \frac{1}{n} e_{n-1} \oplus 0$ converges to zero in norm as n tends to infinity. This means (cf. XI 2.3 in [12]) that 1 belongs to the essential spectrum of T_1. Similarly, 1 belongs to the essential spectrum of T_2. So each of T_1, T_2 has a hypercyclic subspace.

To show (ii) assume, to the contrary, that there exists a closed, infinite dimensional subspace Z of $\ell_2 \oplus \ell_2$ such that every non-zero vector $(x, y) \in Z$ is hypercyclic for $(I + B_{\mathrm{w}}) \oplus B_2$ and $B_2 \oplus (I + B_{\mathrm{w}})$. In particular, both x and y must be hypercyclic for B_2.

Now, a simple Hilbert space argument shows that (at least) one of the coordinate projections $P_1(Z)$ and $P_2(Z)$ must contain a closed infinite dimensional subspace. Indeed, given an orthonormal sequence in Z one can find a subsequence such that its sequence (x_n) of i-th coordinate projections ($i = 1$ or 2) is linearly independent, bounded, and bounded away from zero. Next one can find a subsequence (x_{n_k}) of (x_n) that is equivalent as a basic sequence to an orthonormal sequence, what gives that $P_i(Z)$ contains the closed linear span of the sequence (x_{n_k}).

But this implies that B_2 has a hypercyclic subspace, which is not the case ([25], Theorem 3.4). So T_1 and T_2 have no common hypercyclic subspace.

EXAMPLE 2.2. Let $X = H$ be a separable, infinite-dimensional Hilbert space, and let S_H be the unit sphere of H. Let (w_n) be a sequence of positive scalars satisfying

$$\lim_{n\to\infty} \inf_k \Big(\prod_{j=1}^{n} w_{k+j} \Big)^{1/n} \leqslant 1 \quad \text{and} \quad \limsup \prod_{j=1}^{n} w_j = \infty.$$

For each h in S_H, let $\{e(h)_n : n \geqslant 1\}$ be a basis of H with $e(h)_1 = h$, and let $T_h : H \to H$ be the corresponding unilateral weighted backward shift defined by

$$T_h e(h)_n = \begin{cases} 0 & \text{if } n = 1, \\ w_n e(h)_{n-1} & \text{if } n \geqslant 2. \end{cases} \tag{2.2}$$

So T_h has a hypercyclic subspace ([23], Corollary 2.3). Also, notice that $\mathcal{F} = \{T_h : h \in S_H\}$ satisfies that for all $0 \neq y$ in H,

$$T_{\frac{y}{\|y\|}} y = 0.$$

That is, $\mathcal{F}$ is a family of operators, each one having a hypercyclic subspace, but such that there is no hypercyclic vector common to all members of $\mathcal{F}$.

Let us also observe that in [1] the authors mention that there is no common hypercyclic vector for the family of hypercyclic operators $\{\lambda B \oplus \delta B : |\lambda|, |\delta| > 1\}$. It is easy to see that no operator in this family admits a hypercyclic subspace.

3. A SUFFICIENT CONDITION FOR A COMMON HYPERCYCLIC SUBSPACE

We prove the main result in the more general setting of universality. Given a sequence $\mathcal{F} = \{T_j\}_{j\in\mathbb{N}}$ of bounded operators acting on a Banach space X, we say that a vector $x \in X$ is *universal* for $\mathcal{F}$ if $\{Tx : T \in \mathcal{F}\}$ is dense in X; the set of such universal vectors is denoted $HC(\mathcal{F})$.The sequence $\mathcal{F}$ is said to be *universal* (respectively, *densely universal*) provided $HC(\mathcal{F})$ is non-empty (respectively, dense in X). $\mathcal{F}$ is called *hereditarily universal* (respectively, *hereditarily densely universal*) provided $\{T_{n_k}\}_{k\in\mathbb{N}}$ is universal (respectively, densely universal) for each increasing sequence (n_k) of positive integers. For more on the notion of universality, see [15] and [19]. A result similar to the following theorem is proved in [10] for a (single) sequence of universal operators in the context of Fréchet spaces.

THEOREM 3.1. *Let $T_{n,j}$ $(n, j \in \mathbb{N})$ be bounded operators on a Banach space X, and let Y be a closed subspace of X of infinite dimension. Suppose that for each $n \in \mathbb{N}$*

(i) *$\{T_{n,j}\}_{j\in\mathbb{N}}$ is hereditarily densely universal, and*

(ii) *$\lim_{j\to\infty} \|T_{n,j}x\| = 0$ for each x in Y.*

Then there exists a closed, infinite dimensional subspace X_1 of X such that $\{T_{n,j}x\}_{j\in\mathbb{N}}$ is dense in X for each non-zero $x \in X_1$ and $n \in \mathbb{N}$. That is, X_1 is a universal subspace of $\{T_{n,j}\}_{j\in\mathbb{N}}$ for each $n \in \mathbb{N}$.

LEMMA 3.2. *Let $T_{n,j}$ $(n, j \in \mathbb{N})$ be bounded operators on a Banach space X such that for each fixed integer n the family $\{T_{n,j}\}_{j\geqslant 1}$ is densely universal. Then the set $\bigcap_{n=1}^{\infty} HC(\{T_{n,j}\}_{j\geqslant 1})$ of common universal vectors to every sequence $\{T_{n,j}\}_{j\in\mathbb{N}}$ is dense in X.*

Proof. $\bigcap_{n=1}^{\infty} HC(\{T_{n,j}\}_{j\geqslant 1})$ is a countable intersection of dense G_δ subsets of the Baire space X ([18], Satz 1.2.2). ∎

Proof of Theorem 3.1. Reducing the subspace Y if necessary, we may assume it has a normalized Schauder basis $(e_j)_j$. Let (e_j^*) be its associated sequence in Y^*

of coordinate functionals, that is, such that $e_j^*(e_i) = \delta_{i,j}$ for $i, j \in \mathbb{N}$. Let $A(Y, X)$ denote the norm closure (in $L(X, Y)$) of the subspace

$$\Big\{ \sum_{j=1}^{n} x_j e_j^*(\cdot) : n \in \mathbb{N}, x_1, \dots, x_n \in X \Big\}.$$

For each T in $B(X)$, define $L_T : A(Y, X) \to A(Y, X)$ by $L_T V := TV$. We make use of the following lemma, whose proof follows that of Theorem 3.1. Analogous versions of this lemma are proved in [10] for several operator ideals (nuclear, compact, approximable), in a more general context, by using tensor product techniques developed in [24].

LEMMA 3.3. *Suppose $\{T_j\}_{j\in\mathbb{N}}$ is a sequence of bounded operators on X that is hereditarily densely universal. Then $\{L_{T_{r_j}}\}_{j\geqslant 1}$ is a hereditarily densely universal sequence of operators on $A(Y, X)$, for some increasing sequence (r_j) of positive integers.*

Now, notice that by (i) and Lemma 3.3, for each fixed $n \in \mathbb{N}$ there exists a sequence of positive integers $(r_{n,j})_j$ such that the sequence of operators $\{L_{T_{n,r_{n,j}}}\}_{j\in\mathbb{N}}$ is hereditarily densely universal on the Banach space $A(Y, X)$. By Lemma 3.2, there exists V in $A(Y, X)$ that is universal for every sequence $\{L_{T_{n,r_{n,j}}}\}_{j\in\mathbb{N}}$, and hence universal for every $\{L_{T_{n,j}}\}_{j\in\mathbb{N}}$, too ($n \in \mathbb{N}$). Multiplying V by a non-zero scalar if necessary, we may assume that $\|V\| < \frac{1}{2}$. Consider now $X_1 := (i + V)(Y)$, where $i : Y \to X$ is the inclusion. For each $x \in Y$, $\|(i + V)x\| \geqslant \|x\| - \|Vx\| \geqslant \frac{1}{2}\|x\|$. So $i + V$ is bounded below and X_1 is closed and of infinite dimension. Notice that $\{T_{n,j}Vx\}_{j\in\mathbb{N}}$ is dense in X for every $0 \neq x \in Y$ and every $n \in \mathbb{N}$. Indeed, given $\epsilon > 0$, let $z \in X$ be arbitrary, and let S be a finite rank operator in $A(Y, X)$ such that $Sx = z$. By Lemma 3.3, for each n there is some $T_{n,j}$ such that $\|T_{n,j}V - S\| < \frac{\epsilon}{\|x\|}$. In particular, $\|T_{n,j}Vx - Sx\| = \|T_{n,j}Vx - z\| < \epsilon$. The theorem now follows from condition (ii). ∎

Proof of Lemma 3.3. Since $\{T_j\}_{j\in\mathbb{N}}$ is hereditarily densely universal on X, it follows from Theorem 2.2 of [7] that there exists a dense subspace X_0 of X, an increasing sequence of positive integers (r_j) and (possibly discontinuous) linear mappings $S_j : X_0 \to X$ ($j \in \mathbb{N}$) such that

$$T_{r_j}, S_j, \text{ and } (T_{r_j}S_j - I) \underset{j\to\infty}{\longrightarrow} 0 \tag{3.1}$$

pointwise on X_0. Now, consider

$$A_0 := \{V \in A(Y, X) : V(Y) \subset X_0 \text{ and } \dim(V(Y)) < \infty\}.$$

Then A_0 is dense in $A(Y, X)$, and it follows from (3.1) that

$$L_{T_{r_j}}, L_{S_j}, \text{ and } [L_{T_{r_j}}L_{S_j} - I] \underset{j\to\infty}{\longrightarrow} 0$$

pointwise on A_0. So $\{L_{T_{r_j}}\}_{j\geqslant 1}$ is hereditarily densely universal on $A(Y, X)$, by Theorem 2.2 of [7]. ∎

REMARK 3.4. An alternative constructive proof of Theorem 3.1 may be done with the arguments from Theorem 2.2 in [25]. The proof here is much simpler, and follows arguments from [10] and [11].

COROLLARY 3.5. *Let T_l ($l \in \mathbb{N}$) be operators acting on a Banach space X. Suppose there exists a closed, infinite dimensional subspace Y of X, increasing sequences $(n_{l,q})_q$ of positive integers, and scalars $c_{l,q}$ such that for $l \in \mathbb{N}$*

(i) $\{c_{l,q} T_l^{n_{l,q}}\}_{q\in\mathbb{N}}$ *is hereditarily universal, and*

(ii) $\lim_{q\to\infty} \|c_{l,q} T_l^{n_{l,q}} x\| = 0$ *for each x in Y.*

Then there exists a closed, infinite dimensional subspace X_1 of X such that $\{c_{l,q} T_l^{n_{l,q}} x\}_{q\in\mathbb{N}}$ is dense in X for each non-zero $x \in X_1$ and each $l \in \mathbb{N}$. That is, X_1 is a supercyclic subspace for T_l for every $l \in \mathbb{N}$. Moreover X_1 is a hypercyclic subspace for T_l for every $l \in \mathbb{N}$ if the constants $c_{l,q}$ are equal to one.

In virtue of Theorem 3.1 and Example 2.1 it is natural to ask:

PROBLEM 2. Let T_1, T_2 be two hereditarily hypercyclic operators acting on a Banach space X, with a common hypercyclic subspace. Must there exist sequences $(n_{l,q})_q$ $(l = 1, 2)$ and a closed infinite dimensional subspace Y of X such that $\{T_l^{n_{l,q}}\}_q$ is hereditarily universal and $T_l^{n_{l,q}} \underset{q\to\infty}{\to} 0$ pointwise on Y $(l = 1, 2)$?

4. AN APPLICATION TO COUNTABLE FAMILIES OF OPERATORS

We now apply Theorem 3.1 to show the following extension of Theorem 4.1 in [22] to countable families of operators.

THEOREM 4.1. *Let $\mathcal{F} = \{T_l = U_l + K_l : l \in \mathbb{N}\}$ be a family of operators acting on a common Banach space X. Suppose that for each $l \in \mathbb{N}$*

(i) $\|U_l\| \leqslant 1$, *K_l is compact, and*

(ii) $\{T_l^{n_{l,q}}\}_{q\geqslant 1}$ *is hereditarily universal, for some increasing sequence $(n_{l,q})_{q\geqslant 1}$ of positive integers.*

Then the operators in $\mathcal{F}$ have a common hypercyclic subspace.

To show Theorem 4.1, we make use of the three lemmas below. Lemma 4.2 and Lemma 4.3 follow from slight modifications of a proof by Mazur ([14], p. 38–39) and of a proof by Bernal-González and Calderón-Moreno ([6], Theorem 3.1), respectively. Lemma 4.4 is proved at the end of this section.

LEMMA 4.2. *Let (X_n) be a sequence of closed, finite-codimensional subspaces of X, with $X_n \supseteq X_{n+1}$ $(n \geqslant 1)$. Then there exists a normalized basic sequence (e_n) such that e_n belongs to X_n for all $n \geqslant 1$.*

LEMMA 4.3. *Let $T_{l,j}$ $(l, j \in \mathbb{N})$ be bounded operators on a Banach space X such that for each $l \in \mathbb{N}$ the family $\{T_{l,j}\}_j$ is hereditarily densely universal. Then there exists* *a dense manifold* X_0 *of* X *and, for each* $l \in \mathbb{N}$*, an increasing sequence of positive integers* $(r_{l,q})_q$ *such that*

$$\lim_{q\to\infty} \|T_{l,r_{l,q}}x\| = 0 \quad (x \in X_0).$$

Moreover, X_0 *may be chosen such that each non-zero vector of* X_0 *is universal for* $\{T_{l,j}\}_{j\geqslant 1}$*, for each* $l \in \mathbb{N}$.

LEMMA 4.4. *Let* X *and* Z *be Banach spaces, and let* $K_{l,n} : X \to Z$ *be compact operators* $(l, n \geqslant 1)$*. Given* $\epsilon > 0$*, there exist closed linear subspaces* X_n *of finite codimension in* X $(n \geqslant 1)$ *such that:*

(i) $X_n \supseteq X_{n+1}$*;*

(ii) $\|K_{l,n}x\| \leqslant \epsilon\|x\|$ $(x \in X_n, 1 \leqslant l \leqslant n)$.

Proof of Theorem 4.1. Notice that for each $l \in \mathbb{N}$, $\{T_l^{n_{l,q}}\}_{q\geqslant 1}$ must be hereditarily densely universal ([8], Lemma 2.5). Hence, by Theorem 3.1 it suffices to get a closed, infinite dimensional subspace Y of X and subsequences $(m_{l,q})_q$ of $(n_{l,q})_q$ such that

$$\lim_{q\to\infty} \|T_l^{m_{l,q}}x\| = 0 \quad (x \in Y, l \in \mathbb{N}).$$

For each pair of positive integers n and l, let $K_{l,n}$ be the compact operators defined by $T_l^n = (U_l + K_l)^n = U_l^n + K_{l,n}$. Apply Lemma 4.4 to get closed, finite codimensional subspaces X_n of X satisfying

$$\text{(4.1)} \qquad \begin{cases} \text{(a)} \quad X_n \supseteq X_{n+1}, \\ \text{(b)} \quad \|K_{l,n}x\| \leqslant \|x\| \ (x \in X_n, 1 \leqslant l \leqslant n). \end{cases}$$

By Lemma 4.2, we can pick a normalized basic sequence (e_n) in X such that $e_n \in X_n$ $(n \in \mathbb{N})$. Let $K > 0$ be the basis constant of (e_n), and pick a decreasing sequence of positive scalars, (ϵ_m), such that $\sum_{n=1}^{\infty} \epsilon_n < \frac{1}{2K}$. By Lemma 4.3 (applied to the operators $T_{l,j} = T_l^{n_{l,j}}$ $l, j \in \mathbb{N}$), there exist subsequences $(\widetilde{n}_{l,q})_q$ of $(n_{l,q})_q$ and a dense subspace X_0 of X such that

$$\text{(4.2)} \qquad \lim_{q\to\infty} \|T_l^{\widetilde{n}_{l,q}}x\| = 0 \quad (x \in X_0).$$

Pick a sequence (z_m) in X_0 such that

$$\text{(4.3)} \qquad \|e_n - z_n\| < \frac{\epsilon_n}{\max\{\|T_l^i\| : l, i \leqslant n.\}}.$$

Notice that $\|e_n - z_n\| < \epsilon_n$ $(n \geqslant 1)$ and, because (e_n) is normalized, $|e_n^*(x)| \leqslant 2K\|x\|$ $(n \geqslant 1)$ for all x in $Y_0 = \overline{\text{span}\{e_1, e_2, \ldots\}}$, where (e_n^*) is the sequence of functional coefficients associated with the Schauder basis (e_n) of Y_0. Hence $\sum_{n=1}^{\infty} \|e_n^*\|\|e_n - z_n\| < 2K \sum_{n=1}^{\infty} \epsilon_n < 1$, and so any subsequence (z_{n_k}) of (z_m) is equivalent to the corresponding basic sequence (e_{n_k}) ([14], p. 46). We let $Y := \overline{\text{span}\{z_{n_k} : k \geqslant 1\}}$, where $(z_{n_k}) \subseteq (z_n)$ is defined as follows. Let $n_0 := 1$. For

$l \in \mathbb{N}$, choose $m_{l,1}$ in $(\widetilde{n}_{l,q})$ such that $\|T_l^{m_{l,1}} z_{n_0}\| < \frac{\epsilon_{n_0}}{2}$. Also, let $n_1 := m_{1,1}$. Next, for each $l \in \mathbb{N}$, since z_{n_0}, $z_{n_1} \in X_0$, we may apply (4.2) to get $m_{l,2} \in (\widetilde{n}_{l,q})_q$ which satisfies the following conditions:

$$\begin{cases} m_{l,2} & > \max\{2, n_1, m_{l,1}\} \\ \|T_l^{m_{l,2}} z_{n_i}\| & < \frac{\epsilon_{n_i}}{2^2} \quad i = 0, 1. \end{cases}$$

Also, let $n_2 := \max_{1 \leqslant l \leqslant 2} \{m_{l,2}\}$. Continuing this process we get, for each $l \in \mathbb{N}$, an integer $m_{l,s}$ in $(\widetilde{n}_{l,q})_q$ such that

$$\begin{cases} \text{(i) } m_{l,s} & > \max\{s, n_{s-1}, m_{l,s-1}\}, \\ \text{(ii) } \|T_l^{m_{l,s}} z_{n_i}\| & < \frac{\epsilon_{n_i}}{2^s} \quad i = 0, \dots, s-1, \end{cases} \tag{4.4}$$

where $n_r = \max_{1 \leqslant l \leqslant r} \{m_{l,r}\}$ for each $r \in \mathbb{N}$. It suffices to show that $T_l^{m_{l,s}} \underset{s \to \infty}{\to} 0$ pointwise on Y ($l \in \mathbb{N}$). Let $0 \neq z = \sum_{j=1}^{\infty} \alpha_j z_{n_j}$ in Y, $l \in \mathbb{N}$ be fixed, and $s \geqslant l$ be arbitrary. Then

$$T_l^{m_{l,s}} z = \sum_{j=1}^{s-1} \alpha_j T_l^{m_{l,s}} z_{n_j} + \sum_{j=s}^{\infty} \alpha_j T_l^{m_{l,s}} (z_{n_j} - e_{n_j}) + T_l^{m_{l,s}} \Big(\sum_{j=s}^{\infty} \alpha_j e_{n_j} \Big). \tag{4.5}$$

Notice that $|\alpha_j| \leqslant 2L\|z\|$ $(1 \leqslant j)$, where L is the basis constant of (z_{n_k}). By (4.4(ii)),

$$\Big\| \sum_{j=1}^{s-1} \alpha_j T_l^{m_{l,s}} z_{n_j} \Big\| < \sum_{j=1}^{s-1} |\alpha_j| \frac{\epsilon_{n_j}}{2^s} \leqslant \frac{L\|z\|}{2^{s-1}} \sum_{j=1}^{s-1} \epsilon_{n_j}. \tag{4.6}$$

Also, by (4.4(i)) and (4.3)

$$\Big\| \sum_{j=s}^{\infty} \alpha_j T_l^{m_{l,s}} (z_{n_j} - e_{n_j}) \Big\| \leqslant 2L\|z\| \sum_{j=s}^{\infty} \epsilon_{n_j}. \tag{4.7}$$

Finally, since $X_{n_s} \subseteq X_{m_{l,s}}$ and $\|U_l\| \leqslant 1$, by (4.1(b))

$$\begin{aligned} \Big\| T_l^{m_{l,s}} \sum_{j=s}^{\infty} \alpha_j e_{n_j} \Big\| &= \Big\| (U_l^{m_{l,s}} + K_{l,m_{l,s}}) \Big(\sum_{j=s}^{\infty} \alpha_j e_{n_j} \Big) \Big\| \\ &\leqslant 2 \Big\| \sum_{j=s}^{\infty} \alpha_j e_{n_j} \Big\| \quad (s \geqslant l). \end{aligned} \tag{4.8}$$

So by (4.5), (4.6), (4.7), and (4.8), $\lim_{s \to \infty} \|T_l^{m_{l,s}} z\| = 0$. We finish the proof of Theorem 4.1 by showing Lemma 4.4.

Proof of Lemma 4.4. Let $n \geqslant 1$ and $\epsilon > 0$ be fixed. Because each $K_{l,n}^* : Z^* \to X^*$ is compact, there exist $x_{l,n,1}^*, \dots, x_{l,n,k_{l,n}}^*$ in X^* such that

$$K_{l,n}^*(B_{Z^*}) \subseteq \bigcup_{i=1}^{k_{l,n}} B(x_{l,n,i}^*, \epsilon). \tag{4.9}$$

For each positive integer s, let $X_s := \bigcap_{n=1}^{s} \bigcap_{l=1}^{n} \bigcap_{i=1}^{k_{l,n}} \mathrm{Ker}(x^*_{l,n,i})$. So each X_s is closed and of finite codimension in X, and $X_s \supseteq X_{s+1}$ $(s \geqslant 1)$. Now, let $x \in X_n$, and let $1 \leqslant l \leqslant n$ be fixed. By the Hahn-Banach theorem, there is a functional z^* of norm one such that $\|K_{l,n}x\| = \langle K_{l,n}x, z^*\rangle$. By (4.9), we may choose $1 \leqslant j \leqslant k_{l,n}$ such that $\|K^*_{l,n}z^* - x^*_{l,n,j}\| < \epsilon$. Hence, because x is in $X_n \subseteq \mathrm{Ker}(x^*_{l,n,j})$, $\|K_{l,n}x\| = \langle x, K^*_{l,n}z^* - x^*_{l,n,j}\rangle \leqslant \epsilon\|x\|$. ∎

The proof of Theorem 4.1 is now complete. ∎

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