J. OPERATOR THEORY
54:2(2005), 269–290

AN INTERTWINING PROPERTY FOR POSITIVE TOEPLITZ OPERATORS

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Communicated by Şerban Strătilă

ABSTRACT. We obtain a necessary and sufficient condition for the existence of an operator V satisfying the operator equation $VT_\Gamma = T_\Gamma S$. Here T_Γ is a positive Toeplitz operator and S is the canonical unilateral shift acting on the Hardy space $H^2(\mathcal{U})$.

KEYWORDS: *Positive Toeplitz operator, factorization, outer spectral factor.*

MSC (2000): Primary 47A62, 47B35; Secondary 47A56, 47A57.

1. INTRODUCTION

We consider the problem of existence of an operator V satisfying the operator equation

$$VT_\Gamma = T_\Gamma S \tag{1.1}$$

where T_Γ is a given positive Toeplitz operator corresponding to (operator valued) symbol Γ and S is the canonical unilateral shift on the Hardy space. This is motivated by the hyper-weighted Sarason problem considered in [2]. In case an operator V satisfying (1.1) exists, the hyper-weighted Sarason problem with weight Γ can be solved by using the Treil-Volberg generalization of the commutant lifting theorem. A necessary and sufficient condition for the existence of such an operator V satisfying (1.1) in case Γ is scalar valued was obtained in Proposition 6.3 in [2]. In this note, we generalize this result to the case when Γ is possibly an operator valued function. The organization is as follows: In Section 2, we establish some notation and review the notion of the *maximal outer spectral factor* for a positive Toeplitz operator T_Γ in [7]. In Section 3, we state and prove our main result providing a necessary and sufficient condition for the existence of an operator V as in (1.1). In Section 4, we provide a sufficient condition for the existence of an operator V as in (1.1) which resembles more closely the condition that we obtained in [2] when Γ is a scalar valued function. Subsequently, in Section 5, we show that in fact, this condition is also necessary in case the function Γ is matrix-valued and acts on a finite dimensional space. Finally, in Section 6, we conclude with a couple of questions arising naturally in this context, the answer to which is as yet unknown to us.

2. PRELIMINARIES

We begin by recalling some well established facts and notation. For any Hilbert space $\mathcal{Y}$ and integer $p \geqslant 1$, the Banach space $L^p(\mathcal{Y})$ denotes the Lebesgue space of all $\mathcal{Y}$ valued, strongly measurable functions u on the unit circle $\mathbb{T} = \{e^{it} : 0 \leqslant t < 2\pi\}$ such that $\int_0^{2\pi} \|u(e^{it})\|^p dt$ is finite. The norm for a function $u \in L^p(\mathcal{Y})$ is given by

$$\|u\|_p = \|u\| = \Big(\frac{1}{2\pi}\int_0^{2\pi} \|u(e^{it})\|^p dt\Big)^{1/p}.$$

Let $H^p(\mathcal{Y})$ denote the Hardy subspace of $L^p(\mathcal{Y})$ consisting of all functions h in $L^p(\mathcal{Y})$ with Fourier series expansions of the form $\sum_{n=0}^{\infty} a_n e^{int}$. Recall that such an h can also be viewed as the analytic function h in the open unit disc $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ given by $h(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$. For convenience, $\mathcal{Y}$ will be identified with the subspace of $H^p(\mathcal{Y})$ formed by the constant functions in $H^p(\mathcal{Y})$.

For two separable Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, let $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ be the set of all bounded linear operator from $\mathcal{U}$ to $\mathcal{Y}$. Moreover, $L^\infty(\mathcal{U}, \mathcal{Y})$ denotes the set of all essentially bounded measurable functions with respect to the strong operator topology from the unit circle $\mathbb{T}$ to $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. The norm for a function Ω in $L^\infty(\mathcal{U}, \mathcal{Y})$ is given by

$$\|\Omega\|_\infty = \text{essential sup}\,\{\|\Omega(e^{it})\| : 0 \leqslant t < 2\pi\}.$$

Furthermore, $H^\infty(\mathcal{U}, \mathcal{Y})$ denotes the subspace of $L^\infty(\mathcal{U}, \mathcal{Y})$ consisting of all functions with Fourier series expansion of the form $\Omega(e^{it}) = \sum_{n=0}^{\infty} \Omega_n e^{int}$ where Ω_n are bounded linear operators from $\mathcal{U}$ into $\mathcal{Y}$ for all integers $n \geqslant 0$. A function Ω in $H^\infty(\mathcal{U}, \mathcal{Y})$ can also be identified with its analytic extension in the unit disc given by the Taylor series expansion $\Omega(\lambda) = \sum_{n=0}^{\infty} \Omega_n \lambda^n$ for $\lambda \in \mathbb{D}$. In other words, $H^\infty(\mathcal{U}, \mathcal{Y})$ can be viewed as the set of all uniformly bounded analytic functions on $\mathbb{D}$ with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. In order to simplify notation, for $1 \leqslant p < \infty$, the spaces $L^p(\mathbb{C})$ and $H^p(\mathbb{C})$ will be denoted as L^p and H^p respectively and the spaces $L^\infty(\mathbb{C}, \mathbb{C})$ and $H^\infty(\mathbb{C}, \mathbb{C})$ will be denoted as L^∞ and H^∞ respectively.

For any Ω in $L^\infty(\mathcal{U},\mathcal{Y})$, let M_Ω be the multiplication operator from $L^2(\mathcal{U})$ to $L^2(\mathcal{Y})$ given by

$$(M_\Omega v)(\mathrm{e}^{\mathrm{i}t}) = \Omega(\mathrm{e}^{\mathrm{i}t})v(\mathrm{e}^{\mathrm{i}t}) \quad (v \in L^2(\mathcal{U}),\ t \in [0,2\pi)).$$

The Toeplitz operator T_Ω associated with the symbol Ω is the operator from $H^2(\mathcal{U})$ to $H^2(\mathcal{Y})$ defined by

$$(T_\Omega u)(\lambda) = P_+\Omega(\lambda)u(\lambda) \quad (\lambda \in \mathbb{D}, u \in H^2(\mathcal{U})),$$

where P_+ is the orthogonal projection from $L^2(\mathcal{Y})$ onto $H^2(\mathcal{Y})$. In other words, T_Ω is the compression of M_Ω to the appropriate Hardy subspaces. Notice that if Ω is in $H^\infty(\mathcal{U},\mathcal{Y})$, then the projection P_+ in the above definition of a Toeplitz operator is redundant, that is, $T_\Omega = M_\Omega | H^2(\mathcal{U})$. In this case, the Toeplitz operator is referred to as the analytic Toeplitz operator.

A function Θ in $H^\infty(\mathcal{U},\mathcal{Y})$ is said to be *inner* if $\Theta(\mathrm{e}^{\mathrm{i}t})$ is almost everywhere an isometry, or equivalently, if T_Θ is an isometry from $H^2(\mathcal{U})$ to $H^2(\mathcal{Y})$. A function Ω in $H^\infty(\mathcal{U},\mathcal{Y})$ is said to be *outer* if $T_\Omega H^2(\mathcal{U})$ is dense in $H^2(\mathcal{Y})$.

For any Hilbert space $\mathcal{U}$, let $U_\mathcal{U}$ denote multiplication by $\mathrm{e}^{\mathrm{i}t}$ on $L^2(\mathcal{U})$. In other words, $U_\mathcal{U} = M_\Omega$ where $\Omega(\lambda) = \lambda$ and we will refer to the unitary operator $U_\mathcal{U}$ as the canonical bilateral shift. Let $S_\mathcal{U} = U_\mathcal{U} | H^2(\mathcal{U})$ be the canonical unilateral shift on the Hardy space $H^2(\mathcal{U})$, that is, $(S_\mathcal{U} u)(\lambda) = \lambda u(\lambda)$. This is precisely the analytic Toeplitz operator associated to the symbol $\Omega(\lambda) = \lambda$. Notice that $S_\mathcal{U}$ is an isometry and its range consists of all functions $h \in H^2(\mathcal{U})$ with Fourier series expansion of the form $h(\lambda) = \sum\limits_{n=1}^{\infty} a_n\lambda^n$ for $\lambda \in \mathbb{D}$. The orthogonal projection onto the range of $S_\mathcal{U}$ is given by $S_\mathcal{U} S_\mathcal{U}^*$. The Hilbert space $\mathcal{U}$ can be regarded as the subspace of constant functions in $H^2(\mathcal{U})$ and it is the range of the orthogonal projection $I - S_\mathcal{U} S_\mathcal{U}^*$.

An isometry S on a Hilbert space $\mathcal{H}$ is said to be a shift if S^{*n} converges to 0 strongly. Assume that S is a shift and set $\mathcal{U} = \mathcal{H} \ominus S\mathcal{H}$. Then the *Fourier transform* $\widehat{F} : \mathcal{H} \to H^2(\mathcal{U})$ is defined by

$$\widehat{F}\Big(\sum_{n=0}^{\infty} S^n u_n\Big) = \sum_{n=0}^{\infty} \lambda^n u_n \quad (u_n \in \mathcal{U}).$$

Finally, $\widehat{F}$ is a unitary operator intertwining S with the canonical unilateral shift $S_\mathcal{U}$.

Recall that for $\Gamma \in L^\infty(\mathcal{U},\mathcal{Y})$, the Toeplitz operator T_Γ from $H^2(\mathcal{U})$ to $H^2(\mathcal{Y})$ satisfies the property

$$S_\mathcal{Y}^* T_\Gamma S_\mathcal{U} = T_\Gamma. \tag{2.1}$$

Conversely, if an operator Y from $H^2(\mathcal{U})$ to $H^2(\mathcal{Y})$ satisfies the relation

$$S_\mathcal{Y}^* Y S_\mathcal{U} = Y$$

then $Y = T_\Gamma$ for some $\Gamma \in L^\infty(\mathcal{U},\mathcal{Y})$.

On the other hand, an operator Y from $H^2(\mathcal{U})$ to $H^2(\mathcal{Y})$ satisfies the relation

$$YS_{\mathcal{U}} = S_{\mathcal{Y}}Y \tag{2.2}$$

if and only if Y is an analytic Toeplitz operator, that is, $Y = T_F$ for some F in $H^\infty(\mathcal{U}, \mathcal{Y})$. For more details on the notations and for the proofs of facts mentioned, see Chapter V in [7] or Chapter I in [5]. At this point, we would like to introduce a class of operator valued functions which will be needed later.

DEFINITION 2.1. By $H^2(\mathcal{U}, \mathcal{Y})$ we denote the set of all operator valued analytic functions on the unit disc with Taylor expansions

$$\Psi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \Psi_n \quad (\Psi_n \in \mathcal{L}(\mathcal{U}, \mathcal{Y}),\ \lambda \in \mathbb{D}),$$

such that for $u \in \mathcal{U}$, the formula

$$(\Psi u)(\lambda) = \Psi(\lambda)u \quad (\lambda \in \mathbb{D})$$

defines a bounded linear operator Ψ from $\mathcal{U}$ to $H^2(\mathcal{Y})$.

REMARK 2.2. It is easy to see that if Z is a bounded linear operator from $\mathcal{U}$ to $H^2(\mathcal{Y})$, then there exists a function $\Psi \in H^2(\mathcal{U}, \mathcal{Y})$ such that

$$(Zu)(\lambda) = \Psi(\lambda)u \quad (u \in \mathcal{U},\ \lambda \in \mathbb{D}).$$

Recall that a Toeplitz operator T_Γ on $H^2(\mathcal{U})$ is positive if and only if its symbol Γ is a.e. a positive operator on $\mathcal{U}$. We now introduce the notion of *maximal outer spectral factor* for the positive Toeplitz operator T_Γ. This discussion is taken from Chapter V of [7] and is included here for the sake of completion.

DEFINITION 2.3. An outer function B in $H^\infty(\mathcal{U}, \mathcal{F})$ for some Hilbert space $\mathcal{F}$ is said to be the maximal outer spectral factor for the positive Toeplitz operator T_Γ if it satisfies the following two properties:
(i) $\Gamma(e^{it}) \geqslant B(e^{it})^* B(e^{it})$ a.e.
(ii) If Θ is any function $H^\infty(\mathcal{E}, \mathcal{F}_1)$ for some Hilbert space $\mathcal{F}_1$ such that $\Gamma(e^{it}) \geqslant \Theta(e^{it})^*\Theta(e^{it})$ almost everywhere, then

$$B(e^{it})^* B(e^{it}) \geqslant \Theta(e^{it})^*\Theta(e^{it}) \quad \text{a.e.} \tag{2.3}$$

It is easy to show that the maximal outer spectral factor of T_Γ is unique up to a constant unitary factor from the left. See page 200–201 in [7] for more details. We now discuss briefly how to obtain the maximal outer spectral factor B. This discussion, as well as the notation established here, will be useful for the proof of our main result.

Let $\mathcal{H}$ be any separable Hilbert space. For notational convenience, henceforth, the canonical bilateral and unilateral shifts $U_{\mathcal{H}}$ and $S_{\mathcal{H}}$ acting on $L^2(\mathcal{H})$ and $H^2(\mathcal{H})$ respectively, will simply be referred to as U and S. The underlying Hilbert space $\mathcal{H}$ will be inferred from context in each case. Let N denote the function defined by $N(e^{it}) = \Gamma(e^{it})^{1/2}$. Clearly, $M_\Gamma = M_N^* M_N = M_N^2$ and the subspace

$\mathcal{N} = \overline{M_N H^2(\mathcal{U})}$ is invariant to U and $U|\mathcal{N}$ is an isometry. By employing the Wold decomposition for the isometry $U|\mathcal{N}$, we obtain $\mathcal{N} = \mathcal{N}_{\mathrm{s}} \oplus \mathcal{N}_{\mathrm{u}}$ where each of the spaces $\mathcal{N}_{\mathrm{s}}$ and $\mathcal{N}_{\mathrm{u}}$ is reducing for $U|\mathcal{N}$. Moreover, $U|\mathcal{N}_{\mathrm{s}}$ is an unilateral shift and hence is unitarily equivalent to (via the Fourier transform described above) the canonical unilateral shift S on $H^2(\mathcal{F})$ where $\mathcal{F} = \mathcal{N} \ominus U\mathcal{N}$. The operator $U|\mathcal{N}_u$ is a unitary operator and the space

$$\mathcal{N}_{\mathrm{u}} = \bigcap_{n \geqslant 0} \mathrm{e}^{\mathrm{i}nt} \overline{M_N H^2(\mathcal{U})}.$$

One can verify that the operator

$$X = P_{\mathcal{N}_{\mathrm{s}}} M_N | H^2(\mathcal{U}) \tag{2.4}$$

intertwines $S = U|H^2(\mathcal{U})$ with $U|\mathcal{N}_{\mathrm{s}}$ which implies that $\widehat{F}X$ intertwines the canonical unilateral shift on $H^2(\mathcal{U})$ with that on $H^2(\mathcal{F})$. Consequently,

$$\widehat{F}Xh = T_B h \quad (h \in H^2(\mathcal{U})), \tag{2.5}$$

for some $B \in H^\infty(\mathcal{U}, \mathcal{F})$ and B is outer because the range of X is dense in $\mathcal{N}_{\mathrm{s}}$. The function B is precisely the maximal outer spectral factor of Γ.

Since the Fourier transform $\widehat{F}$ in (2.5) is an unitary operator, $\|Xh\| = \|T_B h\|$ for $h \in H^2(\mathcal{U})$ and the space $\mathcal{N}_{\mathrm{u}}$ is reducing for the isometry $U|\mathcal{N}$. Let $\mathcal{M}$ be the subspace of $H^2(\mathcal{U})$ defined by

$$\mathcal{M} := \{h \in H^2(\mathcal{U}) : \|M_N h\| = \|T_B h\|\}.$$

It follows that

$$\mathcal{M} = \{h \in H^2(\mathcal{U}) : P_{\mathcal{N}_{\mathrm{u}}} M_N h = 0\}$$

is invariant to S. Thus, by the Beurling-Lax-Halmos theorem, there exists an inner function $\Omega \in H^\infty(\mathcal{Y}, \mathcal{U})$ such that

$$\mathcal{M} = T_\Omega H^2(\mathcal{Y}) \tag{2.6}$$

for an adequate Hilbert space $\mathcal{Y}$ of dimension less than or equal to that of $\mathcal{U}$. It is also easy to see that $h \in \mathcal{M}$ if and only if

$$N(\mathrm{e}^{\mathrm{i}t})h(\mathrm{e}^{\mathrm{i}t}) = (Xh)(\mathrm{e}^{\mathrm{i}t}) \quad \text{a.e.}$$

or equivalently, since $X^*X = T_B^* T_B$,

$$(\Gamma(\mathrm{e}^{\mathrm{i}t}) - B(\mathrm{e}^{\mathrm{i}t})^* B(\mathrm{e}^{\mathrm{i}t}))h(\mathrm{e}^{\mathrm{i}t}) \quad \text{a.e.}$$

Thus we have

$$\mathcal{M} := \{h \in H^2(\mathcal{U}) : (\Gamma(\mathrm{e}^{\mathrm{i}t}) - B^*(\mathrm{e}^{\mathrm{i}t} B(\mathrm{e}^{\mathrm{i}t}))h(\mathrm{e}^{\mathrm{i}t}) \text{ a.e.}\} \tag{2.7}$$

REMARK 2.4. The space $\mathcal{M}$ can be alternatively characterized as

$$\mathcal{M} = \{h \in H^2(\mathcal{U}) : T_\Gamma h = T_B^* T_B h\}. \tag{2.8}$$

To see this, first note that for h in $H^2(\mathcal{U})$ we have $(T_\Gamma h, h) = (M_\Gamma h, h) = \|M_N h\|^2$. Consequently, from the definition of the space $\mathcal{M}$, h belongs to $\mathcal{M}$ if and only if

$$(T_\Gamma h, h) = \|T_B h\|^2 = (T_B^* T_B h, h). \tag{2.9}$$

Assume first that h satisfies the relation $T_\Gamma h = T_B^* T_B h$. Clearly, in this case, h satisfies (2.9) and hence belongs to $\mathcal{M}$.

Conversely, assume now that h belongs to $\mathcal{M}$. In this case, due to (2.9), we have $((T_\Gamma - T_B^* T_B)h, h) = 0$. Recall however that B is the maximal outer spectral factor of Γ satisfying the relation

$$\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant B(\mathrm{e}^{\mathrm{i}t})^* B(\mathrm{e}^{\mathrm{i}t}) \quad \text{a.e.},$$

which in turn implies that the operator $T_\Gamma - T_B^* T_B$ is positive. Using now the fact that for a positive operator X on a Hilbert space, $(Xw, w) = 0$ for a vector w if and only if $Xw = 0$, it follows that h satisfies $T_\Gamma h = T_B^* T_B h$.

3. MAIN RESULT

We will henceforth let Γ be a function in $L^\infty(\mathcal{U}, \mathcal{U})$ such that $\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant 0$ almost everywhere. In other words, we assume $T_\Gamma \geqslant 0$. In our subsequent discussion, we will impose the condition that the positive Toeplitz operator T_Γ has dense range, and therefore zero kernel. It is easy to see that under this assumption, if there exists an operator V satisfying $VT_\Gamma = T_\Gamma S$, then it is unique. The proposition below gives a sufficient condition for T_Γ to have this property. Henceforth, for an operator A, $\ker A$ will denote its kernel.

PROPOSITION 3.1. *Let Γ be a function in $L^\infty(\mathcal{U}, \mathcal{U})$ such that $\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant 0$ almost everywhere. Moreover, assume that $\ker \Gamma(\mathrm{e}^{\mathrm{i}t})$ equals zero on a set of positive measure on the unit circle. Then the range of T_Γ is dense in $H^2(\mathcal{U})$, (i.e., $\overline{T_\Gamma H^2(\mathcal{U})} = H^2(\mathcal{U})$) or equivalently, since T_Γ is selfadjoint, $\ker T_\Gamma = \{0\}$.*

Proof. Since $T_\Gamma \geqslant 0$, the operator T_Γ is selfadjoint. Hence it is sufficient to show that T_Γ is one to one. Assume that h in $H^2(\mathcal{U})$ is in the kernel of T_Γ. Then

$$0 = (T_\Gamma h, h) = \frac{1}{2\pi} \int_0^{2\pi} (\Gamma(\mathrm{e}^{\mathrm{i}t}) h(\mathrm{e}^{\mathrm{i}t}), h(\mathrm{e}^{\mathrm{i}t}))\, \mathrm{d}t\,.$$

Because $\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant 0$ almost everywhere, this implies that $(\Gamma(\mathrm{e}^{\mathrm{i}t}) h(\mathrm{e}^{\mathrm{i}t}), h(\mathrm{e}^{\mathrm{i}t}))_{\mathcal{U}} = 0$ almost everywhere. Using the fact that the kernel of $\Gamma(\mathrm{e}^{\mathrm{i}t})$ is zero on a set of positive measure, we see that $h(\mathrm{e}^{\mathrm{i}t}) = 0$ on a set of positive measure Δ. In particular, $(h(\mathrm{e}^{\mathrm{i}t}), u)_{\mathcal{U}} = 0$ on Δ for all u in $\mathcal{U}$. An application of the Cauchy-Schwartz inequality shows that $(h(\mathrm{e}^{\mathrm{i}t}), u)_{\mathcal{U}}$ is a scalar valued function in H^1. Hence $(h(\mathrm{e}^{\mathrm{i}t}), u)_{\mathcal{U}} = 0$ almost everywhere for all u in $\mathcal{U}$. Since $\mathcal{U}$ is separable, it implies that $h = 0$. Here we have used the fact that if a scalar valued function

f in H^1 vanishes on a set of positive measure on the unit circle, it must be zero almost everywhere on the circle. This completes the proof. ∎

We are now ready to state and prove our main result. We follow the same notations as in the previous section.

THEOREM 3.2. *Let Γ be a function in $L^\infty(\mathcal{U},\mathcal{U})$ such that $\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant 0$ almost everywhere. Moreover, assume that the kernel of T_Γ is zero. Let B in $H^\infty(\mathcal{U},\mathcal{F})$ be the maximal outer factor corresponding to T_Γ. As before, let*

$$\mathcal{M} = \{f \in H^2(\mathcal{U}) : (T_\Gamma f, f) = \|T_B f\|^2\} = \{f \in H^2(\mathcal{U}) : T_\Gamma f = T_B^* T_B f\}.$$

Then there exists an operator V satisfying $VT_\Gamma = T_\Gamma S$ if and only if there exists a function Φ in $H^2(\mathcal{U},\mathcal{U})$ which satisfies the following three properties:

$$\Phi(0) = I_{\mathcal{U}}, \quad \Phi u \in \mathcal{M} \quad (u \in \mathcal{U}), \quad \text{and} \quad B(\lambda)\Phi(\lambda) = B(0) \quad (\lambda \in \mathbb{D}). \tag{3.1}$$

Moreover, in this case, the adjoint $V^*(= V_\Phi^*)$ of $V(= V_\Phi)$ is defined by

$$V^* h = S^* g, \quad (h \in H^2(\mathcal{U})) \quad \text{where } g(\lambda) = h(\lambda) - \Phi(\lambda)h(0), \quad (\lambda \in \mathbb{D}). \tag{3.2}$$

We want to point out here that the equivalent condition in (3.1) for the existence of an operator V satisfying $VT_\Gamma = T_\Gamma S$ involves only the maximal outer spectral factor of Γ.

Before we prove the theorem, we will need the following lemma.

LEMMA 3.3. *Let g be in $H^2(\mathcal{U})$. Then $T_\Gamma g$ belongs to $\mathcal{U}$ if and only if*

$$g \in \mathcal{M} \quad \text{and} \quad T_B g \in \mathcal{F}. \tag{3.3}$$

Proof. First assume that the two conditions in (3.3) hold. To verify that $T_\Gamma g$ belongs to $\mathcal{U}$ it suffices to show that $(T_\Gamma g, Sh) = 0$ for all $h \in H^2(\mathcal{U})$. Indeed for an arbitrary $h \in H^2(\mathcal{U})$,

$$(T_\Gamma g, Sh) = (T_B^* T_B g, Sh) = (T_B g, T_B Sh) = (T_B g, S T_B h) = 0$$

where the first equality follows from (2.8) and the last equality follows from the fact that $T_B g \in \mathcal{F}$. Conversely, assume that $T_\Gamma g = b \in \mathcal{U}$ for some $g \in H^2(\mathcal{U})$. Setting $N(\mathrm{e}^{\mathrm{i}t}) = \Gamma(\mathrm{e}^{\mathrm{i}t})^{1/2}$ a.e. one obtains that M_N is a positive operator on $L^2(\mathcal{U})$ and $M_\Gamma = M_N^* M_N = M_N^2$. For all $h \in H^2(\mathcal{U})$ we have

$$(M_N g, U M_N h) = (M_N g, M_N S h) = (M_\Gamma g, Sh) = (T_\Gamma g, Sh) = 0 \tag{3.4}$$

where the second last equality follows from the fact that $(M_\Gamma h_1, h_2) = (T_\Gamma h_1, h_2)$ for h_1, h_2 in $H^2(\mathcal{U})$ and the last equality follows from the fact that $T_\Gamma g \in \mathcal{U}$. Recall that $\mathcal{N} = \overline{M_N H^2(\mathcal{U})}$. From (3.4) we conclude that $M_N g$ is orthogonal to $U\mathcal{N}$ and thus $M_N g$ belongs to $\mathcal{F} = \mathcal{N} \ominus U\mathcal{N} \subset \mathcal{N}_{\mathrm{s}}$. This implies that

$$M_N g = P_{\mathcal{N}_{\mathrm{s}}} M_N g = Xg \quad \text{and} \quad Xg \in \mathcal{F}. \tag{3.5}$$

Recalling that $T_B g = \widehat{F} X g$ it is thus clear that $T_B g = \widehat{F} X g$ is in $\mathcal{F} \subset H^2(\mathcal{F})$. This proves the second part of (3.3).

Let us now prove that the first part of (3.3) holds as well. To that end, let $h \in H^2(\mathcal{U})$ be arbitrary and now using (3.5) and the definition of X in (2.4) we have

$$\begin{aligned}(T_\Gamma g, h) &= (M_\Gamma g, h) = (M_N g, M_N h) = (P_{\mathcal{N}_s} M_N g, P_{\mathcal{N}_s} M_N h)\\ &= (Xg, Xh) = (\widehat{F}Xg, \widehat{F}Xh) = (T_B g, T_B h) = (T_B^* T_B g, h).\end{aligned}$$

Since $h \in H^2(\mathcal{U})$ was arbitrary, it follows that $T_\Gamma g = T_B^* T_B g$ and hence g is in $\mathcal{M}$. This concludes the proof of the lemma. ∎

REMARK 3.4. If Φ is as in (3.1), then for any vector $u \in \mathcal{U}$, we must have

$$T_\Gamma \Phi u \in \mathcal{U}.$$

To see this, let $g = \Phi u$ and note that for $\lambda \in \mathbb{D}$ we have

$$(T_B g)(\lambda) = (T_B \Phi u)(\lambda) = B(\lambda)\Phi(\lambda)u = B(0)u \in \mathcal{F}. \tag{3.6}$$

Since g satisfies the conditions in (3.3), the previous lemma yields $T_\Gamma \Phi u \in \mathcal{U}$.

REMARK 3.5. If there exists a function Φ is as in (3.1), then the kernel of $B(0)$ is zero. Indeed, if u belongs to kernel of $B(0)$ then from (3.6) we can infer that $T_B \Phi u = 0$. On the other hand, since $\Phi u \in \mathcal{M}$, from (2.8) we must have

$$T_\Gamma \Phi u = T_B^* T_B \Phi u = 0.$$

Using now the hypothesis that the kernel of T_Γ is trivial, we have that $\Phi u = 0$. Since $\Phi(0) = I$ we conclude that $u = 0$ thus completing the proof.

Proof of Theorem 3.2. First assume that an operator valued function as in (3.1) exists. Let $V : H^2(\mathcal{U}) \to H^2(\mathcal{U})$ be the operator whose adjoint is defined by the formula

$$(V^* h)(\lambda) = \frac{h(\lambda) - \Phi(\lambda)h(0)}{\lambda} \quad (h \in H^2(\mathcal{U}),\ \lambda \in \mathbb{D}). \tag{3.7}$$

Since $\Phi(0) = I$ and $\Phi \in H^2(\mathcal{U},\mathcal{U})$ it is clear that V^* is a bounded linear operator on $H^2(\mathcal{U})$. Hence V is also a bounded linear operator on $H^2(\mathcal{U})$. For any $h \in H^2(\mathcal{U})$, the vector g defined by

$$g(\lambda) = h(\lambda) - \Phi(\lambda)h(0) \quad (\lambda \in \mathbb{D}) \tag{3.8}$$

belongs to the range of S thus satisfying $SS^*g = g$, and moreover, $V^*h = S^*g$. We note here that due to Remark 3.4, the vector $T_\Gamma \Phi h(0)$ belongs to $\mathcal{U}$ and therefore

$$S^* T_\Gamma g = S^* T_\Gamma h - S^* T_\Gamma \Phi h(0) = S^* T_\Gamma h. \tag{3.9}$$

Consequently, using (2.1), for $h \in H^2(\mathcal{U})$ and g as in (3.8) we have

$$T_\Gamma V^* h = T_\Gamma S^* g = S^* T_\Gamma S S^* g = S^* T_\Gamma g = S^* T_\Gamma h,$$

where the last equality follows from (3.9). Thus, $T_\Gamma V^* = S^* T_\Gamma$ which is equivalent to $V T_\Gamma = T_\Gamma S$ since T_Γ is self-adjoint.

Conversely, assume now that V is an operator on $H^2(\mathcal{U})$ satisfying $VT_\Gamma = T_\Gamma S$. Multiplying both sides of this equality by S^* and again using the fact that $S^*T_\Gamma S = T_\Gamma$ it immediately follows that $S^*VT_\Gamma = T_\Gamma$ or equivalently, $(S^*V - I)T_\Gamma = 0$. Now use the fact that due our assumption on Γ, T_Γ has zero kernel and hence dense range (being self-adjoint). It follows that

$$S^*V = I = S^*S$$

and therefore,

$$S^*(V - S) = 0. \tag{3.10}$$

Now letting $V - S = Z$ we have $V = S + Z$. Furthermore, from (3.10) it follows that the range of Z is contained in kernel of S^*. Employing the usual identification of $\mathcal{U}$ in $H^2(\mathcal{U})$ as the space of all constant functions, it is readily seen that kernel of S^* is precisely $\mathcal{U}$. The operator $Z^*|\ker S^*$ is clearly a bounded operator from $\ker S^*(\approx \mathcal{U})$ to $H^2(\mathcal{U})$. By Remark 2.2, there exists a function $\Psi \in H^2(\mathcal{U},\mathcal{U})$ such that

$$(Z^*u)(\lambda) = \Psi(\lambda)u \quad (\lambda \in \mathbb{D},\ u \in \mathcal{U} \approx \ker S^*). \tag{3.11}$$

Recalling once more that $ZH^2(\mathcal{U}) \subset \ker S^*$ we have $Z = (I - SS^*)Z$ or equivalently $Z^* = Z^*(I - SS^*)$. Using now (3.11), it is clear that

$$(Z^*h)(\lambda) = (Z^*(I - SS^*)h)(\lambda) = \Psi(\lambda)h(0), \quad (\lambda \in \mathbb{D},\ h \in H^2(\mathcal{U})). \tag{3.12}$$

The relations $VT_\Gamma = T_\Gamma S$, $V = S + Z$ and $S^*T_\Gamma S = T_\Gamma$ together imply

$$ZT_\Gamma = T_\Gamma S - ST_\Gamma = (I - SS^*)T_\Gamma S. \tag{3.13}$$

By taking adjoint in (3.13) and noting that $(I - SS^*)u = u$ for all $u \in \mathcal{U}$, it follows that

$$S^*T_\Gamma u = T_\Gamma Z^*u \quad (u \in \mathcal{U}).$$

Now using $S^*T_\Gamma S = T_\Gamma$ we have,

$$S^*T_\Gamma(u - SZ^*u) = 0 \quad (u \in \mathcal{U}). \tag{3.14}$$

Due to the fact that Z is a bounded operator, there exists a function Φ in $H^2(\mathcal{U},\mathcal{U})$ such that

$$(u - SZ^*u)(\lambda) = \Phi(\lambda)u \quad (\lambda \in \mathbb{D},\ u \in \mathcal{U}), \tag{3.15}$$

and since $(u - SZ^*u)(0) = u$, we must have $\Phi(0) = I$. From (3.14) and (3.15), it is clear that $T_\Gamma \Phi u$ belongs to $\mathcal{U}$. Invoking Lemma 3.3, it now follows that Φu is in $\mathcal{M}$ for all $u \in \mathcal{U}$ and $T_B\Phi u$ is in $\mathcal{F}$ and therefore

$$(T_B\Phi u)(\lambda) \equiv (T_B\Phi u)(0) \quad (\lambda \in \mathbb{D}).$$

Consequently, since $\Phi(0) = I$, for all $u \in \mathcal{U}$ we have

$$\begin{aligned} B(0)u &= B(0)\Phi(0)u \\ &= (T_B\Phi u)(0) = (T_B\Phi u)(\lambda) \\ &= B(\lambda)(\Phi u)(\lambda) = B(\lambda)\Phi(\lambda)u. \end{aligned}$$

Finally, note that (3.15) can also be written as

$$u - \lambda\Psi(\lambda)u = \Phi(\lambda)u \quad (u \in \mathcal{U}, \lambda \in \mathbb{D})$$

where Ψ is as in (3.11). This, together with (3.12) and (3.7) readily yields (3.2). The proof of the theorem is now complete. ∎

We now proceed to discuss a certain minimality property in the class of all positive, operator valued symbols Γ admitting an intertwining operator V satisfying $VT_\Gamma = T_\Gamma S$.

PROPOSITION 3.6. *Let $\Gamma \in L^\infty(\mathcal{U},\mathcal{U})$ with $\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant 0$ a.e. and such that T_Γ is injective (i.e.,* $\ker T_\Gamma = \{0\}$*). Let B in $H^\infty(\mathcal{U},\mathcal{F})$ denote the maximal outer spectral factor of Γ. Assume that there exists $\Phi \in H^2(\mathcal{U},\mathcal{U})$ satisfying the three conditions in* (3.1)*. Moreover, let*

$$\Phi = \omega\Phi_o \tag{3.16}$$

where ω in $H^\infty(\mathcal{F}',\mathcal{U})$ is an inner function and Φ_o in $H^2(\mathcal{U},\mathcal{F}')$ is an outer function. Then, the following conditions hold:

(i) *$\omega(0)$ is invertible and $\omega(0)^{-1} = \Phi_o(0)$.*

(ii) *The function $B_1 = B\omega$ in $H^\infty(\mathcal{F}',\mathcal{U})$ is outer and there exists a bounded operator V_1 on $H^2(\mathcal{F}')$ such that $VT_{\Gamma_1} = T_{\Gamma_1}S$ where $\Gamma_1 = B_1^*B_1$.*

(iii) *$\Gamma_1(\mathrm{e}^{\mathrm{i}t}) = \omega(\mathrm{e}^{\mathrm{i}t})^*\Gamma(\mathrm{e}^{\mathrm{i}t})\omega(\mathrm{e}^{\mathrm{i}t})$ a.e.*

Conversely, let B_1 be an outer function in $H^\infty(\mathcal{F}',\mathcal{U})$ and assume that there exists an operator V_1 such that $V_1T_{\Gamma_1} = T_{\Gamma_1}S$ where $\Gamma_1(\mathrm{e}^{\mathrm{i}t}) = B_1(\mathrm{e}^{\mathrm{i}t})^*B(\mathrm{e}^{\mathrm{i}t})$ a.e. Moreover, suppose B is an outer function such that

$$B_1(\lambda) = B(\lambda)\omega(\lambda) \quad (\lambda \in \mathbb{D}), \quad \omega \text{ is inner and } \omega(0) \text{ is invertible.} \tag{3.17}$$

Then any positive symbol Γ having B as the maximal outer spectral factor and satisfying the relation

$$\Gamma_1(\mathrm{e}^{\mathrm{i}t}) = \omega(\mathrm{e}^{\mathrm{i}t})^*\Gamma(\mathrm{e}^{\mathrm{i}t})\omega(\mathrm{e}^{\mathrm{i}t}) \quad a.e. \tag{3.18}$$

will admit an intertwining operator V satisfying $VT_\Gamma = T_\Gamma S$.

Proof. Let Γ be a positive symbol and assume that there exists Φ in $H^2(\mathcal{U},\mathcal{U})$ satisfying the three conditions in (3.1). We will first show that $\omega(0)$ is invertible. Since Φ_o is outer, the operator $\Phi_o(0)$ has dense range. From (3.16), we have $\omega(0)\Phi_o(0) = \Phi(0) = I$ which implies that $\Phi_o(0)$ is invertible and $\Phi_o(0)^{-1} = \omega(0)$. Define $\Phi_1(\lambda) = \Phi_o(\lambda)\omega(0)$. Clearly, $\Phi_1(0) = I$ and Φ_1 is in $H^2(\mathcal{F}',\mathcal{F}')$ and

$$\begin{aligned} B_1(\lambda)\Phi_1(\lambda) &= B(\lambda)\omega(\lambda)\Phi_o(\lambda)\omega(0) \\ &= B(\lambda)\Phi(\lambda)\omega(0) = B(0)\omega(0) = B_1(0). \end{aligned} \tag{3.19}$$

Note also that since B is outer, the operator $B(0)$ has dense range which together with (3.19) implies that B_1 is outer. Since $\Gamma_1 = B_1^*B_1$ the corresponding space

$$\mathcal{M}_1 = \{f \in H^2(\mathcal{F}') : T_{\Gamma_1}f = T_{B_1}^*T_{B_1}f\} = H^2(\mathcal{F}').$$

Now invoking Theorem 3.2, the conclusion in part (ii) of the proposition follows.

Finally, in order to prove the assertion in part (iii), note that since $\Phi u \in \mathcal{M}$, $(u \in \mathcal{U})$, by (2.7) we have

$$(\Gamma(\mathrm{e}^{\mathrm{i}t}) - B^*(\mathrm{e}^{\mathrm{i}t})B(\mathrm{e}^{\mathrm{i}t}))\omega(\mathrm{e}^{\mathrm{i}t})\Phi_{\mathrm{o}}(\mathrm{e}^{\mathrm{i}t}) = 0 \quad \text{a.e.} \tag{3.20}$$

However, since Φ_{o} is outer, from (3.20) it follows that

$$\omega(\mathrm{e}^{\mathrm{i}t})(\Gamma(\mathrm{e}^{\mathrm{i}t}) - B^*(\mathrm{e}^{\mathrm{i}t})B(\mathrm{e}^{\mathrm{i}t}))\omega(\mathrm{e}^{\mathrm{i}t}) = 0 \quad \text{a.e.}$$

from which we immediately obtain

$$\Gamma_1(\mathrm{e}^{\mathrm{i}t}) = \omega(\mathrm{e}^{\mathrm{i}t})^*\Gamma(\mathrm{e}^{\mathrm{i}t})\omega(\mathrm{e}^{\mathrm{i}t}) \quad \text{a.e.}$$

Let us now prove the converse. To that end, let B be the maximal outer spectral factor for Γ and assume that B satisfies (3.17). By Theorem 3.2, there exists an operator valued function Φ_1 in $H^2(\mathcal{F}', \mathcal{F}')$ such that

$$B_1(\lambda)\Phi(\lambda) = B_1(0) \quad (\lambda \in \mathbb{D}). \tag{3.21}$$

Define the function

$$\Phi(\lambda) = \omega(\lambda)\Phi_1(\lambda)\omega(0)^{-1} \quad (\lambda \in \mathbb{D}). \tag{3.22}$$

Then, since $B_1 = B\omega$, for $\lambda \in \mathbb{D}$, we have

$$B(\lambda)\Phi(\lambda) = B(\lambda)\omega(\lambda)\Phi_1(\lambda)\omega(0)^{-1} = B_1(\lambda)\Phi_1(\lambda)\omega(0)^{-1} = B(0).$$

We now claim that the range of T_ω is contained in $\mathcal{M}$ where $\mathcal{M}$ is as in (2.8). By (2.7), it is enough to verify

$$(\Gamma(\mathrm{e}^{\mathrm{i}t}) - B^*(\mathrm{e}^{\mathrm{i}t})B(\mathrm{e}^{\mathrm{i}t}))\omega(\mathrm{e}^{\mathrm{i}t}) = 0 \quad \text{a.e. in } t.$$

Since B is the maximal outer spectral factor for Γ, we have

$$\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant B^*(\mathrm{e}^{\mathrm{i}t})B(\mathrm{e}^{\mathrm{i}t}) \quad \text{a.e. in } t.$$

Consequently,

$$\Gamma_1(\mathrm{e}^{\mathrm{i}t}) = \omega^*(\mathrm{e}^{\mathrm{i}t})\Gamma(\mathrm{e}^{\mathrm{i}t})\omega(\mathrm{e}^{\mathrm{i}t}) \geqslant \omega^*(\mathrm{e}^{\mathrm{i}t})B^*(\mathrm{e}^{\mathrm{i}t})B(\mathrm{e}^{\mathrm{i}t})\omega(\mathrm{e}^{\mathrm{i}t}) = \Gamma_1(\mathrm{e}^{\mathrm{i}t}) \quad \text{a.e. in } t.$$

This implies that

$$\omega^*(\mathrm{e}^{\mathrm{i}t})(\Gamma(\mathrm{e}^{\mathrm{i}t}) - B^*(\mathrm{e}^{\mathrm{i}t})B(\mathrm{e}^{\mathrm{i}t}))\omega(\mathrm{e}^{\mathrm{i}t}) = 0 \quad \text{a.e. in } t$$

from which it immediately follows that the range of T_ω is contained in $\mathcal{M}$. This shows that Φu, $(u \in \mathcal{U})$ is in $\mathcal{M}$ where Φ is as defined in (3.15). Invoking (3.1), the proof of the proposition is now complete. ∎

REMARK 3.7. Suppose Γ is a positive symbol such that T_Γ is injective and let B denote the maximal outer spectral factor of Γ. Moreover, assume that there exists an operator V such that $VT_\Gamma = T_\Gamma S$. In this case, any symbol Γ_1 with

$$\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant \Gamma_1(\mathrm{e}^{\mathrm{i}t}) \geqslant B^*(\mathrm{e}^{\mathrm{i}t})B(\mathrm{e}^{\mathrm{i}t})$$

will admit an intertwining operator V_1 satisfying $VT_{\Gamma_1} = T_{\Gamma_1}S$. To see this, note that by Theorem 3.2, there exists a function $\Phi \in H^2(\mathcal{U},\mathcal{U})$ as in (3.1). By Remark 3.5, this implies that $\ker B(0) = \{0\}$. Thus T_B, and consequently T_{Γ_1}, has trivial kernel. Now we may apply Theorem 3.2 to obtain the conclusion.

4. A SUFFICIENT CONDITION

Let $\Gamma \in L^\infty(\mathcal{U},\mathcal{U})$ be a positive symbol. In this section, we provide a sufficient condition for the existence of an intertwining operator V on $H^2(\mathcal{U})$ satisfying $VT_\Gamma = T_\Gamma S$. This condition closely resembles the necessary and sufficient condition we obtained in [2] for a scalar valued symbol Γ. In the next section, we will show that the condition we obtain here is also necessary in case the dimension of the fiber space $\mathcal{U}$ is finite.

THEOREM 4.1. *Let Γ be a function in $L^\infty(\mathcal{U},\mathcal{U})$ such that $\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant 0$ almost everywhere and, as before, assume that the kernel of T_Γ is zero. In this case, if $\|\Gamma(\mathrm{e}^{\mathrm{i}t})^{-1}\|$ exists a.e. and $\|\Gamma(\cdot)^{-1}\| \in L^1(\mathbb{T})$, then there exists an operator V satisfying $VT_\Gamma = T_\Gamma S$. In fact, if B is the maximal outer spectral factor for Γ, then we have $\Gamma(\mathrm{e}^{\mathrm{i}t}) = B(\mathrm{e}^{\mathrm{i}t})^*B(\mathrm{e}^{\mathrm{i}t})$ a.e. in t and moreover, $B(\lambda)^{-1}$ exists for all λ in $\mathbb{D}$ and the function $\lambda \to B(\lambda)^{-1}$ is in $H^2(\mathcal{U},\mathcal{U})$.*

Before proving the theorem, we will need a couple of lemmas. The lemma given below is well known but we include the proof here for the sake of completeness.

LEMMA 4.2. *Let Γ be a function in $L^\infty(\mathcal{U},\mathcal{U})$ such that $\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant 0$ a.e. and moreover assume that $\|\Gamma(\mathrm{e}^{\mathrm{i}t})^{-1}\|$ exists a.e. and $\|\Gamma(\cdot)^{-1}\| \in L^1(\mathbb{T})$. Under these assumptions, we have $\Gamma(\mathrm{e}^{\mathrm{i}t}) = B(\mathrm{e}^{\mathrm{i}t})^*B(\mathrm{e}^{\mathrm{i}t})$ a.e. in t where B is the maximal outer spectral factor of Γ.*

Proof. Let $\gamma(\mathrm{e}^{\mathrm{i}t}) = \|\Gamma(\mathrm{e}^{\mathrm{i}t})^{-1}\| \in L^1(\mathbb{T})$ and B denotes the maximal outer spectral factor of Γ as discussed before. Recall that if A is an invertible operator, then $\frac{1}{\|A^{-1}\|} \leqslant \|A\|$ which implies that

$$\frac{1}{\gamma(\cdot)} \leqslant \|\Gamma(\cdot)\| \leqslant \|T_\Gamma\|. \tag{4.1}$$

For t belonging to the set $\{s : \gamma(\mathrm{e}^{\mathrm{i}s}) \leqslant 1\}$, due to (4.1), we have the inequality

$$0 \leqslant \log\Big(\frac{1}{\gamma(\mathrm{e}^{\mathrm{i}t})}\Big) \leqslant \log\|T_\Gamma\|.$$

On the other hand, for t in the set $\{s : \gamma(\mathrm{e}^{\mathrm{i}s}) \geqslant 1\}$ we have the inequality

$$\Big|\log\Big(\frac{1}{\gamma(\mathrm{e}^{\mathrm{i}t})}\Big)\Big| = |\log(\gamma(\mathrm{e}^{\mathrm{i}t}))| \leqslant \gamma(\mathrm{e}^{\mathrm{i}t}).$$

Therefore, the function $\log\left(\frac{1}{\gamma}\right) \in L^1(\mathbb{T})$ and using a well known classical result (see [6]) there exists an outer function $\phi \in H^2$ such that

$$\frac{1}{\gamma(e^{it})} = |\phi(e^{it})|^2. \tag{4.2}$$

Recall that if A is a positive, invertible operator then

$$(Au, u) \geqslant \eta(u, u), \quad \eta = \frac{1}{\|A^{-1}\|}, \quad \forall u.$$

This fact along with the relation (4.2) and the definition of γ imply that

$$(\Gamma(e^{it})u(e^{it}), u(e^{it})) \geqslant (\phi(e^{it})u(e^{it}), \phi(e^{it})u(e^{it})) \quad (u \in H^2(\mathcal{U})). \tag{4.3}$$

and therefore, the map W from $\mathcal{N} := \overline{\Gamma^{1/2}H^2(\mathcal{U})} (\subset L^2(\mathcal{U}))$ to $H^2(\mathcal{U})$ defined by

$$W(\Gamma^{1/2}(\cdot)u(\cdot)) = (\phi u)(\cdot) \quad (u \in H^2(\mathcal{U}))$$

is a well defined contraction. As usual, we denote by U the canonical bilateral shift on $L^2(\mathcal{U})$ and $S = U|H^2(\mathcal{U})$ is the canonical unilateral shift. As discussed in Section 2, the space $\overline{\Gamma^{1/2}H^2(\mathcal{U})}$ is invariant to U, and moreover, it is easy to verify that $WU|\mathcal{N} = SW$. It follows that $W(U^n\overline{\Gamma^{1/2}H^2(\mathcal{U})}) \subset S^nH^2(\mathcal{U})$. Since $\bigcap\limits_{n=0}^{\infty} S^nH^2(\mathcal{U}) = \{0\}$, we conclude that the space $\bigcap\limits_{n=0}^{\infty} U^n\overline{\Gamma^{1/2}H^2(\mathcal{U})}$ is contained in the kernel of W.

We claim that the kernel of W is $\{0\}$. To see this, let $v \in \ker W$ and let $u_n \in H^2(\mathcal{U})$ such that $v_n(e^{it}) = \Gamma^{1/2}(e^{it})u_n(e^{it})$ converges to v. From the definition of W, this means that ϕu_n converges to 0 in $H^2(\mathcal{U})$. By passing through a subsequence if necessary, we may without loss of generality assume that $\phi(e^{it})u_n(e^{it}) \to 0$ in $\mathcal{U}$ a.e. and since $\phi \neq 0$ a.e. in t it follows that $u_n(e^{it}) \to 0$ a.e. However, since $\sup\limits_t \|\Gamma(e^{it})\| < \infty$ it follows that $v_n(e^{it}) \to 0$ a.e. in t. This shows that $v = 0$ and hence kernel of W is $\{0\}$.

We showed before that the kernel of W contains the space $\bigcap\limits_{n=0}^{\infty} U^n\overline{\Gamma^{1/2}H^2(\mathcal{U})}$. We thus conclude that $\bigcap\limits_{n=0}^{\infty} U^n\overline{\Gamma^{1/2}H^2(\mathcal{U})} = \{0\}$ and consequently (see Proposition 4.2 of [7]),

$$\Gamma(e^{it}) = B(e^{it})^*B(e^{it}). \tag{4.4}$$

This concludes the proof of the lemma. ∎

Before proceeding with the next lemma, we will need the following elementary fact the proof of which is omitted.

FACT. *Let X be a bounded operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$ with dense range. Assume moreover that the operator $Y = X^*X$ is invertible. In this case, X is invertible and $\|X^{-1}\|^2 = \|Y^{-1}\|$.*

LEMMA 4.3. *Let Γ be as in Lemma 4.2. Then, $B(\lambda)^{-1}$ exists for all λ in $\mathbb{D}$ and the function $\lambda \to B(\lambda)^{-1}$ is in $H^2(\mathcal{U},\mathcal{U})$.*

Proof. Since B is an outer function, $B(e^{it})$ has dense range a.e. in t. Invoking the Fact mentioned above, $B(e^{it})^{-1}$ exists a.e. in t and

$$\|B(e^{it})^{-1}\|^2 \leqslant \|\Gamma(e^{it})^{-1}\|. \tag{4.5}$$

Let $\Sigma : H^2(\mathcal{U}) \to L^1(\mathcal{U})$ be the map defined by

$$(\Sigma f)(e^{it}) = B(e^{it})^{-1} f(e^{it}) \quad (f \in H^2(\mathcal{U})). \tag{4.6}$$

Let us first show that Σ is a bounded linear operator from $H^2(\mathcal{U})$ to $L^1(\mathcal{U})$. To see this, note that for any $f \in H^2(\mathcal{U})$ we have

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \|B(e^{it})^{-1} f(e^{it})\|\mathrm{d}t &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \|B(e^{it})^{-1}\|\|f(e^{it})\|\mathrm{d}t \\ &\leqslant \Big(\frac{1}{2\pi}\int_0^{2\pi} \|\Gamma(e^{it})^{-1}\|\mathrm{d}t\Big)^{1/2} \|f\|_{H^2(\mathcal{U})}. \end{aligned} \tag{4.7}$$

The inequality in the last line in (4.7) follows from Cauchy-Schwartz inequality and (4.5). Clearly, from (4.7), it follows that Σ defines a bounded operator from $H^2(\mathcal{U})$ to $L^1(\mathcal{U})$ with $\|\Sigma\| \leqslant \nu := \Big(\frac{1}{2\pi}\int_0^{2\pi} \|\Gamma(e^{it})^{-1}\|\mathrm{d}t\Big)^{1/2}$.

We now claim that the range of Σ is included in $H^1(\mathcal{U})$. Indeed, for any $g \in H^2(\mathcal{U})$ from (4.6) we have $\Sigma(T_B g) = g$. Since Σ is a bounded operator from $H^2(\mathcal{U})$ to $L^1(\mathcal{U})$ and T_B has dense range in $H^2(\mathcal{U})$ (B being outer), the claim immediately follows.

On the other hand, for any $u \in \mathcal{U}$, using once again (4.5), we obtain the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} \|B(e^{it})^{-1}u\|^2\mathrm{d}t \leqslant \|u\|^2 \frac{1}{2\pi}\int_0^{2\pi} \|\Gamma(e^{it})^{-1}\|\mathrm{d}t$$

which in turn implies that the map $\Sigma|\mathcal{U}$ is in fact a bounded operator from $\mathcal{U}$ to $H^2(\mathcal{U})$ with norm less than or equal to $\nu := \Big(\frac{1}{2\pi}\int_0^{2\pi} \|\Gamma(e^{it})^{-1}\|\mathrm{d}t\Big)^{1/2}$. Therefore, by Remark 2.2, there exists a function $\Phi \in H^2(\mathcal{U},\mathcal{U})$ such that

$$(\Sigma u)(\lambda) = (\Phi u)(\lambda) := \Phi(\lambda)u \quad (\lambda \in \mathbb{D},\ u \in \mathcal{U}). \tag{4.8}$$

By taking boundary values, we immediately have

$$(\Phi u)(e^{it}) = B(e^{it})^{-1}u \text{ a.e. in } t \quad (u \in \mathcal{U}). \tag{4.9}$$

Due to (4.9) we have

$$(T_B \Phi u)(e^{it}) = u \quad \text{a.e. in } t. \tag{4.10}$$

Referring back to (4.8), we see that for all $u \in \mathcal{U}$ the function $v(\lambda) = B(\lambda)\Phi(\lambda)u$ is in $H^2(\mathcal{U})$ and by taking boundary values, in view of (4.10), we get $v(\mathrm{e}^{\mathrm{i}t}) = u$ a.e. in t. This implies $v(\lambda) \equiv u$ for all $\lambda \in \mathbb{D}$ and consequently

$$B(\lambda)\Phi(\lambda) = I \quad (\lambda \in \mathbb{D}). \tag{4.11}$$

To complete the proof of the lemma, we now need to show that $B(\lambda)^{-1}$ exists for all $\lambda \in \mathbb{D}$. In fact, we will show that

$$\Phi(\lambda) = B(\lambda)^{-1} \quad (\lambda \in \mathbb{D}).$$

Let us first remark that it suffices to show that $B(0)$ is invertible. To see this, note that if $B(0)$ is invertible, $B(\lambda)$ will be invertible as well for all λ sufficiently close to zero. From (4.11) and the uniqueness of inverse, it follows that $B(\lambda)^{-1} \equiv \Phi(\lambda)$ for all λ sufficiently close to zero which in turn implies that

$$\Phi(\lambda)B(\lambda) = I, \quad |\lambda| \text{ sufficiently small.}$$

By analyticity, we can conclude that $\Phi(\lambda)B(\lambda) = I$ for all $\lambda \in \mathbb{D}$ and thus it follows that $\Phi(\lambda) = B(\lambda)^{-1}$, $(\lambda \in \mathbb{D})$.

In order to complete the proof, we now need to show that $B(0)$ is invertible. Since B is in $H^\infty(\mathcal{U})$, from (4.11) it follows that

$$\|B\|_\infty \|\Phi(0)u\| \geqslant \|u\| \quad (u \in \mathcal{U}).$$

This implies that $\Phi(0)$ is bounded below. We now claim that $\Phi(0)$ has dense range. Let $u_0 \in \mathcal{U}$ be such that u_0 is orthogonal to $\overline{\Phi(0)\mathcal{U}}$. As shown before, the range of the operator Σ is included in $H^1(\mathcal{U})$ and moreover, the range of the operator $\Sigma|\mathcal{U}$ is included in $H^2(\mathcal{U})$. So, for an arbitrary vector $u \in \mathcal{U}$ we have

$$(S^n \Sigma u, u_0)_{H^2(\mathcal{U})} = \frac{1}{2\pi} \int_0^{2\pi} (\mathrm{e}^{\mathrm{i}nt} B(\mathrm{e}^{\mathrm{i}t})^{-1} u, u_0) \mathrm{d}t = 0 \quad (n \geqslant 1). \tag{4.12}$$

On the other hand, due to (4.8),

$$(\Sigma u, u_0)_{H^2(\mathcal{U})} = (\Phi(0)u, u_0) = 0 \tag{4.13}$$

where the last equality in the above line follows from the assumption that u_0 is orthogonal to $\overline{\Phi(0)\mathcal{U}}$. Recalling that finite linear combinations of the form $\sum_{n=0}^{N} S^n u_n$ $(u_n \in \mathcal{U})$ are dense in $H^2(\mathcal{U})$ and that $\Sigma : H^2(\mathcal{U}) \to H^1(\mathcal{U})$ is a continuous linear operator, the equalities in (4.12) and (4.13) together imply that, for any $w \in H^2(\mathcal{U})$

$$\frac{1}{2\pi} \int_0^{2\pi} ((\Sigma w)(\mathrm{e}^{\mathrm{i}t}), u_0) \mathrm{d}t = 0. \tag{4.14}$$

Letting $w = T_B u_0$ from (4.14) we have

$$0 = \frac{1}{2\pi} \int_0^{2\pi} ((\Sigma w)(\mathrm{e}^{\mathrm{i}t}), u_0) \mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} (B(\mathrm{e}^{\mathrm{i}t})^{-1} B(\mathrm{e}^{\mathrm{i}t}) u_0, u_0) \mathrm{d}t = \|u_0\|^2 = 0.$$

Thus $u_0 = 0$, and consequently, $\Phi(0)$ has dense range. We have thus shown that $\Phi(0)$ is bounded below and has dense range, and is therefore invertible. This finishes the proof of the lemma. ∎

Proof of Theorem 4.1. By Lemma 4.2, we have

$$\Gamma(\mathrm{e}^{\mathrm{i}t}) = B(\mathrm{e}^{\mathrm{i}t})^* B(\mathrm{e}^{\mathrm{i}t}), \tag{4.15}$$

where B is the maximal outer spectral factor of Γ. Lemma 4.3 yields that $B(\lambda)^{-1}$ exists for all $\lambda \in \mathbb{D}$ and the function $\lambda \to B(\lambda)^{-1}$ is in $H^2(\mathcal{U},\mathcal{U})$. Take $\Phi(\lambda) = B(\lambda)^{-1}B(0)$ and note that $\Phi \in H^2(\mathcal{U},\mathcal{U})$. The proof of the theorem is now complete. ∎

5. FINITE DIMENSIONAL CASE

In [2] it was proven that if a is a scalar-valued positive function on the unit circle (i.e., $\mathcal{U}$ is one-dimensional), then there exists an operator V on H^2 satisfying $VT_a = T_a S$ if and only if the function $\frac{1}{a}$ is in L^1. In this section, we obtain a complete analogue of that result in case $\mathcal{U}$ is finite dimensional. To be precise, we show that if dimension of $\mathcal{U}$ is finite, there exists an operator V such that $VT_\Gamma = T_\Gamma S$ if and only if $\Gamma(\mathrm{e}^{\mathrm{i}t})^{-1}$ exists a.e. in t and $\|\Gamma(\cdot)^{-1}\|$ is in L^1. The sufficiency of this condition was already established in Theorem 4.1. We now show its necessity if the dimension of $\mathcal{U}$ is finite.

THEOREM 5.1. *Let $\mathcal{U}$ be finite-dimensional and Γ in $L^\infty(\mathcal{U},\mathcal{U})$ with $\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant 0$ and such that* $\ker T_\Gamma = \{0\}$. *If there exists an operator V satisfying $VT_\Gamma = T_\Gamma S$ then $\Gamma(\mathrm{e}^{\mathrm{i}t}) = B(\mathrm{e}^{\mathrm{i}t})^* B(\mathrm{e}^{\mathrm{i}t})$ a.e. in t for some outer function B in $H^\infty(\mathcal{U},\mathcal{U})$ such that $B(\lambda)^{-1}$ exists $(\lambda \in \mathbb{D})$ with $B(\cdot)^{-1} \in H^2(\mathcal{U},\mathcal{U})$. Moreover, $\Gamma(\mathrm{e}^{\mathrm{i}t})^{-1}$ exists a.e. in t and $\|\Gamma(\cdot)^{-1}\| \in L^1(\mathbb{T})$.*

Proof. Let B be the maximal outer spectral factor of Γ. Applying Theorem 3.2 we see that there exists a function $\Phi \in H^2(\mathcal{U},\mathcal{U})$ such that (3.1) holds. Recall that Ω defined in (2.6) is an inner function and $\mathcal{M} = T_\Omega H^2(\mathcal{Y})$ where $\mathcal{Y}$ is an adequate Hilbert space with dimension less than or equal to that of $\mathcal{U}$ and that $\Phi u \in \mathcal{M}$ for all $u \in \mathcal{U}$. This implies that

$$\Phi(\lambda) = \Omega(\lambda)\Psi(\lambda) \quad (\lambda \in \mathbb{D}), \tag{5.1}$$

for some $\Psi \in H^2(\mathcal{U},\mathcal{Y})$. Since $\Phi(0) = I_{\mathcal{U}}$, from (5.1) it follows that $I_{\mathcal{U}} = \Omega(0)\Psi(0)$ which immediately implies that $\Omega(0)\mathcal{Y} = \mathcal{U}$. Consequently, dimension of $\mathcal{Y}$ equals dimension of $\mathcal{U}$. Since Ω is inner and $\mathcal{U}$ is finite dimensional it follows that $\Omega(\mathrm{e}^{\mathrm{i}t})$ is unitary a.e. in t. From (2.6) and (2.8), it follows that

$$(\Gamma(\mathrm{e}^{\mathrm{i}t}) - B(\mathrm{e}^{\mathrm{i}t})^* B(\mathrm{e}^{\mathrm{i}t}))\Omega(\mathrm{e}^{\mathrm{i}t}) = 0 \quad \text{a.e. in } t.$$

Since $\Omega(\cdot)$ is unitary a.e. we must have

$$\Gamma(\mathrm{e}^{\mathrm{i}t}) = B(\mathrm{e}^{\mathrm{i}t})^* B(\mathrm{e}^{\mathrm{i}t}) \quad \text{a.e. in } t. \tag{5.2}$$

Recall that by Remark 3.5, the kernel of $B(0)$ is trivial and moreover, by the previous discussion, we know $\mathcal{U}$ and $\mathcal{Y}$ have the same (finite) dimension. So, $B(0)$ is invertible. From (3.1), it follows that $\Phi_1(\cdot) = \Phi(\cdot)B(0)^{-1}$ is the inverse of $B(\cdot)$. Since $\Phi \in H^2(\mathcal{U},\mathcal{U})$ so must be Φ_1 and by finite dimensionality, each entry in the matrix of Φ_1 must be in H^2. Therefore, $\Phi_1(\mathrm{e}^{\mathrm{i}t})$ exists a.e. and by (5.2)

$$\Gamma(\mathrm{e}^{\mathrm{i}t})^{-1} = \Phi_1(\mathrm{e}^{\mathrm{i}t})\Phi_1(\mathrm{e}^{\mathrm{i}t})^* \quad \text{a.e.} \tag{5.3}$$

Moreover, since each entry in the matrix of $\Phi_1 \in H^2$ from (5.3) it follows that each entry in the matrix $\Gamma(\cdot)^{-1}$ is in $L^1(\mathbb{T})$ and denoting by $\alpha_{ij}(\cdot)$ the $(i,j)^{\text{th}}$ entry of $\Gamma(\cdot)^{-1}$ we have

$$\|\Gamma(\mathrm{e}^{\mathrm{i}t})^{-1}\| \leqslant \Big(\sum_{i,j}|\alpha_{ij}(\mathrm{e}^{\mathrm{i}t})|^2\Big)^{1/2} \leqslant \sum_{ij}|\alpha_{ij}(\mathrm{e}^{\mathrm{i}t})| \in L^1(\mathbb{T}).$$

The first inequality in the above line follows from the fact that the norm of a matrix is dominated by its Hilbert-Schmidt norm, whereas, the second inequality is elementary. This finishes the proof of the theorem. ∎

Using Theorem 4.1 and Theorem 5.1, in case $\mathcal{U}$ is finite dimensional, we can restate our main theorem as follows.

THEOREM 5.2. *Let Γ be in $L^\infty(\mathcal{U},\mathcal{U})$ such that $\Gamma(\mathrm{e}^{\mathrm{i}t}) \geqslant 0$ a.e. in t and the kernel of T_Γ is $\{0\}$. Then the following statements are equivalent.*

(i) *There exists a bounded operator V on $H^2(\mathcal{U})$ such that $VT_\Gamma = T_\Gamma S$.*

(ii) *There exists an outer function B in $H^\infty(\mathcal{U},\mathcal{U})$ such that $\Gamma(\mathrm{e}^{\mathrm{i}t}) = B^*(\mathrm{e}^{\mathrm{i}t})B(\mathrm{e}^{\mathrm{i}t})$ a.e. in t. Moreover, $B(\lambda)^{-1}$ exists for λ in $\mathbb{D}$ and $B(\cdot)^{-1}$ is in $H^2(\mathcal{U},\mathcal{U})$.*

(iii) *$\Gamma(\mathrm{e}^{\mathrm{i}t})^{-1}$ exists a.e. in t and $\|\Gamma(\mathrm{e}^{\mathrm{i}t})^{-1}\|$ is in $L^1([0,2\pi))$.*

EXAMPLE 5.3. We provide an example to show that the sufficient condition in Theorem 4.1 is not necessary in general. Let $\{r_n\}_{n=0}^\infty$ be a dense sequence of rational numbers in the interval $[0,2\pi]$. For $n \geqslant 0$ and $\lambda \in \mathbb{D}$ define the outer functions in H^∞

$$b_n(\lambda) = \exp\Big[\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}\theta}+\lambda}{\mathrm{e}^{\mathrm{i}\theta}-\lambda}k_n(\theta)\mathrm{d}\theta\Big],\ k_n(\theta) = \log|\theta - r_n|^{1/4}\ (\theta \in [0,2\pi]). \tag{5.4}$$

The functions in (5.4) are well defined since $k_n \in L^1[0,2\pi]$, $n = 0,1,2,\ldots$. The functions b_n, $n \geqslant 0$ defined in (5.4) are outer functions (see [6]) and

$$|b_n(\mathrm{e}^{\mathrm{i}\theta})| = |\theta - r_n|^{1/4} \quad \text{a.e. in } t \tag{5.5}$$

which implies that

$$b_n \in H^\infty, \quad \|b_n\| \leqslant 2\pi \quad (n \geqslant 0). \tag{5.6}$$

Note also that the function $\phi_n = \frac{1}{b_n}$ is analytic in $\mathbb{D}$ and by (5.5), for all $n \geqslant 0$, we have

$$\int_0^{2\pi} |\phi_n(e^{in\theta})|^2 d\theta = \int_0^{2\pi} \frac{1}{|\theta - r_n|^{1/2}} d\theta \leqslant \int_{-2\pi}^{4\pi} \frac{1}{\sqrt{|u|}} du := K < \infty \quad n \geqslant 0,$$

where K is a constant independent of n. Consequently,

$$\phi_n \in H^2, \quad \|\phi_n\|_{H^2} \leqslant K \quad (n \geqslant 0). \tag{5.7}$$

On the other hand, ϕ_n is not in H^∞. Moreover, since $\{r_n\}$ is dense in $[0, 2\pi]$, by (5.5) we have

$$\sup_n |\phi_n(e^{i\theta})| = \sup_n \frac{1}{|\theta - r_n|^{1/4}} = \infty \quad \text{a.e. in } t \tag{5.8}$$

Let $\mathcal{U} = \ell^2 = \left\{ \mathbf{u} = (u_0, u_1, \ldots) : \|\mathbf{u}\|^2 = \sum_{i=0}^{\infty} |u_i|^2 < \infty \right\}$. Note that due to (5.6), the operator valued function B defined by

$$B(\lambda) = \text{diagonal}((b_n(\lambda))) \quad (\lambda \in \mathbb{D}) \tag{5.9}$$

is in $H^\infty(\mathcal{U}, \mathcal{U})$ and is outer since each b_n is an outer function in H^∞. Let

$$\Phi(\lambda) = \text{diagonal}((\phi_n(\lambda))) \quad (\lambda \in \mathbb{D}). \tag{5.10}$$

Clearly, $B(\lambda)\Phi(\lambda) = I$ for all $\lambda \in \mathbb{D}$. Moreover, for $\mathbf{u} \in \mathcal{U}$ we have

$$\|\Phi(\lambda)\mathbf{u}\|^2_{H^2(\mathcal{U})} = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^{\infty} |\phi_n(e^{it})|^2 |u_n|^2 dt = \sum_{n=0}^{\infty} |u_n|^2 \frac{1}{2\pi} \int_0^{2\pi} |\phi_n(e^{it})|^2 dt \leqslant \frac{K}{2\pi} \|\mathbf{u}\|^2,$$

where to obtain the last inequality above, we used (5.7). This immediately shows that $\Phi \in H^2(\mathcal{U}, \mathcal{U})$. On the other hand, $\Phi(e^{it})$ exists a.e. in t for $t \in [0, 2\pi]$ but due to (5.8), $\|\Phi(e^{it})\| = \infty$ a.e. in t. Thus if we define $\Gamma(e^{it}) = B(e^{it})^* B(e^{it})$ then for this Γ there exists a Φ satisfying (3.1) but $\|\Gamma(e^{it})^{-1}\| = \infty$ a.e. in t which means that $\|\Gamma(\cdot)^{-1}\| \notin L^1$.

6. SOME RELATED PROBLEMS

Comparing the results in case the fiber space is finite dimensional (namely, Theorem 5.2) with the infinite dimensional case (namely, Theorem 3.2) one is led naturally to the following questions.

(i) In view of Theorem 5.2, we can ask the following question: With Γ as in Theorem 3.2, if there exists an operator V on $H^2(\mathcal{U})$ satisfying $VT_\Gamma = T_\Gamma S$, does that imply that Γ can be factorized as in part (ii) of Theorem 5.2?

(ii) Is Φ in Theorem 3.2 outer? In other words, is it true that $H^2(\mathcal{U})$ is the closed linear space generated by the sets

$$S^n(\Phi(\cdot)\mathcal{U}), \quad n = 0, 1, 2, \ldots?$$

Note that in view of Proposition 3.6, a positive answer to the question in (ii) automatically implies a positive answer to the preceding one.

Due to Theorem 3.2, for an operator V to exist satisfying $VT_\Gamma = T_\Gamma S$, it is necessary that there exists an operator valued analytic function Φ in $H^2(\mathcal{U},\mathcal{U})$ such that

$$B(\lambda)\Phi(\lambda) = B(0), \quad \Phi(0) = I. \tag{6.1}$$

The condition $\Phi(0) = I$ immediately implies that $\Phi(\lambda)$ is invertible for $|\lambda|$ small enough and the inverse is analytic in a neighborhood of the origin. Consequently, taking $\Psi(\lambda) = \Phi(\lambda)^{-1}$ we have

$$B(\lambda) = B(0)\Psi(\lambda) \quad (|\lambda| \text{ small}). \tag{6.2}$$

It is natural to ask if the conditions $B(\cdot)$ outer in $H^\infty(\mathcal{U},\mathcal{U})$, $\ker B(0) = \{0\}$ and (6.2) together imply that $\Psi(\lambda)$ could be extended analytically in the whole unit disc. The following example ([1]) shows that this may not be possible.

EXAMPLE 6.1. Let $\mathcal{U} = \ell^2_+(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C} \oplus \cdots$ and let $B \in H^\infty(\mathcal{U},\mathcal{U})$ be the diagonal matrix defined by

$$B(\lambda) = \bigoplus_{n=0}^{\infty} b_n(\lambda), \quad b_n(\lambda) = \Big(\frac{1+\lambda^n}{2}\Big)^{2^n} \quad (\lambda \in \mathbb{D}).$$

Clearly, B is an outer function in $H^\infty(\mathcal{U},\mathcal{U})$. If an analytic function Ψ exists such that

$$B(\lambda) = B(0)\Psi(\lambda) \quad (\lambda \in \mathbb{D}) \tag{6.3}$$

then we must have

$$\Psi(\lambda) = \bigoplus_{n=0}^{\infty} \psi_n(\lambda), \quad \psi_n(\lambda) = (1+\lambda^n)^{2^n} \quad (\lambda \in \mathbb{D}). \tag{6.4}$$

Note that for $|\lambda| \leqslant \frac{1}{2}$, we have

$$|\psi_n(\lambda)| \leqslant \Big(1 + \Big(\frac{1}{2}\Big)^n\Big)^{2^n}. \tag{6.5}$$

Since $\lim_{n\to\infty} \big(1 + (\frac{1}{2})^n\big)^{2^n} = e$, from (6.5), it follows that

$$\sup_{|\lambda|\leqslant\frac{1}{2}} \|\Psi(\lambda)\| < \infty.$$

This shows that the factorization in (6.3) holds for all $|\lambda| \leqslant \frac{1}{2}$ where Ψ is the analytic operator valued function defined in (6.4).

However, if λ is a real number between $\frac{1}{2}$ and 1,

$$\psi_n(\lambda) = [(1+\lambda^n)^{(1/\lambda)^n}]^{(2\lambda)^n} \to \infty.$$

This means that for all real λ, $\frac{1}{2} < \lambda < 1$, we have $\|\Psi(\lambda)\| = \infty$. Consequently, the operator valued analytic function $\Psi(\cdot)$ cannot be extended on the entire unit disc such that (6.3) holds.

It is however easy to see that in Example 6.1 presented above, there does not exist a function Φ in $H^2(\mathcal{U},\mathcal{U})$ satisfying (6.1). One could hope that if there exists a function Φ in $H^2(\mathcal{U},\mathcal{U})$ satisfying (6.1), then there exists a function Ψ in $H^\infty(\mathcal{U},\mathcal{U})$ satisfying (6.3) for all λ in $\mathbb{D}$. Again, in the the next example ([1]) we show that this is not necessarily the case.

EXAMPLE 6.2. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ be any two sequences of strictly positive real numbers such that

$$\min\{\alpha_n, \beta_n\} > 0, \ \max\left\{\alpha_n, \frac{\alpha_n}{\beta_n}\right\} < 1 (n \in \mathbb{N}), \text{ and } \lim_{n\to\infty} \max\{\alpha_n, \beta_n\} = 0. \tag{6.6}$$

For instance, one can take $\beta_n = \frac{1}{n+1}$, $\alpha_n = \frac{1}{2}\beta_n$, $n = 1, 2, \ldots$. For λ in $\mathbb{D}$ and $n = 1, 2, \ldots$, define

$$b_n(\lambda) = \frac{\alpha_n + \beta_n}{2 + \beta_n} \frac{1 + \gamma_n \lambda}{1 + (\alpha_n - 1)\lambda}, \quad \text{where } \gamma_n := \frac{\alpha_n - \beta_n + \alpha_n\beta_n}{\alpha_n + \beta_n}. \tag{6.7}$$

It is easy to check that

$$b_n(\lambda) = f_n(\theta(\lambda)) \tag{6.8}$$

where

$$f_n(\lambda) = \frac{\lambda + 1 + \beta_n}{2 + \beta_n} \quad \text{and} \quad \theta(\lambda) = \frac{\lambda - (1 - \alpha_n)}{1 - (1 - \alpha_n)\lambda}.$$

Since θ is inner and f_n is in H^∞ with $\|f_n\|_\infty \leqslant 1$, the relation (6.8) implies that b_n is in H^∞ with $\|b_n\|_\infty \leqslant 1$. As in Example 6.1, let $\mathcal{U} = \ell^2_+(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C} \oplus \cdots$ and let $B \in H^\infty(\mathcal{U},\mathcal{U})$ be the diagonal matrix defined by

$$B(\lambda) = \bigoplus_{n=0}^{\infty} b_n(\lambda).$$

For all $\lambda \in \mathbb{D}$ and $n = 1, 2, \ldots$

$$\phi_n(\lambda) = \frac{b_n(0)}{b_n(\lambda)} = \frac{1 + (\alpha_n - 1)\lambda}{1 + \gamma_n\lambda}, \quad \psi_n(\lambda) = \frac{1}{\phi_n(\lambda)}. \tag{6.9}$$

Since by our choice, $\alpha_n < 1$ (see (6.6)) it is easy to verify that $|\gamma_n| < 1$ for all $n \geqslant 1$. This immediately implies that for each n, the linear, fractional transformation ϕ_n extends analytically to a neighborhood of the disc. This means that ϕ_n takes the unit circle to another circle in the complex plane and moreover, the points $\phi_n(1)$ and $\phi_n(-1)$ are diametrically opposite points on that image circle. However, both $\phi_n(1)$ and $\phi_n(-1)$ are real and since $\alpha_n < 1$, we also have $\phi_n(1) = \frac{\alpha_n}{1+\gamma_n} < \phi_n(-1) = \frac{2-\alpha_n}{1-\gamma_n}$, $n = 1, 2, \ldots$. Thus, due to the geometry described above and the maximum modulus principle, we have $\sup_{|\lambda|\leqslant 1} |\phi_n(\lambda)| = \frac{2-\alpha_n}{1-\gamma_n}$ and $\inf_{|\lambda|\leqslant 1} |\phi_n(\lambda)| = \frac{\alpha_n}{1+\gamma_n}$, $n = 1, 2, \ldots$ that is,

(6.10) $$\sup_{|\lambda|\leqslant 1} |\phi_n(\lambda)| = \frac{2-\alpha_n}{1-\gamma_n} = 1 + \frac{\alpha_n}{\beta_n}$$

and

(6.11) $$\sup_{|\lambda|\leqslant 1} |\psi_n(\lambda)| = \frac{1+\gamma_n}{\alpha_n} = \frac{2+\beta_n}{\alpha_n+\beta_n}, \quad n = 1, 2, \ldots.$$

Define the diagonal matrix

$$\Phi(\lambda) = \bigoplus_{n=1}^{\infty} \phi_n(\lambda) \quad (\lambda \in \mathbb{D}).$$

Clearly, for all $\lambda \in \mathbb{D}$, $B(\lambda)\Phi(\lambda) = B(0)$ and on account of (6.10) and (6.6) it follows that $\|\Phi\|_\infty \leqslant 2$.

On the other hand, if there exists a function Ψ in $H^\infty(\mathcal{U},\mathcal{U})$ such that $B(\lambda) = B(0)\Psi(\lambda)$ for all λ in $\mathbb{D}$, then we must have

$$\Psi(\lambda) = \bigoplus_{n=1}^{\infty} \psi_n(\lambda) \quad (\lambda \in \mathbb{D}),$$

where ψ_n is as in (6.9). This is clearly not possible on account of (6.11) and the fact that $\alpha_n + \beta_n \to 0$ as $n \to \infty$.

As yet we do not know if Example 6.2 can be modified to display the singular feature of Example 6.1. However, from the discussion in Section 4 and Section 5, it follows that in case $\dim(\mathcal{U}) < \infty$, if there exists an operator valued analytic function Φ in $H^2(\mathcal{U},\mathcal{U})$ satisfying (6.1), then

(6.12) $$B(\lambda) = B(0)\Psi(\lambda) \quad (\lambda \in \mathbb{D}) \text{ for some } \Psi \text{ in } H^\infty(\mathcal{U},\mathcal{U}).$$

It is thus natural to try to show that the questions in (i) and (ii) above have a positive answer in case (6.12) holds. In this situation, the questions in (i) and (ii) can be reformulated as follows. Assume that (6.12) holds and there exists Φ in $H^2(\mathcal{U},\mathcal{U})$ such that (6.1) is satisfied. If there exists a function $\Phi \in H^2(\mathcal{U},\mathcal{U})$ as in Theorem 3.2, then we have

(6.13) $$\Psi(\lambda)\Phi(\lambda) = \Phi(\lambda)\Psi(\lambda) = I \quad (\lambda \in \mathbb{D}).$$

Indeed, since $\ker B(0) = \{0\}$ we have $\Psi(\lambda)\Phi(\lambda) = I$ $(\lambda \in \mathbb{D})$. However, since $\Phi(0) = I$, $\Phi(\lambda)^{-1}$ exists for λ sufficiently close to zero. By uniqueness of inverse, we must have $\Psi(\lambda) = \Phi(\lambda)^{-1}$ for λ close to zero, which implies that

(6.14) $$\Phi(\lambda)\Psi(\lambda) = I, \quad |\lambda| \text{ small}.$$

By analyticity, we have (6.14) for all λ in $\mathbb{D}$. In other words, we have a function $\Psi \in H^\infty(\mathcal{U},\mathcal{U})$ and a function $\Phi \in H^2(\mathcal{U},\mathcal{U})$ such that

$$\Psi(\lambda)\Phi(\lambda) = \Phi(\lambda)\Psi(\lambda) = I \quad (\lambda \in \mathbb{D}).$$

Does this imply that Φ is outer where, in this case, the notion of outer is as described in question (ii)?

Acknowledgements. The first named author's research is supported in part by Junior Research Fellowship (Summer 2003) from UNC-Charlotte.

REFERENCES

[1] H. BERCOVICI, private communication.

[2] A. BISWAS, C. FOIAŞ, A.E. FRAZHO, Weighted commutant lifting, *Acta Sci. Math. (Szeged)* **65**(1999), 805–834.

[3] A. BISWAS, C. FOIAŞ, A.E. FRAZHO, Weighted variants of the three chains completion theorem, in *Recent Advances in Operator Theory and Related Topics (Széged, 1999)*, Oper. Theory Adv. Appl., vol. 127, Birkhäuser, Basel 2001, pp. 127–144.

[4] C. FOIAŞ, A.E. FRAZHO, *The Commutant Lifting Approach to Interpolation Problems,* Oper. Theory Adv. Appl., vol. 44, Birkhäuser, Basel 1990.

[5] C. FOIAŞ, A.E. FRAZHO, I. GOHBERG, M.A. KAASHOEK, *Metric Constrained Interpolation, Commutant Lifting and Systems*, Oper. Theory Adv. Appl., vol. 100, Birkhäuser, Basel 1998.

[6] K. HOFFMAN, *Banach Spaces of Analytic Functions,* Englewood Cliffs, Prentice Hall, New Jersey 1962.

[7] B. SZ.-NAGY, C. FOIAŞ, *Harmonic Analysis of Operators on Hilbert Space,* North Holland Publishing Co., Amsterdam-Budapest 1970.

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Received November 10, 2003.