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A LARGE WEAK OPERATOR CLOSURE FOR THE ALGEBRA GENERATED BY TWO ISOMETRIES

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ABSTRACT. Given two (or n) isometries on a Hilbert space $\mathcal{H}$, such that their ranges are mutually orthogonal, one can use them to generate a C^*-algebra. If the ranges sum to $\mathcal{H}$, then this C^*-algebra, the Cuntz C^*-algebra $\mathcal{O}_n$, is unique up to $*$-isomorphism. However if we omit to close under the $*$ operation and merely consider the norm closed algebra generated by such isometries, that is certainly not unique; even if we take the weak operator topology (WOT) closure of such an algebra, it is still not unique. Among the possible WOT closures that one might conceivably get, the largest is the von Neumann algebra that will be obtained if it should chance to happen that the WOT closed algebra generated is, in fact, closed under hermitian conjugation after all. In this paper we show that this largest possible case can indeed happen (a problem which was posed by Davidson); we exhibit pairs of isometries S_0, S_1 with disjoint ranges, such that the WOT closed algebra generated by S_0 and S_1 is the whole of $\mathcal{B}(\mathcal{H})$.

KEYWORDS: *Isometry, Cuntz algebra, C^*-algebra, operator algebra, free semigroup.*

MSC (2000): Primary 47L80; Secondary 47L55.

1. INTRODUCTION

Following Davidson we define the *free semigroup algebra* generated by finitely many isometries $S_0, S_1, \ldots, S_{n-1}$ with orthogonal ranges to be the unital WOT-closed subalgebra $\overline{\mathcal{A}}^{\mathrm{WOT}}$, where $\mathcal{A} = \mathrm{Alg}\{S_0, S_1, \ldots, S_{n-1}\}$. In general, the archetypal example of this is the left regular representation of the free semigroup on n generators, with S_i being the representation of the ith generator. However, we are going to take $n = 2$, and we are going to need isometries S_0, S_1 with $S_0S_0^* + S_1S_1^* = I$; so for our purposes a standard example that's closer to the one we need is obtained by regarding $L_2([0, \frac{1}{2}])$ and $L_2([\frac{1}{2}, 1])$ as complementary subspaces of $L_2([0,1])$ in the obvious way. We may then define S_0 to be the natural isometry $L_2([0,1]) \to L_2([0, \frac{1}{2}])$ with $(S_0 f)(x) = \sqrt{2} f(2x)$, while S_1 can be the natural isometry $L_2([0,1]) \to L_2([\frac{1}{2}, 1])$ with $(S_1 f)(x) = \sqrt{2} f(2x - 1)$.

What we do in the paper is (up to unitary equivalence) a matter of making this definition slightly more complicated with a phase change $e^{i\Theta(x)}$, i.e. defining

$$(S_0 f)(x) = \sqrt{2}e^{i\Theta(x)} f(2x) \tag{1.1}$$

and

$$(S_1 f)(x) = \sqrt{2}e^{i\Theta(x)} f(2x-1) \tag{1.2}$$

for a suitable measurable function Θ. However, the function Θ is very much dependent on the binomial expansion of $x \in [0,1]$, so in fact we prefer to couch everything in terms of a binomial expansion $\mathbf{x} \in \Omega = \{0,1\}^{\mathbb{N}}$.

In this paper, therefore, the underlying Hilbert space $\mathcal{H}$ is in fact $L_2(\Omega, \mathbb{P})$, where the probability measure $\mathbb{P}(E) = \lambda(\beta(E))$, λ Lebesgue measure on $[0,1]$ and $\beta : \Omega \to [0,1]$ the binomial expansion with $\beta(\mathbf{x}) = \sum\limits_{n=1}^{\infty} 2^{-n} x_n$. The appropriate unitary equivalent to (1.1) and (1.2) is given below in equation (4.3).

For a general introduction to the theory of free semigroup algebras, and a much more comprehensive set of references, we refer the reader to Davidson's forthcoming survey article [4]. The Cuntz algebra itself is studied in many papers starting with Cuntz [2]; an application to theoretical physics is developed by Bratteli and Jorgensen ([1] and several other papers). The standard examples $\mathfrak{L}_n$ of free semigroup algebras have been very much studied, yielding information about their structure such as the hyperreflexivity result of Davidson and Pitts [6]. In the more usual case when the free semigroup algebra is *not* a von Neumann algebra, it sometimes has analytic properties; these are investigated by Popescu ([9] and several other papers). Ideas involving the relations $S_i^* S_i = I$, $S_i^* S_j = 0$ for $i \neq j$ on general Banach spaces are used in some of the ramifications of the celebrated Gowers-Maurey constructions (see [7], [8]). In those papers the operators involved are defined using "spreads", i.e. increasing bijections between infinite subsets of $\mathbb{N}$. The same relations also turn up in an investigation of traces in the Banach space setting by Dales, Laustsen and Read [3].

2. PRELIMINARY DEFINITIONS

There are two preliminary choices that we make before defining our operators S_0 and S_1. First, we choose (once and for all) a bijection $\psi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, such that writing $\psi(n) = (p_n, q_n)$, we have $p_n \leqslant n$, $q_n \leqslant n$ for all n.

Second, we choose a strictly increasing sequence $(b_k)_{k=1}^{\infty}$ of positive integers, which must satisfy the mild growth conditions (6.3) and (6.13). The reader may easily see that $b_k = 10k^2$ (say) satisfies both these conditions, and may assume $b_k = 10k^2$ throughout if desired; but all you really need is a sequence — **any** sequence — that grows moderately quickly.

3. NOTATION

Let $\Omega = \{0,1\}^{\mathbb{N}}$. Let $X_i : \Omega \to \{0,1\}$ be the map that picks the ith coordinate, and more generally for $0 \leqslant n < m$ let $\tau_{n,m} : \Omega \to \{0,1\}^{m-n}$ be the map sending the sequence $\mathbf{x} = (x_i)_{i=1}^{\infty} \in \Omega$ to $(x_{n+1}, x_{n+2}, \dots, x_m) \in \{0,1\}^{n-m}$. For convenience we'll write τ_n for $\tau_{0,n}$. (It is one of the personal conventions of the present author that Greek "τ" often stands for "truncation").

Let $\mathbb{P}$ be the Borel probability measure on Ω such that with respect to $\mathbb{P}$, the X_i are independent, and for all i, $\mathbb{P}(X_i = 1) = \frac{1}{2}$. Obviously this is equivalent to the definition $\mathbb{P}(E) = \lambda(\beta(E))$ given in the Introduction.

Let $\mathcal{F}$ denote the Borel sets of Ω ; let $\mathcal{F}_{n,m} \subset \mathcal{F}$ denote the finite σ-field generated by $\{X_i : i = n+1, \dots, m\}$; and let $\mathcal{F}_n = \mathcal{F}_{0,n}$.

Let $\mathcal{H}$ be the Hilbert space $L_2(\Omega, \mathcal{F}, \mathbb{P})$. Let $\beta : \Omega \to [0,1]$ be the binomial expansion $\mathbf{x} \to \sum\limits_{i=1}^{\infty} 2^{-i} x_i$, and let $\beta_{n,m} : \Omega \to [0,1]$ be the related map with $\beta_{n,m}(\mathbf{x}) = \sum\limits_{i=n+1}^{m} 2^{n-i} x_i$.

If $\mathbf{x} \in \{0,1\}^m$ ($m \in \mathbb{N}$) and $\mathbf{y} \in \{0,1\}^n$ ($n \in \mathbb{N} \cup \{\infty\}$) we write $(\mathbf{x}|\mathbf{y})$ for the sequence $\mathbf{z} \in \{0,1\}^{n+m}$ with $z_i = x_i$ ($i \leqslant m$) or y_{i-m} ($i > m$). We write $\mathbf{0}_n, \mathbf{1}_n$ (respectively $\mathbf{0}, \mathbf{1}$) for the sequences in $\{0,1\}^n$ (respectively Ω) consisting of all zeros and all ones, and we write just 0,1 for the sequences $\mathbf{0}_1, \mathbf{1}_1$. Of particular significance to us is the sequence $(0|\mathbf{1}_{b-2}|0) \in \{0,1\}^b$. We shall use it to "code" the function Θ in the main definition that follows.

For $l \in \mathbb{N}$, and $i \geqslant 0$ we define $\Gamma_{i,l} \subset \Omega$ to be the set of $\mathbf{x} \in \Omega$ such that we have

$$\tau_{i,i+b_l}(\mathbf{x}) = (0|\mathbf{1}_{b_l-2}|0); \tag{3.1}$$

if $i < 0$ we define $\Gamma_{i,l} = \emptyset$. Note that for $i \geqslant 0$ we have

$$\mathbb{P}(\Gamma_{i,l}) = 2^{-b_l}. \tag{3.2}$$

For $N, l \in \mathbb{N}$ we then define

$$G_{N,l} = \bigcup_{i=N+2-p_l-b_l}^{N+1-b_l} \Gamma_{i,l}. \tag{3.3}$$

Note that $G_{N,l} \in \mathcal{F}_{N+1}$ and

$$\mathbb{P}(G_{N,l}) \leqslant p_l \cdot 2^{-b_l} \leqslant l \cdot 2^{-b_l}, \tag{3.4}$$

because as i ranges from $N - p_l - b_l + 2$ to $N - b_l + 1$, there are p_l posible values of i, each involving an event of probability either zero or 2^{-b_l}.

Also, let

$$E_{N,k} = \Gamma_{N,k} \setminus \Big(\bigcup_{l=k}^{\infty} G_{N,l}\Big) \in \mathcal{F}_{N+b_k}. \tag{3.5}$$

We then define $\mathcal{E}_{N,k} = \tau_{N+b_k}(E_{N,k}) \subset \{0,1\}^{N+b_k}$, so that the number $|\mathcal{E}_{N,k}|$ of elements in $\mathcal{E}_{N,k}$ is $2^{N+b_k}\mathbb{P}(E_{N,k})$.

If $r \in \mathbb{N}$ and $\delta \in \{0,1\}^r$ we write Ω_δ for $\{\mathbf{x} \in \Omega : \tau_r(\mathbf{x}) = \delta\}$; and if $r \leqslant N + b_k$ we write $E_{N,k,\delta} = E_{N,k} \cap \Omega_\delta$, $\mathcal{E}_{N,k,\delta} = \tau_{N+b_k}(E_{N,k,\delta})$. Once again, $|\mathcal{E}_{N,k,\delta}| = 2^{N+b_k}\mathbb{P}(E_{N,k,\delta})$, and we shall get an estimate for this number later on in Lemma 5.3. In the meanwhile, we note that in the above situation, if $\mathbf{x} \in \Gamma_{N,k} \cap \Omega_\delta$ then exactly $r + b_k$ of the coordinates x_i are determined (the ones from 1 to r and the ones from $N+1$ to $N+b_k$), hence $\mathbb{P}(\Gamma_{N,k} \cap \Omega_\delta) = 2^{-r-b_k}$; and

$$|\mathcal{E}_{N,k,\delta}| \leqslant 2^{N+b_k}\mathbb{P}(\Gamma_{N,k} \cap \Omega_\delta) = 2^{N-r}. \tag{3.6}$$

4. THE MAIN DEFINITION

DEFINITION 4.1. For each $k \in \mathbb{N}$, let us define a map $\theta_k : \Omega \to \mathbb{R}$ by

$$\theta_k(\mathbf{x}) = \begin{cases} 0 & \text{unless } \mathbf{x} \in \Gamma_{0,k}, \\ 2\pi \cdot \beta_{b_k, b_k+p_k}(\mathbf{x}) & \text{if } \mathbf{x} \in \Gamma_{0,k}. \end{cases} \tag{4.1}$$

Then, define the map $\Theta : \Omega \to \mathbb{R}$ by

$$\Theta(\mathbf{x}) = \sum_{k=1}^{\infty} q_k \theta_k(\mathbf{x}). \tag{4.2}$$

Note that because the sets $\Gamma_{0,k}$ are disjoint, the sum (4.2) never has more than one nonzero term; and for future reference, we note that if $x_1 = 1$ then $\Theta(\mathbf{x}) = 0$. There is no doubt about the convergence of the sum (4.2).

Let $L : \Omega \to \Omega$ denote the left shift, with $(L\mathbf{x})_i = x_{i+1}$ for all $i \in \mathbb{N}$ and $\mathbf{x} \in \Omega$; and let $\varepsilon \in \{0,1\}$. We define an isometry $S_\varepsilon \in \mathcal{B}(\mathcal{H})$ by

$$(S_\varepsilon f)(\mathbf{x}) = \begin{cases} \sqrt{2}\mathrm{e}^{\mathrm{i}\Theta(\mathbf{x})} f(L\mathbf{x}) & \text{if } x_1 = \varepsilon, \\ 0 & \text{otherwise}; \end{cases} \tag{4.3}$$

$(f \in \mathcal{H}, \mathbf{x} \in \Omega)$.

We shall show that the subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ generated by S_0 and S_1 is WOT-dense in $\mathcal{B}(\mathcal{H})$(in fact weak* dense). The general outline of the proof is to exhibit a collection of finite rank operators in the weak* closure $\overline{\mathcal{A}}^{\mathrm{w}^*}$, such that their norm closed linear span contains every compact operator. Since the compact operators are weak* dense in $\mathcal{B}(\mathcal{H})$, this establishes that $\overline{\mathcal{A}}^{\mathrm{w}^*} = \overline{\mathcal{A}}^{\mathrm{WOT}} = \mathcal{B}(\mathcal{H})$.

5. PROVING THE MAIN RESULT

Having defined our operators S_0 and S_1, we make the obvious remarks that they are indeed isometries with orthogonal ranges, and $S_0^*S_0 + S_1^*S_1 = I$. Indeed,

$S_\varepsilon S_\varepsilon^*$ is the projection onto the set of all functions supported on $\{\mathbf{x} \in \Omega : x_1 = \varepsilon\}$ $(\varepsilon = 0, 1)$.

If $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{0,1\}^n$ is a finite sequence, we define

$$S_{\boldsymbol{\varepsilon}} = S_{\varepsilon_1} \circ S_{\varepsilon_2} \circ \cdots \circ S_{\varepsilon_n}; \tag{5.1}$$

note that

$$(S_{\boldsymbol{\varepsilon}} f)(\mathbf{x}) = \begin{cases} 2^{\frac{n}{2}} \mathrm{e}^{\mathrm{i}\sum_{=r}^{0} n-1 \Theta(L^r \mathbf{x})} \cdot f(L^n \mathbf{x}) & \text{if } \tau_n(\mathbf{x}) = \boldsymbol{\varepsilon}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Our next step is to define certain linear combinations of the products $S_{\boldsymbol{\varepsilon}}$, which will be used to establish that the WOT closure (indeed, the w* closure) of $\mathcal{A}$ is all of $\mathcal{B}(\mathcal{H})$.

DEFINITION 5.1. For each $N, k \in \mathbb{N}$ define a function $\phi_{N,k} : \{0,1\}^{N+b_k} \to \mathbb{R}$,

$$\phi_{N,k}(\boldsymbol{\varepsilon}) = \sum_{i=0}^{N-1} \Theta(L^i(\boldsymbol{\varepsilon}|\mathbf{0})). \tag{5.3}$$

For each $r < N$ and $\boldsymbol{\delta} \in \{0,1\}^r$ we then define an operator $T_{N,k,\boldsymbol{\delta}} \in \mathcal{A}$ by

$$T_{N,k,\boldsymbol{\delta}} = 2^{\frac{1}{2}(b_k - N)} \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}_{N,k,\boldsymbol{\delta}}} \mathrm{e}^{-\mathrm{i}\phi_{N,k}(\boldsymbol{\varepsilon})} \cdot S_{\boldsymbol{\varepsilon}}. \tag{5.4}$$

Now each $S_{\boldsymbol{\varepsilon}}$ is an isometry; and for different $\boldsymbol{\varepsilon}$ of the same length they have orthogonal ranges, so

$$\Big\| \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}_{N,k,\boldsymbol{\delta}}} S_{\boldsymbol{\varepsilon}} \Big\| \leqslant \sqrt{|\mathcal{E}_{N,k,\boldsymbol{\delta}}|} \leqslant 2^{\frac{1}{2}(N-r)} \tag{5.5}$$

by (3.6), whence

$$\|T_{N,k,\boldsymbol{\delta}}\| \leqslant 2^{\frac{1}{2}(b_k - r)}, \tag{5.6}$$

a result that is, significantly, independent of N. So a weak* limit point of the operators $T_{N,k,\boldsymbol{\delta}}$ must exist as $N \to \infty$. We shall find out what it is. Let us find such a limit point and call it $T_{k,\boldsymbol{\delta}}$ — for the moment passing over the small matter that there is, in fact, just one possible limit point because the sequence is weak* convergent.

For $r \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \{0,1\}^r$, let's write $\chi_{r,\boldsymbol{\alpha}}$ for the characteristic function

$$\chi_{r,\boldsymbol{\alpha}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \tau_r(\mathbf{x}) = \boldsymbol{\alpha}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

Let us determine the product $\langle S_{\boldsymbol{\varepsilon}} \chi_{r,\boldsymbol{\alpha}}, \chi_{r,\boldsymbol{\gamma}} \rangle$ for each $\boldsymbol{\varepsilon} \in \mathcal{E}_{N,k}$, $r < N$ and $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \{0,1\}^r$. Now by (5.2),

$$S_{\varepsilon}\chi_{r,\boldsymbol{\alpha}}(\mathbf{x}) = \begin{cases} 2^{\frac{N+b_k}{2}} e^{i\sum_{j=0}^{N+b_k-1}\Theta(L^j\mathbf{x})}\chi_{r,\boldsymbol{\alpha}}(L^{N+b_k}\mathbf{x}) & \text{if } \tau_{N+b_k}(\mathbf{x}) = \boldsymbol{\varepsilon}, \\ 0 & \text{otherwise}, \end{cases} \tag{5.8}$$

$$= \begin{cases} 2^{\frac{N+b_k}{2}} e^{i\sum_{j=0}^{N+b_k-1}\Theta(L^j\mathbf{x})} & \text{if } \tau_{N+b_k+r}(\mathbf{x}) = (\boldsymbol{\varepsilon}|\boldsymbol{\alpha}), \\ 0 & \text{otherwise}. \end{cases}$$

We can deal with most of the $e^{i\Theta}$ factors as follows:

LEMMA 5.2. *For all $N, k \in \mathbb{N}$ and $\boldsymbol{\varepsilon} \in \mathcal{E}_{N,k}$, and for all $\mathbf{y} \in \Omega$, we have*

$$\sum_{i=0}^{N-1} \Theta(L^i(\boldsymbol{\varepsilon}|\mathbf{y})) = \sum_{i=0}^{N-1} \Theta(L^i(\boldsymbol{\varepsilon}|\mathbf{0})) = \phi_{N,k}(\boldsymbol{\varepsilon}). \tag{5.9}$$

Furthermore, $\Theta(L^i(\boldsymbol{\varepsilon}|\mathbf{y})) = 0$ for $N+1 \leqslant i < N+b_k-1$, and $\Theta(L^N(\boldsymbol{\varepsilon}|\mathbf{y})) = 2\pi q_k \sum_{j=1}^{p_k} 2^{-j}y_j$, and $\Theta(L^{N+b_k-1}(\boldsymbol{\varepsilon}|\mathbf{y})) = \Theta(0|\mathbf{y})$.

Obviously the last statement follows because every $\boldsymbol{\varepsilon} \in \mathcal{E}_{N,k}$ has length $N + b_k$ and has its last coordinate $\varepsilon_{N+b_k} = 0$; we defer the rest of the proof to the Appendix.

Using Lemma 5.2, we find that the second row in (5.8) is equal to

$$\begin{cases} 2^{\frac{N+b_k}{2}} e^{2\pi i q_k \sum_{j=1}^{p_k} 2^{-j}\alpha_j} \cdot e^{i\phi_{N,k}(\boldsymbol{\varepsilon})} \cdot e^{i\Theta(L^{N+b_k-1}\mathbf{x})} & \text{if } \tau_{N+b_k+r}(\mathbf{x}) = (\boldsymbol{\varepsilon}|\boldsymbol{\alpha}), \\ 0 & \text{otherwise}. \end{cases} \tag{5.10}$$

So for a vector $\mathbf{x} = (\boldsymbol{\varepsilon}|\boldsymbol{\alpha}|\mathbf{y})$ we have (using the last part of Lemma 5.2)

$$e^{-i\phi_{N,k}(\boldsymbol{\varepsilon})} S_{\boldsymbol{\varepsilon}}\chi_{r,\boldsymbol{\alpha}}(\mathbf{x}) = 2^{\frac{N+b_k}{2}} e^{2\pi i q_k \sum_{j=1}^{p_k} 2^{-j}\alpha_j} \cdot e^{i\Theta(0|\boldsymbol{\alpha}|\mathbf{y})}; \tag{5.11}$$

and $S_{\boldsymbol{\varepsilon}}\chi_{r,\boldsymbol{\alpha}}(\mathbf{x}) = 0$ for any vector $\mathbf{x}$ not of this form. Hence, the inner product $\langle e^{-i\phi_{N,k}(\boldsymbol{\varepsilon})} S_{\boldsymbol{\varepsilon}}\chi_{r,\boldsymbol{\alpha}}, \chi_{r,\gamma}\rangle$ is equal to zero if $(\gamma_1, \gamma_2, \dots, \gamma_r) \neq (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$. Otherwise, it is equal to

$$2^{-\frac{2r+N+b_k}{2}} e^{2\pi i q_k \sum_{j=1}^{p_k} 2^{-j}\alpha_j} \cdot \int e^{i\Theta(0|\boldsymbol{\alpha}|\mathbf{y})} d\mathbb{P}(\mathbf{y}), \tag{5.12}$$

(for the Radon-Nikodym derivative for $d\mathbb{P}(\mathbf{y})$ versus $d\mathbb{P}(\mathbf{x}) = d\mathbb{P}(\boldsymbol{\varepsilon}|\boldsymbol{\alpha}|\mathbf{y})$ gives us a factor of 2^{-r-N-b_k}). Now let $\delta \in \{0,1\}^{p_k}$, and let's assume that $r > p_k$. Summing over all $\boldsymbol{\varepsilon} \in \mathcal{E}_{N,k,\delta}$ we count 1 for each element of $\mathcal{E}_{N,k,\delta}$ that starts with the sequence γ; obviously the first p_k elements of γ must be equal to the sequence δ or we get zero; more generally we get

$$\langle T_{N,k,\delta}\chi_{r,\boldsymbol{\alpha}}, \chi_{r,\gamma}\rangle = 2^{\frac{1}{2}(b_k-N)} \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}_{N,k,\delta}} \langle e^{-i\phi_{N,k}(\boldsymbol{\varepsilon})} S_{\boldsymbol{\varepsilon}}\chi_{r,\boldsymbol{\alpha}}, \chi_{r,\gamma}\rangle \tag{5.13}$$

$$= \begin{cases} 0 & \text{if } (\gamma_1, \gamma_2, \dots, \gamma_{p_k}) = (\delta_1, \delta_2, \dots, \delta_{p_k}), \\ 2^{-N-r} e^{2\pi i q_k \sum_{j=1}^{p_k} 2^{-j}\alpha_j} \cdot I_{\boldsymbol{\alpha}} \cdot |\mathcal{E}_{N,k,\gamma}| & \text{otherwise}, \end{cases}$$

where $I_{\boldsymbol{\alpha}} = \int \mathrm{e}^{\mathrm{i}\Theta(0|\boldsymbol{\alpha}|\mathbf{y})} \mathrm{d}\mathbb{P}(\mathbf{y})$. This is now the moment to estimate the size $|\mathcal{E}_{N,k,\gamma}|$ of the set $\mathcal{E}_{N,k,\gamma}$.

LEMMA 5.3. *For each $k \in \mathbb{N}$ there is a constant $\eta_k > 0$ with the property that for every $r \in \mathbb{N}$ and $\boldsymbol{\gamma} \in \{0,1\}^r$, we have*

$$\lim_{N\to\infty} |\mathcal{E}_{N,k,\gamma}| \cdot 2^{r-N} = \eta_k. \tag{5.14}$$

Once again we defer this fairly routine proof to the Appendix. Taking the limit of expression in the second row of (5.13) as $N \to \infty$, we find that for every $r > p_k$, $\boldsymbol{\alpha}, \boldsymbol{\gamma} \in \{0,1\}^r$ and $\boldsymbol{\delta} \in \{0,1\}^{p_k}$ we have $\langle T_{k,\delta}\chi_{r,\boldsymbol{\alpha}}, \chi_{r,\gamma}\rangle =$

$$\begin{cases} 0 & \text{unless } (\gamma_1, \gamma_2, \dots, \gamma_{p_k}) = (\delta_1, \delta_2, \dots, \delta_{p_k}), \\ 2^{-2r}\eta_k \cdot \mathrm{e}^{2\pi \mathrm{i} q_k \sum_{j=1}^{p_k} 2^{-j}\alpha_j} \cdot I_{\boldsymbol{\alpha}} & \text{otherwise.} \end{cases} \tag{5.15}$$

But there can be only one operator $T_{k,\delta}$ that satisfies equation (5.15) for all $r, \boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. It is the operator given by

$$T_{k,\delta}(f) = \eta_k \langle f, \overline{g}_k\rangle \chi_{p_k,\delta}, \tag{5.16}$$

where

$$g_k(\mathbf{y}) = \mathrm{e}^{2\pi \mathrm{i} q_k \sum_{j=1}^{p_k} 2^{-j} y_j} \mathrm{e}^{\mathrm{i}\Theta(0|\mathbf{y})}. \tag{5.17}$$

Formulae (5.16) and (5.17) remove the shackles from our whole situation. The operator $T_{k,\delta}$ has rank 1, and we claim that the norm closed linear span of all these operators is the full algebra $\mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$. For we can vary k in such a way that p_k remains constant and q_k varies from 1 to 2^{p_k}; now inverting the finite Fourier transform that's happening in (5.17), we can find (for each $\boldsymbol{\zeta} \in \{0,1\}^{p_k}$) a finite linear combination of the operators $T_{k,\delta}$ equal to $R_{\zeta,\delta}$, where

$$R_{\zeta,\delta}(f) = \langle f, \overline{\psi}_{p_k,\zeta}\rangle \chi_{p_k,\delta} \tag{5.18}$$

and

$$\psi_{p_k,\zeta}(\mathbf{y}) = \chi_{p_k,\zeta}(\mathbf{y}) \cdot \mathrm{e}^{\mathrm{i}\Theta(0|\mathbf{y})}. \tag{5.19}$$

But it is obvious that the norm-closed linear span of these rank 1 operators is $\mathcal{K}(\mathcal{H})$. The weak* closure is therefore all of $\mathcal{B}(\mathcal{H})$, and so, $\overline{\mathcal{A}}^{\mathrm{w}^*} = \overline{\mathcal{A}}^{\mathrm{WOT}} = \mathcal{B}(\mathcal{H})$ as claimed.

6. APPENDIX

In this section we give the proofs of two lemmas which we deferred in the main text of the paper. We begin by proving the three clauses of Lemma 5.2 which were not proved in the main text.

Proof of certain clauses of Lemma 5.2. The main part of this is to prove equation (5.9), and it is enough if we show that for all i with $0 \leqslant i \leqslant N-1$ we have $\Theta(L^i(\boldsymbol{\varepsilon}|\mathbf{y})) = \Theta(L^i(\boldsymbol{\varepsilon}|\mathbf{0}))$. The sum of these values will then certainly be $\phi_{N,k}(\boldsymbol{\varepsilon})$ by Definition 5.1. Write then $\mathbf{x} = (\boldsymbol{\varepsilon}|\delta)$, $\mathbf{x}' = (\boldsymbol{\varepsilon}|\mathbf{0})$, $\mathbf{y} = L^i(\mathbf{x})$ and $\mathbf{y}' = L^i(\mathbf{x}')$. Now $\boldsymbol{\varepsilon} \in \mathcal{E}_{N,k}$ by hypothesis, so $\mathbf{x}, \mathbf{x}'$ are both in $E_{N,k} \subset \Gamma_{N,k}$. It follows from (3.1) that $x_{N+1} = x'_{N+1} = 0$. Accordingly, $y_{N+1-i} = y'_{N+1-i} = 0$ with $2 \leqslant N+1-i \leqslant N+1$. We know from Definition 4.1 that $\Theta(\mathbf{y}) = \sum_{l=1}^{\infty} q_l \theta_l(\mathbf{y})$, and that $\theta_l(\mathbf{y}) = 0$ unless $\tau_{b_l}(\mathbf{y}) = (0|\mathbf{1}_{b_l-2}|0)$. Since $y_{N+1-i} = 0$ we deduce that

$$b_l \leqslant N+1-i \leqslant N+1. \tag{6.1}$$

We may suppose, then, that $\tau_{b_l}(\mathbf{y}) = \tau_{b_l}(\mathbf{y}') = (0|\mathbf{1}_{b_l-2}|0)$ for a (necessarily unique) value of l, otherwise $\Theta(\mathbf{y}) = \Theta(\mathbf{y}') = 0$ (for $y_j = y'_j = \varepsilon_j$ when $j \leqslant N+b_k$). Suppose such a value l exists. If $l \geqslant k$ then by (3.5), in order to have $\mathbf{x}, \mathbf{x}' \in E_{N,k}$ we must have $\mathbf{x}, \mathbf{x}' \notin G_{N,l}$. Accordingly, since $\tau_{i,i+b_l}(\mathbf{x}) = (0|\mathbf{1}_{b_l-2}|0)$, we must have $i \notin [N+2-p_l-b_l, N+1-b_l]$; but by (6.1) we don't have $i > N+1-b_l$ either; therefore $i \leqslant N+1-p_l-b_l$, but then

$$\Theta(\mathbf{y}) = q_l\theta_l(\mathbf{y}) = 2\pi\beta_{b_l,b_l+p_l}(\mathbf{y}) = 2\pi\beta_{b_l,b_l+p_l}(\mathbf{y}') = \Theta(\mathbf{y}'), \tag{6.2}$$

because β_{b_l,b_l+p_l} is a $\mathcal{F}_{b_l+p_l}$-measurable function, and $\mathbf{y}$ and $\mathbf{y}'$ agree in their first $N+b_k-i$ places; in particular they agree for their first $N+1-i$ places; and $b_l+p_l \leqslant N+1-i$.

To handle the case $l < k$, we assume (as a mild condition of rapid increase on the sequence (b_k)) that

$$b_l + l < b_{l+1} \tag{6.3}$$

for all $l \in \mathbb{N}$. Since $p_l \leqslant l$ this ensures that $p_l + b_l < b_k$ for all $l < k$. So $\mathbf{y}$ and $\mathbf{y}'$ (which agree up to position $N+b_k-i$) agree up to position b_l+p_l ($i < N, l < k$). Once again we get $\Theta(\mathbf{y}) = \Theta(\mathbf{y}')$ by the method of (6.2). Thus we have established formula (5.9).

The remaining clauses of Lemma 5.2 are relatively easy; since $\mathbf{x}, \mathbf{x}' \in E_{N,k} \subset \Gamma_{N,k}$, it follows that $x_j = x'_j = 1$ for $N+2 \leqslant j < N+b_k$, for

$$\tau_{N,N+b_k}(\mathbf{x}) = \tau_{N,N+b_k}(\mathbf{x}') = (0|\mathbf{1}_{b_k-2}|0). \tag{6.4}$$

For $N+1 \leqslant i < N+b_k-1$ the vector $L^i(\mathbf{x})$ begins with one of the sequence of 1's that are involved in the $\mathbf{1}_{b_k-2}$ part of the sequence; and so $\Theta(L^i(\mathbf{x})) = 0$, likewise $\Theta(L^i(\mathbf{x}')) = 0$; for as remarked in Definition 4.1, in order to have $\Theta(\mathbf{z}) \neq 0$ one must have $z_1 = 0$.

Finally (6.4) tells us that the vectors $L^N\mathbf{x}, L^N\mathbf{x}'$ begin with the key sequence $(0|\mathbf{1}_{b_k-2}|0)$; accordingly (4.2) gives

$$\Theta(L^N\mathbf{x}) = q_k\theta_k(L^N\mathbf{x}) = 2\pi q_k\beta_{b_k,b_k+p_k}(L^N\mathbf{x}) = 2\pi q_k\sum_{j=1}^{p_k}2^{-j}x_{N+b_k+j} \tag{6.5}$$

$$= 2\pi q_k\sum_{j=1}^{p_k}2^{-j}y_j$$

(for $\mathbf{x} = (\boldsymbol{\varepsilon}|\mathbf{y})$). This completes the proof of Lemma 5.2, except for the last clause whose proof was given in the main text. ∎

We continue by proving Lemma 5.3. The chief interesting thing about this lemma is the fact that η_k does not depend on γ; and that, as the reader may have guessed, is a probabilistic independence argument.

Proof of Lemma 5.3. Fix $m \geqslant k$ and consider the set

$$C_{N,k,m} = \Gamma_{N,k} \setminus \Big(\bigcup_{l=k}^{m} G_{N,l}\Big). \tag{6.6}$$

Looking at the definitions ((3.3) and (3.1)) we note that $\Gamma_{N,k} \in \mathcal{F}_{N,N+b_k}$ while $G_{N,l} \in \mathcal{F}_{N-p_l-b_l,N+b_l}$. Furthermore, $\Gamma_{N,k} = L\Gamma_{N,k+1}$; and $G_{N,l} = LG_{N+1,l}$ provided $N - p_l - b_l \geqslant 0$ (so that all the $\Gamma_{i,l}$ involved are nonempty). Since $p_m \leqslant m$, it follows from condition (6.3) that the sequence $p_m + b_m$ is increasing, so we find that $C_{N,k,m} \in \mathcal{F}_{N-p_m-b_m,N+b_m}$. Also $\mathbb{P}(C_{N,k,m})$ is independent of N provided $N > p_m + b_m$, because $\Gamma_{N,k} = L\Gamma_{N+1,k}$ for all N, $G_{N,l} = LG_{N+1,l}$ provided $N > p_l + b_l$ (see (3.3) and (3.1)), and hence $C_{N,k,m} = LC_{N+1,k,m}$. Let the common asymptotic value of $\mathbb{P}(C_{N,k,m})$ be $\zeta_{k,m}$; plainly the sequence $\zeta_{k,m}$ decreases with m, so let

$$\zeta_k = \lim_{m\to\infty}\zeta_{k,m}. \tag{6.7}$$

Now by (3.5) and (6.6), for any m we have

$$\mathbb{P}(C_{N,k,m}) \geqslant \mathbb{P}(E_{N,k}) \geqslant \mathbb{P}(\Gamma_{N,k}) - \sum_{l=k}^{\infty}\mathbb{P}(\Gamma_{N,k}\cap G_{N,l}) \tag{6.8}$$

$$\geqslant \mathbb{P}(\Gamma_{N,k}) - \mathbb{P}(G_{N,k}\cap\Gamma_{N,k}) - \sum_{l=k+1}^{\infty}\mathbb{P}(G_{N,l}).$$

By (3.4) and (3.2) this is at least

$$2^{-b_k} - \mathbb{P}(G_{N,k}\cap\Gamma_{N,k}) - \sum_{l=k+1}^{\infty} l\cdot 2^{-b_l}. \tag{6.9}$$

Now for $i \in (N+1-p_k-b_k, N+1-b_k]$ the intervals $(i, i+b_k]$ and $(N, N+b_k]$ can overlap in at most one point; hence (referring to (3.1)) we see that $\mathbb{P}(\Gamma_{i,k} \cap$

$\Gamma_{N,k}) \leqslant 2^{1-2b_k}$ (for the coordinates of $\mathbf{x} \in \Gamma_{i,k} \cap \Gamma_{N,k}$ are determined uniquely in at least $2b_k - 1$ places), and summing over these values of i and using (3.3) we have

$$\mathbb{P}(G_{N,k} \cap \Gamma_{N,k}) \leqslant \sum_{i=N+2-p_k-b_k}^{N+1-b_k} \mathbb{P}(\Gamma_{i,k} \cap \Gamma_{N,k}) \leqslant p_k \cdot 2^{1-2b_k} \leqslant k \cdot 2^{1-2b_k}. \tag{6.10}$$

Substituting this inequality into (6.9) we obtain

$$\mathbb{P}(E_{N,k}) \geqslant 2^{-b_k} - k \cdot 2^{1-2b_k} - \sum_{l=k+1}^{\infty} l \cdot 2^{-b_l}, \tag{6.11}$$

and hence

$$\zeta_k \geqslant 2^{-b_k} - k \cdot 2^{1-2b_k} - \sum_{l=k+1}^{\infty} l \cdot 2^{-b_l}. \tag{6.12}$$

It is another mild condition of rapid increase on the sequence b_k that

$$2^{-b_k} - k \cdot 2^{1-2b_k} - \sum_{l=k+1}^{\infty} l \cdot 2^{-b_l} > 0 \tag{6.13}$$

for all k. That ensures the important fact that $\zeta_k > 0$. Now $C_{N,k,m} \in \mathcal{F}_{N-p_m-b_m,N+b_m}$ and $\Omega_\gamma \in \mathcal{F}_r$. For large N those two σ-fields are *independent*. Accordingly, for large enough N we have

$$\mathbb{P}(\Omega_\gamma \cap C_{N,k,m}) = \zeta_{k,m} \cdot \mathbb{P}(\Omega_\gamma) = 2^{-r}\zeta_{k,m}. \tag{6.14}$$

Now $E_{N,k,\gamma} = \Omega_\gamma \cap E_{N,k}$, thus

$$\Omega_\gamma \cap C_{N,k,m} \supset E_{N,k,\gamma} = \Omega_\gamma \cap C_{N,k,m} \setminus \Big(\bigcup_{l=m+1}^{\infty} G_{N,l} \Big), \tag{6.15}$$

and hence, for large enough N we have

$$2^{-r}\zeta_{k,m} \geqslant \mathbb{P}(E_{N,k,\gamma}) \geqslant 2^{-r}\zeta_{k,m} - \sum_{l=m+1}^{\infty} l \cdot 2^{-b_l} \tag{6.16}$$

by (3.4) again. Plainly we have $\lim_{N\to\infty} \mathbb{P}(E_{N,k,\gamma}) = 2^{-r}\zeta_k$ and hence $\lim |\mathcal{E}_{N,k,\gamma}| \cdot 2^{-N-b_k} = \lim \mathbb{P}(E_{N,k,\gamma}) = 2^{-r}\zeta_k$, or

$$\lim_{N\to\infty} |\mathcal{E}_{N,k,\gamma}| \cdot 2^{r-N} = 2^{b_k}\zeta_k, \tag{6.17}$$

establishing (5.14) with $\eta_k = 2^{b_k}\zeta_k > 0$. Thus the lemma is proved. ∎

7. CONCLUSIONS

Given any free semigroup algebra $\mathcal{B}$, there is always a maximal von Neumann algebra $\mathcal{N} \subset \mathcal{B}$, namely $\mathcal{N} = \mathcal{B} \cap \mathcal{B}^*$; and there is always the enveloping von Neumann algebra $\mathcal{M} \supset \mathcal{B}$. Usually we expect that $\mathcal{N}$ is rather small; but here we have $\mathcal{N} = \mathcal{M}$. It would be nice to know if this is general, i.e. if every enveloping von Neumann algebra $\mathcal{M}$ for a free semigroup algebra $\mathcal{B}$ is in fact itself a free semigroup algebra. Presumably a proof of such a result would involve perturbing the generators in a manner similar to the introduction of the function Θ here. But if pursuing this simple-mindedly, one will always need an extra generator if $\sum\limits_{i=1}^{n} S_i S_i^* < I$. A more thorough examination of possible changes in the number of generators might lead in the general direction of the notorious question of whether von Neumann algebras generated by free groups on n generators are, in fact, $*$-isomorphic for $n \geqslant 2$. When one has a free semigroup algebra with $\sum\limits_{i=1}^{n} S_i S_i^* = I$, I guess one is usually free to change the number of generators as one wishes subject to $n \geqslant 2$. After all, we got our example for $n = 2$ from the binary expansion on $[0, 1]$. Real numbers in $[0, 1]$ also have a decimal expansion — which suggests that any example along these lines might be varied so one gets the same semigroup algebra from a closely related set of 10 generators. But what happens when $\sum\limits_{i=1}^{n} S_i S_i^* < I$ is another matter.

Obviously the von Neumann algebra $\mathcal{N}$ is a unitary invariant associated with the choice of the generators; I wonder which von Neumann algebras can be so obtained, and how they can be "positioned" within the free semigroup algebra as a whole.

REFERENCES

[1] O. BRATTELI, P. JORGENSEN, Compactly supported wavelets and representations of the Cuntz relations, *Appl. Comput. Harmon. Anal.* **145**(1997), 323–373.

[2] J. CUNTZ, Simple C^*-algebras generated by isometries, *Comm. Math. Phys.* **57**(1977), 173–185.

[3] H.G. DALES, N.J. LAUSTSEN, C.J. READ, A properly infinite Banach $*$-algebra with a non-zero, bounded trace, *Studia Math.* **155**(2003), 107–129.

[4] K.R. DAVIDSON, Free semigroup algebras: a survey, in *Proceedings of IWOTA 2000*, to appear.

[5] K.R. DAVIDSON, $\mathcal{B}(\mathcal{H})$ is a free semigroup algebra, preprint.

[6] K.R. DAVIDSON, D.R. PITTS, Invariant subspaces and hyper-reflexivity for free semigroup algebras, *Proc. Lond. Math. Soc.* **78**(1999), 401–430.

[7] W.T. GOWERS, B. MAUREY, The unconditional basic sequence problem, *J. Amer. Math. Soc.* **6**(1993), 851–874.

[8] W.T. GOWERS, B. MAUREY, Banach spaces with small spaces of operators, *Math. Ann.* **307**(1997), 543–568.

[9] G. POPESCU, Functional calculus for noncommuting operators, *Michigan Math. J.* **42**(1995), 345–356.

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ADDED IN PROOFS. This construction has now been rather ingeniously simplified by K.R. Davidson; see [5] for details.