J. OPERATOR THEORY
54:2(2005), 339–361

A PROBABILISTIC INDEX FOR COMPLETELY POSITIVE MAPS AND AN APPLICATION

ROLF GOHM

Communicated by Kenneth R. Davidson

ABSTRACT. The probabilistic index of a completely positive map is defined in analogy with a formula of M. Pimsner and S. Popa for conditional expectations. As an application, we describe a new strategy for computing the Jones index of the range of certain endomorphisms. We use extended transition operators to associate to an endomorphism a unital completely positive map acting on a finite dimensional matrix algebra. Then the index to be computed equals the probabilistic index of this map. For a class of examples we get a complete classification.

KEYWORDS: *Adapted endomorphism, extended transition, completely positive, probabilistic index.*

MSC (2000): 46L53, 46L37, 47B65.

1. INTRODUCTION

In [20], M. Pimsner and S. Popa studied a notion of index for subfactors which became known as Pimsner–Popa index or probabilistic index. For II_1-factors it coincides with the Jones index introduced by V. Jones in [12]. Many generalizations have been based on it, in particular by interpreting it as an index of a conditional expectation, see [15].

We are mainly interested in the original setting, namely in the following formula: Let $\mathcal{A}_0 \subset \mathcal{A}$ be an inclusion of II_1-factors. Then the probabilistic index $\pi(E)$ of the trace-preserving conditional expectation $E : \mathcal{A} \to \mathcal{A}_0$ can be computed by

$$\pi(E)^{-1} = \inf_{0 \neq a \in \mathcal{A}_+} \frac{\|E(a)\|_2^2}{\|a\|_2^2},$$

with $\|\cdot\|_2$ the norm defined by the trace. See Section 4 and 2.2 of [20], for more details.

The notion of probabilistic index for a completely positive map which we introduce in this paper, see Definition 4.1, is a very natural generalization of this formula. Its elementary properties are developed in Section 4. The paper is arranged in such a way that we can give in the end an application of this new concept to a well known problem, namely the computation of the Jones index of the range of certain endomorphisms of the hyperfinite II_1-factor. Let us describe this problem.

One of the connections between the noncommutative theory of probability and the theory of operator algebras lies in the fact that the time evolution of a noncommutative stationary stochastic process is an endomorphism of an operator algebra, see for example [17]. In recent time the author has studied the question how the probabilistic notion of adaptedness of a process with respect to a filtration manifests itself on the level of endomorphisms of operator algebras. See [8] for some general theory, here we concentrate on the following very interesting class of examples.

Think of the hyperfinite II_1-factor $\mathcal{R}$ as a weak closure with respect to the trace of an infinite tensor product of copies $(M_d)_n$ of M_d, the algebra of complex $d \times d$-matrices ($d \geqslant 2$):

$$\mathcal{R} = \Big(\bigotimes_{n=0}^{\infty} (M_d)_n \Big)^{-}.$$

This tensor product structure may be interpreted as a filtration. Motivated by the above considerations we call a (unital normal $*$-) endomorphism $\alpha : \mathcal{R} \to \mathcal{R}$ *adapted* if for all $N \in \mathbb{N}_0$

$$\alpha\Big(\bigotimes_{n=0}^{N} (M_d)_n \Big) \subset \bigotimes_{n=0}^{N+1} (M_d)_n.$$

It is not difficult to show that such an endomorphism admits a product representation in the following sense (see 4.5.2 of [8], or 5.1.6 of [14]):

$$\alpha = \lim_{N\to\infty} \mathrm{Ad}(U_1 \cdots U_N),$$

where U_n is a unitary in $(M_d)_{n-1} \otimes (M_d)_n$ (naturally embedded in $\mathcal{R}$), so that $\mathrm{Ad}(U_n) = U_n \cdot U_n^*$ is an automorphism of $(M_d)_{n-1} \otimes (M_d)_n$. Given any sequence of unitaries $U_n \in M_d \otimes M_d$, such a limit always exists pointwise in the σ-weak topology (because on localized elements only finitely many factors in the product act nontrivially) and it defines an adapted endomorphism. Let us call the particular case when all U_n are equal to some fixed $U \in M_d \otimes M_d$ the *homogeneous* case and denote it by α_U.

In fact, it is possible to arrive at these endomorphisms in quite different ways. As analyzed by J. Cuntz in [4], [5], any endomorphism of a Cuntz algebra $\mathcal{O}_d$ is induced by a unitary element $U \in \mathcal{O}_d$ and may be indexed λ_U. Now $\mathcal{O}_d$ contains an infinite tensor product of M_d in a natural way, and if $U \in (M_d)_0 \otimes (M_d)_1$ then it can be checked that the restriction of λ_U to this subalgebra (and weak closure with respect to the trace) yields α_U. See 1.2 of [5].

This point of view also shows that the α_U are of some interest in algebraic quantum field theory [10]. It was R. Longo who in [18] started to use the theory of sectors to study properties of these endomorphisms. In particular he posed the problem of computing the index of the range which equals the square of the statistical dimension. Partial results are obtained in [18], [13] and refinements of these methods and many more results along these lines are contained in [3], [2]. The concept of "localized endomorphism" as defined by R. Conti and C. Pinzari in [3] is a slightly more general version of what we have called "homogeneous adapted endomorphism" above, and the reader can find in their introduction a discussion how this is related to the concept of "localized endomorphism" in algebraic quantum field theory [10].

As V. Jones and V.S. Sunder write in 5.1.6 of [14], "it would be very interesting to determine the exact dependence of the index on the initial sequence of unitary elements". Despite the progress sketched above one cannot say that this goal is fully achieved and it may well be worth to consider reformulations of the problem. The methods we present here come from the probabilistic interpretation sketched in the beginning. In the monograph [8] we give ample evidence that the study of noncommutative stationary processes greatly benefits from the systematic use of certain operators which are not contained in the original von Neumann algebras. Our description of this approach in Section 2 is concise but selfcontained for our purposes here. In detail, we associate to an adapted endomorphism α a family of unital completely positive maps, all acting on finite dimensional spaces. In Theorem 2.3 it is shown how their asymptotic properties determine whether α is an automorphism (i.e. surjective) or not. In the homogeneous case the result is especially nice: Surjectivity corresponds to the existence of an absorbing vector state for the associated completely positive map.

In Section 3 we review the concept of an extended transition operator from [8], [9] and derive some new aspects of it, namely an interesting interplay between these operators and the positive cone of the von Neumann algebra or the GNS-space. In this way we get a close connection between the algebraic level and the level of GNS-spaces, which may be seen as a part of spatial theory.

The connection between all these topics is finally established in Section 5. In Theorem 5.1 we show that the Jones index $[\mathcal{R} : \alpha(\mathcal{R})]$ of the range of an adapted endomorphism α equals the probabilistic index of a unital completely positive map X_α acting on M_d. Thus we have a reduction to a finite dimensional problem. Note that this does not depend on localizability assumptions for the conditional expectation onto $\alpha(\mathcal{R})$ which underly many results in [3].

In Section 6 we show how the computation can actually be done for a class of real orthogonal 4×4-matrices. Even in this low dimensional case the complete classification seems to be new. These computations are elementary but need some work because X_α is obtained as a limit and because evaluation of the probabilistic index requires a detailed understanding of the completely positive map. But we think that we have achieved a separation of the relevant problems in such a way that in each part we can profit from independent theoretical progress. It is an interesting question, for example, how other properties of the endomorphism α reflect themselves in the finite dimensional map X_α.

Some conventions: For operator algebraic terminology we refer to [23]. The algebras considered in this paper are von Neumann algebras and maps between them are assumed to be normal. In particular, by the notation $T : (\mathcal{A}, \phi_\mathcal{A}) \to (\mathcal{B}, \phi_\mathcal{B})$ we mean a normal unital completely positive map $T : \mathcal{A} \to \mathcal{B}$ which respects the normal states $\phi_\mathcal{A}$ of $\mathcal{A}$ and $\phi_\mathcal{B}$ of $\mathcal{B}$ in the sense that $\phi_\mathcal{B} \circ T = \phi_\mathcal{A}$. We denote by Tr the non-normalized trace for trace class operators, while tr always denotes a tracial state, i.e. $\mathrm{tr}(\mathbf{1}) = 1$. The norm $\|\cdot\|_2$ is defined by $\|a\|_2^2 = \mathrm{tr}(a^*a)$. The positive cone in a von Neumann algebra $\mathcal{A}$ is denoted $\mathcal{A}_+$, and $\mathcal{A}_+^1$ consists of the normalized elements in $\mathcal{A}_+$ in the sense that $\|a\|_2 = 1$.

2. AN APPROACH VIA GNS-SPACES

$$\text{Suppose } \mathcal{R} = \Big(\bigotimes_{n=0}^{\infty} (M_d)_n \Big)^- \text{ is the hyperfinite II}_1\text{-factor and}$$

$$\alpha = \lim_{N\to\infty} \mathrm{Ad}(U_1 \cdots U_N) \quad (\text{with unitaries } U_n \in (M_d)_{n-1} \otimes (M_d)_n)$$

is an adapted endomorphism, as described in Section 1. By the GNS-construction with respect to the tracial state we obtain a Hilbert space $\mathcal{H}$ with a cyclic and separating vector Ω. If $(\mathcal{H}_n, \Omega_n)$ is obtained by GNS-construction from $(M_d)_n$ with the tracial state then we can think of $\mathcal{H}$ as an infinite tensor product of Hilbert spaces $\mathcal{H}_n$ along the sequence of unit vectors Ω_n. Because the trace is invariant for the endomorphism α we can define an extension to an isometry v on $\mathcal{H}$. With unitaries $u_n \in \mathcal{B}(\mathcal{H})$ given by

$$u_n a\Omega := \mathrm{Ad}\, U_n(a)\Omega, \quad n \in \mathbb{N},\ a \in \mathcal{R}$$

we have

$$v a\Omega := \alpha(a)\Omega = \lim_{N\to\infty} u_1 \cdots u_N a\Omega.$$

If q_N is the orthogonal projection from $\mathcal{H}$ onto the subspace $\bigotimes_{n=0}^{N} \mathcal{H}_n$ and e_N is the orthogonal projection onto $v\Big(\bigotimes_{n=0}^{N} \mathcal{H}_n\Big)$ then, taking into account that $v|_{\bigotimes_{n=0}^N \mathcal{H}_n} = u_1 \cdots u_{N+1}|_{\bigotimes_{n=0}^N \mathcal{H}_n}$, we have

$$e_N = u_1 \cdots u_{N+1}\, q_N\, u_{N+1}^* \cdots u_1^*.$$

We want to compute $\|E(a)\|_2$, where $E : \mathcal{R} \to \alpha(\mathcal{R})$ is the trace-preserving conditional expectation onto $\alpha(\mathcal{R})$ and $\|\cdot\|_2$ is defined by $\|a\|_2^2 = \mathrm{tr}(a^*a)$. Let $e \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection onto the closure of $\alpha(\mathcal{R})\Omega$, i.e. $E(a)\Omega = ea\Omega$ for all

$a \in \mathcal{R}$. Then

$$\begin{aligned}\|E(a)\|_2 &= \|E(a)\Omega\| = \|ea\Omega\| \\ &= \lim_{N\to\infty} \|e_N a\Omega\| \;=\; \lim_{N\to\infty} \|q_N\, u^*_{N+1} \cdots u^*_1 a\Omega\|.\end{aligned}$$

The following lemma prepares a reformulation of this equality in terms of completely positive maps.

LEMMA 2.1. *Let $\mathcal{K}_1, \mathcal{K}_2$ be Hilbert spaces, $\Omega_2 \in \mathcal{K}_2$ a unit vector. If q is the orthogonal projection from $\mathcal{K}_1 \otimes \mathcal{K}_2$ onto $\mathcal{K}_1 \otimes \Omega_2$ and $v : \mathcal{K}_1 \to \mathcal{K}_1 \otimes \mathcal{K}_2$ is an isometry, then for all $\xi \in \mathcal{K}_1$*

$$\|q\, v\, \xi\|^2 \;=\; \langle \Omega_2\, , \mathrm{Tr}_1(v\, p_\xi\, v^*)\, \Omega_2 \rangle.$$

Here p_ξ denotes the one-dimensional projection onto $\mathbb{C}\xi$, and Tr_1 denotes the partial trace evaluated on $\mathcal{K}_1$, i.e. $\mathrm{Tr}_1(\rho_1 \otimes \rho_2) = \mathrm{Tr}(\rho_1)\rho_2$.

Proof. Choose an ONB (δ_i) of $\mathcal{K}_2$ with $\delta_0 = \Omega_2$. Note that $v\, p_\xi\, v^*$ is the one-dimensional projection onto $\mathbb{C}v\xi$. Thus if we expand $v\xi = \sum_i \xi_i \otimes \delta_i$ with $\xi_i \in \mathcal{K}_1$, then we get

$$\langle \Omega_2\, , \mathrm{Tr}_1(v\, p_\xi\, v^*)\, \Omega_2 \rangle \;=\; \Big\langle \delta_0, \sum_{i,j} \langle \xi_i, \xi_j \rangle \delta_j \Big\rangle\, \langle \delta_i, \delta_0 \rangle \;=\; \langle \xi_0, \xi_0 \rangle \;=\; \|q\, v\, \xi\|^2. \quad ∎$$

PROPOSITION 2.2. *For all $a \in \bigotimes_{n=0}^{M} (M_d)_n$ with $\|a\|_2 = 1$*

$$\|E(a)\|_2^2 \;=\; \lim_{N\to\infty} \langle \Omega_N, F_N \cdots F_{M+1}(\rho)\, \Omega_N \rangle.$$

Here $\rho := \mathrm{Tr}_{0,\ldots,M-1}(u^*_M \cdots u^*_1\, p_{a\Omega}\, u_1 \cdots u_M)$ is a positive operator with $\mathrm{Tr}(\rho) = 1$ (a so-called density operator) on $\mathcal{H}_M$ and $p_{a\Omega}$ is the one-dimensional projection onto $\mathbb{C}a\Omega$. Further for all $n \in \mathbb{N}$ the map $F_n(\cdot) := \mathrm{Tr}_{n-1}(v_n \cdot v_n^*)$ is completely positive, mapping density operators on $\mathcal{H}_{n-1}$ into density operators on $\mathcal{H}_n$, where v_n is an isometry from $\mathcal{H}_{n-1}$ into $\mathcal{H}_{n-1} \otimes \mathcal{H}_n$ given by $v_n\, \xi := u_n^*(\xi \otimes \Omega_n)$ for $\xi \in \mathcal{H}_{n-1}$. The subscripts of Tr *indicate the collection of indices of those $\mathcal{H}_n$ where the partial trace is evaluated.*

Proof. For $M \leqslant N$ use Lemma 2.1 with $\mathcal{K}_1 := \bigotimes_{n=0}^{N} \mathcal{H}_n$ and $\mathcal{K}_2 := \mathcal{H}_{N+1}$ to obtain

$$\begin{aligned}&\|q_N\, u^*_{N+1} \cdots u^*_1 a\Omega\|^2 \\ &\qquad = \langle \Omega_{N+1}, \mathrm{Tr}_{0,\ldots,N}(u^*_{N+1} \cdots u^*_1\, p_{a\Omega}\, u_1 \cdots u_{N+1})\Omega_{N+1} \rangle \\ &\qquad = \langle \Omega_{N+1}, \mathrm{Tr}_{M,\ldots,N}(u^*_{N+1} \cdots u^*_{M+1}\, \rho\, u_{M+1} \cdots u_{N+1})\Omega_{N+1} \rangle \\ &\qquad = \langle \Omega_{N+1}, F_{N+1} \cdots F_{M+1}(\rho)\, \Omega_{N+1} \rangle.\end{aligned}$$

Combining this with the computations in the beginning of this section yields the result. ∎

Note that for the index computations by the Pimsner–Popa formula (as explained in Section 1) we need to control $\|E(a)\|_2$ for positive $a \in \mathcal{R}$. It will be shown in the following sections how this can be done. If we only want to know whether α is an automorphism we only have to check whether $\|E(a)\|_2 = 1$ for all $a \in \mathcal{R}$ with $\|a\|_2 = 1$. Criteria for this can thus be derived directly from Proposition 2.2.

THEOREM 2.3. *The adapted endomorphism α is an automorphism if and only if for all $M \in \mathbb{N}_0$ and all density operators ρ_M on $\mathcal{H}_M$*

$$\lim_{N\to\infty} (F_N \cdots F_{M+1}(\rho_M) - p_{\Omega_N}) = 0.$$

Here p_{Ω_N} denotes the one-dimensional projection onto $\mathbb{C}\Omega_N$. Note that the spaces are finite dimensional and we can interpret the limit in many equivalent ways, for example by operator norms getting small.

We remark that only tails $(F_n)_{n\geqslant M}$, with M arbitrarily large, are relevant in this criterion. If α is homogeneous, $\alpha = \alpha_U$, then identifying all $\mathcal{H}_n$ with $\mathcal{H}_0$ and all Ω_n with Ω_0 we can also identify all F_n with $F := F_1$, mapping the space of density operators on $\mathcal{H}_0$ into itself. Then the criterion can be simplified as follows:

COROLLARY 2.4. *The homogeneous adapted endomorphism α_U is an automorphism if and only if the vector state given by Ω_0 is absorbing for F, in the sense that for all $x \in \mathcal{B}(\mathcal{H}_0)$ and all density operators ρ on $\mathcal{H}_0$ we have*

$$\lim_{N\to\infty} \operatorname{Tr}(F^N(\rho)\, x) \;=\; \langle \Omega_0, x\Omega_0 \rangle.$$

Because in the setting of Corollary 2.4 we have to consider powers of a single completely positive map F we can use spectral theory to check whether we have an automorphism or not. See [8] for more details about connections with ergodic theory.

Proof. α is an automorphism if and only if $\|E(a)\|_2 = 1$ for all $a \in \mathcal{R}$ with $\|a\|_2 = 1$. It is enough to check this for $a \in \bigotimes_{n=0}^{M} (M_d)_n$ for all M. Running in Proposition 2.2 through all $a \in \bigotimes_{n=0}^{M} (M_d)_n$ with $\|a\|_2 = 1$ we get all one-dimensional projections on $\bigotimes_{n=0}^{M} \mathcal{H}_n$ as $p_{a\Omega}$ and (for $M \geqslant 1$) also as $u_M^* \cdots u_1^*\, p_{a\Omega}\, u_1 \cdots u_M$ and thus all density operators on $\mathcal{H}_M$ as ρ. Summarizing, α is an automorphism if and only if

$$\lim_{N\to\infty} \langle \Omega_N, F_N \cdots F_{M+1}(\rho_M)\, \Omega_N \rangle \;=\; 1$$

for all M and all density operators ρ_M on $\mathcal{H}_M$. Now Theorem 2.3 and Corollary 2.4 are a consequence of the following folklore result about trace class operators which we only state below. A detailed proof is written down in A.5.3 of

[8]. Compare further III.5.11 of [23]. The lemma also indicates the correct notions of convergence to be used here if one considers infinite dimensional generalizations. ∎

LEMMA 2.5. *Consider sequences $(\mathcal{K}_n)$ of Hilbert spaces, (Ω_n) of unit vectors, (ρ_n) of density matrices such that $\Omega_n \in \mathcal{K}_n$ and ρ_n on $\mathcal{K}_n$ for all n. Then for $N \to \infty$ the following assertions are equivalent:*

(i) $\langle \Omega_N, \rho_N \Omega_N \rangle \to 1$.

(ii) $\| \rho_N - p_{\Omega_N} \|_1 \to 0$ $(\|\rho\|_1 := \mathrm{Tr}|\rho|)$.

(iii) *For all uniformly bounded sequences (x_n) with $x_n \in \mathcal{B}(\mathcal{K}_n)$ for all n we have* $\mathrm{Tr}(\rho_N x_N) - \langle \Omega_N, x_N \Omega_N \rangle \to 0$.

3. EXTENDED TRANSITION AND POSITIVITY

Consider von Neumann algebras $\mathcal{A}, \mathcal{B}, \mathcal{C}$ on Hilbert spaces $\mathcal{G}, \mathcal{H}, \mathcal{K}$ with cyclic vectors $\Omega_\mathcal{G}, \Omega_\mathcal{H}, \Omega_\mathcal{K}$. The normal states on $\mathcal{A}, \mathcal{B}, \mathcal{C}$ induced by these vectors are denoted $\phi_\mathcal{A}, \phi_\mathcal{B}, \psi$. Further suppose that

$$j : (\mathcal{B}, \phi_\mathcal{B}) \to (\mathcal{A} \otimes \mathcal{C}, \phi_\mathcal{A} \otimes \psi)$$

is a (normal unital $*$-)homomorphism. Here we use von Neumann tensor products and the notation introduced at the end of Section 1. It is convenient to assume that $\phi_\mathcal{A}$ and $\phi_\mathcal{B}$ are faithful, and we do that from now on. Then the vectors $\Omega_\mathcal{G}$ and $\Omega_\mathcal{H}$ are also separating and j is injective.

We can extend j to a map

$$\begin{aligned} v : \mathcal{H} &\to \mathcal{G} \otimes \mathcal{K}, \\ b\Omega_\mathcal{H} &\mapsto j(b)\, \Omega_\mathcal{G} \otimes \Omega_\mathcal{K}, \end{aligned}$$

which is easily checked to be isometric and which will be called the associated isometry. Then we can define a normal unital completely positive map

$$\begin{aligned} Z : \mathcal{B}(\mathcal{G}) &\to \mathcal{B}(\mathcal{H}), \\ x &\mapsto v^* \,(x \otimes \mathbf{1})\, v. \end{aligned}$$

Note further that $v\Omega_\mathcal{H} = \Omega_\mathcal{G} \otimes \Omega_\mathcal{K}$ and thus $\langle \Omega_\mathcal{G}, x\Omega_\mathcal{G} \rangle = \langle \Omega_\mathcal{H}, Z(x)\Omega_\mathcal{H} \rangle$.

Operators Z of this type have been studied in [8], [9] and have been called "extended transition operators". They play an interesting role in the spatial theory of noncommutative Markov processes and their name is derived from the fact that if we think of j as a dilation of a transition operator from $\mathcal{B}$ to $\mathcal{A}$, then Z extends the dual transition operator on the commutants. For a survey on this type of noncommutative Markov processes we refer to [17], further probabilistic background and details of the extension theory mentioned above can be found in [8], [9]. Here we shall be concerned with another property of this class of operators:

THEOREM 3.1. *Let $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be an extended transition operator as introduced above. Suppose that there exists a conditional expectation Q from $\mathcal{A} \otimes \mathcal{C}$ onto $j(\mathcal{B})$ with invariant state $\phi_{\mathcal{A}} \otimes \psi$. Then for any $X : (\mathcal{A}, \phi_{\mathcal{A}}) \to (\mathcal{A}, \phi_{\mathcal{A}})$ there exists a unique $\widehat{Z}(X) : (\mathcal{B}, \phi_{\mathcal{B}}) \to (\mathcal{B}, \phi_{\mathcal{B}})$ such that for some contraction $x \in \mathcal{B}(\mathcal{G})$ the following equations are valid:*

$$xa\Omega_{\mathcal{G}} = X(a)\Omega_{\mathcal{G}} \qquad \text{for all } a \in \mathcal{A} \tag{3.1}$$

$$Z(x)b\Omega_{\mathcal{H}} = \widehat{Z}(X)(b)\Omega_{\mathcal{H}} \qquad \text{for all } b \in \mathcal{B}. \tag{3.2}$$

Explicitly: $\widehat{Z}(X) \;=\; j^{-1}Q\,(X \otimes \mathrm{Id})\,j$, *where* Id *is the identity on* $\mathcal{C}$.

Proof. The Kadison-Schwarz inequality for X tells us that $X(a^*)\,X(a) \leqslant X(a^*a)$, and using the $\phi_{\mathcal{A}}$-invariance this implies that there is a contraction $x \in \mathcal{B}(\mathcal{G})$ which is uniquely determined by the first equation.

Further we get

$$\begin{aligned} Z(x)b\Omega_{\mathcal{H}} &= v^*\,(x \otimes \mathbf{1})\,vb\Omega_{\mathcal{H}} \\ &= v^*\,(x \otimes \mathbf{1})\,j(b)\,\Omega_{\mathcal{G}} \otimes \Omega_{\mathcal{K}} \\ &= v^*\,X \otimes \mathrm{Id}(j(b))\,\Omega_{\mathcal{G}} \otimes \Omega_{\mathcal{K}} \\ &= (j^{-1}Q\,(X \otimes \mathrm{Id})\,j)(b)\,\Omega_{\mathcal{H}}, \end{aligned}$$

because $v^*(a \otimes c)\;\Omega_{\mathcal{G}} \otimes \Omega_{\mathcal{K}} \;=\; j^{-1}Q(a \otimes c)\,\Omega_{\mathcal{H}}$ for $a \in \mathcal{A}$, $c \in \mathcal{C}$. Because $\Omega_{\mathcal{H}}$ is separating, $\widehat{Z}(X)$ is uniquely determined by the second equation and thus we have

$$\widehat{Z}(X) \;=\; j^{-1}Q\,(X \otimes \mathrm{Id})\,j.$$

From the properties of the factors of this product it is then easily checked that indeed $\widehat{Z}(X)$ is a normal unital completely positive map with invariant state $\phi_{\mathcal{B}}$. ∎

REMARK 3.2. In [8], [9] we also considered non-unital homomorphisms j with $j(\mathbf{1}) \geqslant \mathbf{1} \otimes p_{\Omega_{\mathcal{K}}}$, where $p_{\Omega_{\mathcal{K}}}$ is the one-dimensional projection onto $\mathbb{C}\Omega_{\mathcal{K}}$. Then all the arguments above still work except that $\widehat{Z}(X)$ may fail to be unital.

On the other hand, the following special unital case is particularly convenient and will in fact be the only one which appears in the applications in Section 6: Let us call the homomorphism $j : (\mathcal{B}, \phi_{\mathcal{B}}) \to (\mathcal{A} \otimes \mathcal{C}, \phi_{\mathcal{A}} \otimes \psi)$ *automorphic* if $\mathcal{A} = \mathcal{B}$, $\mathcal{G} = \mathcal{H}, \phi_{\mathcal{A}} = \phi_{\mathcal{B}} =: \phi$ and there is a (normal $*$-)automorphism $\beta : (\mathcal{A} \otimes \mathcal{C}, \phi \otimes \psi) \to (\mathcal{A} \otimes \mathcal{C}, \phi \otimes \psi)$ such that

$$j(a) \;=\; \beta(a \otimes \mathbf{1}) \quad \text{for all } a \in \mathcal{A}.$$

Then Theorem 3.1 can be simplified as follows:

COROLLARY 3.3. *In the automorphic case the conditional expectation Q always exists, namely $Q = \beta\,Q_0\,\beta^{-1}$, where $Q_0 : (\mathcal{A} \otimes \mathcal{C}, \phi \otimes \psi) \to (\mathcal{A}, \phi)$ is the conditional expectation determined by $Q_0(a \otimes c) = a\,\psi(c)$ ("slice map"). Then we have*

$$\widehat{Z}(X) \;=\; Q_0\,\beta^{-1}\,(X \otimes \mathrm{Id})\;\beta.$$

Proof. Immediate from the definition of automorphic and Theorem 3.1. ∎

The preceding results show that an extended transition operator Z can also be interpreted as a map $\widehat{Z}$ between spaces of completely positive maps. This is a second kind of positivity which must be clearly distinguished from the complete positivity of Z itself. To make this more precise, we give

DEFINITION 3.4. Let $\mathcal{A}$ be a von Neumann algebra with a faithful normal state ϕ. A normal completely positive map $X : \mathcal{A} \to \mathcal{A}$ is called (ϕ-)*doubly positive* if additionally

$$\phi(a^* X(a)) \geqslant 0 \quad \text{for all } a \in \mathcal{A}.$$

EXAMPLE 3.5. Let A be a $d \times d$-matrix with nonnegative real entries. Thinking of it as a positive map on the commutative algebra $\mathbb{C}^d$ with the arithmetic mean as the state, we can check that in this case double positivity means that A is also positive semidefinite. For example $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is such a matrix. In the theory of matrices "doubly positive", or more precisely: "doubly nonnegative", is a well established terminology for that (but beware: "completely positive" is used with a different meaning), see [1]. Some examples of doubly positive operators acting on noncommutative algebras can be seen in Remark 4.9.

PROPOSITION 3.6. *Let $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be an extended transition operator corresponding to some $j : (\mathcal{B}, \phi_{\mathcal{B}}) \to (\mathcal{A} \otimes \mathcal{C}, \phi_{\mathcal{A}} \otimes \psi)$. If $X : (\mathcal{A}, \phi_{\mathcal{A}}) \to (\mathcal{A}, \phi_{\mathcal{A}})$ is $\phi_{\mathcal{A}}$-doubly positive then $\widehat{Z}(X) : (\mathcal{B}, \phi_{\mathcal{B}}) \to (\mathcal{B}, \phi_{\mathcal{B}})$ is $\phi_{\mathcal{B}}$-doubly positive.*

Proof. It suffices to check the additional property in Definition 3.4. If $xa\Omega_{\mathcal{G}} = X(a)\Omega_{\mathcal{G}}$ for all $a \in \mathcal{A}$, then by assumption

$$0 \leqslant \phi_{\mathcal{A}}(a^* X(a)) = \langle a\Omega_{\mathcal{G}}, xa\Omega_{\mathcal{G}} \rangle, \quad \text{i.e. } x \geqslant 0.$$

Then also $Z(x) \geqslant 0$ and for all $b \in \mathcal{B}$

$$\phi_{\mathcal{B}}(b^* \widehat{Z}(X)(b)) = \langle b\Omega_{\mathcal{H}}, Z(x) b\Omega_{\mathcal{H}} \rangle \geqslant 0.$$

This shows that $\widehat{Z}(X)$ is $\phi_{\mathcal{B}}$-doubly positive. ∎

4. A PROBABILISTIC INDEX FOR COMPLETELY POSITIVE MAPS

DEFINITION 4.1. Let $\mathcal{A}$ be a von Neumann algebra with a faithful normal tracial state and let $S : \mathcal{A} \to \mathcal{A}$ be a normal unital completely positive map. Then $\pi(S)$, the *probabilistic index* of S, is defined by

$$\pi(S)^{-1} := \inf_{0 \neq a \in (\mathcal{A} \otimes M_n)_+,\, n \in \mathbb{N}} \frac{\operatorname{tr}(a \cdot S \otimes \mathrm{Id}_n(a))}{\operatorname{tr}(a^2)}.$$

Introducing the notation $(\mathcal{A} \otimes M_n)^1_+$ for positive elements a with $\mathrm{tr}(a^2) = 1$ we can also write $\pi(S)^{-1} := \inf \mathrm{tr}(a \cdot S \otimes \mathrm{Id}_n(a))$, the infimum over $a \in (\mathcal{A} \otimes M_n)^1_+$ and all $n \in \mathbb{N}$. Obviously we have $\pi(S) \in [1, \infty]$ and $\pi(S) = 1$ if and only if $S = \mathrm{Id}$. We now discuss further elementary properties of this concept in a series of remarks.

REMARK 4.2. Suppose that $a, b \in (\mathcal{A} \otimes M_n)_+$. Then

$$\mathrm{tr}((a+b) \cdot S \otimes \mathrm{Id}_n(a+b)) \geqslant \mathrm{tr}(a \cdot S \otimes \mathrm{Id}_n(a)) + \mathrm{tr}(b \cdot S \otimes \mathrm{Id}_n(b)),$$

because the mixed terms are nonnegative. If further $0 \neq a, b$ and $\mathrm{tr}(ab) = 0$ then $\mathrm{tr}(a^2) + \mathrm{tr}(b^2) = \mathrm{tr}((a+b)^2)$ and

$$\begin{aligned}
&\frac{\mathrm{tr}((a+b) \cdot S \otimes \mathrm{Id}_n(a+b))}{\mathrm{tr}((a+b)^2)} \\
&\qquad \geqslant \frac{\mathrm{tr}(a \cdot S \otimes \mathrm{Id}_n(a))}{\mathrm{tr}(a^2)} \, \frac{\mathrm{tr}(a^2)}{\mathrm{tr}((a+b)^2)} + \frac{\mathrm{tr}(b \cdot S \otimes \mathrm{Id}_n(b))}{\mathrm{tr}(b^2)} \, \frac{\mathrm{tr}(b^2)}{\mathrm{tr}((a+b)^2)} \\
&\qquad \geqslant \min\left\{ \frac{\mathrm{tr}(a \cdot S \otimes \mathrm{Id}_n(a))}{\mathrm{tr}(a^2)}, \ \frac{\mathrm{tr}(b \cdot S \otimes \mathrm{Id}_n(b))}{\mathrm{tr}(b^2)} \right\}.
\end{aligned}$$

Using the spectral theorem for positive operators we infer that

$$\pi(S)^{-1} = \inf_{0 \neq p} \frac{\mathrm{tr}(p \cdot S \otimes \mathrm{Id}_n(p))}{\mathrm{tr}(p)},$$

where p ranges over all (nonzero) projections of $\mathcal{A} \otimes M_n$ for all $n \in \mathbb{N}$. If all such projections are orthogonal sums of minimal ones then it is enough to consider the infimum for minimal projections.

REMARK 4.3. If $\mathcal{A}$ is a II_1-factor and $S = E$ is a trace-preserving conditional expectation onto a II_1-subfactor $\mathcal{A}_0$, then $\pi(S)$ coincides with the probabilistic index or Pimsner–Popa index introduced in [20]. In 2.2 of [20], it is shown that this also coincides with the Jones index $[\mathcal{A} : \mathcal{A}_0]$ introduced in [12]. In fact, by 2.2 of [20], we have

$$[\mathcal{A} : \mathcal{A}_0]^{-1} = \inf_{0 \neq a \in \mathcal{A}_+} \frac{\|E(a)\|_2^2}{\|a\|_2^2} = \inf_{0 \neq a \in \mathcal{A}_+} \frac{\mathrm{tr}(a \cdot E(a))}{\mathrm{tr}(a^2)}$$

and similarly $\pi(S)^{-1} =$

$$\inf_{0 \neq a \in (\mathcal{A} \otimes M_n)_+,\, n \in \mathbb{N}} \frac{\mathrm{tr}(a \cdot E \otimes \mathrm{Id}_n(a))}{\mathrm{tr}(a^2)} = \inf_{n \in \mathbb{N}} [\mathcal{A} \otimes M_n : \mathcal{A}_0 \otimes M_n] = [\mathcal{A} : \mathcal{A}_0]^{-1},$$

where the last inequality follows from 2.1.15 of [12].

REMARK 4.4. There is another situation when no amplification of S is needed in Definition 4.1: If $\mathcal{A}$ is commutative then

$$\pi(S)^{-1} = \inf_{0 \neq a \in \mathcal{A}_+} \frac{\mathrm{tr}(a \cdot S(a))}{\mathrm{tr}(a^2)}.$$

Proof. Identify $\mathcal{A}$ with $L^\infty(\Omega,\Sigma,\mu)$ for a probability space (Ω,Σ,μ). The probability measure μ represents the trace. Then

$$\mathcal{A}\otimes M_n = L^\infty(\Omega,\Sigma,\mu)\otimes M_n \simeq L^\infty(\Omega,\Sigma,\mu;M_n),$$

i.e. M_n-valued functions. Let $\widetilde{p}$ be a projection-valued function which yields a good approximation of the infimum in Remark 4.2. We can approximate $\widetilde{p}$ by a step function with only finitely many projections as values (for details in measure theory we refer to [21]). Using Remark 4.2 we infer that we can find a nonzero projection $p\otimes q\in\mathcal{A}\otimes M_n$ which also yields a good approximation of the infimum. But now we get

$$\frac{\operatorname{tr}(p\otimes q\cdot S\otimes\mathrm{Id}_n(p\otimes q))}{\operatorname{tr}(p\otimes q)} = \frac{\operatorname{tr}(p\cdot S(p))}{\operatorname{tr}(p)},$$

which shows that we do not need to consider amplifications of S to compute the index. Note that if $\mathcal{A}=\mathbb{C}^d$ with the arithmetic mean as the trace and $S:\mathcal{A}\to\mathcal{A}$ is considered as a stochastic matrix, then $\pi(S)^{-1}$ is nothing but the smallest diagonal entry. ∎

REMARK 4.5. To see an example where amplification is needed to get the correct value of the probabilistic index, consider the trace-preserving conditional expectation $P_0: M_2\to M_2,\ a\mapsto \operatorname{tr}(a)\,\mathbf{1}$. Then $\inf\operatorname{tr}(p\,P_0(p))=\frac{1}{2}$ if the infimum is computed for one-dimensional projections in M_2. This is easily checked by using the explicit parameterization as a Bloch sphere, see for example [19]. But $\pi(P_0)=4$, as can be seen by applying Proposition 4.8 below. This is also the correct value for the index of the inclusion $\mathbb{C}\subset M_2$ in the sense of Chapter 2 in [7].

REMARK 4.6. For $S: M_n\to M_n$ we have

$$\pi(S)^{-1} = \inf_{a\in(M_n\otimes M_n)^1_+}\operatorname{tr}(a\cdot S\otimes\mathrm{Id}_n(a)).$$

In fact, if $m\geqslant n$ then for any $\eta\in\mathbb{C}^n\otimes\mathbb{C}^m$ there exists $\eta'\in\mathbb{C}^n\otimes\mathbb{C}^n$ and an isometry $w:\mathbb{C}^n\to\mathbb{C}^m$ such that $\eta=(\mathbf{1}\otimes w)\eta'$, see for example Lemma 2.2.1 of [6]. Thus for any one-dimensional projection $p\in M_n\otimes M_m$ there exists a one-dimensional projection $p'\in M_n\otimes M_n$ with $p=(\mathbf{1}\otimes w)p'(\mathbf{1}\otimes w^*)$. But then

$$\operatorname{Tr}(p\cdot S\otimes\mathrm{Id}_m(p)) = \operatorname{Tr}(p'\cdot S\otimes\mathrm{Id}_n(p')).$$

The result follows from this because by Remark 4.2 considering minimal projections is enough.

REMARK 4.7. Recall from Definition 3.4 that we call S (tr-)doubly positive if it additionally satisfies $\operatorname{tr}(a^*\,S(a))\geqslant 0$ for all $a\in\mathcal{A}$. If the (unital) map S is also trace-preserving and (tr-)doubly positive then we get an upper bound

$$\pi(S)\leqslant\pi(P_0),$$

where $P_0 : \mathcal{A} \to \mathcal{A},\ a \mapsto \operatorname{tr}(a)\,\mathbf{1}$ is the conditional expectation onto the constants. Note that P_0 is itself trace-preserving and (tr-)doubly positive.

Proof. Decompose $a \in (\mathcal{A} \otimes M_n)_+$ as $a = a_0 + a_1$ with $a_0 = P_0 \otimes \mathrm{Id}_n(a)$ and $a_1 = a - P_0 \otimes \mathrm{Id}_n(a)$. Note that the a_i are selfadjoint and satisfy $\operatorname{tr}(a_0 \cdot a_1) = 0$. Now we compute

$$\operatorname{tr}(a \cdot S\otimes\mathrm{Id}_n(a)) = \operatorname{tr}(a \cdot S\otimes\mathrm{Id}_n(a_0)) + \operatorname{tr}(a_0 \cdot S\otimes\mathrm{Id}_n(a_1)) + \operatorname{tr}(a_1 \cdot S\otimes\mathrm{Id}_n(a_1)).$$

The first summand is

$$\operatorname{tr}(a \cdot (S\otimes\mathrm{Id}_n)(P_0\otimes\mathrm{Id}_n)(a)) = \operatorname{tr}(a \cdot P_0\otimes\mathrm{Id}_n(a)).$$

For the second summand we find

$$\begin{aligned}\operatorname{tr}(P_0\otimes\mathrm{Id}_n(a) \cdot S\otimes\mathrm{Id}_n(a_1)) &= \operatorname{tr}(S\otimes\mathrm{Id}_n(P_0\otimes\mathrm{Id}_n(a) \cdot a_1))\\ &= \operatorname{tr}((P_0\otimes\mathrm{Id}_n)(a) \cdot a_1) = 0.\end{aligned}$$

Finally the third summand is nonnegative because if S is (tr-)doubly positive then $S\otimes\mathrm{Id}_n$ is $(\operatorname{tr} \otimes \operatorname{tr}_n)$-doubly positive. In fact, writing $b = \sum_{i,j} b_{ij}\otimes e_{ij}$ with $b_{ij} \in \mathcal{A}$ and matrix units $e_{ij} \in M_n$ we get

$$(\operatorname{tr}\otimes\operatorname{tr}_n)(b^* \cdot S\otimes\mathrm{Id}_n(b)) = \sum_{i,j,k,\ell} \operatorname{tr}(b_{ij}^* S(b_{k\ell}))\operatorname{tr}_n(e_{ij}^* e_{k\ell}) = \frac{1}{n}\sum_{i,j} \operatorname{tr}(b_{ij}^* S(b_{ij})) \geqslant 0. \quad \blacksquare$$

We now compute $\pi(S)$ for a class of examples.

PROPOSITION 4.8. *Let $S : M_2 \to M_2$ be a unital completely positive map with the Pauli matrices*

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

as eigenvectors, i.e., $S\,\sigma_i = \lambda_i\,\sigma_i$ with $\lambda_0 = 1,\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. With $\lambda_{\min} := \min\{\lambda_1, \lambda_2, \lambda_3\}$ we get

$$\pi(S)^{-1} \;=\; \min\;\Big\{\frac{1}{2}(1+\lambda_{\min}),\ \frac{1}{4}\sum_{i=0}^{3}\lambda_i\Big\}.$$

REMARK 4.9. Note that any $S : M_2 \to M_2$ as in Proposition 4.8 is automatically trace-preserving. It is well known (see Appendix B of [16]) that a unital linear map on M_2 with the Pauli matrices as eigenvectors is completely positive if and only if the tuple $(\lambda_1, \lambda_2, \lambda_3)$ of eigenvalues is contained in the (real) tetrahedron with corners at

$$(1,1,1),\ (1,-1,-1),\ (-1,1,-1),\ (-1,-1,1).$$

As it should be, this implies that the formula for $\pi(S)^{-1}$ in Proposition 4.8 always yields values in the interval $[0,1]$.

If $M_2 \ni a = \sum_{i=0}^{3} \alpha_i \sigma_i$ for some numbers α_i, then from $\mathrm{tr}(\sigma_i \sigma_j) = \delta_{ij}$ we infer that $\mathrm{tr}(a^* S(a)) = \sum_{i=0}^{3} \lambda_i |\alpha_i|^2$. Thus S is (tr-)doubly positive if and only if we have additionally $\lambda_1, \lambda_2, \lambda_3 \geqslant 0$. In this case $1 \leqslant \pi(S) \leqslant \pi(P_0) = 4$ by Remark 4.7. On the other hand, if we drop double positivity we can easily write down examples with $\pi(S) = \infty$. For example, check that $S : a \mapsto \sigma_1 a \sigma_1$ has $\lambda_1 = 1, \lambda_2 = \lambda_3 = -1$ and then use Proposition 4.8.

Proof. Using Remark 4.6 we have

$$\pi(S)^{-1} = \inf_{a \in (M_2 \otimes M_2)^1_+} \mathrm{tr}(a \cdot S \otimes \mathrm{Id}_2(a)).$$

Now write $a \in (M_2 \otimes M_2)^1_+$ in the form

$$a = \sum_{i=0}^{3} \sigma_i \otimes \rho_i$$

with $\rho_i \in M_2$. We have $\rho_i = \rho_i^*$ and $\sum_{i=0}^{3} \mathrm{tr}(\rho_i^2) = 1$. Then

$$\pi(S)^{-1} = \inf_{a \in (M_2 \otimes M_2)^1_+} \mathrm{tr}(a \cdot S \otimes \mathrm{Id}_2(a)) = \inf_{a \in (M_2 \otimes M_2)^1_+} \sum_{i=0}^{3} \lambda_i \, \mathrm{tr}(\rho_i^2). \quad \blacksquare$$

Therefore Proposition 4.8 is an immediate consequence of the following

LEMMA 4.10. *Consider the set*

$$M := \{(\mathrm{tr}(\rho_0^2), \mathrm{tr}(\rho_1^2), \mathrm{tr}(\rho_2^2), \mathrm{tr}(\rho_3^2))\}_{a \in (M_2 \otimes M_2)^1_+}$$

as a subset of the hyperplane $x_0 + x_1 + x_2 + x_3 = 1$ *in* $\mathbb{R}^4$. *Then the convex hull of* M *is the polyhedron with corners at*

$$(1,0,0,0),\ \frac{1}{2}(1,1,0,0),\ \frac{1}{2}(1,0,1,0),\ \frac{1}{2}(1,0,0,1),\ \frac{1}{4}(1,1,1,1).$$

Proof. Let us first check that the given corners belong to the set M. In fact, we find the following correspondences:

$$\begin{aligned} a_0 &= \sigma_0 \otimes \sigma_0 \rhd (1,0,0,0), \\ a_1 &= \frac{1}{\sqrt{2}}(\sigma_0 \otimes \sigma_0 + \sigma_1 \otimes \sigma_1) \rhd \frac{1}{2}(1,1,0,0), \\ a_2 &= \frac{1}{\sqrt{2}}(\sigma_0 \otimes \sigma_0 + \sigma_2 \otimes \sigma_2) \rhd \frac{1}{2}(1,0,1,0), \\ a_3 &= \frac{1}{\sqrt{2}}(\sigma_0 \otimes \sigma_0 - \sigma_3 \otimes \sigma_3) \rhd \frac{1}{2}(1,0,0,1), \\ a_4 &= \frac{1}{2}(\sigma_0 \otimes \sigma_0 + \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3) \rhd \frac{1}{4}(1,1,1,1). \end{aligned}$$

$a_0, \dots, a_4$ are multiples of projections, normalized so that they belong to $(M_2 \otimes M_2)^1_+$.

It remains to show that any element of M is a convex combination of these corners. To see that, we derive some properties shared by all elements $(\alpha, \beta, \gamma, \delta)$ of M:

(i) $\alpha, \beta, \gamma, \delta \geqslant 0$,
(ii) $\alpha + \beta + \gamma + \delta = 1$,
(iii) $\alpha \geqslant \beta,\ \gamma,\ \delta$,
(iv) $\alpha + \beta \geqslant \gamma + \delta, \quad \alpha + \gamma \geqslant \beta + \delta, \quad \alpha + \delta \geqslant \beta + \gamma$.

In fact, (i) and (ii) are immediate from the definition. To see (iii) and (iv) let us write $a \in (M_2 \otimes M_2)^1_+$ as a block matrix $\begin{pmatrix} A & B^* \\ B & C \end{pmatrix}$ with $A, B, C \in M_2$. Then

$$\rho_0 = \frac{1}{2}(A + C), \quad \rho_1 = \frac{1}{2}(A - C), \quad \rho_2 = \frac{1}{2}(B + B^*), \quad \rho_3 = \frac{1}{2\mathrm{i}}(B - B^*).$$

Because $A, C \geqslant 0$ we have $\rho_0 \pm \rho_1 \geqslant 0$. We conclude that

$$\mathrm{tr}(\rho_0^2 - \rho_1^2) \ = \ \mathrm{tr}((\rho_0 + \rho_1)(\rho_0 - \rho_1)) \geqslant 0,$$

which is $\alpha \geqslant \beta$. The other inequalities $\alpha \geqslant \gamma$ and $\alpha \geqslant \delta$ in (iii) follow by applying automorphisms of M_2 which permute the Pauli matrices.

Further, see 3.5.15 of [11], for a general inequality for such block matrices which specialized to the trace norm $\| \cdot \|_2$ yields

$$\mathrm{tr}(BB^*) \leqslant \|A\|_2 \, \|C\|_2.$$

Using this we get

$$\begin{aligned}
\gamma + \delta &= \mathrm{tr}(\rho_2^2) + \mathrm{tr}(\rho_3^2) \\
&= \frac{1}{4}\big(\,\mathrm{tr}((B + B^*)^2) - \mathrm{tr}((B - B^*)^2)\big) \\
&= \frac{1}{2}\mathrm{tr}(BB^* + B^*B) \ = \ \mathrm{tr}(BB^*) \\
&\leqslant \|A\|_2 \, \|C\|_2 \ \leqslant \ \frac{1}{2}(\|A\|_2^2 + \|C\|_2^2) \\
&= \frac{1}{2}(\mathrm{tr}(A^2) + \mathrm{tr}(C^2)) \ = \ \mathrm{tr}(\rho_0^2) + \mathrm{tr}(\rho_1^2) \ = \ \alpha + \beta.
\end{aligned}$$

Again by applying automorphisms of M_2 which permute the Pauli matrices we also get the other inequalities in (iv).

Now start with any $(\alpha, \beta, \gamma, \delta) \in M$, and without restriction of generality assume that β is the minimal number in $\beta,\ \gamma,\ \delta$. Define $\alpha' := \alpha - \beta,\ \gamma' := \gamma - \beta,\ \delta' := \delta - \beta$. Then

$$(\alpha, \beta, \gamma, \delta) \ = \ \beta(1,1,1,1) + \gamma'(1,0,1,0) + \delta'(1,0,0,1) + (\alpha' - \gamma' - \delta')(1,0,0,0).$$

Using properties (i), (ii), (iii), (iv) above we can easily check that this presents $(\alpha, \beta, \gamma, \delta)$ as a convex combination of the corners given in Lemma 4.10. Thus Lemma 4.10 is proved. ∎

5. COMPUTATION OF $[\mathcal{R} : \alpha\mathcal{R}]$

Putting together the definitions and results from the previous sections we can describe a strategy for the computation of $[\mathcal{R} : \alpha\mathcal{R}]$ for an adapted endomorphism α. This is summarized in

THEOREM 5.1. *Let $\alpha = \lim_{N\to\infty} \mathrm{Ad}(U_1 \cdots U_N)$ be an adapted endomorphism of $\mathcal{R} = \Big(\bigotimes_{n=0}^{\infty} (M_d)_n\Big)^{-}$. Then there exists a unital completely positive and trace preserving map $X_\alpha : M_d \to M_d$ such that*

$$[\mathcal{R} : \alpha\mathcal{R}] \;=\; \pi(X_\alpha).$$

Explicitly: Using the maps F_n from Section 2 and the notation from Section 3 (see also Remark 5.3 below), the limit

$$X_\alpha^{(M)} := \lim_{N\to\infty} \widehat{F}^*_{M+1} \cdots \widehat{F}^*_N(P_N)$$

(with $P_N : (M_d)_N \to (M_d)_N$, $a \mapsto \mathbf{tr}(a)\mathbf{1}$) exists for all $M \in \mathbb{N}_0$, and we can take for X_α any accumulation point of the $X_\alpha^{(M)}$ (as maps on M_d). If α is homogeneous then we define

$$X_\alpha := X_\alpha^{(0)} = \lim_{N\to\infty} (\widehat{F}^*)^N(P_0).$$

Proof. From Proposition 2.2, for $a \in \bigotimes_{n=0}^{M} (M_d)_n$ with $\|a\|_2 = 1$ we have

$$\|E(a)\|_2^2 \;=\; \lim_{N\to\infty} \langle \Omega_N, F_N \cdots F_{M+1}(\rho)\, \Omega_N\rangle$$

with $\rho := \mathrm{Tr}_{0,\dots,M-1}(u_M^* \cdots u_1^*\, p_{a\Omega}\, u_1 \cdots u_M)$ and $F_n(\cdot) := \mathrm{Tr}_{n-1}(v_n \cdot v_n^*)$, where $v_n := u_n^*|_{\mathcal{H}_{n-1}}$. We transform this as follows:

$$\begin{aligned}
&\langle \Omega_N, F_N \cdots F_{M+1}(\rho)\, \Omega_N\rangle \\
&\qquad = \mathrm{Tr}(p_{\Omega_N}\, F_N \cdots F_{M+1}(\rho)) \;=\; \mathrm{Tr}(F^*_{M+1} \cdots F^*_N(p_{\Omega_N})\, \rho) \\
&\qquad = \mathrm{Tr}(\mathbf{1}_{0,\dots,M-1} \otimes F^*_{M+1} \cdots F^*_N(p_{\Omega_N})\; u_M^* \cdots u_1^*\, p_{a\Omega}\, u_1 \cdots u_M) \\
&\qquad = \langle \xi, \mathbf{1}_{0,\dots,M-1} \otimes F^*_{M+1} \cdots F^*_N(p_{\Omega_N})\xi\rangle,
\end{aligned}$$

where p_{Ω_N} is the one-dimensional projection onto $\mathbb{C}\Omega_N$, $\xi := u_M^* \cdots u_1^* a\Omega = \mathrm{Ad}(U_M^* \cdots U_1^*)(a)\Omega$ and F_n^* is the adjoint of F_n with respect to the duality given by Tr. Explicitly:

$$\begin{aligned} F_n^* : \mathcal{B}(\mathcal{H}_n) &\to \mathcal{B}(\mathcal{H}_{n-1}), \\ x &\mapsto v_n^* \,(\mathbf{1} \otimes x)\, v_n. \end{aligned}$$

In fact, if $x \in \mathcal{B}(\mathcal{H}_n)$ then

$$\mathrm{Tr}(F_n(\rho) \cdot x) \;=\; \mathrm{Tr}(\mathrm{Tr}_{n-1}(v_n \rho v_n^*) \cdot x) \;=\; \mathrm{Tr}(v_n \rho v_n^* \cdot \mathbf{1} \otimes x) \;=\; \mathrm{Tr}(\rho \cdot v_n^* \,\mathbf{1} \otimes x\, v_n).$$

Because v_n is an isometric extension of $\mathrm{Ad}\, U_n^*|_{(M_d)_{n-1}}$ it turns out that F_n^* is an extended transition operator as studied in Section 3. See Remark 5.3 below for additional details which are helpful for later computations but are not needed in this proof. Note further that

$$p_{\Omega_N} a \Omega_N \;=\; P_N(a)\Omega_N \quad \text{for all } a \in (M_d)_N.$$

We conclude from Theorem 3.1 that for $b \in (M_d)_M$

$$F_{M+1}^* \cdots F_N^*(p_{\Omega_N}) b \Omega_M \;=\; X_{MN}(b)\Omega_M$$

with a unital completely positive and trace preserving map

$$X_{MN} := \widehat{F}_{M+1}^* \cdots \widehat{F}_N^*(P_N) : (M_d)_M \to (M_d)_M.$$

Because $v_{N+1}\Omega_N = \Omega_N \otimes \Omega_{N+1}$ we get $F_{N+1}^*(p_{\Omega_{N+1}}) \geqslant p_{\Omega_N}$ and thus

$$F_{M+1}^* \cdots F_{N+1}^*(p_{\Omega_{N+1}}) \geqslant F_{M+1}^* \cdots F_N^*(p_{\Omega_N}),$$

which means that $x_{MN} := F_{M+1}^* \cdots F_N^*(p_{\Omega_N})$ is (for $N \to \infty$) an increasing sequence of positive operators. It is bounded by $\mathbf{1}$ and thus it converges to a positive operator $x_\alpha^{(M)} \in \mathcal{B}(\mathcal{H}_M)$. Then for all $b \in (M_d)_M$ we get

$$X_{MN}(b)\Omega_M = x_{MN} b \Omega_M \overset{N\to\infty}{\longrightarrow} x_\alpha^{(M)} b \Omega_M =: X_\alpha^{(M)}(b)\Omega_M.$$

Here $X_\alpha^{(M)}(b)$ is well defined because Ω_M is separating, and as a limit of unital completely positive and trace preserving maps, $X_\alpha^{(M)}$ is also a unital completely positive and trace preserving map from $(M_d)_M$ to $(M_d)_M$.

Now recall that for $a \in \bigotimes_{n=0}^{M} (M_d)_n$ with $\|a\|_2 = 1$ we have $\|E(a)\|_2^2 =$

$$\lim_{N\to\infty} \langle \Omega, \mathrm{Ad}(U_M^* \cdots U_1^*)(a^*) \mathbf{1}_{0,\ldots,M-1} \otimes F_{M+1}^* \cdots F_N^*(p_{\Omega_N}) \mathrm{Ad}(U_M^* \cdots U_1^*)(a)\Omega \rangle.$$

Varying a in $\Big(\bigotimes_{n=0}^{M} (M_d)_n\Big)_+^1$ we notice that also $\mathrm{Ad}(U_M^* \cdots U_1^*)(a)$ takes all values in $\Big(\bigotimes_{n=0}^{M} (M_d)_n\Big)_+^1$ and thus with $\inf = \inf_{a \in (\bigotimes_{n=0}^{M} (M_d)_n)_+^1}$

$$\begin{aligned} \inf \|E(a)\|_2^2 &= \inf \lim_{N\to\infty} \langle \Omega, a\, \mathbf{1}_{0,\ldots,M-1} \otimes F_{M+1}^* \cdots F_N^*(p_{\Omega_N})\; a\Omega \rangle \\ &= \inf\; \langle \Omega, a\, \mathbf{1}_{0,\ldots,M-1} \otimes X_\alpha^{(M)}(a)\Omega \rangle \qquad (5.1) \\ &= \inf\; \mathrm{tr}(a\, \mathbf{1}_{0,\ldots,M-1} \otimes X_\alpha^{(M)}(a)). \end{aligned}$$

Let us first consider the homogeneous case. Then the $X_\alpha^{(M)}$ can all be identified with $X_\alpha := X_\alpha^{(0)}$ on $(M_d)_0$. Considering $M \to \infty$ for $\Big(\bigotimes_{n=0}^{M}(M_d)_n\Big)^1_+$ we get

$$\inf_{a\in\mathcal{R}^1_+} \|E(a)\|_2^2 = \inf_{M\in\mathbb{N}_0,\, a\in(\bigotimes_{n=0}^{M}(M_d)_n)^1_+} \operatorname{tr}(a \cdot \mathbf{1}_{0,\dots,M-1} \otimes X_\alpha(a)).$$

The left hand side is the Pimsner–Popa index, which is equal to the Jones index $[\mathcal{R} : \alpha\mathcal{R}]$, see Section 1, Remark 3.3, and first of all 2.2 of [20], for a proof of this equality. The right hand side is $\pi(X_\alpha)$, the probabilistic index of X_α, see Definition 4.1. Thus for the homogeneous case the proof is finished.

In the inhomogeneous case let the $X_\alpha^{(M)}$ all act on M_d and then let X_α be any accumulation point of the $X_\alpha^{(M)}$. Such points exist because the set of unital completely positive maps on M_d is compact. Equation 5.1 above shows that

$$\inf_{a\in(\bigotimes_{n=0}^{M}(M_d)_n)^1_+} \operatorname{tr}(a \cdot \mathbf{1}_{0,\dots,M-1} \otimes X_\alpha^{(M)}(a))$$

is decreasing with M, and then similar arguments as above make clear that $[\mathcal{R} : \alpha\mathcal{R}] = \pi(X_\alpha)$. ∎

REMARK 5.2. Using Proposition 3.6 we conclude that X_α is (tr-)doubly positive. By Remark 4.7 this yields the upper bound

$$[\mathcal{R} : \alpha\mathcal{R}] = \pi(X_\alpha) \leqslant \pi(P_0).$$

For example, for $d = 2$ we have $\pi(P_0) = 4$, see Remark 4.5 and Proposition 4.8. Of course the inequality $[\mathcal{R} : \alpha\mathcal{R}] \leqslant d^2$ is well known by other arguments, see [18], [14].

REMARK 5.3. Let us describe F_n^* as an extended transition operator in more detail. As noted in the proof of Theorem 5.1 we have

$$F_n^*(x) = v_n^* \,(\mathbf{1} \otimes x)\, v_n,$$

where v_n extends $\operatorname{Ad} U_n^*|_{(M_d)_{n-1}}$. Identifying all $(M_d)_n$ with M_d, we can identify F_n^* with an extended transition operator $Z_n^\sharp$ which is given in an automorphic way, as in Corollary 3.3. With canonical unit vectors (δ_i) and

$$\operatorname{flip} : \mathbb{C}^d \to \mathbb{C}^d, \quad \delta_i \otimes \delta_j \mapsto \delta_j \otimes \delta_i,$$

the defining automorphism $\beta_n^\sharp$ is given explicitly as

$$\beta_n^\sharp : M_d \otimes M_d \to M_d \otimes M_d, \quad \beta_n^\sharp = \operatorname{Ad}(U_n^\sharp) \quad \text{with } U_n^\sharp := \operatorname{flip} \circ U_n^*.$$

6. A CLASS OF EXAMPLES

We now present a class of examples where we can obtain a complete classification of the occurring index values. This is based on the following computations.

LEMMA 6.1. *Suppose that $X : M_2 \to M_2$ is a unital completely positive map with $X\sigma_i = \lambda_i \sigma_i$, $i = 0, 1, 2, 3$ (as in Proposition 4.8). Further let $U \in M_2 \otimes M_2 \simeq M_4$ be a real orthogonal matrix with*

$$
\begin{aligned}
U = & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \alpha & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} a & 0 \\ 0 & \delta \end{pmatrix} \\
& + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & b \\ \gamma & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \beta \\ c & 0 \end{pmatrix} \\
& \simeq \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & d & c & 0 \\ 0 & b & a & 0 \\ \gamma & 0 & 0 & \delta \end{pmatrix}.
\end{aligned}
$$

Form the extended transition operator Z for the automorphism $\beta = \mathrm{Ad}\, U$, see Corollary 3.3 (automorphic case). Then $\widehat{Z}(X)\sigma_i = \lambda'_i \sigma_i$ with

$$
\begin{pmatrix} \lambda'_0 \\ \lambda'_1 \\ \lambda'_2 \\ \lambda'_3 \end{pmatrix} = A \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix},
$$

where A is a stochastic 4×4-matrix whose rows A_0, A_1, A_2, A_3 are given by

$$
\begin{aligned}
A_0 &= (1, 0, 0, 0), \\
A_1 &= \Big(\frac{1}{4}(\alpha^2 + b^2 - \gamma^2 - d^2)^2,\ \frac{1}{4}(\alpha^2 - b^2 - \gamma^2 + d^2)^2, \\
&\qquad (\alpha\gamma + bd)^2,\ (\alpha\gamma - bd)^2 \Big), \\
A_2 &= \Big(\frac{1}{4}(\alpha c + \beta d + \gamma a + \delta b)^2,\ \frac{1}{4}(\alpha c + \beta d - \gamma a - \delta b)^2, \\
&\qquad \frac{1}{2}(\alpha a + \beta b)^2 + \frac{1}{2}(\gamma c + \delta d)^2,\ 0 \Big), \\
A_3 &= \Big(\frac{1}{4}(\alpha c - \beta d - \gamma a + \delta b)^2,\ \frac{1}{4}(\alpha c - \beta d + \gamma a - \delta b)^2, \\
&\qquad 0,\ \frac{1}{2}(\alpha a - \beta b)^2 + \frac{1}{2}(\gamma c - \delta d)^2 \Big).
\end{aligned}
$$

Proof. This is a lengthy but straightforward computation with 4×4-matrices. We omit writing it down but indicate what has to be done and explain the specific features of the solution: We have to put the Pauli matrices σ_i into the formula $\widehat{Z}(X) = Q_0\, \beta^{-1}(X \otimes \mathrm{Id})\, \beta$ obtained in Corollary 3.3. Explicitly:

$$
\widehat{Z}(X)\sigma_i = Q_0\, U^*[X \otimes \mathrm{Id}(U\, \sigma_i \otimes \mathbf{1}\, U^*)]U.
$$

By its special form U is an even transformation with respect to the $\mathbb{Z}_2$-grading of $\mathbb{C}^2 \otimes \mathbb{C}^2$ which is given with the canonical unit vectors $\{\delta_0, \delta_1\}$ as

$$\mathbb{C}^2 \otimes \mathbb{C}^2 = \operatorname{span}\{\delta_0 \otimes \delta_0, \delta_1 \otimes \delta_1\} \oplus \operatorname{span}\{\delta_0 \otimes \delta_1, \delta_1 \otimes \delta_0\}.$$

This prevents $\widehat{Z}(X)$ from mixing up σ_1 with σ_2, σ_3. Because the entries of U are real numbers, $\widehat{Z}(X)$ also does not mix up σ_2 and σ_3 and thus has the same eigenvectors as X. To obtain the formulas above it is finally necessary to insert the orthogonality relations for the entries of U. ∎

LEMMA 6.2. *Given a unitary $U \in M_2 \otimes M_2$, let $Z^\sharp$ be the extended transition operator belonging to the automorphism $\beta^\sharp = \operatorname{Ad} U^\sharp$, where $U^\sharp = \operatorname{flip} \circ U^*$ (compare Remark* 5.3*). Suppose that $X : M_2 \to M_2$ is a unital completely positive map with $X\sigma_i = \lambda_i \sigma_i$, $i = 0, 1, 2, 3$. Then in the following two cases we have $\widehat{Z}^\sharp(X)\sigma_i = \lambda_i^\sharp \sigma_i$ so that*

$$\begin{pmatrix} \lambda_0^\sharp \\ \lambda_1^\sharp \\ \lambda_2^\sharp \\ \lambda_3^\sharp \end{pmatrix} = A^\sharp \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

with a stochastic matrix $A^\sharp$:

$$\text{(I) } \textit{For} \quad U = \begin{pmatrix} c_1 & 0 & 0 & s_1 \\ 0 & s_2 & c_2 & 0 \\ 0 & -c_2 & s_2 & 0 \\ s_1 & 0 & 0 & -c_1 \end{pmatrix} \quad \textit{or} \quad U = \begin{pmatrix} c_1 & 0 & 0 & s_1 \\ 0 & s_2 & c_2 & 0 \\ 0 & c_2 & -s_2 & 0 \\ -s_1 & 0 & 0 & c_1 \end{pmatrix}$$

$$\textit{we have} \quad A^\sharp = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c_-^2 & c_+^2 & s_+^2 & s_-^2 \\ 0 & s_{12}^2 & c_{12}^2 & 0 \\ 0 & s_{21}^2 & 0 & c_{21}^2 \end{pmatrix}.$$

$$\text{(II) } \textit{For} \quad U = \begin{pmatrix} c_1 & 0 & 0 & s_1 \\ 0 & s_2 & c_2 & 0 \\ 0 & -c_2 & s_2 & 0 \\ -s_1 & 0 & 0 & c_1 \end{pmatrix} \quad \textit{or} \quad U = \begin{pmatrix} c_1 & 0 & 0 & s_1 \\ 0 & s_2 & c_2 & 0 \\ 0 & c_2 & -s_2 & 0 \\ s_1 & 0 & 0 & -c_1 \end{pmatrix}$$

$$\textit{we have} \quad A^\sharp = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c_-^2 & c_+^2 & s_+^2 & s_-^2 \\ s_{21}^2 & 0 & c_{21}^2 & 0 \\ s_{12}^2 & 0 & 0 & c_{12}^2 \end{pmatrix}.$$

Here we have used the following notation: Take angles ϕ_1, ϕ_2 in the interval $(-\frac{\pi}{2}, +\frac{\pi}{2}]$ and $c_i := \cos\phi_i$, $s_i := \sin\phi_i$ for $i = 1, 2$ and

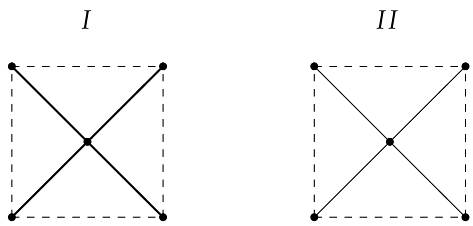
$$\begin{aligned} c_{12} &:= \cos(\phi_1+\phi_2), \quad s_{12} := \sin(\phi_1+\phi_2),\\ c_{21} &:= \cos(\phi_1-\phi_2), \quad s_{21} := \sin(\phi_1-\phi_2),\\ c_+ &:= \frac{1}{2}[\cos(2\phi_1)+\cos(2\phi_2)], \quad s_+ := \frac{1}{2}[\sin(2\phi_1)+\sin(2\phi_2)],\\ c_- &:= \frac{1}{2}[\cos(2\phi_1)-\cos(2\phi_2)], \quad s_- := \frac{1}{2}[\sin(2\phi_1)-\sin(2\phi_2)]. \end{aligned}$$

Proof. Forming $U^\sharp$ from U means transposition followed by an exchange of the second and third row. Now we can apply Lemma 6.1 for $U^\sharp$ instead of U. With some elementary trigonometry we get the formulas above, and we have taken the opportunity to present $A^\sharp$ in a more readable way by distinguishing two cases. ∎

THEOREM 6.3. *Consider the homogeneous adapted endomorphism α_U with U as in Cases* (I) *and* (II) *of Lemma* 6.2, $\phi_1, \phi_2 \in (-\frac{\pi}{2}, +\frac{\pi}{2}]$.

If $\phi_1 \neq \pm\phi_2$ then α_U is an automorphism (i.e. $[\mathcal{R} : \alpha_U\mathcal{R}] = 1$). Suppose that $\phi_1 = \pm\phi_2$. Then in Case (I) *we always have $[\mathcal{R} : \alpha_U\mathcal{R}] = 4$. In Case* (II) *we have $[\mathcal{R} : \alpha_U\mathcal{R}] = 4$ if $\phi_1 = \phi_2 = 0$ or $\phi_1 = \phi_2 = \frac{\pi}{2}$ and we have $[\mathcal{R} : \alpha_U\mathcal{R}] = 2$ otherwise.*

REMARK 6.4. Note that because only $\mathrm{Ad}\,U$ $(= \mathrm{Ad}(-U))$ is relevant for α_U, choosing angles $\phi_1, \phi_2 \in (-\frac{\pi}{2}, +\frac{\pi}{2}]$ in Cases (I) and (II) covers all homogeneous adapted endomorphisms which are defined by an orthogonal matrix U of the form given in Lemma 6.1. The following pictures summarize the content of Theorem 6.3: for the two Cases (I) and (II) the dashed lines enclose the ϕ_1, ϕ_2-square $[-\frac{\pi}{2}, +\frac{\pi}{2}]^2$. We leave it white when the index equals 1, thin colouring indicates that the index equals 2 and thick colouring that the index equals 4.



Proof. By Theorem 5.1 (and Remark 5.3) we have $[\mathcal{R} : \alpha\mathcal{R}] = \pi(X_\alpha)$ with

$$X_\alpha = \lim_{N\to\infty} (\widehat{Z}^\sharp)^N(P_0) : M_2 \to M_2.$$

Using Lemma 6.2, we start with $\lambda_0 = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ corresponding to P_0 and conclude that the Pauli matrices are eigenvectors of X_α with eigenvalues

$$\begin{pmatrix} \lambda_0^\alpha \\ \lambda_1^\alpha \\ \lambda_2^\alpha \\ \lambda_3^\alpha \end{pmatrix} = \lim_{N\to\infty} (A^\sharp)^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Finally we can use Proposition 4.8 to get the value of $\pi(X_\alpha)$ from the λ_i^α.

With the notation in Lemma 6.2 we can check that for $\phi_1, \phi_2 \in (-\frac{\pi}{2}, +\frac{\pi}{2}]$

$$\begin{aligned} c_- = 0 &\Leftrightarrow \cos(2\phi_1) = \cos(2\phi_2) \;\Leftrightarrow\; \phi_1 = \pm\phi_2, \\ s_{12} = 0 &\Leftrightarrow \sin(\phi_1 + \phi_2) = 0 \;\Leftrightarrow\; \phi_1 = \phi_2 = \frac{\pi}{2} \text{ or } \phi_1 = -\phi_2, \\ s_{21} = 0 &\Leftrightarrow \sin(\phi_1 - \phi_2) = 0 \;\Leftrightarrow\; \phi_1 = \phi_2. \end{aligned}$$

Thus if $\phi_1 \neq \pm\phi_2$ then c_-, s_{12}, $s_{21} \neq 0$. Looking at the Markov chain associated to the stochastic matrix $A^\sharp$, we then observe that in both Cases (I) and (II) we have paths of nonvanishing probability connecting any state to the absorbing state belonging to row 0. Then it is well known (see for example [22]) that

$$\lim_{N\to\infty} (A^\sharp)^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

This means that $X_\alpha = \mathrm{Id}$ and $\pi(X_\alpha) = 1$.

Now assume that $\phi_1 = \pm\phi_2$. This can be analyzed in a similar way. We have $c_- = 0$ which for Case (I) immediately implies that

$$\lim_{N\to\infty} (A^\sharp)^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which means that $\pi(X_\alpha) = 4$. Now consider Case (II). If $\phi_1 = \phi_2 = 0$ or $\phi_1 = \phi_2 = \frac{\pi}{2}$ then $c_- = s_{21} = s_{12} = 0$ and

$$\lim_{N\to\infty} (A^\sharp)^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{i.e. } \pi(X_\alpha) = 4.$$

For other angles we have $c_- = 0$ and either $s_{21} \neq 0,\ s_{12} = 0$ or $s_{21} = 0,\ s_{12} \neq 0$. Then either

$$\lim_{N\to\infty} (A^\sharp)^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ * \\ 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \lim_{N\to\infty} (A^\sharp)^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ * \\ 0 \\ 1 \end{pmatrix},$$

which by Proposition 4.8 always yields $\pi(X_\alpha) = 2$. ∎

REMARK 6.5. For Case (II) with $\phi_1 = \phi_2 = \frac{\pi}{4}$ the value 2 of the index has been computed by R. Conti and F. Fidaleo in 4.2 of [2], using the theory of sectors. It is an example of a braided endomorphism.

REMARK 6.6. Similar as in the proof of Theorem 6.3 we can also use Proposition 4.8 and Lemma 6.2 together with Theorem 5.1 to determine index values for inhomogeneous adapted endomorphisms $\alpha = \lim_{N\to\infty} \mathrm{Ad}(U_1 \cdots U_N)$ where the U_n are orthogonal matrices of the form given in Lemma 6.1 or 6.2. Then the problem can be reduced to the study of asymptotics for inhomogeneous Markov chains, see [22].

Acknowledgements. This work was supported by DFG (Deutsche Forschungsgemeinschaft). Further I would like to thank the referee for helpful remarks.

REFERENCES

[1] A. BERMAN, Complete positivity, *Linear Algebra Appl.* **107**(1988), 57–63.

[2] R. CONTI, F. FIDALEO, Braided endomorphisms of Cuntz algebras, *Math. Scand.* **87**(2000), 93–114.

[3] R. CONTI, C. PINZARI, Remarks on the index of endomorphisms of Cuntz algebras, *J. Funct. Anal.* **142**(1996), 369–405.

[4] J. CUNTZ, Automorphisms of certain simple C^*-algebras, in *Quantum Fields — Algebras, Processes (Proc. Sympos., Univ. Bielefeld, Bielefeld, 1978)*, Springer, Vienna, 1980, pp. 187–196.

[5] J. CUNTZ, Regular actions of Hopf algebras on the C^*-algebra generated by a Hilbert space, in *Operator Algebras, Mathematical Physics, and Low Dimensional Topology*, Res. Notes Math., vol. ??, A.K. Peters, Wellesley, 1993, pp. ??.

[6] E. EFFROS, Z. RUAN, *Operator Spaces*, London Math. Soc. Monographs (N.S.), vol. 23, Clarendon Press, Oxford 2000.

[7] F. GOODMAN, P. DE LA HARPE, V. JONES, *Coxeter Graphs and Towers of Algebras*, Springer-Verlag, Berlin 1989.

[8] R. GOHM, *Elements of a Spatial Theory for Non-Commutative Stationary Processes with Discrete Time Index*, Habilitationsschrift (2002). A revised version appeared as *Noncommutative Stationary Processes*, Lecture Notes in Math., vol. 1839, Springer-Verlag, Berlin 2004.

[9] R. GOHM, A duality between extension and dilation, in *Advances in Quantum Dynamics, AMS-IMS-SIAM Joint Summer Research Conference 2002*, Contemp. Math., vol. 335, Amer. Math. Soc., Providence RI 2003, pp. 139–147.

[10] R. HAAG, *Local Quantum Physics*, Springer-Verlag, Berlin 1992.

[11] R. HORN, C. JOHNSON, *Topics in Matrix Analysis*, Cambridge Univ. Press, Cambridge 1991.

[12] V.F.R. JONES, Index for subfactors, *Invent. Math.* **71**(1983), 1–25.

[13] V.F.R. JONES, On a family of almost commuting endomorphisms, *J. Funct. Anal.* **119**(1994), 84–90.

[14] V.F.R. JONES, V.S. SUNDER, *Introduction to Subfactors*, Cambridge Univ. Press, Cambridge 1997.

[15] H. KOSAKI, Extension of Jones' theory on index of arbitrary factors, *J. Funct. Anal.* **66**(1986), 123–140.

[16] C. KING, M.B. RUSKAI, Minimal entropy of states emerging from noisy quantum channels, *IEEE Trans. Inform. Theory* **47**(2001), 192–209.

[17] B. KÜMMERER, Survey on a theory of non-commutative stationary Markov processes, in *Quantum Probability and Applications.* III, Lecture Notes in Math., vol. 1303, Springer-Verlag, Berlin 1988, pp. 154–182.

[18] R. LONGO, A duality for Hopf algebras and for subfactors. I, *Comm. Math. Phys.* **159**(1994), 123–150.

[19] M.A. NIELSEN, I.L. CHUANG, *Quantum Computation and Quantum Information*, Cambridge Univ. Press, Cambridge 2000.

[20] M. PIMSNER, S. POPA, Entropy and index for subfactors, *Ann. Sci. École Norm. Sup. (4)*, **19**(1986), 57–106.

[21] W. RUDIN, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, 1987.

[22] E. SENETA, *Non-Negative Matrices and Markov Chains*, Springer Ser. Statist., Springer-Verlag, Berlin 1981.

[23] M. TAKESAKI, *Theory of Operator Algebras.* I, Springe-Verlag, Berlin 1979.

ROLF GOHM, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF READING, WHITEKNIGHTS, PO BOX 220, BERKSHIRE, RG6 6AX, UK

E-mail address: r.gohm@reading.ac.uk

Received December 12, 2003.