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INFINITE STABLE RANK FOR A CONTINUOUS FIELD ALGEBRA WITH FIBRES $\mathcal{K}$

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ABSTRACT. We give an example of a Dixmier-Douady bundle with fibres isomorphic to the compact operators, such that the C^*-algebra of sections has infinite stable rank.

KEYWORDS: *C^*-algebras, stable rank, infinite-dimensional.*

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1. INTRODUCTION

The Gelfand theorem [10] suggests that a C^*-algebra can be thought of as a noncommutative generalization of a topological space. This point of view has been systematically developed by Connes [6] and many others. One of the key properties of a topological space is its covering dimension, or Lebesgue dimension. Stable rank is one of several generalizations of the Lebesgue dimension of topological spaces to the setting of C^*-algebras, and is defined to be the Bass stable rank of the unital ring obtained by unitizing a C^*-algebra. The seminal work on stable rank of C^*-algebras is probably that of Rieffel [23]. In recent years, stable rank has been of increasing interest for the classification program for C^*-algebras, probably because of an example [26] of two C^*-algebras with the same Elliott invariant but different stable ranks. In this paper we give a counterexample that is closely tied to the Dixmier-Douady classification of bundles of elementary algebras. Recall that Dixmier and Douady [9] first showed (using Michael's selection theorem as has been done several times in classification problems) that locally trivial Hilbert bundles over compact finite-dimensional paracompact spaces are in fact trivial (this is nowadays a corollary of Kuiper's theorem, which implies contractibility of the structure group of a Hilbert bundle; the restriction on the dimension can then be dropped). They then considered not necessarily locally trivial bundles with fibres $\mathcal{K}(H)$ and structure group $B(H)^{-1}$, such that sections by projections of rank 1 exist locally. The global sections of the bundle form a C^*-algebra, and:

(1) These algebras are classified up to stable isomorphism by the third Čech cohomology class defined by their transition functions [9].

(2) If the base space is finite-dimensional (and paracompact) then the bundle is necessarily locally trivial ([8], Theorem 10.8.8).

(3) More recently, see Proposition 1.12 in [21], it has been noted that the algebra of sections is stable if and only if the bundle is locally trivial (c.f. [7]).

It is of great interest to find an algebraic characterization of the above class of algebras. Dixmier ([8], Theorem 10.9.5 and [7]) did this by showing that the above algebras of sections are exactly the separable, $\aleph_0$-homogeneous, continuous trace C^*-algebras with Hausdorff spectrum. Naturally, the main case of interest has been the locally trivial case. Nevertheless, local triviality can be dropped, and Lee [14] showed that any C^*-algebra with Hausdorff spectrum and all representations infinite-dimensional is the section algebra of some (possibly not locally trivial) bundle with varying fibres, leading to the theory of operator fields. For some interesting results about obtaining local multiplier algebras and quasi-standard C^*-algebras as operator fields, see Section 3.5 and 3.6 of [2]. Returning to the topic of stable rank, one can pose questions such as the following:

QUESTION 1.1. Let X be a second countable, σ-compact metric space with dimension k (possibly infinite). Suppose that A is a maximal full algebra of operator fields over X with fibre algebras, say, $\{A_t\}_{t\in X}$. Then is it the case that the stable rank of A satisfies the inequality

$$\mathrm{sr}(A) \leqslant \sup_{t\in X} \mathrm{sr}(C([0,1]^k) \otimes A_t) \ ?$$

This is indeed the case if the space X has finite dimension [19], [18]. The result was used to compute the stable rank of the universal C^*-algebra of an arbitrary finitely generated torsion-free two-step nilpotent group Γ. Roughly speaking, Ng and Sudo showed that the stable rank of $C^*(\Gamma)$ is controlled by the ordinary topological dimension of the one-dimensional irreducible representations of Γ [19], [18].

2. MAIN RESULT

In this paper, we contribute modestly to the recent trend for examples of C^*-algebras that cause difficulty for the classification program due to what are in some sense higher-dimensional phenomena. (For example, see references [27] and [24], and for an earlier application, Dixmier and Douady ([8], Section 10.10.9).) Our counterexample gives a negative answer to the following two questions:

(i) Given that the Dixmier-Douady invariant classified continuous trace C^*-algebras with finite-dimensional Hausdorff spectrum, and more generally classified continuous trace algebras (with Hausdorff spectrum) up to stable isomorphism, is it possible that it classifies all $\aleph_0$-homogeneous continuous trace algebras with Hausdorff spectrum?

(ii) Given that [23] the stable rank of $A \otimes \mathcal{K}$ is less than or equal to 2, is it true that the section algebra of a bundle with fibres $\mathcal{K}$ will have stable rank less than or equal to 2?

We exhibit a not locally trivial bundle with fibre $\mathcal{K}$ and infinite stable rank. This algebra is immediately a counterexample for (ii), and since it has the same Dixmier-Douady invariant as its stabilization, it is a counterexample to (i). Dixmier and Douady already had examples of not locally trivial bundles that are not isomorphic to their stabilization, so question (i) is purely rhetorical. A counterexample to (ii) is perhaps not very surprising, except that we can arrange for the stable rank to be infinite. Given the growing role of stable rank in the classification program, we hope our example may however be of interest.

Specifically, we aim to show the following:

THEOREM 2.1. *There is a compact, second countable topological space Z and a (not locally trivial) bundle over Z such that:*

(i) *every fibre of the bundle is isomorphic to $\mathcal{K}$, the algebra of compact operators over a separable infinite-dimensional Hilbert space;*

(ii) *the stable rank of the section algebra is infinite.*

3. REMARKS ON DIFFERENTIAL TOPOLOGY

The key fact that we shall need is Proposition 4.3. Before stating it, let us digress in order to give a quick exposé of the parts of differential topology that we shall need. References for this section are [14] and [5].

By a generalization of the classical vector calculus, one can define cohomology groups in terms of differential forms. Explicit calculations show that the compactly supported top cohomology group of Euclidean space is singly generated, and by the Leray-Hirsch theorem [5], in general, if E is a vector bundle, there exists a cohomology class on the total space of E whose restriction to the fibres is a generator of the fibre's top cohomology group. More precisely,

PROPOSITION 3.1 ([5], Proposition 6.18). *There is a unique cohomology class in $H^n(E)$ of the total space E of an oriented real vector bundle of rank n that restricts to the generator of $H^n(V)$ on each fibre V. The cohomology is defined to have compact support in the fibre direction.*

The above class is sometime called the *Thom class* of E. It follows from the definition that the Thom class behaves well with respect to fibrewise direct sum of vector bundles:

PROPOSITION 3.2 ([5], Proposition 6.19). *The Thom class of $E \oplus F$ is the wedge product of pullbacks $\pi_1^*(\mathrm{Thom}(E)) \wedge \pi_2^*(\mathrm{Thom}(F))$ where the π_i are the projection maps from E and F to the common base space M.*

We can now define the Euler class to be a restriction of the form defining the Thom class. More precisely, the Euler class is the pullback of the Thom class by the zero section, $z : M \longrightarrow E$, regarded as a map into the total space E of a vector bundle. The Euler class thus inherits the above property of Thom classes, so that the Euler class of a fibrewise direct sum of vector bundles $\xi_1 \oplus \xi_2$ is the wedge product of the Euler classes $e(\xi_1)$ and $e(\xi_2)$. Since the dimension of the Euler class depends on the real dimension of the vector bundle, and we shall be using complex dimensions, it follows that the Euler class lives in $H^{2n}(M)$ where n is the complex dimension of the vector bundle of interest.

Moreover, it is readily seen that if the vector bundle has a nowhere-zero cross section, its Euler class is zero, and conversely. The Euler class can readily be computed in most cases by means of an alternative construction involving a curvature form.

Just as in cohomology we have the Thom isomorphism (or Poincaré Lemma) relating the cohomology of the total space with that of the base space, in topological K-theory we have Bott periodicity, giving an isomorphism of $K_0(X)$ with $K_0(S^2X)$. The image of the generator of $K_0(\mathrm{pt}) \cong \mathbb{Z}$ under this isomorphism is a vector bundle over S^2, called the Bott bundle (of S^2), and the Euler class of the Bott bundle is canonically given by an element of the image of the Chern map from K-theory to cohomology.

4. PROOF OF MAIN RESULT

We borrow from the arguments of Villadsen and of Rørdam [24], [25], [27]. The first two lemmas are from Villadsen.

Suppose that $\mathcal{M}$ is a finite dimensional, compact, connected and orientable differentiable manifold. Suppose that L is some finite dimensional normed real vector space, and let B be the closed unit ball in L. Let $\pi_1 : B \times \mathcal{M} \to B$ and $\pi_2 : B \times \mathcal{M} \to \mathcal{M}$ be the natural projection maps. Let $C(B \times \mathcal{M}, L)$ be the space of continuous functions from $B \times \mathcal{M}$ into L, and let $C(B \times \mathcal{M}, L)$ have the supremum norm. Now let $B_{\mathcal{M}}$ be the open unit ball in $C(B \times \mathcal{M}, L)$, with centre π_1. With this terminology, we now can state the following lemma of Villadsen's (the lemma is similar to the Lefschetz theorem ([16], Theorem 7.3) and is therefore likely to have a Morse-theoretical proof):

LEMMA 4.1 ([27], Theorem 1). *There is a dense subset of $B_{\mathcal{M}}$ consisting of func-*

tions f such that if π *is the restriction of* π_2 *to* $\mathcal{N} := f^{-1}\{0\}$*, then the induced map* $\pi^* : H^*(\mathcal{M}) \to H^*(\mathcal{N})$ *is injective.*

Recall that $Lg_n(A)$, as defined by Bass [4], [3], [23], is the set of left-invertible n-tuples of elements of the unitization $\widetilde{A}$ of A. The stable rank, by definition, is the smallest integer such that $Lg_n(A)$ is dense in $\widetilde{A}^n$.

LEMMA 4.2 ([27], Proposition 5). *Suppose that* A *is a unital* C^**-algebra,* p *a projection in* A*, and* $(b_1, b_2, \ldots, b_n) \in Lg_n(A)$ *such that* $pb_ip = 0$ *for* $1 \leqslant i \leqslant n$*. Then* p *is Murray-von Neumann equivalent to a subprojection of* $n(1-p)$ *in* $A \otimes \mathcal{K}$*. Here,* 1 *is the unit of* A*,* $n(1-p)$ *is the direct sum of* n *copies of* $(1-p)$*, and* $\mathcal{K}$ *is the algebra of compact operators over a separable infinite dimensional Hilbert space.*

In the next proposition, γ is the K-theoretical Bott bundle over S^2. One could use copies of $\mathbb{C}P^n$ instead of the S^2 in the next proposition (as in the early paper [13]).

PROPOSITION 4.3. *Let* $Z := \left(\prod_{n=1}^{\infty} \mathbb{D}\right) \times \left(\prod_{m=1}^{\infty} S^2\right)$*. There exists a sequence* (p_k) *of nowhere zero one dimensional projections in* $C(Z) \otimes \mathcal{K}$ *such that:*

(i) *the* p_k*s are mutually orthogonal;*

(ii) *for each* k*, there is a projection* r_k *in* $C(S^2) \otimes \mathcal{K}$ *such that* $p_k(\vec{x}, \vec{y}) = r_k(y_k)$ *for all* $\vec{x} \in \left(\prod_{n=1}^{\infty} \mathbb{D}\right)$ *and* $\vec{y} \in \left(\prod_{m=1}^{\infty} S^2\right) = Z$ *where* y_j *denotes the coordinate belonging to the* j*th copy of* S^2*; moreover, the* K_0*-class of* r_k *corresponds to the* K_0*-class of the Bott bundle* γ *over* S^2*;*

(iii) *there is a trivial one-dimensional projection* $\theta_1 \in C(Z) \otimes \mathcal{K}$ *such that* θ_1 *is orthogonal to every* p_k*;*

(iv) $P = \sum_{k=1}^{\infty} p_k$ *converges in the strict topology in* $\mathcal{M}(C(Z) \otimes \mathcal{K})$*, where* $\mathcal{M}(C(Z) \otimes \mathcal{K})$ *is the multiplier algebra of* $C(Z) \otimes \mathcal{K}$*, and where* $\mathcal{K}$ *is the algebra of compact operators over a separable infinite dimensional Hilbert space.*

We now begin the proof of Theorem 2.1:

Proof. Let θ_1 be as in part (iii) of the proposition and let P be as in part (iv) of the proposition. Let $Q = \theta_1 \oplus P$. Let $A = Q(C(Z) \oplus \mathcal{K})Q$, where $\mathcal{K}$ is the algebra of operators over a separable infinite dimensional Hilbert space. Here, $A = Q(C(Z) \otimes \mathcal{K})Q$ is the intersection of $C(Z) \otimes \mathcal{K}$ with the hereditary subalgebra of $\mathcal{M}(C(Z) \otimes \mathcal{K})$ generated by Q, and thus is a hereditary subalgebra of $C(Z) \otimes \mathcal{K}$. For the purposes of computing the stable rank of A, one must by definition consider its minimal unitization $\widetilde{A}$. Since Q certainly acts as the unit on A, we may as well take $\widetilde{A}$ to be $A \oplus \mathbb{C}Q$. We shall show that for any integer n, the stable rank is larger than n. Thus, we shall show that $Lg_n(A)$ is not dense in $\widetilde{A}^n$ for any n.

The projection Q is a strictly continuous [1] function from Z into the projections of $B(H)$, therefore, A is the algebra generated by the sections $\{QfQ : f \in C(Z) \otimes \mathcal{K}\}$. These sections satisfy the axioms of Dixmier and Douady ([9], p. 268), hence, (the presence of the one-dimensional constant projection θ_1 insures that the so-called Fell's condition is satisfied) by the results discussed in the introduction, the algebra A is actually the section algebra of a bundle with fibres each equal to $\mathcal{K}$ or $M_n(\mathbb{C})$ for some n. However, since the projection $P \in \mathcal{M}(C(Z) \otimes \mathcal{K})$ is not of finite rank at any point in Z, it follows that the fibres of A must be infinite-dimensional.

Since A is nonunital, as explained earlier the stable rank of A is the stable rank of $\widetilde{A} \cong A + \mathbb{C}Q$, the unitization of A.

To show that the stable rank of A is infinite, we must show that it is not bounded by any finite n. So fix an integer $n \geqslant 1$. Let $i_1 : \mathbb{D}^n \times \prod_{m=1}^{\infty} S^2 \to Z = \Big(\prod_{n=1}^{\infty} \mathbb{D}\Big) \times \Big(\prod_{m=1}^{\infty} S^2\Big)$ be the continuous embedding given by $i_1(x_1, x_2, \ldots, x_n, y_1, y_2, y_3, \ldots) = (x_1, x_2, \ldots, x_n, x_n, x_n, x_n, \ldots, y_1, y_2, y_3, \ldots)$ for $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots) \in \mathbb{D}^n \times \prod_{m=1}^{\infty} S^2$, where $\mathbb{D}^n$ is the n-fold Cartesian product of $\mathbb{D}$ with itself, x_i is in the ith copy of $\mathbb{D}$ in $\mathbb{D}^n$, and y_j is in the jth copy of S^2. Now i_1 induces a surjective $*$-homomorphism $\Psi : C(Z) \otimes \mathcal{K} \to C\Big(\mathbb{D}^n \times \prod_{m=1}^{\infty} S^2\Big) \otimes \mathcal{K}$.

This in turn, by the noncommutative Tietze extension theorem, induces a strictly continuous surjective map, still denoted Ψ, from $\mathcal{M}(C(Z) \otimes \mathcal{K})$ onto $\mathcal{M}\Big(C\Big(\mathbb{D}^n \times \prod_{m=1}^{\infty} S^2\Big) \otimes \mathcal{K}\Big)$. Denoting the image of A under Ψ by B, we see that the image of $\widetilde{A}$ is $B + \mathbb{C}\Psi(Q)$, which is (isomorphic to) the unitization of B since Ψ is a homomorphism. We can as well take $\widetilde{B}$ to be $\widetilde{B} \cong B + \mathbb{C}\Psi(Q) = \Psi(Q)\Big(C\Big(\mathbb{D}^n \times \prod_{m=1}^{\infty} S^2\Big) \otimes \mathcal{K}\Big)\Psi(Q)$.

By one of Rieffel's results [23], passing to quotients can only decrease the stable rank. Hence, to show that the stable rank of $\widetilde{A}$ is greater than n, it suffices to show that the stable rank of the quotient $B + \mathbb{C}\Psi(Q)$ is greater than n.

The element $\Psi(\theta_1)$ is still a trivial one-dimensional projection in $C\Big(\mathbb{D}^n \times \prod_{m=1}^{\infty} S^2\Big) \otimes \mathcal{K}$, which we again denote by θ_1. Also, since Ψ is strictly continuous, $\Psi(Q) = \theta_1 \oplus \sum_{k=1}^{\infty} \Psi(p_k)$, where the the latter sum converges in the strict topology. Now for each i, let $a_i \in C(\mathbb{D}^n)$ be the projection onto the ith coordinate. Let $\pi_1 : \mathbb{D}^n \times \prod_{m=1}^{\infty} S^2 \to \mathbb{D}^n$ be the natural projection. Then $(a_1 \circ \pi_1)\theta_1$, $(a_2 \circ \pi_1)\theta_1, \ldots, (a_n \circ \pi_1)\theta_1$ are all elements of B. We will show that $((a_1 \circ \pi_1)\theta_1, (a_2 \circ \pi_1)\theta_1, \ldots, (a_n \circ \pi_1)\theta_1)$ has distance at least one from $Lg_n(B^+)$.

So, suppose to the contrary, that $(c_1, c_2, \dots, c_n) \in Lg_n(\widetilde{B})$ is such that $\|(a_i \circ \pi_1)\theta_1 - c_i\| < 1$ for every i. Since $\theta_1 c_i \theta_1 \in C\Big(\mathbb{D}^n \times \prod_{m=1}^{\infty} S^2\Big) \otimes \mathcal{K}$ for each i, and since in particular θ_1 is a one-dimensional projection, there must exist a function $g_i \in C\Big(\mathbb{D}^n \times \prod_{m=1}^{\infty} S^2\Big)$ such that

$$\theta_1 c_i \theta_1 = g_i \theta_1 \tag{4.1}$$

for every i.

To simplify notation, let Z_n be $\mathbb{D}^n \times \prod_{m=1}^{\infty} S^2$, and for every positive integer M, let $Z_{n,M}$ be $\mathbb{D}^n \times \prod_{m=1}^{M} S^2$. By the definition of the product topology, $C(Z_n)$ is an inductive limit $C(Z_n) = \overline{\bigcup_{M=1}^{\infty} C(Z_{n,M})}$ with the obvious connecting maps. Let us denote the connecting maps by $\phi_{M_1,M_2} : C(Z_{n,M_1}) \to C(Z_{n,M_2})$, and $\phi_M : C(Z_{n,M}) \to C(Z_n)$. Choosing an integer M sufficiently large, we can find a $h_i \in C(Z_{n,M})$ for $1 \leqslant i \leqslant M$ such that:

(i) $\|\phi_M(h_i) - g_i\| < \varepsilon$ for all i;

(ii) $\|a_i \circ \pi_1^M - h_i\| < \varepsilon$ for all i, where $\pi_1^M : Z_{n,M} = \mathbb{D}^n \times \prod_{m=1}^{M} S^2 \to \mathbb{D}^n$ is the natural projection.

Note that $a_i \circ \pi_1 = \phi_N(a_i \circ \pi_1^N)$ for every integer N, and also $(a_1 \circ \pi_1^M, a_2 \circ \pi_1^M, \dots, a_n \circ \pi_1^M) = \pi_1^M$. Hence, by Lemma 4.1, choosing ε sufficiently small, we may assume that $(h_1, h_2, \dots, h_n)$ is in the dense subset of $B_{\prod_{m=1}^{M} S^2}$ of functions whose zero sets satisfy the conclusion of Lemma 4.1. More specifically, $B_{\prod_{m=1}^{M} S^2}$ is as in Lemma 4.1, with $\mathcal{M}$ being $\prod_{m=1}^{M} S^2$, and with the real vector space $L = \mathbb{C}^n$ given the supremum norm.

Now for each i, let d_i be $c_i + (\phi_M(h_i) - g_i)\theta_1$. As pointed out above, we may assume that $(d_1, d_2, \dots, d_n)$ is in $Lg_n(\widetilde{B})$. Also, from equation (4.1), $\theta_1 d_i \theta_1$ is equal to $\phi_M(h_i)\theta_1$.

In order to make effective use of Lemma 4.1, let W be the closed subset of Z_n given by $W := \{z \in Z_n : \phi_M(h_1)(z) = \phi_M(h_2)(z) = \cdots = \phi_M(h_n)(z) = 0\}$. Hence, by Lemma 4.2, we have that $\theta_1|_W$ is Murray-von Neumann equivalent in $B^+|_W = B|_W + \mathbb{C}\Phi(Q)|_W$ to a subprojection of $\Psi(P)|_W$, where $P = Q - \theta_1$ is as in part (iv) of Proposition 4.3. Since, as pointed out earlier, $\Psi : \mathcal{M}(A) \to \mathcal{M}(B)$ is a strictly continuous surjection, we have that $\Psi(P) = \sum_{k=1}^{\infty} \Psi(p_k)$, where the sum converges in the strict topology in the multiplier algebra $\mathcal{M}(B)$. It follows from the definition of strict convergence of elements of a bundle that we still have strict convergence if we restrict the base space to the closed subset $W \subset Z$.

Hence, $\Psi(P)|_W = \sum_{k=1}^{\infty} \Psi(p_k)|_W$, where the sum converges in the strict topology in the multiplier algebra $\mathcal{M}(B|_W)$.

To simplify notation, let us write θ_1 for the trivial projection $\theta_1|_W$. Now, by Lemma 4.2, let $\vartheta_1 \in B|_W$ be a subprojection of $\Phi(P)|_W$ such that in $B|_W$ the projection ϑ_1 is Murray-von Neumann equivalent to θ_1. Hence, both $\sum_{k=1}^{\infty} (\Psi(p_k)|_W)\vartheta_1$, $\sum_{k=1}^{\infty} \vartheta_1(\Psi(p_k)|_W)$ and $\sum_{k=1}^{\infty} (\Psi(p_k)|_W)\vartheta_1(\Psi(p_k)|_W)$ converge in norm as elements of $B|_W$ to ϑ_1. Moreover, since we have equivalence of the infinite sum to a proper subprojection, there will be some large positive integer N such that θ_1 is Murray-von Neumann equivalent in $B|_W$ to a proper subprojection of $\sum_{k=1}^{N} (\Psi(p_k)|_W)$. Increasing N if necessary, we may assume that $N \geqslant M$.

We can, since $N > M$, choose in the definition of W to have $W = W_N \times \prod_{m=N+1}^{\infty} S^2$, where $W_N = \{z \in Z_{n,N} : \phi_{M,N}(h_1)(z) = \phi_{M,N}(h_2)(z) = \cdots = \phi_{M,N}(h_n)(z) = 0\}$. Now for each k, let r_k be as in part (ii) of Proposition 4.3. Then, by our construction of Ψ, for each k, $\Psi(p_k)(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots) = r_k(y_k)$ for all $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots) \in Z_n$. Hence, by the proposition, in $K^0(W_N)$, ρ_1 is isomorphic to a subbundle of the vector bundle $(\pi_2^N)^*(\gamma^{\otimes N})|_{W_N}$, where $\pi_2^N : Z_{n,N} \to \prod_{m=1}^{N} S^2$ is, as before, the natural projection map; γ is the Bott bundle over S^2, and $\gamma^{\otimes N}$ is the N-fold fibrewise Cartesian product of γ with itself. Also, ρ_1 is the trivial one-dimensional complex line bundle over W_N.

Then the Euler class $e((\pi_2^N)^*(\gamma^{\otimes N})|_{W_N})$ is zero. Now let $i_2 : W_N \to Z_{n,N}$ be the natural inclusion map, so that $\pi^N = \pi_2^M \circ i_2$ is the restriction of π_2^N to W_N and $(\pi_2^N)^*(\gamma^{\otimes N}|_{W_N}) = (\pi^N)^*(\gamma^{\otimes N})$. By the naturality of the Euler classes, $(\pi^N)^*(e(\gamma^{\otimes N}))$ is still zero. However, $e(\gamma^{\otimes N}) = \prod_{m=1}^{N} e(\gamma)$, where the product on the right hand side is the cup product, and since the Euler class generates the top cohomology group, $\prod_{m=1}^{N} e(\gamma)$ is a generator of the top cohomology group of $\gamma^{\otimes N}$. By the Kunneth formula, $H^{2n}(S^{2n}) \cong H^2(S^2) \otimes H^2(S^2) \otimes \cdots \otimes H^2(S^2) = H^2(S^2)^{\otimes n}$. Since $e(\gamma)$ is a generator of the torsion-free group $H^2(S^2) \cong \mathbb{R}$, it is clear that $e(\gamma)^{\otimes n}$ is of infinite degree. Thus, in particular, $e(\gamma^{\otimes N})$ is nonzero. Hence, to get a contradiction, it suffices to show that $(\pi^N)^* : H^*\Big(\prod_{m=1}^{N} S^2\Big) \to H^*(W_N)$ is injective and hence has no kernel. But by our choice of the h_is and by Lemma 4.1, we have that $(\pi^M)^* : H^*\Big(\prod_{m=1}^{M} S^2\Big) \to H^*(W_M)$, which is constructed in exactly the same way, is injective. Hence, by the naturality of the Kunneth formula, $(\pi^N)^*$ is also injective. This gives us the contradiction. ∎

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