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ON QUASINILPOTENT OPERATORS. III

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ABSTRACT. This paper is a continuation of our study [9], [3] of quasinilpotent operators in which we obtain some new hyperinvariant-subspace theorems for such operators. We also prove a structure theorem about certain quasinilpotent operators and reduce the hyperinvariant subspace problem for quasinilpotent operators to a special subcase.

KEYWORDS: *Quasinilpotent operators, invariant subspaces, hyperinvariant subspaces.*

MSC (2000): 47A15, 47B10.

1. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. In [9] and [3] we modified and extended a technique introduced by P. Enflo in [1] involving some "extremal vectors" to produce a nontrivial hyperinvariant subspaces (n.h.s.) for some quasinilpotent operators in $\mathcal{L}(\mathcal{H})$. In this paper we introduce a different construction that leads to the existence of a n.h.s. for some additional classes of quasinilpotent operators. In particular, if $Q \in \mathcal{L}(\mathcal{H})$ is a given quasinilpotent operator, we consider two sequences of projections associated with Q that converge in the weak operator topology (WOT). In certain cases it can be shown that one of these sequences has a subsequence that converges in the strong operator topology (SOT), and in these cases we get a n.h.s. for Q. In Section 4 we prove a modest structure theorem for a special class of quasinilpotent operators. Finally, in Section 5 we show, using techniques from [5], that the general hyperinvariant subspace problem for quasinilpotent operators in $\mathcal{L}(\mathcal{H})$ can be reduced to a subclass that is perhaps more amenable to study. In particular, this produces a new structure theorem for quasinilpotent operators (Theorem 5.6) that may be useful in the future.

2. THE MAIN CONSTRUCTION

As usual, $\mathbb{N}$ ($\mathbb{N}_0$) will denote the set of positive (nonnegative) integers, and $\mathbb{Z}$ the set of integers. The ideal of compact operators in $\mathcal{L}(\mathcal{H})$ will be denoted by $\mathbb{K}$, and the Calkin map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbb{K}$ by π. If $T \in \mathcal{L}(\mathcal{H})$, the spectrum, point spectrum, and essential (Calkin) spectrum will be written as $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$, and $\sigma_{\mathrm{e}}(T)$, respectively, and $r(T)$ will denote the spectral radius of T. As usual, $\|T\|_{\mathrm{e}}$ and $r_{\mathrm{e}}(T)$ will denote the essential norm and essential spectral radius of T. Finally, we write $\{T\}' = \{S \in \mathcal{L}(\mathcal{H}) : ST = TS\}$ for the commutant of T. The main construction will be presented in several steps, in only the last of which the quasinilpotence of the operator plays a role.

2.1. DEFINITIONS AND NOTATION. Let $T \notin \mathbb{C}1_{\mathcal{H}}$ be an arbitrary quasiaffinity (i.e., $\ker T = \ker T^* = (0)$) in $\mathcal{L}(\mathcal{H})$ such that $\|T\| = 1$ (a harmless normalization), and let $E^{(n)}$be the spectral measure associated with the operator T^nT^{*n}, so

$$T^nT^{*n} = \int\limits_{[0,\|T^n\|^2]} \lambda \mathrm{d}E^{(n)}, \quad n \in \mathbb{N}, \tag{2.1}$$

and define

$$E^{(n)}_{\lambda} = E^{(n)}([0,\lambda]), \quad E^{(n)}_{\lambda^-} = E^{(n)}([0,\lambda)), \quad 0 \leqslant \lambda \leqslant \|T^n\|^2,\ n \in \mathbb{N}. \tag{2.2}$$

Moreover, fix an arbitrary $0 < \theta < 1$ and an arbitrary unit vector x_0 in $\mathcal{H}$, and define

$$\lambda_n = \lambda_n(T, \theta, x_0) = \inf\{\lambda \in [0, \|T^n\|^2] : \|E^{(n)}([0,\lambda])x_0\| \geqslant \theta\}, \quad n \in \mathbb{N}. \tag{2.3}$$

Then $\lambda_n > 0$ for $n \in \mathbb{N}$ (since $\ker(T^{*n}) = (0)$) and the space

$$\mathcal{M}_n := (1_{\mathcal{H}} - E^{(n)}_{\lambda_n^-})\mathcal{H} = E^{(n)}([\lambda_n, \|T^n\|^2])\mathcal{H}, \quad n \in \mathbb{N},$$

is a reducing (spectral) subspace for T^nT^{*n} such that $T^nT^{*n}|\mathcal{M}_n$ is an invertible operator. Thus there exists a unique $x_n \in \mathcal{M}_n$ such that

$$T^nT^{*n}x_n = (1_{\mathcal{H}} - E^{(n)}_{\lambda_n^-})x_0, \quad n \in \mathbb{N}; \tag{2.4}$$

namely,

$$x_n = \Big(\int\limits_{[\lambda_n, \|T^n\|^2]} \frac{1}{\lambda} \mathrm{d}E^{(n)} \Big) x_0, \quad n \in \mathbb{N}. \tag{2.5}$$

We define

$$y_n = T^{*n}x_n, \quad z_n = E^{(n+1)}_{\lambda_{n+1}} x_0, \quad n \in \mathbb{N}, \tag{2.6}$$

and fix an arbitrary contraction S in $\{T\}'$.

LEMMA 2.1. *With the notation as above, for each $n \in \mathbb{N}$ we have:*

(i) $\|x_0 - T^n y_n\| = \|E^{(n)}_{\lambda_n^-} x_0\| \leqslant \theta < 1$;

(ii) $\langle z_n, x_0\rangle = \|E^{(n+1)}_{\lambda_{n+1}} x_0\|^2 \geqslant \theta^2$;

(iii) $\limsup_n (\lambda_n)^{1/n} \leqslant r(T)^2$.

Consequently, if T is quasinilpotent, then $(\lambda_n)^{1/n} \to 0$.

LEMMA 2.2. *With the notation as above and S any contraction in $\{T\}'$, we have:*

(iv) $|\langle ST^n y_n, T^* z_n\rangle| = |\langle Sy_n, (T^*)^{n+1} z_n\rangle| \leqslant \|y_n\| \|(T^*)^{n+1} z_n\| \leqslant (\lambda_{n+1}/\lambda_n)^{1/2}$;

(v) $|\langle S(1_{\mathcal{H}} - E^{(n)}_{\lambda_n^-})x_0, T^* E^{(n+1)}_{\lambda_{n+1}} x_0\rangle| \leqslant (\lambda_{n+1}/\lambda_n)^{1/2}$.

Proof of Lemmas 2.1 *and* 2.2. All of the projection valued functions $\lambda \to E^{(n)}_\lambda$ and $\lambda \to E^{(n)}_{\lambda^-}$ are clearly monotone increasing (i.e., non-decreasing) and $E^{(n)}_{\lambda^-} \leqslant E^{(n)}_\lambda$ for all $\lambda \in [0, \|T^n\|^2]$ and $n \in \mathbb{N}$. The conclusions (i) and (ii) are consequences of the fact (cf. [6]) that spectral measures are inner and outer regular, and, as noted above, since T^n is a quasiaffinity, $\lambda_n > 0$ for $n \in \mathbb{N}$. Also (iii) is a direct consequence of the fact that $\lambda_n \leqslant \|T^n\|^2$. Furthermore, in (iv) only the last inequality needs proof, and it follows from (2.6) and the calculation

$$\begin{aligned}
\|y_n\|^2 \|(T^*)^{n+1} z_n\|^2 &= \langle T^n T^{*n} x_n, x_n\rangle \langle T^{n+1}(T^*)^{n+1} z_n, z_n\rangle \\
&= \Big(\int\limits_{[\lambda_n, \|T^n\|^2]} \frac{1}{\lambda} \mathrm{d}E^{(n)}_{x_0,x_0}\Big)\Big(\int\limits_{[0,\lambda_{n+1}]} \lambda \mathrm{d}E^{(n+1)}_{x_0,x_0}\Big) \\
&\leqslant \Big(\frac{1}{\lambda_n} \int\limits_{[\lambda_n, \|T^n\|^2]} \mathrm{d}E^{(n)}_{x_0,x_0}\Big)\Big(\lambda_{n+1} \int\limits_{[0,\lambda_{n+1}]} \mathrm{d}E^{(n+1)}_{x_0,x_0}\Big) \\
&\leqslant \frac{\lambda_{n+1}}{\lambda_n}, \quad n \in \mathbb{N},
\end{aligned}$$

where $E^{(n)}_{x_0,x_0}$ is the measure on $[0, \|T^n\|^2]$ defined for every Borel set $\mathcal{B} \subset [0, \|T^n\|^2]$ by

$$E^{(n)}_{x_0,x_0}(\mathcal{B}) = \langle E^{(n)}(\mathcal{B}) x_0, x_0\rangle, \quad n \in \mathbb{N}. \tag{2.7}$$

Finally, (v) is just (iv) rewritten using (2.4) and (2.6). ∎

To continue we need the following elementary and well known lemma.

LEMMA 2.3. *If $\{\lambda_n\}_{n\in\mathbb{N}}$ is any sequence of positive numbers, then there exists a subsequence $\{n_m\}$ of $\mathbb{N}$ such that*

$$\limsup_m \Big(\frac{\lambda_{n_m+1}}{\lambda_{n_m}}\Big) \leqslant \liminf_n (\lambda_n)^{1/n}. \tag{2.8}$$

We continue with the basic construction in the form of another lemma.

LEMMA 2.4. *With the notation as established above, let $T \notin \mathbb{C}1_{\mathcal{H}}$ be a quasiaffinity of norm one. Then there exists a subsequence $\{n_m\} \subset \mathbb{N}$ such that* (2.8) *holds and the sequences $\{E^{(n_m)}_{\lambda^-_{n_m}}\}_{m\in\mathbb{N}}$ and $\{E^{(n_m+1)}_{\lambda_{n_m+1}}\}_{m\in\mathbb{N}}$ of projections from* (2.2) *and* (2.3) *converge in the weak operators topology* (WOT) *to positive semidefinite operator $P^-_0 \neq 1_{\mathcal{H}}$ and $P_0 \neq 0$, respectively. Moreover, if we define*

$$s_m = (1_{\mathcal{H}} - E^{(n_m)}_{\lambda^-_{n_m}})x_0, \quad t_m = T^* E^{(n_m+1)}_{\lambda_{n_m}+1} x_0, \quad m \in \mathbb{N},$$

then the following are valid:

(A) *the sequences $\{s_m\}_{m\in\mathbb{N}}$ and $\{t_m\}_{m\in\mathbb{N}}$ are weakly convergent to the nonzero vectors $(1_{\mathcal{H}} - P^-_0)x_0$ and $T^*P_0x_0$, respectively, and*

(B) *the inequality $|\langle Ss_m, t_m\rangle| \leqslant \liminf\limits_n (\lambda_n)^{1/2n}$ is valid.*

Proof. Using Lemma 2.3 and the compactness of the closed unit ball in $\mathcal{L}(\mathcal{H})$ in the WOT, one can easily obtain, by dropping down to three successive subsequences, a subsequence $\{n_m\}$ of $\mathbb{N}$ such that (2.8) holds and the sequences $\{E^{(n_m)}_{\lambda^-_{n_m}}\}_{m\in\mathbb{N}}$ and $\{E^{(n_m+1)}_{\lambda_{n_m+1}}\}_{m\in\mathbb{N}}$ converge in the WOT to P^-_0 and P_0, respectively. The fact that $(1_{\mathcal{H}} - P^-_0)x_0$ and P_0x_0 are nonzero is a consequence of the inequalities (i) and (ii) of Lemma 2.1. Finally, (B) is a consequence of (2.8) and (iv) and (v) in Lemma 2.2. ∎

The results obtained in [9] and [3] followed from Corollary 3.3 in [3] (and its earlier version in [9]). To see that the present construction yields those same results, we immediately establish that corollary. Here, for the first time, the quasinilpotence of the operator under consideration comes into play.

COROLLARY 2.5 ([3], Corollary 3.3). *With Q a quasinilpotent quasiaffinity of norm one in $\mathcal{L}(\mathcal{H})$ and the notation as in the above lemmas, let $\{A_{k,m}\}$ and $\{B_{k,m}\}$ be any doubly indexed sequences of contractions in $\mathcal{L}(\mathcal{H})$. Then*

$$\lim_m \langle Q^{n_m} A_{k,m} y_{n_m}, Q^* z_{n_m}\rangle = 0, \quad k \in \mathbb{N},$$

and

$$\lim_m \Big\langle \Big(\frac{Q^{n_m}}{\|Q^{n_m}\|}\Big) B_{k,m} Q^{n_m} y_{n_m}, Q^* z_{n_m} \Big\rangle = 0, \quad k \in \mathbb{N}.$$

Proof. We have

$$|\langle Q^{n_m} A_{k,m} y_{n_m}, Q^* z_{n_m}\rangle| \leqslant \|y_{n_m}\| \|(Q^*)^{n_m+1} z_{n_m}\| \leqslant \Big(\frac{\lambda_{n_m+1}}{\lambda_{n_m}}\Big)^{1/2}$$

from Lemma 2.2, and, similarly,

$$\Big|\Big\langle \Big(\frac{Q^{n_m}}{\|Q\|^{n_m}}\Big) B_{k,m} Q^{n_m} y_{n_m}, Q^* z_{n_m} \Big\rangle\Big| \leqslant \|y_{n_m}\| \|(Q^*)^{n_m+1} z_{n_m}\| \leqslant \Big(\frac{\lambda_{n_m+1}}{\lambda_{n_m}}\Big)^{1/2},$$

so the result follows from the fact that the sequence $\{\lambda_{n_m+1}/\lambda_{n_m}\}_{m\in\mathbb{N}}$ converges to zero. ∎

3. HYPERINVARIANT SUBSPACES

We now begin to use the above construction to obtain some new hyperinvariant-subspace results for quasinilpotent operators in $\mathcal{L}(\mathcal{H})$. In all that follows, the notation $(\mathcal{Q})$ will consistently be used to denote the set of all nonzero quasinilpotent operators in $\mathcal{L}(\mathcal{H})$. What makes this technique (and its predecessor from [1]) useful is that sometimes one of the sequences of vectors from (A) of Lemma 2.4 can be made to converge strongly.

THEOREM 3.1. *Suppose that $Q \in (\mathcal{Q})$ and note that, without loss of generality, we may suppose that Q is a quasiaffinity of norm one. With the notation as established in Lemma* 2.4 *above, if either of the sequences of vectors $\{s_m\}$ or $\{t_m\}$ has a subsequence that converges strongly (which will certainly be the case if either of the sequences $\{E^{(n_m)}_{\lambda^-_{n_m}}\}_{m\in\mathbb{N}}$ or $\{E^{(n_m+1)}_{\lambda_{n_m+1}}\}_{m\in\mathbb{N}}$ has a subsequence that converges in the* SOT*), then the operator Q has a* n.h.s.

Proof. One knows that if $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are sequences in $\mathcal{H}$ converging weakly to u_0 and v_0, respectively, such that $\{\langle u_n, v_n\rangle\}_{n\in\mathbb{N}} \to 0$ and either $\{u_n\}_{n\in\mathbb{N}}$ or $\{v_n\}_{n\in\mathbb{N}}$ converges in norm, then $\langle u_0, v_0\rangle = 0$. Therefore, since Q is quasinilpotent, by using the subsequence $\{n_m\}$ from Lemmas 2.2 and 2.4 and Lemma 2.1(iii), one gets immediately that

$$\langle S(1_{\mathcal{H}} - P_0^-)x_0, Q^*P_0x_0\rangle = 0,$$

for every contraction S from $\{Q\}'$. This shows that $(\{Q\}'(1_{\mathcal{H}} - P_0^-)x_0)^-$ is orthogonal to the nonzero vector $Q^*P_0x_0$ and thus is the desired n.h.s. for Q. ∎

COROLLARY 3.2. *Suppose that $Q \in (\mathcal{Q})$, $\|Q\| = 1$, and with the notation as in Lemma* 2.4, *either some subsequence of the sequence $\{1_{\mathcal{H}} - E^{(n_m)}_{\lambda^-_{n_m}}\}_{m\in\mathbb{N}}$ or some subsequence of the sequence $\{E^{(n_m)}_{\lambda_{n_m}}\}_{m\in\mathbb{N}}$ is monotone increasing or monotone decreasing. Then Q has a* n.h.s.

Proof. One knows that monotone sequences of projections converge in the SOT. ∎

THEOREM 3.3. *Suppose that$Q \in (\mathcal{Q})$ and that there exists a finite dimensional subspace $\mathcal{M} \neq (0)$ of $\mathcal{H}$ that is invariant for (equivalently, reduces) each member of the sequence $\{Q^nQ^{*n}\}_{n\in\mathbb{N}}$. Then Q has a* n.h.s.

Proof. Without loss of generality we may suppose that Q is a quasiaffinity of norm one. By the spectral theorem, $\mathcal{M}$ reduces all of the projections that are values of any of the spectral measures $E^{(n)}$. Thus we choose a vector x_0 in $\mathcal{M}$ and observe that each vector s_m (from Theorem 3.1) belongs to $\mathcal{M}$. Since $\mathcal{M}$ is finite dimensional, the convergence of the sequence $\{s_m\}_{m\in\mathbb{N}}$ to $(1_{\mathcal{H}} - P_0^-)x_0$

coming from Lemma 2.4 is norm convergence, and the result follows from Theorem 3.1. ∎

COROLLARY 3.4. *Suppose that* $Q \in (\mathcal{Q})$ *and that the operators in the sequence* $\{Q^n Q^{*n}\}_{n\in\mathbb{N}}$ *have a common eigenvector* x_0. *Then* Q *has a* n.h.s.

Proof. Take $\mathcal{M} = \mathbb{C}x_0$ in Theorem 3.3. ∎

This result solves Problem 4.1 of [3]. We will reconsider quasinilpotent operators with this property in Section 4.

THEOREM 3.5. *Suppose* $Q \in (\mathcal{Q})$ *is such that* Q^*Q *has a cyclic vector and* Q^*Q *commutes with* QQ^*. *Suppose also that some* $Q^{k_0}Q^{*k_0}$ *has an eigenvalue of finite multiplicity. Then* Q *has a* n.h.s.

Proof. Without loss of generality we may assume that Q is a quasiaffinity of norm one. It follows from Proposition 3.4 in [12] that the sequence $\{Q^n Q^{*n}\}_{n\in\mathbb{N}}$ consists of mutually commuting operators and the finite dimensional eigenspace of $Q^{k_0}Q^{*k_0}$ is a n.h.s. for $Q^{k_0}Q^{*k_0}$, and thus reduces all the commuting operators $Q^k Q^{*k}$, $k \in \mathbb{N}$. The result thus follows from Theorem 3.3. ∎

REMARK 3.6. Note that to apply Theorem 3.5, it is not necessary that the eigenvalue of $Q^{k_0}Q^{*k_0}$ of finite multiplicity be an isolated point of $\sigma(Q^{k_0}Q^{*k_0})$.

The following construction shows that we are not operating in a vacuum, i.e., that there are operators in $(\mathcal{Q})$ other than weighted unilateral or bilateral shifts to which the above results may apply. Let $((0,1], \mathcal{B}, \mu)$ be the measure space consisting of Lebesgue measure on the Borel subsets of the half-open unit interval, $(0,1]$, and let $\alpha \in (0,1)$ be irrational. Consider the invertible, ergodic, measure preserving transformation T_α on $((0,1], \mathcal{B}, \mu)$ defined by

$$T_\alpha(x) = \begin{cases} x+\alpha & x \in (0, 1-\alpha], \\ x-1+\alpha & x \in (1-\alpha, 1]; \end{cases}$$

write $T_\alpha^{(k)}$ for the k-fold composition of T_α with itself, and recall the following result [13].

PROPOSITION 3.7 (Weyl). *Suppose* $\varphi : (0,1] \to (0,1]$ *is Riemann integrable, and* $\alpha \in (0,1)$ *is irrational. Then the sequence of functions*

$$c_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} (\varphi \circ T_\alpha^{(k)})(x), \quad x \in (0,1],\ n \in \mathbb{N},$$

converges uniformly on $(0,1]$ *to the constant function* $\int_0^1 \varphi \, \mathrm{d}\mu$.

For $\alpha \in (0,1)$, let $U_\alpha \in \mathcal{L}(L^2(0,1], \mathcal{B}, \mu)$ denote the unitary operator defined by

$$(U_\alpha g)(x) = g(T_\alpha(x)), \quad g \in L^2(0,1], x \in (0,1].$$

Moreover, for $\varphi \in L^\infty((0,1],\mathcal{B},\mu)$, let $M_\varphi \in \mathcal{L}(L^2(0,1],\mathcal{B},\mu)$ denote, as usual, the operator of multiplication by φ.

THEOREM 3.8. *Suppose $\alpha \in (0,1)$ is irrational, φ is a homeomorphism of $(0,1]$ onto itself and $\int_{(0,1]} \ln \varphi \mathrm{d}\mu = -\infty$. Then the operator $Q = M_\varphi U_\alpha$ in $\mathcal{L}(L^2(0,1],\mathcal{B},\mu)$ is quasinilpotent and has the property that the sequence $\{Q^n Q^{*n}\}_{n\in\mathbb{N}}$ consists of mutually commuting operators.*

Proof. An easy calculation shows that

$$QQ^* = M_{\varphi^2}$$

and a somewhat tedious calculation shows that, with $\psi = \varphi^2$,

$$Q^n Q^{*n} = M_{\psi(\psi\circ T_\alpha)(\psi\circ T_\alpha^{(2)})\cdots(\psi\circ T_\alpha^{(n-1)})}, \quad n \in \mathbb{N},$$

and

$$\begin{aligned}\|Q^n\| &= \|M_{\varphi(\varphi\circ T_\alpha)(\varphi\circ T_\alpha^{(2)})\cdots(\varphi\circ T_\alpha^{(n-1)})}\| \\ &= \|\varphi(\varphi\circ T_\alpha)(\varphi\circ T_\alpha^{(2)})\cdots(\varphi\circ T_\alpha^{(n-1)})\|_\infty.\end{aligned}$$

Thus $\{Q^n Q^{*n}\}_{n\in\mathbb{N}}$ consists of mutually commuting operators. Moreover, since φ is a homeomorphism of $(0,1]$ and for each $k \in \mathbb{N}$, $T_\alpha^{(k)}$ has both right and left hand limits at each point $x \in (0,1)$ and one-sided limits at 0 and 1, the product

$$\varphi(\varphi\circ T_\alpha)(\varphi\circ T_\alpha^{(2)})\cdots(\varphi\circ T_\alpha^{(n-1)})$$

has these same properties. Thus, for each $n \in \mathbb{N}$, we may choose $x_n \in (0,1]$ so that

$$\lim_{x\to x_n} \varphi(x)((\varphi\circ T_\alpha)(x))\cdots((\varphi\circ T_\alpha^{(n-1)})(x)) = \|\varphi(\varphi\circ T_\alpha)\cdots(\varphi\circ T_\alpha^{(n-1)})\|_\infty.$$

To show that Q is quasinilpotent it clearly suffices to show that

$$\lim_n \frac{1}{n}\ln\|Q^n\| = -\infty,$$

or, equivalently, that

$$\limsup_{n\to\infty}\Big\{\frac{1}{n}\ln\lim_{x\to x_n}\{\varphi(x)((\varphi\circ T_\alpha)(x))\cdots((\varphi\circ T_\alpha^{(n-1)})(x))\}\Big\} = -\infty. \tag{3.1}$$

But the left hand side of (3.1) is clearly equal to

$$\limsup_{n\to\infty}\Big\{\lim_{x\to x_n}\frac{1}{n}\sum_{k=0}^{n-1}\ln\{(\varphi\circ T_\alpha^{(k)})(x)\}\Big\}. \tag{3.2}$$

Let M be any positive number. It suffices to show that there exists $N_1 \in \mathbb{N}$ such that for all $n \geqslant N_1$ and all $x \in (0,1]$,

$$\frac{1}{n}\sum_{k=0}^{n-1}\ln\{(\varphi\circ T_\alpha^{(k)})(x)\} \leqslant -M.$$

Choose next a monotone sequence $\varepsilon_m \searrow 0$, and define, for each $m \in \mathbb{N}$, the function φ_m by

$$\varphi_m(x) = \max\{\varphi(x), \varepsilon_m\}.$$

Obviously, $\{\varphi_m\}_{m\in\mathbb{N}}$ is a monotone decreasing sequence of Riemann integrable functions that converges to φ pointwise on $(0,1]$, so by the Monotone Convergence Theorem,

$$\lim_{m\to\infty} \int\limits_{(0,1]} \ln \varphi_m \mathrm{d}\mu = \int\limits_{(0,1]} \ln \varphi \mathrm{d}\mu = -\infty.$$

Choose $N_0 \in \mathbb{N}$ sufficiently large that for $m \geqslant N_0$,

$$\int\limits_{(0,1]} \ln \varphi_m \mathrm{d}\mu \leqslant -2M.$$

By Proposition 3.7, the sequence of functions

$$\Big\{ c_{n,N_0} = \frac{1}{n} \sum_{k=0}^{n-1} \ln\{(\varphi_{N_0} \circ T_\alpha^{(k)})\} \Big\}_{n\in\mathbb{N}}$$

converges uniformly on $(0,1]$ to

$$\int\limits_{(0,1]} \ln \varphi_{N_0} \mathrm{d}\mu \leqslant -2M.$$

Thus there exists $N_1 \in \mathbb{N}$ such that for all $n \geqslant N_1$ and all $x \in (0,1]$, $c_{n,N_0}(x) \leqslant -M$. Since $\varphi \leqslant \varphi_{N_0}$ this gives us that for all $n \geqslant N_1$ and all $x \in (0,1]$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \ln\{(\varphi \circ T_\alpha^{(k)})(x)\} \leqslant -M,$$

so (3.1) is valid. ∎

REMARK 3.9. The operator $Q = U_\alpha M_\varphi$ of Theorem 3.8 is amenable to be studied via the construction of Section 2. Moreover it is clear from the definitions of φ and T_α above that $\sigma_{\mathrm{p}}(Q^n Q^{*n}) = \emptyset$ for all $n \in \mathbb{N}$. Thus the only results of this paper that might be applicable to produce a n.h.s. for Q are Theorem 3.1 and Corollary 3.2. Unfortunately, despite considerable effort, the authors were unable to decide whether either result is applicable to this operator Q, due to the difficulty in calculating the sequences of projections $\{E^{(n)}_{\lambda_n^-}\}$ and $\{E^{(n)}_{\lambda_n}\}$. Thus we pose, once again, the following problem.

PROBLEM 3.10. Show that every $Q \in (\mathcal{Q})$ such that the operators in the family $\{Q^n Q^{*n}\}_{n\in\mathbb{N}}$ all commute with one another has a n.h.s. In this connection, see Corollary 3.5 and Theorem 5.5.

4. A STRUCTURE THEOREM

In this section we establish a structure theorem which shows that we can say more about the operators treated in Corollary 3.4. If $\mathcal{M}, \mathcal{N} \in \mathrm{Lat}(Q)$ with $\mathcal{M} \supset \mathcal{N}$, we write, as usual, $Q_{\mathcal{M}\ominus\mathcal{N}}$ for the compression of Q to the semi-invariant subspace $\mathcal{M} \ominus \mathcal{N}$.

THEOREM 4.1. *Suppose $Q \in (\mathcal{Q})$ is quasiaffinity. Suppose also that the operators in the sequence $\{Q^n Q^{*n}\}_{n\in\mathbb{N}}$ have a common eigenvector w_0. Then the subspace $\mathcal{N} = (\{Q\}' w_0)^-$ is a* n.h.s. *for Q, and there exists a (strictly) larger $\mathcal{M} \in Lat(Q)$ such that if we write $\mathcal{H} = \mathcal{N} \oplus (\mathcal{M} \ominus \mathcal{N}) \oplus \mathcal{M}^\perp$, the corresponding operator matrix for Q has the form*

$$Q = \begin{pmatrix} Q|\mathcal{N} & Q_{12} & Q_{13} \\ 0 & Q_{\mathcal{M}\ominus\mathcal{N}} & 0 \\ 0 & 0 & Q_{33} \end{pmatrix},$$

where $Q_{\mathcal{M}\ominus\mathcal{N}}$ is a backward weighted shift of multiplicity 1.

Proof. We have, by hypothesis,

$$Q^n Q^{*n} w_0 = \mu_n w_0, \quad \|w_0\| = 1,\ n \in \mathbb{N}, \tag{4.1}$$

and since Q is a quasinilpotent quasiaffinity, we know that $0 < \mu_n \leqslant \|Q^n\|^2$ and that $(\mu_n)^{1/n} \to 0$. Choose now, by Lemma 2.3, a subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}} \subset \{\mu_n\}$ such that $\mu_{n_k+1}/\mu_{n_k} \to 0$ as $k \to \infty$. We first show that if X is arbitrary in $\{Q\}'$, then $\langle Xw_0, Q^* w_0\rangle = 0$. We write

$$\begin{aligned}
|\langle Xw_0, Q^*w_0\rangle|^2 &= \frac{1}{\mu_{n_k}^2} |\langle XQ^{n_k}Q^{*n_k}w_0, Q^*w_0\rangle|^2 \\
&= \frac{1}{\mu_{n_k}^2} |\langle XQ^{*n_k}w_0, (Q^*)^{n_k+1}w_0\rangle|^2 \\
&\leqslant \frac{1}{\mu_{n_k}^2} \|X\|^2 \|Q^{*n_k}w_0\|^2 \|(Q^*)^{n_k+1}w_0\|^2 \\
&= \frac{1}{\mu_{n_k}^2} \|X\|^2 \langle Q^{n_k}Q^{*n_k}w_0, w_0\rangle \langle Q^{n_k+1}(Q^*)^{n_k+1}w_0, w_0\rangle \\
&= \frac{1}{\mu_{n_k}^2} \|X\|^2 \mu_{n_k}\mu_{n_k+1} \\
&= \frac{\mu_{n_k+1}}{\mu_{n_k}} \|X\|^2,
\end{aligned}$$

so we conclude that

$$\langle Xw_0, Q^*w_0\rangle = 0, \quad X \in \{Q\}'. \tag{4.2}$$

Since $Q^*w_0 \neq 0$, this proves that $\mathcal{N} = (\{Q\}'w_0)^-$ is a n.h.s. for Q. Moreover, since

$$\langle Xw_0, Q^{*k+1}w_0\rangle = \langle Q^k Xw_0, Q^*w_0\rangle = 0, \quad k \in \mathbb{N}_0, X \in \{Q\}', \tag{4.3}$$

we see that the (nonzero) cyclic invariant subspace

$$\mathcal{C} := \bigvee_{n\in\mathbb{N}} (Q^*)^n w_0$$

for Q^* satisfies $\mathcal{C} \subset \mathcal{N}^\perp$. We next show that $\mathcal{M} = \mathcal{N} \oplus \mathcal{C}$ also belongs to $\mathrm{Lat}(Q)$, i.e., that $Q\mathcal{C} \subset \mathcal{N} \oplus \mathcal{C}$. An easy computation based on (4.1) and (4.3) (with $X = I$) shows that

$$\langle Q^{*j} w_0, Q^{*k} w_0 \rangle = 0, \quad j,k \in \mathbb{N},\ j \neq k,$$

and thus that $\{Q^{*n}w_0/\|Q^{*n}w_0\|\}_{n\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{C}$. Furthermore, the computation

$$Q^{k+1}Q^{*k+1}w_0 = \frac{\mu_{k+1}}{\mu_k} Q^k Q^{*k} w_0, \quad k \in \mathbb{N}_0,$$

together with the fact that Q^k is a quasiaffinity, gives that

$$Q\Big(\frac{Q^{*k+1}w_0}{\|Q^{*k+1}w_0\|}\Big) = \frac{\mu_{k+1}}{\mu_k} \frac{\|Q^{*k}w_0\|}{\|Q^{*k+1}w_0\|} \frac{Q^{*k}w_0}{\|Q^{*k}w_0\|}, \quad k \in \mathbb{N},$$

which shows, simultaneously, that $Q\mathcal{C} \subset \mathcal{C} \vee \mathbb{C}w_0$, and that $Q_{\mathcal{C}} = Q_{\mathcal{M}\ominus\mathcal{N}}$ is a backward weighted unilateral shift of multiplicity one with weight sequence

$$\Big\{ \frac{\mu_{k+1}}{\mu_k} \frac{\|Q^{*k}w_0\|}{\|Q^{*k+1}w_0\|} \Big\}.$$

Of course, this also shows that relative to the decomposition $\mathcal{H} = \mathcal{N} \oplus (\mathcal{M} \ominus \mathcal{N}) \oplus \mathcal{M}^\perp$, the matrix for Q is as indicated above. ∎

5. A REDUCTION

In this section we will show that the question whether every operator in $(\mathcal{Q})$ has a n.h.s. can be reduced to that question for a special subclass of $(\mathcal{Q})$. However we must begin with some preparatory material. If n is any cardinal number satisfying $1 \leqslant n \leqslant \aleph_0$, we will denote by $\mathcal{H}^{(n)}$ the direct sum of n copies of $\mathcal{H}$ indexed by the appropriate initial segment of $\mathbb{N}_0$. Moreover for any T in $\mathcal{L}(\mathcal{H})$ we will denote by $T^{(n)}$ the direct sum (ampliation) of n copies of T acting on $\mathcal{H}^{(n)}$ in the obvious fashion. We recall from [5] that if T_1 and T_2 are operators in $\mathcal{L}(\mathcal{H})$ and there exist cardinal numbers m and n with $1 \leqslant m, n \leqslant \aleph_0$ such that $T_1^{(m)}$ is quasisimilar to $T_2^{(n)}$, then T_1 and T_2 are said to be *ampliation quasisimilar* (notation: $T_1 \overset{\mathrm{a}}{\sim} T_2$). The following is Theorem 2.4 of [5].

PROPOSITION 5.1. *The relation $\overset{\mathrm{a}}{\sim}$ is an equivalence relation on $\mathcal{L}(\mathcal{H})$. Moreover, if T_1 and T_2 are operators in $\mathcal{L}(\mathcal{H})$ and $T_1 \overset{\mathrm{a}}{\sim} T_2$, then T_1 has a* n.h.s. *if and only if T_2 has a* n.h.s.

We will also need a deep theorem of Apostol-Herrero [2], [7] from the theory of closures of similarity orbits of operators. Let us call a Q_0 in $(\mathcal{Q})$ a *universal quasinilpotent* if for every Q in $(\mathcal{Q})$ there exists a sequence $\{S_n\}$ of invertible operators in $\mathcal{L}(\mathcal{H})$ such that

$$\lim_n \|S_n Q_0 S_n^{-1} - Q\| = 0. \tag{5.1}$$

THEOREM 5.2 (Apostol-Herrero). *If $Q_0 \in (\mathcal{Q})$ and no (positive, integral) power of Q_0 is a nonzero compact operator, then Q_0 is a universal quasinilpotent.*

COROLLARY 5.3. *Every Q in $(\mathcal{Q})$ such that Q is not a universal quasinilpotent quasiaffinity has a* n.h.s.

Proof. If Q is not a quasiaffinity then either kernel Q or kernel Q^* is an n.h.s. for Q, so we may suppose that Q is a quasiaffinity. If $Q^{n_0} = K \in \mathbb{K}$ for some $n_0 \in \mathbb{N}$, then there are two cases to consider. If $K = 0$, the kernel of Q is a n.h.s. for Q, and if $K \neq 0$, then Q has a n.h.s. by Lomonosov's theorem [10]. ∎

As a last tool we need a well known fact from Fredholm theory; cf., for example, Chapter I of [11].

LEMMA 5.4. *Let P_0 be an arbitrary compact, positive semidefinite, quasiaffinity in $\mathcal{L}(\mathcal{H})$ and let $\{P_k\}_{k\in\mathbb{N}}$ be an arbitrary sequence of positive semidefinite operators in $\mathcal{L}(\mathcal{H})$ such that $\|P_k - P_0\| \to 0$. Then for every $\varepsilon > 0$, there exists $K_\varepsilon \in \mathbb{N}$ such that for all $k \geqslant K_\varepsilon$, $r_{\mathrm{e}}(P_k) < \varepsilon$ and $\sigma(P_k) \cap (\varepsilon, +\infty)$ is a nonempty set consisting of a finite number of eigenvalues of P_k (including $\lambda_k = \|P_k\|$), each of finite multiplicity.*

The following corollary may be useful in attempts to solve the invariant subspace problem for operators in $(\mathcal{Q})$, and complements Corollary 5.3 above.

COROLLARY 5.5. *Let Q be a universal element of $(\mathcal{Q})$ and let $0 < \delta < \widetilde{\delta}$. Then there exists $\widehat{Q} = \widehat{U}\widehat{P}$ (polar decomposition) similar to Q such that*

$$0 < \|\widehat{P}\|_{\mathrm{e}} < \delta < \|\widehat{P}\| < \widetilde{\delta},$$

and consequently $\sigma(\widehat{P}) \cap (\delta, \widetilde{\delta})$ consists of a finite number of eigenvalues of $\widehat{P}$, each of finite multiplicity.

Proof. Let Q_1 be a compact operator in $(\mathcal{Q})$ such that $\|Q_1\| \in (\delta, \widetilde{\delta})$. Since Q is universal, there exists a sequence $\{S_k\}$ of invertible operators in $\mathcal{L}(\mathcal{H})$ such that

$$\|S_k Q S_k^{-1} - Q_1\| \longrightarrow 0$$

and therefore that, if we write $P_k := (S_k Q S_k^{-1})^*(S_k Q S_k^{-1})$, then $\|P_k - Q_1^* Q_1\| \to 0$. Thus

$$\|P_k\|^{1/2} \longrightarrow \|(Q_1^* Q_1)^{1/2}\| = \|Q_1\| \in (\delta, \widetilde{\delta}),$$

and since $r_{\mathrm{e}}(P_k) \to r_{\mathrm{e}}(Q_1^* Q_1) = 0$, the result follows from Lemma 5.4. ∎

Let us denote the class of quasinilpotent quasiaffinities Q in $\mathcal{L}(\mathcal{H})$ such that $(Q^*Q)^{1/2}$ has an infinite dimensional reducing subspace $\mathcal{M}$ with $Q^*Q|\mathcal{M}$ compact by $(\mathcal{CRQ})$.

The following is our reduction of the hyperinvariant subspace problem for quasinilpotent operators.

THEOREM 5.6. *Every universal quasiaffinity Q in $(\mathcal{Q})$ is ampliation quasisimilar to a quasiaffinity $\widehat{Q} \in (\mathcal{CRQ})$. Consequently (by Proposition* 5.1 *and Corollary* 5.3*), if every quasiaffinity $\widehat{Q} \in (\mathcal{CRQ})$ has a* n.h.s.*, then every $Q \in (\mathcal{Q})$ has a* n.h.s.

Proof. We begin with an arbitrary nonzero quasinilpotent operator Q in $\mathcal{L}(\mathcal{H})$ for which we wish to find a n.h.s., and by virtue of Corollary 5.3 we may suppose that Q is a quasiaffinity and universal. Choose monotone sequences $\{\delta_n\}$ and $\{\widetilde{\delta}_n\}$ of positive numbers tending to zero and satisfying

$$\widetilde{\delta}_0 > \delta_0 > \widetilde{\delta}_1 > \delta_1 > \cdots > \widetilde{\delta}_n > \delta_n > \cdots . \tag{5.2}$$

Then, by Corollary 5.5, there exists a sequence $\{S_k\}$ of invertible operators in $\mathcal{L}(\mathcal{H})$ such that, if we write $S_kQS_k^{-1} = U_kP_k$ (polar decomposition), $k \in \mathbb{N}_0$, we have that $\sigma(P_k) \cap (\delta_k, \widetilde{\delta}_k)$ is a finite nonempty set of eigenvalues of P_k, each of finite multiplicity. Since $\|S_kQS_k^{-1}\| = \|P_k\| < \widetilde{\delta}_0$ for all $k \in \mathbb{N}_0$, the operator

$$\widehat{Q} = \bigoplus_{k\in\mathbb{N}_0} S_kQS_k^{-1} = \bigoplus_{n\in\mathbb{N}_0} U_kP_k \in \mathcal{L}(\mathcal{H}^{(\aleph_0)}). \tag{5.3}$$

Moreover, since for every $\varepsilon > 0$, $\widehat{Q}$ can be written as $\widehat{Q} = Q_\varepsilon^{(1)} \oplus Q_\varepsilon^{(2)}$, where $Q_\varepsilon^{(1)} \in (\mathcal{Q})$ and $\|Q_\varepsilon^{(2)}\| < \varepsilon$, it is obvious that $\widehat{Q}$ is quasinilpotent. Clearly $Q \overset{\text{a}}{\sim} \widehat{Q}$, so by Proposition 5.1, Q has a n.h.s. if and only if $\widehat{Q}$ does. Finally, from (5.3) we see that

$$(\widehat{Q}^*\widehat{Q})^{1/2} = \bigoplus_{k\in\mathbb{N}_0} P_k, \tag{5.4}$$

and since $\|P_k\| \to 0$ and each P_k has a finite dimensional reducing subspace, namely, the spectral subspace of P_k corresponding to the finite Borel set $\sigma(P_k) \cap (\delta_k, \widetilde{\delta}_k)$ from above, we get the desired conclusion. ∎

REMARK 5.7. Note that the spectral projection of the operator $(\widehat{Q}^*\widehat{Q})^{1/2}$ in (5.4) corresponding to the Borel set $(\delta_0, \widetilde{\delta}_0)$ is a finite rank projection on the span of the eigenspaces of P_0 corresponding to the eigenvalues lying in $(\delta_0, \widetilde{\delta}_0)$. Thus $(\widehat{Q}^*\widehat{Q})^{1/2}$ has a finite dimensional (reducing) *spectral* subspace.

REMARK 5.8. It is obvious that the quasinilpotent operator $Q = U_\alpha M_\varphi$ in Theorem 3.8 has the property that $Q^*Q = M_{\varphi^2}$ has no eigenvalues, so certainly not every quasinilpotent quasiaffinity belongs to $(\mathcal{CRQ})$. Moreover, it is known (cf., e.g., Chapter IX of [11]) that *not* every quasinilpotent operator Q in $\mathcal{L}(\mathcal{H})$ commutes with a nonzero compact operator. Theorem 5.6 naturally brings to mind the question whether if $S \overset{\mathrm{a}}{\sim} T$ and T commutes with a nonzero compact operator, does S necessarily commute with a nonzero compact operator? The following theorem shows that the answer is yes.

THEOREM 5.9. *Suppose* $S, T \in \mathcal{L}(\mathcal{H})$, $S \overset{\mathrm{a}}{\sim} T$, *and* $K \in \mathbb{K}\backslash(0)$ *satisfies* $TK = KT$. *Then there exists* $\widetilde{K} \in \mathbb{K}\backslash(0)$ *such that* $S\widetilde{K} = \widetilde{K}S$.

Proof. By definition, there are cardinal numbers m and n satisfying $1 \leqslant m, n \leqslant \aleph_0$ such that $S^{(m)}$ is quasisimilar to $T^{(n)}$. Since $TK = KT$, it is obvious that there exists a nonzero compact operator $K_1 \in \mathcal{L}(\mathcal{H}^{(n)})$ such that $K_1 T^{(n)} = T^{(n)} K_1$. Furthermore an easy calculation shows that quasisimilarity preserves the property of commuting with a nonzero compact operator, and thus there is such a $K_2 \in \mathcal{L}(\mathcal{H}^{(m)})$ that commutes with $S^{(m)}$. Finally, another easy computation shows that $\{S^{(m)}\}'$ consists of all $m \times m$ operator matrices (Z_{ij}) acting on $\mathcal{H}^{(m)}$ in the obvious way such that $Z_{ij} \in \{S\}'$ for all i, j. Thus $(Z_{ij}) = K_2$ is such a matrix, and clearly then every nonzero Z_{ij} is compact and commutes with S. ∎

Theorem 5.9 would seem to indicate that the result established in Theorem 5.6 may be near best possible.

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