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SECTIONAL CURVATURE AND COMMUTATION OF PAIRS OF SELFADJOINT OPERATORS

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To the memory of Angel Rafael Larotonda

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ABSTRACT. The space $\mathcal{G}^+$ of postive invertible operators of a C^*-algebra $\mathcal{A}$, with the appropriate Finsler metric, behaves like a (non positively curved) symmetric space. Among the characteristic properties of such spaces, one has that two selfadjoint elements $x, y \in \mathcal{A}$ (regarded as tangent vectors at $a \in \mathcal{G}^+$) verify that

$$\|x - y\|_a \leqslant d(\exp_a(x), \exp_a(y)).$$

In this paper we investigate the ocurrence of the equality

$$\|x - y\|_a = d(\exp_a(x), \exp_a(y)).$$

If $\mathcal{A}$ has a trace, and the trace is used to measure tangent vectors then, as in the finite dimensional classical setting, this equality is equivalent to the fact that x and y commute. In arbitrary C^*-algebras, when the usual C^*-norm is used, the equality is equivalent to a weaker condition. We introduce in $\mathcal{G}^+$ an analogous of the sectional curvature for pairs of selfadjoint operators, and study the vanishing of this invariant.

KEYWORDS: *Positive operator, selfadjoint operator, sectional curvature.*

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1. INTRODUCTION

Let $\mathcal{A}$ be a unital C^*-algebra, let $\mathcal{G}$ the group of invertibles of $\mathcal{A}$, and let $\mathcal{G}^+$ the set of *positive* elements of $\mathcal{G}$. The space $\mathcal{G}^+$ admits a rich geometric structure [2]. It is a differentiable manifold modelled in $\mathcal{A}$ (in fact an open subset of the real Banach space $\mathcal{A}_h$ of hermitian elements of $\mathcal{A}$), carries a transitive left action of the Banach-Lie group $\mathcal{G}$,

$$g \cdot a = gag^*, \quad g \in \mathcal{G}, a \in \mathcal{G}^+$$

and an invariant Finsler metric $\| \cdot \|_a$,

$$\|x\|_a = \|a^{-1/2} x a^{-1/2}\|,$$

for $x \in \mathcal{A}_h$, which identifies with the tangent space $(T\mathcal{G}^+)_a$. The connection induced by the action in a natural way, which is compatible with the metric, is given by

$$\frac{\mathrm{D}X}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t} - \frac{1}{2}(\dot{\gamma}\gamma^{-1}X + X\gamma^{-1}\dot{\gamma}),$$

where X is a tangent (i.e. hermitian) field along $\gamma \in \mathcal{G}^+$. The curvature tensor is given by

$$R_a(x,y)z = -\frac{1}{4}a[[a^{-1}x, a^{-1}y], a^{-1}z],$$

for $x, y, z \in \mathcal{A}_h$.

If $\mathcal{A} = M_n(\mathbb{C})$ is the algebra of $n \times n$ matrices, and one replaces the C^*-norm by the trace norm, then the connection is the Levi-Civita connection of the (Riemannian) metric $g_a(x,y) = \mathrm{tr}(xa^{-1}ya^{-1})$ (with corresponding morm $g_a(x,x)^{1/2} = \|a^{-1/2}xa^{-1/2}\|_2$). This is the well known metric which makes the set $M_n^+(\mathbb{C})$ a symmetric space [4].

Remarkably, in the infinite dimensional, non Riemannian setting, the space $\mathcal{G}^+$ has many of the structural and characteristic properties of the classical case. Let us make a short list:

(i) Two elements in $\mathcal{G}^+$ are joined by a unique minimizing geodesic [2]. More precisely, if $a, b \in \mathcal{G}^+$,

$$\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$$

is the unique geodesic with $\gamma_{a,b}(0) = a$ and $\gamma_{a,b}(1) = b$, and

$$d(a,b) = \mathrm{length}(\gamma_{a,b}) = \|\log(a^{-1/2}ba^{-1/2})\|.$$

Here log denotes the smooth logarithm of positive invertible elements.

(ii) The space $\mathcal{G}^+$ is complete, in both senses of the word: geodesics are defined for all time $t \in \mathbb{R}$, and the geodesic distance d is complete [2].

(iii) If δ and γ are two geodesics of $\mathcal{G}^+$, then the map

$$t \mapsto d(\delta(t), \gamma(t))$$

is convex [3].

(iv) If J is a Jacobi field along a geodesic γ, then the map

$$t \mapsto \|J(t)\|_{\gamma(t)}$$

is convex [3].

(v) If $x, y \in \mathcal{A}_h$ (regarded as tangent vectors at $a \in \mathcal{G}^+$), then [3]

$$\|x - y\|_a \leqslant d(\exp_a(x), \exp_a(y)). \tag{1.1}$$

In this paper we study the following problem related to the inequality (1.1): for which $x, y \in \mathcal{G}^+$ does one have *equality*,

$$\|x - y\|_a = d(\exp_a(x), \exp_a(y))?$$

In geometric terms: in which directions is the exponential map isometric? One can use the action, which is isometric, and reduce the problem to the case $a = 1$.

In the finite dimensional case, for the trace induced metric, the answer is that this equality holds if and only if x and y commute. One readily sees that this condition is sufficient in the infinite dimensional case.

In Section 2 we consider a C^*-algebra $\mathcal{A}$ with a faithful trace τ, and pose the problem for the metric induced by τ. This metric is not complete (if $\mathcal{A}$ is not finite dimensional), nevertheless, and quite unsurprisingly, the answer here is the same as in the finite dimensional case: the exponential does not distort the metric if and only if x and y commute. Although the metric is not a Riemannian metric strictly speaking, the methods here are essentially Riemannian.

In Section 3 we consider the question for the Finsler metric based on the C^*-norm. Riemannian methods are no longer available, and the answer is also less transparent: $\|x - y\| = d(\mathrm{e}^x, \mathrm{e}^y)$ if and only if there is a multiplicative functional φ in the C^*-algebra $\mathcal{A}_{x,y}$ generated by x and y such that $|\varphi(v)| = \|v\|_a$, where v is the velocity vector of the geodesic joining $a = \mathrm{e}^x$ and $b = \mathrm{e}^y$. This condition is weaker than commutation. For example, if $x = p$ and $y = q$ are projections, it is equivalent to $\|p - q\| = 1$.

One proves that non-distortion of the metric implies commutation in the finite dimensional case by means of the sectional curvature: commutation is equivalent to vanishing of the sectional curvature. In Section 4, motivated by an observation by J. Milnor in [6] (page 101), that the sectional curvature can be obtained as a limit involving norms and distances (and not the inner product), we define an analogous of the sectional curvature. We characterize the vanishing of this number. In general, it is a condition which is weaker than the non-distortion of the metric. However, if $x = p$ and $y = q$ are projections, these two notions coincide.

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2. DISTORTION IN THE 2-NORM AND COMMUTATIVITY.

Throughout this section the C^*-algebra $\mathcal{A}$ is assumed to have a faithful trace τ. We shall consider the 2-norm $\|\cdot\|_2$:

$$\|a\|_2 = \tau(a^*a)^{1/2}.$$

The trace τ enables one to define inner products in the tangent bundle of $\mathcal{G}^+$, which mimic the natural Riemannian metric on the space of positive definite finite matrices. Namely, if x, y are selfadjoint elements of $\mathcal{A}$, regarded as tangent vectors at the point $a \in \mathcal{G}^+$, put

$$g_a(x, y) = \tau(xa^{-1}ya^{-1}) = \tau(a^{-1}xa^{-1}y).$$

This metric varies smoothly with a. Note that g is invariant under the action of $\mathcal{G}$. It is not a Riemannian metric, in that it fails to be complete. Nevertheless we shall treat it as such, and employ the same terminology as in the proper Riemannian case. First note that the Riemannian connection of this metric is precisely the linear connection of $\mathcal{G}^+$ considered in [2], [3], and looks formally identical to the natural connection for positive matrices. If X is a tangent field along a curve $\gamma \in \mathcal{G}^+$ (i.e. a smooth curve of elements in $\mathcal{A}_h$), then the covariant derivative is given by

$$\frac{DX}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t} - \frac{1}{2}(\dot{\gamma}\gamma^{-1}X + X\gamma^{-1}\dot{\gamma}).$$

The proof of this fact is a straightforward verification, and is left to the reader. Therefore, the usual connection of $\mathcal{G}^+$ is the Levi-Civita connection of the metric g. It will become apparent below that the unique geodesic

$$\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2},$$

joining $a, b \in \mathcal{G}^+$, is also minimizing for the metric g.

In this section we prove that if $x, y \in \mathcal{A}_h$, then $\|x - y\|_2 = d_2(\mathrm{e}^x, \mathrm{e}^y)$ if and only if x and y commute. Here $d_2(a, b)$ denotes the geodesic distance, i.e. the minimum of the lengths of all smooth curves joining a and b in $\mathcal{G}^+$, measured with the metric g_a, $a \in \mathcal{G}^+$. Therefore, $d_2(a, b)$ equals the length of the geodesic $\gamma_{a,b}$,

$$\begin{aligned} d_2(a,b) &= \int_0^1 g_{\gamma_{a,b}}(\dot{\gamma_{a,b}}, \dot{\gamma_{a,b}})^{1/2}\mathrm{d}t = \int_0^1 \tau(\dot{\gamma_{a,b}}(\gamma_{a,b})^{-1}\dot{\gamma_{a,b}}(\gamma_{a,b})^{-1})^{1/2}\mathrm{d}t \\ &= \|\log(a^{1/2}ba^{1/2})\|_2. \end{aligned}$$

In our argument, we shall follow ideas contained in [3].

Suppose that γ is a geodesic in $\mathcal{G}^+$ and $J = J(t)$ is a Jacobi field along γ. Then J satisfies the differential equation

$$\frac{D^2 J}{\mathrm{d}t^2} + R(J, \dot{\gamma})\dot{\gamma} = 0.$$

If $x, y \in \mathcal{A}_h$, are regarded as tangent vectors of $\mathcal{G}^+$ at the point a, then the following condition (which is a non positive sectional curvature condition) holds:

$$g_a(R_a(x,y)y, x) = \tau(R_a(x,y)ya^{-1}xa^{-1}) \leqslant 0.$$

The proof of this fact is straightforward. Then

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} g_\gamma(J, J) &= 2\Big\{g_\gamma\Big(\frac{D^2 J}{\mathrm{d}t^2}\Big) + g_\gamma\Big(\frac{DJ}{\mathrm{d}t}, \frac{DJ}{\mathrm{d}t}\Big)\Big\} \\ &= 2\Big\{ - g_\gamma(R_\gamma(J, \dot{\gamma})\dot{\gamma}, J) + g_\gamma\Big(\frac{DJ}{\mathrm{d}t}, \frac{DJ}{\mathrm{d}t}\Big)\Big\} \geqslant 0. \end{aligned}$$

In other words, the smooth function $t \mapsto g_\gamma(J, J)$ is convex. This fact implies that though J may vanish, if $J(t_0) \neq 0$ for some t_0, then $J(t) \neq 0$ for any $t \geqslant t_0$. We shall need convexity of the *norm* of the Jacobi filed (and not of the *square* of the norm just proved).

PROPOSITION 2.1. *Let γ be a geodesic of $\mathcal{G}^+$ and let J a Jacobi field along γ. The map $t \mapsto g_\gamma(J,J)^{1/2}$ is convex.*

Proof. By the above argument, is suffices to prove this assertion for a field J which does not vanish. As in Theorem 1 of [3], by the invariance of the connection and the metric g under the action of $\mathcal{G}$, it suffices to consider the case of a geodesic $\gamma(t) = \mathrm{e}^{tx}$ starting at $1 \in \mathcal{G}^+$ ($x \in \mathcal{A}_h$). For the field $K(t) = \mathrm{e}^{-tx/2}J(t)\mathrm{e}^{-tx/2}$ the Jacobi equation translates into

$$4\ddot{K} = Kx^2 + x^2K - 2xKx. \tag{2.1}$$

Moreover

$$g_\gamma(J,J)^{1/2} = \tau(\gamma^{-1}J\gamma^{-1}J)^{1/2} = \tau(K^2)^{1/2} = \|K\|_2.$$

Let us prove therefore that the map $t \mapsto f(t) = \|K(t)\|_2$ is convex, for any (non vanishing) solution K of (2.1). Note that $f(t)$ is smooth, and $\dot{f} = \tau(K^2)^{-1/2}\tau(K\dot{K})$. Then

$$\ddot{f} = -\tau(K^2)^{-3/2}\tau(K\dot{K})^2 + \tau(K^2)^{-1/2}\{\tau(\dot{K}^2) + \tau(K\ddot{K})\}.$$

Let us multiply this expresion by $\tau(K^2)^{3/2}$ to obtain

$$-\tau(K\dot{K})^2 + \tau(K^2)\tau(\dot{K}^2) + \tau(K^2)\tau(K\ddot{K}). \tag{2.2}$$

The first two terms add up to a non negative number. Indeed,

$$\tau(K\dot{K})^2 \leqslant \tau(K^2)\tau(\dot{K}^2)$$

by the Cauchy-Schwarz inequality for the trace τ. Let us examine the third term $\tau(K^2)\tau(K\ddot{K})$. It suffices to show that $\tau(K\ddot{K})$ is non negative. Using (2.1),

$$\tau(K\ddot{K}) = \frac{1}{4}\{\tau(K^2x^2) + \tau(Kx^2K) - 2\tau(KxKx)\} = \frac{1}{2}\{\tau(K^2x^2) - \tau(KxKx)\}.$$

This number is positive, again by the Cauchy-Schwarz inequality:

$$\tau(KxKx) = \tau((xK)^*Kx) \leqslant \tau((xK)^*xK)^{1/2}\tau((Kx)^*Kx)^{1/2} = \tau(K^2x^2). \quad \blacksquare$$

COROLLARY 2.2. *Let γ_1 and γ_2 be geodesics of $\mathcal{G}^+$. Then the map*

$$t \mapsto d_2(\gamma_1(t), \gamma_2(t)), \quad t \in \mathbb{R}$$

is convex.

Proof. The proof proceeds as the proof of Theorem 2 of [3], replacing the C^*-norm $\|\cdot\|$ by the 2-norm $\|\cdot\|_2$. To carry on this argument, one needs the fact that the geodesics of $\mathcal{G}^+$ are minimizing also for the metric g. This fact is proved in [1]. $\blacksquare$

COROLLARY 2.3. *Let $x, y \in \mathcal{A}_h$, then*

$$\|x - y\|_2 \leqslant d_2(\mathrm{e}^x, \mathrm{e}^y).$$

Proof. See Theorem 3 of [3]. ∎

Our main result follows.

THEOREM 2.4. *Let $x, y \in \mathcal{A}_h$, $a = \mathrm{e}^x$, $b = \mathrm{e}^y$. Then $\|x - y\|_2 = d_2(a, b)$ if and only if x and y commute.*

Proof. Let γ be the geodesic of $\mathcal{G}^+$ with $\gamma(0) = a$ and $\gamma(1) = b$, and let $\delta(t) \in \mathcal{A}_h$ be characterized by the equation $\mathrm{e}^\delta = \gamma$. Let $\mathbb{H}$ be the *real* Hilbert space obtained by completing $\mathcal{A}_h$ in the inner product $\langle v, w\rangle = \tau(vw)$. Applying the above corollaries, we have that

$$\mathrm{length}(\delta) \leqslant d_2(a, b),$$

where δ is regarded as a curve in $\mathbb{H}$, and its length is measured accordingly. Indeed, for any number h,

$$\|\delta(t+h) - \delta(t)\|_2 \leqslant d_2(\gamma(t+h), \gamma(t)).$$

Therefore, if $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ is a partition of the unit interval, then

$$\sum_{i=1}^{n} \|\delta(t_i) - \delta(t_{i-1})\|_2 \leqslant \sum_{i=1}^{n} d_2(\gamma(t_i, t_{i-1})) = d(a, b).$$

Since δ is smooth, the supremum of all such sums on the left hand side of the inequality, taken over all possible partitions of the unit interval, equals the length of δ in $\mathbb{H}$, and our claim is proven.

Clearly $\|x - y\|_2 = d_2(a, b)$ if x and y commute. Suppose that $\|x - y\|_2 = d_2(a, b)$. Then δ, which joins $\delta(0) = x$ and $\delta(1) = y$, satisfies the inequality

$$\|x - y\|_2 \leqslant \mathrm{length}(\delta) \leqslant d_2(a, b) = \|x - y\|_2.$$

This implies (by the uniform convexity of the euclidean norm of $\mathbb{H}$), that

$$x + t(y - x) = \delta(t) = \log(\gamma_{a,b}(t)) = \log(a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}).$$

The map $f(t) = d_2(\mathrm{e}^{tx}, \mathrm{e}^{ty})$ is convex with $f(0) = 0$ and $f(1) = \|x - y\|_2$. Therefore $f(s) = \|sx - sy\|_2$, i.e. sx and sy satisfy the hypothesis of this theorem for $s \in [0, 1]$. Therefore the argument above in fact proves that

$$sx + st(y - x) = \log(a^{s/2}(a^{-s/2}b^s a^{-s/2})^t a^{s/2}, \quad s, t \in [0, 1].$$

Let us compute $\frac{\partial^3}{\partial t^3}$ of both terms of this equality at points $s = 0, t \in [0, 1]$. On the left hand side we obtain zero. After tedious but straightforward computations (which can be easily performed because at such pairs, only the derivatives of log at the origin are involved), one obtains, on the right hand side

$$t^2\Delta + t\Delta, \quad t \in [0, 1]$$

where

$$\Delta = -\frac{1}{2}y^2x - \frac{1}{2}yx^2 + xyx + yxy - \frac{1}{2}x^2y - \frac{1}{2}xy^2.$$

Note that $\Delta = \frac{1}{2}[[x,y], x-y]$. Then, $\Delta = 0$, or equivalently $[[x,y],y] = [[x,y],x]$. Therefore

$$g_1(R_1(x,y)y,x) = \tau(([[x,y],y])x) = \tau(([[x,y],x])x) = 0$$

by the properties of the trace. This implies that

$$\tau(x^2y^2) = \tau(xyxy).$$

This means that we have equality in the Cauchy-Schwarz inequality involving xy and yx:

$$\tau(xyxy) = \tau((yx)^*xy) \leqslant \tau((yx)^*yx)^{1/2}\tau((xy)^*xy)^{1/2} = \tau(x^2y^2) = \tau(xyxy).$$

Therefore xy is a positive multiple of yx: $xy = \alpha yx$, $\alpha \geqslant 0$. If $\alpha = 0$, then $xy = 0 = (xy)^* = yx$. Otherwise $yx = (xy)^* = \alpha(yx)^* = \alpha xy$, i.e. $\alpha = 1$. ∎

REMARK 2.5. The non trivial part of the argument above could be extracted as follows. If the exponential map of $\mathcal{G}^+$ does not distort the distance of x and y (i.e. $\|x-y\|_2 = d_2(\mathrm{e}^x, \mathrm{e}^y)$), then the sectional curvature vanishes at (x,y): $\sec_1(x,y) = 0$.

3. DISTORTION OF THE METRIC IN THE C^*-NORM

Let $a, b \in \mathcal{G}^+$ and let $x, y \in \mathcal{A}_h$ be such that $a = \mathrm{e}^x$ and $b = \mathrm{e}^y$. The geodesic distance $d(a,b)$ is defined as the infimum of the lengths of all smooth curves in $\mathcal{G}^+$ joining a and b, measured with the Finsler metric (introduced and studied in [2], [3]):

$$\|x\|_a = \|a^{-1/2}xa^{-1/2}\|, \quad x \in \mathcal{A}_h,\ a \in \mathcal{G}^+.$$

The next lemma is the key of our exposition in this section. We view $\mathcal{A}$ as represented concretely via its universal representation acting on the Hilbert space $\mathcal{H}$. Recall that any selfadjoint element $x \in \mathcal{A}$ has a unit *norming vector* $\xi \in \mathcal{H}$, which is an eigenvector of x verifying the equation $\|x\xi\| = \|x\|$. Equivalently, $x\xi = \pm\|x\|\xi$.

LEMMA 3.1. *If $\|x-y\| = d(a,b)$ then there exists a norming vector of $x-y$ which is simultaneously an eigenvector of x and y.*

Proof. Let $\gamma(t)$ be the unique geodesic joining a and b, $\gamma(0) = a$ and $\gamma(1) = b$. Explicitely, $\gamma(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$. As in the proof of (2.4), there exists a smooth curve $\delta(t)$ of selfadjoint elements such that $\mathrm{e}^{\delta(t)} = \gamma(t)$. In particular $\delta(0) = x$ and $\delta(1) = y$. Consider the space $\mathcal{A}_h$ with its Banach space flat geometry (geodesics=straight line segments). We claim that

$$\mathrm{length}(\delta) \leqslant \mathrm{length}(\gamma).$$

The proof is similar to the proof of the analogous fact in Theorem 2.4, using the fundamental inequality for the C^*-norm: $\|x-y\| \leqslant d(\mathrm{e}^x, \mathrm{e}^y)$.

Let $\xi \in \mathcal{H}$ be a norming (unit) vector for $x-y$, $(x-y)\xi = -\lambda\xi$, with $\lambda = \pm\|x-y\|$, and consider the smooth curve $\delta(t)\xi$ in $\mathcal{H}$. Then

$$\mathrm{length}(\delta\xi) = \int_0^1 \|\dot{\delta}\xi\| \mathrm{d}t \leqslant \int_0^1 \|\dot{\delta}\| \mathrm{d}t = \mathrm{length}(\delta) \leqslant \mathrm{length}(\gamma).$$

By hypothesis, $\mathrm{length}(\gamma) = d(a,b) = \|x-y\|$. Therefore

$$\mathrm{length}(\delta\xi) \leqslant \|x-y\| = \|(x-y)\xi\|.$$

In other words, $\delta\xi$ is a smooth curve in $\mathcal{H}$ whose length equals the length of the (minimal) line segment $\lambda(t) = [x+t(y-x)]\xi$, and joins the same endpoints. Then

$$[x+t(y-x)]\xi = \delta(t)\xi \tag{3.1}$$

for all $t \in [0,1]$.

On the other hand, it was shown [2] that in general, the function $f(t) = d(\mathrm{e}^{tx}, \mathrm{e}^{ty})$ is convex. In our case $f(0) = 0$ and $f(1) = \|x-y\|$. These facts imply that $f(s) = s\|x-y\|$, or equivalently

$$\|sx - sy\| = d(\mathrm{e}^{sx}, \mathrm{e}^{sy})$$

for all $s \in [0,1]$. Note that the vector ξ is also norming for $sx - sy$, therefore we may apply (3.1) to obtain a refined version of this equality, namely

$$[sx + st(y-x)]\xi = [\log(a^{s/2}(a^{-s/2}b^{s/2}a^{-s/2})^t a^{s/2})]\xi, \tag{3.2}$$

for all $s, t \in [0,1]$.

As in the proof of Theorem 2.4, we compute $\frac{\partial^3}{\partial t^3}$, at pairs $s = 0, t \in [0,1]$. Then

$$0 = t^2\Delta\xi + t\Delta\xi, \quad t \in [0,1],$$

where, as in Theorem 2.4, $\Delta = \frac{1}{2}[[x,y], y-x]$. In this case we have that $\Delta\xi = 0$.

Let us denote by $\mathcal{H}_\lambda$ the space of eigenvectors of $y-x$ associated with the eigenvalue λ. We have just proved that for any vector $\xi \in \mathcal{H}_\lambda$,

$$(y-x)[x,y]\xi = [x,y](y-x)\xi = \lambda[x,y]\xi.$$

That is, $[x,y]$ leaves $\mathcal{H}_\lambda$ invariant. Let $p = P_{\mathcal{H}_\lambda}$ the orthogonal projection onto $\mathcal{H}_\lambda$. We can represent the elements x, y and $[x,y]$ as two by two matrices in terms of p.

$$x = \begin{pmatrix} l & m^* \\ m & n \end{pmatrix}, \quad y = \begin{pmatrix} \lambda + l & m^* \\ m & r \end{pmatrix}$$

and $[x,y]$ a diagonal matrix. This fact implies that $0 = \lambda m + (n-r)m = [\lambda - (r-n)]m$. Now the operator $\lambda + (r-n)$, regarded as an operator acting in $\mathcal{H}_\lambda^\perp$, is injective. Otherwise one could find eigenvectors of $y-x$ in $\mathcal{H}_\lambda^\perp$. Therefore $m = 0$, and x, y leave $\mathcal{H}_\lambda$ invariant. The proof finishes by showing that we can find a common eigenvector for x and y inside $\mathcal{H}_\lambda$. Let $\mathcal{A}''$ be the double commutant of $\mathcal{A}$. Clearly $p \in \mathcal{A}''$. For any selfadjoint element in $\mathcal{A}''$ there exists an eigenvector in $\mathcal{H}$. Indeed, for any selfadjoint element $b \in \mathcal{A}''$, there exists a normal state φ such that $|\varphi(b)| = \|b\|$. Without loss of generality, suppose $\varphi(b) = \|b\|$. The GNS Hilbert space H_φ of φ (with norm $\|\cdot\|_\varphi$ and cyclic vector v_φ) is a subspace of $\mathcal{H}$. Note that

$$\begin{aligned} 0 &\leqslant \|bv_\varphi - \|b\|v_\varphi\|_\varphi^2 = \|bv_\varphi - \|b\|v_\varphi\|^2 = \varphi((b - \|b\|)^2) \\ &= \varphi(b^2) + \|b\|^2 - 2\|b\|\varphi(b) = \varphi(b^2) - \|b\|^2 \leqslant 0, \end{aligned}$$

i.e. v_φ is an eigenvector for b. Therefore there exists an eigenvector ω of $xp = pxp \in \mathcal{A}''$. Necesarilly this vector ω lies in the range of p, which is $\mathcal{H}_\lambda$. Finally ω is an eigenvector of x and $y - x$, and therefore also of y. ∎

REMARK 3.2. In fact, we have also shown that if $\|x - y\| = d(a, b)$, then the spaces of eigenvectors associated to $\pm\|x - y\|$ are invariant for x and y.

Note the following elementary observation. Suppose that ξ is a unit eigenvector of $x = x^* \in \mathcal{A}$, $x\xi = \lambda\xi$. Let $\varphi = \varphi_\xi$ be the state of $\mathcal{A}$ induced by ξ. Then $\varphi(ax) = \lambda\varphi(a)$ for all $a \in \mathcal{A}$. Indeed, $\varphi(ax) = \langle ax\xi, \xi\rangle = \lambda\langle a\xi, \xi\rangle = \lambda\varphi(a)$. Moreover, since $\varphi(x) = \langle x\xi, \xi\rangle = \lambda\|\xi\| = \lambda$, the above formula can be written $\varphi(ax) = \varphi(x)\varphi(a)$. Clearly also $\varphi(xa) = \varphi(x)\varphi(a)$. This motivates the following interpretation of the above lemma:

THEOREM 3.3. *Let $x, y \in \mathcal{A}_h$, and let $a = \mathrm{e}^x$ and $b = \mathrm{e}^y$. Let $\mathcal{A}_{x,y}$ be the unital C^*-algebra generated by x and y.*

(i) *If $\|x - y\| = d(a, b)$, then there exists a multiplicative functional ψ on $\mathcal{A}_{x,y}$ which achieves the norm of $x - y$, i.e. $|\psi(x - y)| = \|x - y\|$. Such functional also achieves the norm of $\Lambda = \log(a^{-1/2}ba^{-1/2})$. This norm equals the speed of the geodesic joining a and b in $\mathcal{A}$ (which lies inside $\mathcal{A}_{x,y}$).*

(ii) *If there exists a multiplicative functional ψ of $\mathcal{A}_{x,y}$ which achieves the norm of Λ, then*

$$\|x - y\| = d(a, b).$$

Proof. Suppose that $\|x - y\| = d(a, b)$. By the above lemma, there exists a unit vector $\xi \in \mathcal{H}$ which is a norming eigenvector for $x - y$ and a common eigenvector for x and y. Let $\varphi = \varphi_\xi$ be the state of $\mathcal{A}$ induced by ξ. Then by the above remark, for all $z \in \mathcal{A}$, $\varphi(xz) = \varphi(zx) = \varphi(x)\varphi(z)$ and $\varphi(yz) = \varphi(zy) = \varphi(y)\varphi(z)$. In particular, the functional φ restricted to $\mathcal{A}_{x,y}$ is multiplicative. Moreover, since ξ norms $x - y$, $\|x - y\|^2 = \|(x - y)\xi\|^2 = \varphi((x - y)^2) = \varphi(x - y)^2$, because φ is multiplicative in $\mathcal{A}_{x,y}$. Then $\|x - y\| = |\varphi(x - y)|$. Finally, by the multiplicativity of φ ($\Lambda \in \mathcal{A}_{x,y}$),

$$|\varphi(\Lambda)| = |\log(\varphi(a^{-1/2}ba^{-1/2})| = |\log(\varphi(\mathrm{e}^y)\varphi(\mathrm{e}^{-x}))| = |\varphi(y) - \varphi(x)| = \|x - y\|.$$

In the reverse direction, suppose that ψ is a multiplicative functional such that $|\psi(\Lambda)| = \|\Lambda\|$. Let γ be the unique geodesic joining $\gamma(0) = a$ and $\gamma(1) = b$,

$\gamma(t) = a^{1/2}\mathrm{e}^{t\Lambda}a^{1/2}$. Then

$$d(a,b) = \text{length}(\gamma) = \int_0^1 \|\dot{\gamma}\|_\gamma \mathrm{d}t = \|\Lambda\| = |\psi(\Lambda)| = |\psi(y) - \psi(x)| \leqslant \|x-y\|.$$

The reverse inequality $\|x-y\| \leqslant d(a,b)$ holds in general. ∎

The conditions stated in the theorem can be interpreted as a weaker form of conmutativity for x and y. Indeed, note that if x and y commute, the algebra $\mathcal{A}_{x,y}$ is abelian and the existence of multiplicative functionals achieving the norm of Λ is automatic.

Suppose now $x = p$, $y = q$ are selfadjoint projections in $\mathcal{A}$. We shall see now that the equality $d(\mathrm{e}^p, \mathrm{e}^q) = \|p-q\|$ has a simpler interpretation.

PROPOSITION 3.4. *$d(\mathrm{e}^p, \mathrm{e}^q) = \|p-q\|$ if and only if either $R(p) \cap \ker(q)$ or $R(q) \cap \ker(p)$ are non trivial, or equivalently, if $\|p-q\| = 1$.*

Proof. If $d(\mathrm{e}^p, \mathrm{e}^q) = \|p-q\|$, by Lemma 3.1 every norming vector ξ of $p - q$ is a common eigenvector of p and q. An eigenvector of p lies either in $R(p)$ or $\ker(p)$. Clearly, if $p \neq q$, ξ cannot lie in the intersection of the ranges (or the kernels) of p and q. Therefore the result follows. On the reverse direction, suppose that $\xi \in R(p) \cap \ker(q)$ with $\|\xi\| = 1$. Then $\mathrm{e}^q\xi = \xi$ and $\mathrm{e}^{-p/2}\xi = \mathrm{e}^{-1/2}\xi$. Then

$$\log(\mathrm{e}^{-p/2}\mathrm{e}^q\mathrm{e}^{-p/2})\xi = -\xi,$$

i.e. $\|\log(\mathrm{e}^{-p/2}\mathrm{e}^q\mathrm{e}^{-p/2})\| \geqslant 1$. Let us show that $\|\log(\mathrm{e}^{-p/2}\mathrm{e}^q\mathrm{e}^{-p/2})\| \leqslant 1$, which ends proof in this case, because $\|p-q\| = 1$. Note that $1 \leqslant \mathrm{e}^q \leqslant \mathrm{e}$, and therefore

$$\mathrm{e}^{-1} \leqslant \mathrm{e}^{-p} \leqslant \mathrm{e}^{-p/2}\mathrm{e}^q\mathrm{e}^{-p/2} \leqslant \mathrm{e}^{1-p} \leqslant \mathrm{e}.$$

Then the spectrum of $\mathrm{e}^{-p/2}\mathrm{e}^q\mathrm{e}^{-p/2}$ is contained in the interval $[\mathrm{e}^{-1}, \mathrm{e}]$. This implies that the spectrum of $\log(\mathrm{e}^{-p/2}\mathrm{e}^q\mathrm{e}^{-p/2}) \subset [-1, 1]$, i.e. $\|\log(\mathrm{e}^{-p/2}\mathrm{e}^q\mathrm{e}^{-p/2})\| \leqslant 1$. If $R(q) \cap \ker(p)$ is non trivial, the proof is analogous. ∎

4. AN ANALOGUE OF THE SECTIONAL CURVATURE

In his optical description of the curvature of a manifold from within [6] (page 101), Milnor recalls that the sectional curvature can be recovered via the limit

$$\langle R_a(x,y)y, x\rangle_a = 6 \lim_{r\to 0^+} \frac{r\|x-y\|_a - d(\exp_a(rx), \exp_a(ry))}{r^2 d(\exp_a(rx), \exp_a(ry))}.$$

Here x, y are tangent vectors at a point a of the manifold. In this section we shall see that this limit makes sense in our (non Riemannian) context for the manifold $\mathcal{G}^+$, with the Finsler metric induced by the C^*-norm. Namely, if $a \in \mathcal{G}^+$ and $x, y \in \mathcal{A}_h$, we define $s_a(x,y)$ as the above limit (we drop the factor 6 for simplicity). Let us first prove that it exists. Suppose that $r > 0$ is close enough to 0 in order that $e^{-rx/2}e^{ry}e^{-rx/2}$ lies within the radius of convergence of the series

$$\log(u) = u - 1 - \frac{1}{2}(u-1)^2 + \frac{1}{3}(u-1)^2 - \cdots$$

Then elementary computations show that

$$\log(e^{-rx/2}e^{ry}e^{-rx/2}) = r(y-x) + r^3\kappa(x,y) + o(r^3),$$

where

$$\kappa(x,y) = \frac{1}{6}yxy + \frac{1}{12}xyx - \frac{1}{12}(xy^2 + y^2x) - \frac{1}{24}(x^2y + yx^2). \tag{4.1}$$

Note that $\kappa(x,y)^* = \kappa(x,y)$.

Denote by $P(x) = \|x\|$. The map P is seldom (Gateaux or Frechet) differentiable. However it is convex, and therefore has a right derivative [7]

$$\partial^+ P_x(z) = \lim_{t\to 0^+} \frac{1}{t}(\|x+tz\| - \|x\|).$$

It also has a continuous (set valued) subdifferential [7]

$$\partial P_x = \{\varphi \in \mathcal{A}_h^* : \varphi(x) = \|x\| \text{ with } \|\varphi\| = 1\}.$$

Here $\mathcal{A}_h^*$ means the real dual space of the real Banach space $\mathcal{A}_h$. These functionals can be extended in a unique way to selfadjoint functionals of $\mathcal{A}$.

PROPOSITION 4.1. *Let $a \in \mathcal{G}^+$ and let $x, y \in \mathcal{A}_h$. The limit*

$$s_a(x,y) = \lim_{r\to 0^+} \frac{r\|x-y\|_a - d(\exp_a(rx), \exp_a(ry))}{r^2 d(\exp_a(rx), \exp_a(ry))}$$

exists and satisfies

$$0 \geqslant s_a(x,y) \geqslant -\frac{\|\kappa(x,y)\|}{\|x-y\|}.$$

Proof. Since the action of $\mathcal{G}$ on $\mathcal{G}^+$ is isometric, it suffices to consider the case $a = 1$. Note that

$$\lim_{r\to 0} \frac{1}{r} d(e^{rx}, e^{ry}) = \lim_{r\to 0} \|y - x + r^2\kappa(x,y) + o(r^2)\| = \|y-x\|.$$

Therefore it suffices to show existence of the limit

$$\lim_{r\to 0^+} \frac{1}{r^3}(r\|x-y\| - \|r(y-x) + r^3\kappa(x,y) + o(r^3)\|),$$

or equivalently

$$\lim_{r\to 0^+} \frac{1}{r^2}(\|x-y\| - \|(y-x) + r^2\kappa(x,y)\|).$$

Clearly this exists [7] and equals $\partial^+ P_{y-x}(\kappa(x,y))$. Since $r\|x-y\| \leqslant d(e^{rx}, e^{ry})$ this limit is non positive. On the other hand

$$\|x-y\| - \|(y-x) + r^2\kappa(x,y)\| \geqslant -r^2\|\kappa(x,y)\|,$$

and therefore $s_1(x,y) \geqslant -\|\kappa(x,y)\|/\|x-y\|$. ∎

Next we shall use an elementary result from [7] (Proposition 2.24), to characterize pairs x,y such that $s_1(x,y)=0$. Obviously, this is the case if the exponential does not distort the metric: if $\|x-y\| = d(\mathrm{e}^x,\mathrm{e}^y)$ then by an argument displayed in the previous section, for $0 \leqslant r \leqslant 1$ one has $r\|x-y\| = d(\mathrm{e}^{rx},\mathrm{e}^{ry})$ as well, and $s_1(x,y)=0$.

THEOREM 4.2. *$s_1(x,y) = 0$ if and only if there exists a selfadjoint linear functional φ in $\mathcal{A}^*$ such that $\|\varphi\| = 1$, $\varphi(y-x) = \|x-y\|$ and $\varphi(\kappa(x,y)) = 0$.*

Proof. By the above computations, since $s(x,y)=0$ then $\partial^+ P_{y-x}(\kappa(x,y)) = 0$. Proposition 2.24 of [7] states that then there exists a linear functional $\varphi \in \partial P_{y-x}$ such that $\varphi(\kappa(x,y)) = 0$. This ends the first part of the proof, since ∂P_{y-x} consists of all selfadjoint normalized functionals of $\mathcal{A}$ which satisfy $\varphi(y-x) = \|x-y\|$.

Suppose now that such a functional φ exists. Since $\varphi(y-x) = \|x-y\| > 0$, if r is small enough, $\varphi(y-x+r^2\kappa(x,y)+o(r^2)) > 0$. Then

$$\begin{aligned}\|x-y\| - \|y-x+r^2\kappa(x,y)+o(r^2)\| &\geqslant \|x-y\| - \varphi(y-x+r^2\kappa(x,y)+o(r^2))\\ &= \varphi(o(r^2)).\end{aligned}$$

Then

$$0 \geqslant \lim_{r\to 0^+} \frac{1}{r^3}(r\|x-y\| - d(\mathrm{e}^{rx},\mathrm{e}^{ry})) \geqslant \lim_{r\to 0^+} \frac{\varphi(o(r^2))}{r^2} = 0,$$

i.e., $s_1(x,y) = 0$. ∎

Note that if $s_1(x,y) = 0$ with $x \neq y$, then either $\kappa(x,y)$ is zero or it is not a multiple of $x-y$.

Suppose that $x = p$ and $y = q$ are projections which are in *generic* position [8], i.e.

$$R(p)\cap\ker(q) = R(q)\cap\ker(p) = R(p)\cap R(q) = \ker(p)\cap\ker(q) = \{0\}.$$

In particular, $\|p-q\| < 1$. Let us compute $s(p,q)$. The C^*-algebra generated by p and q can be faithfully represented as the algebra of 2×2 matrices of continuous functions in the spectrum of $(p-q)^2$, generated by the matrices

$$\begin{pmatrix} 1-t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & t \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which correspond, respectively, to p and q (see for example the paper [8] by N. Vasilevski). Note that the exponentials e^{rp}, e^{rq} as well as $\log(\mathrm{e}^{-rp/2}\mathrm{e}^{rq}\mathrm{e}^{-rp/2})$ lie in the C^*-algebra generated by p and q. Therefore all computations can be done there. For instance,

$$\mathrm{e}^{-rp/2}\mathrm{e}^{rq}\mathrm{e}^{-rp/2} = \begin{pmatrix} 1+t(\mathrm{e}^{-r}-1) & \sqrt{t(1-t)}(\mathrm{e}^{r/2}-\mathrm{e}^{-r/2}) \\ \sqrt{t(1-t)}(\mathrm{e}^{r/2}-\mathrm{e}^{-r/2}) & 1+t(\mathrm{e}^{r}-1) \end{pmatrix}.$$

The eigenvalues of this matrix are e^{λ} and $e^{-\lambda}$, where $\lambda = \arg\cosh(1 + t(\mathrm{ch}(r) - 1))$. Therefore

$$d(e^{rp}, e^{rq}) = \sup_{t \in \operatorname{sp}((p-q)^2)} \arg\cosh(1 + t(\cosh(r) - 1))$$
$$= \arg\cosh(1 + \|p - p\|^2(\cosh(r) - 1)).$$

Then

$$s_1(p,q) = \lim_{r \to 0^+} \frac{r\|p-q\| - \arg\cosh(1 + \|p-q\|(\cosh(r)-1))}{r^3\|p-q\|} = \frac{\|p-q\|^2 - 1}{24\|p-q\|}.$$

If $\|p - q\| = 1$, by the result in the previous section, $s_1(p,q) = 0$. It remains to examine the case $\|p - q\| < 1$ (which is equivalent to $R(p) \cap \ker(q) = R(q) \cap \ker(p) = \{0\}$), but p, q not in generic position. That is, either

$$R(p) \cap R(q) \neq \{0\} \quad \text{or} \quad \ker(p) \cap \ker(q) \neq \{0\}.$$

Suppose that $\mathcal{A}$ is faithfully represented on $\mathcal{H}$, and that $R(p) \cap R(q) = \mathcal{J} \neq \{0\}$. Let $p_0 = P_{\mathcal{J}^\perp}$. Then p_0 (which may lie outside $\mathcal{A}$) commutes with p and q. As before, let $\mathcal{A}_{p,q}$ denote the C^*-algebra generated by p and q. Then the map

$$\mu : \mathcal{A}_{p,q} \to p_0\mathcal{A}_{p,q}, \quad \mu(x) = p_0 x$$

is a $*$-homomorphism which satisfies $\|\mu(p) - \mu(q)\| = \|p - q\|$. We regard $p_0\mathcal{A}_{p,q}$ acting on $\mathcal{J}^\perp$. Note that $\|(\log(e^{-r\mu(p)/2}e^{r\mu(q)}e^{-r\mu(p)/2})\| \leqslant \|\log(e^{-rp/2}e^{rq}e^{-rp/2})\|$. Therefore $s_1(p,q) \leqslant s_1(\mu(p),\mu(q))$. Moreover, $R(\mu(p)) \cap R(\mu(q)) = \{0\}$. Analogously, we can further reduce the algebra $\mathcal{A}_{\mu(p),\mu(q)}$ via the orthogonal projection onto $\ker(p) \cap \ker(q)$, and obtain a $*$-homomorphism $\nu : \mathcal{A}_{p,q} \to \mathcal{A}_{\nu(p),\nu(q)}$ such that $\|\nu(p) - \nu(q)\| = \|p - q\|$, and $\nu(p)$, $\nu(q)$ are in generic position. Then

$$s_1(p,q) \leqslant s_1(\nu(p),\nu(q)) < 0.$$

We can summarize these remarks in the following

THEOREM 4.3. *Let p,q be projections in $\mathcal{A}$. Then the following are equivalent:*

(i) $s_1(p,q) = 0$.
(ii) $\|p - q\| = d(e^p, e^q)$.
(iii) $\|p - q\| = 1$.

In general the equality $\|x - y\| = d(e^x, e^y)$ is a stronger condition than $s_1(x,y) = 0$. This becomes apparent if one states these conditions in terms of functionals of $\mathcal{A}_{x,y}$. The equality $\|x - y\| = d(e^x, e^y)$ implies the existence of a *multiplicative* functional realizing the norm of $x - y$ with an additional property. Whereas $s_1(x,y) = 0$ implies the existence of a *selfadjoint* functional φ which realizes the norm of $x - y$ and satisfies $\varphi(\kappa(x,y)) = 0$: if φ is multiplicative, then authomatically

$$\varphi(\kappa(x,y)) = \varphi\Big(\frac{1}{6}yxy + \frac{1}{12}xyx - \frac{1}{12}(xy^2 + y^2x) - \frac{1}{24}(x^2y + yx^2)\Big) = 0.$$

Let us state a simple example where this discrepancy is made apparent.

EXAMPLE 4.4. Let $\mathcal{A} = M_2(\mathbb{C})$, and let

$$x = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Let $\varphi : M_2(\mathbb{C}) \to \mathbb{C}$, $\varphi(a) = \mathrm{Tr}(\rho a)$, where $\rho = \begin{pmatrix} -\frac{2}{3} & \frac{1}{2} + \mathrm{i}\beta \\ \frac{1}{2} - \mathrm{i}\beta & -\frac{7}{24} \end{pmatrix}$ with $\beta = \sqrt{\frac{10253}{6912}}$. Straightforard computations show that $\|\varphi\| = \|\rho\|_1 = 1$, $\varphi(x-y) = \|x-y\|$ and $\varphi(\kappa(x,y)) = 0$. Then $s_1(x,y) = 0$. However, clearly $\mathcal{A}_{x,y} = M_2(\mathbb{C})$, which has no multiplicative functionals, and therefore $\|x-y\| < d(\mathrm{e}^x, \mathrm{e}^y)$.

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