J. OPERATOR THEORY
55:2(2006), 253–268

ON THE SPECTRAL THEORY OF TENSOR PRODUCT HAMILTONIANS

MONDHER DAMAK

Communicated by Florian-Horia Vasilescu

ABSTRACT. We develop, in this paper the Mourre theory for an abstract class of self-adjoint operators of the form $H_1 \otimes I + I \otimes H_2$, where H_1 and H_2 are two self-adjoint operators acting on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. H_1 and H_2 do not need to be lower semi-bounded. As an example, we consider the time periodic Hamiltonians for which we construct a conjugate operator, prove a Mourre estimate and, finally, study some of its perturbations.

KEYWORDS: *Mourre estimate, conjugate operator, periodic Hamiltonian.*

MSC (2000): Primary 47A10; Secondary 81Q10.

1. INTRODUCTION

This paper is devoted to the Mourre theory for an abstract class of self-adjoint operators of the form $H_1 \otimes I + I \otimes H_2$, where H_1 and H_2 are two self-adjoint operators acting on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. This class will be called the class of *tensor product Hamiltonians*.

The Mourre theory for a self-adjoint operator H_0 acting on some Hilbert space $\mathcal{H}$ is based on the construction of another self-adjoint operator A, called a conjugate operator so that the following estimate holds:

$$E_J(H_0)[H_0, \mathrm{i}A]E_J(H_0) \geqslant c_0\, E_J(H_0) + K, \tag{1.1}$$

where $E_J(H_0)$ denotes the spectral projection on the interval $J \subset \mathbb{R}$ for the operator H_0, c_0 is a positive constant and K is a compact operator. The estimate (1.1) is called a *Mourre estimate*. If $K = 0$ in (1.1), then it is called a *strict Mourre estimate*.

The Mourre estimate has several important consequences for the spectral and scattering theory of H_0. The first one is the discreteness of the point spectrum $\sigma_{\mathrm{p}}(H_0)$ in J, and under some additional assumptions, the existence of the limiting absorption principle, i.e, the existence of the limits $\lim\limits_{\varepsilon\to 0}(H_0 - \lambda \pm \mathrm{i}\varepsilon)^{-1}$, for $\lambda \in J \setminus \sigma_{\mathrm{p}}(H_0)$, as a bounded operator between suitable weighted spaces. The estimates leading to the limiting absorption principle are called *resolvent estimates*. In turn the limiting absorption principle implies that the singular continuous spectrum of H_0, $\sigma_{\rm sc}(H_0)$, is empty in J. Moreover, there exists a natural class of perturbations V for which one can deduce from (1.1) a Mourre estimate for $H = H_0 + V$ with the same conjugate operator A.

However, the most intuitive consequences of the Mourre estimate (1.1) are probably the propagation estimates. They are based on the fact that $[H_0, \mathrm{i}A]$ is the time derivative of $t \mapsto \mathrm{e}^{\mathrm{i}tH_0} A \mathrm{e}^{-\mathrm{i}tH_0}$ at $t = 0$. An example of such a propagation estimate is

$$\|F(A/t < c_0)\, \mathrm{e}^{-\mathrm{i}tH_0}\, E_J^{\rm c}(H_0)\| \ \to 0 \quad \text{when } t \to \pm\infty,$$

where $E_J^{\rm c}(H_0)$ is the spectral projection on the continuous spectral subspace of H_0 in J. Such propagation estimates allow to develop in a very natural way the scattering theory for perturbations $H = H_0 + V$ of H_0. For example, there exists a natural class of perturbations V (the short-range perturbations) for which the local wave operators

$$s - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH}\, \mathrm{e}^{-\mathrm{i}tH_0} E_J^{\rm c}(H_0) =: W^{\pm}$$

can be shown to exist and to be asymptotically complete.

Let us end this very brief overview of the Mourre method by some historical comments and some bibliographical references, which do not intend to be complete. The Mourre method was invented by E. Mourre [12] and subsequently developed and applied in [4], [6], [7], [8], [10], [11], [13], [14], [16]. An essentially optimal version of the Mourre method was developed in [1] and [5]. Recently, in [9], an extension of the Mourre method especially adapted for the study of quantum field Hamiltonians, was proposed. This extension can be used when the commutator $[H_0, \mathrm{i}A]$ is not comparable with H_0 when A is not a self-adjoint operator.

The first purpose of this paper is to prove a strict Mourre estimate for an abstract class of self-adjoint operators. We shall explicitly calculate the function ρ (see Subsection 2.1 for the definition of ρ) of an operator of the form $H_0 = H_1 \otimes I + I \otimes H_2$ in terms of those of H_j assuming that A is similarly decomposable. We obtain the same conclusion as in Theorem 3.4 in [2] without assuming that H_1 and H_2 are bounded from below. As an application we explicitly calculate the function ρ of time periodic relativistic Hamiltonians. In this case $H_1 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}$ in $L^2(\mathbb{T})$ ($\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the torus) and $H_2 = \sqrt{1 + P^2}$ ($P = -\mathrm{i}\nabla$) in $L^2(\mathbb{R}^n)$. In this case, we shall prove that H_0 satisfy a strict Mourre estimate on $\mathbb{R} \setminus \mathbb{Z}$.

The second aim of this work is to extend to a large class of operators (which we will call the class of *dispersive time periodic Hamiltonians*) the spectral analysis initiated in [18]. H_0 will be called dispersive time-periodic if it is a self-adjoint operator in $\mathcal{H} = L^2(\mathbb{T} \times \mathbb{R}^n)$ associated to the sum $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} \otimes 1 + 1 \otimes h(P)$, where $h : X \to \mathbb{R}$ be a real function of class C^2 such that $|\, h''(k) \,| \leqslant c(1 + |\, h'(k) \,|^2)$. We shall prove that H_0 satisfies a strict Mourre estimate on $\mathbb{R} \setminus (\mathbb{Z} + \kappa(h))$, where $\kappa(h)$ denotes the set of critical values of h (see (4.1)). In particular H_0 will not have eigenvalues in $\mathbb{R} \setminus (\mathbb{Z} + \kappa(h))$. As a consequence of the Mourre estimate we prove a strong form of the limiting absorption principle which implies the absence of singular continuous spectrum of H_0 outside $\mathbb{Z} + \kappa(h)$. Finally, we study the spectral structure of the self-adjoint operators of the form $H = H_0 + V$, where V belongs to some perturbation class which is described in our statements.

2. PRELIMINARIES

2.1. In this subsection we recall some facts about the abstract commutator method.

Let H, A be self-adjoint operators in a Hilbert space $\mathcal{H}$. Denote $W_t = \mathrm{e}^{\mathrm{i}At}$ the unitary group in $\mathcal{H}$ generated by A. We say that H is of class $C^1(A)$, and we write $H \in C^1(A)$, if its domain $D(H)$ is invariant under the group W and if for all $u \in D(H)$ the function $t \mapsto \langle W_t u, HW_t u\rangle$ is of class C^1. In this case we denote by $[H, \mathrm{i}A]$ the sesquilinear form on $D(H)$ given by $\langle u, [H, \mathrm{i}A]u\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle W_t u, HW_t u\rangle_{/t=0}$. We equip $D(H)$ with the graph-norm. Then $[H, \mathrm{i}A]$ is a continuous sesquilinear form on $D(H)$ and it is often useful to think of it as a continuous linear operator from $D(H)$ to its adjoint space $D(H)^*$. Analogously, one defines the classes $C^k(A)$, $k \in \mathbb{N}$.

One says that an operator H of class $C^1(A)$ satisfies a Mourre estimate at some point $\lambda \in \mathbb{R}$ if there are an open interval J containing λ, a strictly positive real number c_0, and a compact operator K such that the estimate (1.1) holds. The closed real set

$$\tau_A(H) = \mathbb{R} \setminus \widetilde{\mu}_A(H) = \{\lambda \in \mathbb{R} : \ H \text{ does not satisfy a Mourre estimate at } \lambda\}$$

will be called the set of *A-thresholds of H*. If (1.1) holds with $K = 0$ we define the (closed) set of A-critical points of H by

$$\begin{aligned}\kappa_A(H) &= \mathbb{R} \setminus \mu_A(H)\\ &= \{\lambda \in \mathbb{R} : \ H \text{ does not satisfy a strict Mourre estimate at } \lambda\}.\end{aligned}$$

For the computation of the sets $\tau_A(H)$ and $\kappa_A(H)$ it is convenient to introduce the lower semicontinuous functions $\widetilde{\rho}_H : \mathbb{R} \to (-\infty, +\infty]$ and $\rho_H : \mathbb{R} \to (-\infty, +\infty]$ defined by the following rule. For $\lambda \in \mathbb{R}$ and $\varepsilon > 0$, let $E(\lambda; \varepsilon) = E((\lambda - \varepsilon, \lambda + \varepsilon))$ be the spectral projection on the interval $(\lambda - \varepsilon, \lambda + \varepsilon)$. Then

$$\begin{aligned}\rho_H(\lambda) &= \sup\{a \in \mathbb{R} : \exists \varepsilon > 0 \text{ such that } E(\lambda;\varepsilon)[H, \mathrm{i}A]E(\lambda;\varepsilon) \geqslant aE(\lambda;\varepsilon)\},\\ \widetilde{\rho}_H(\lambda) &= \sup\{\, a \in \mathbb{R} : \ \exists\, \varepsilon > 0 \text{ and a compact operator } K \text{ such that}\\ &\qquad\qquad E(\lambda;\varepsilon)[H, \mathrm{i}A]E(\lambda;\varepsilon) \geqslant aE(\lambda;\varepsilon) + K\}.\end{aligned}$$

Clearly $\tau_A(H)$ is just the set of $\lambda \in \mathbb{R}$ such that $\widetilde{\rho}_H(\lambda) \leqslant 0$ and $\kappa_A(H)$ is the set of $\lambda \in \mathbb{R}$ such that $\rho_H(\lambda) \leqslant 0$.

An important property of the functions $\widetilde{\rho}_H$ and ρ_H is the following (see Theorem 7.2.13 in [1]): $\rho_H(\lambda) = \widetilde{\rho}_H(\lambda)$ with the exception of the points λ which are eigenvalues of H and where $\widetilde{\rho}_H(\lambda) > 0$; at these points one has $\rho_H(\lambda) = 0$. In particular, $\rho_H(\lambda) > 0$ if and only if $\widetilde{\rho}_H(\lambda) > 0$ and $\lambda \notin \sigma_{\rm p}(H)$. In other terms

$$\kappa_A(H) = \tau_A(H) \cup \sigma_{\rm p}(H).$$

We give now an equivalent definition of ρ_H which shows that the supremum in the definition of ρ_H is realized when $\varepsilon \to 0$, (see Lemma 7.2.1 in [1]):

If $\lambda \notin \sigma_{\rm p}(H)$, then $\rho_H(\lambda) = +\infty$. If $\lambda \in \sigma_{\rm p}(H)$, then $\rho_H(\lambda)$ is finite and given by

$$\rho_H(\lambda) = \lim_{\varepsilon \to 0}(\inf\{\langle u, [H, \mathrm{i}A]u\rangle : \|u\| = 1 \, and \, E(\lambda; \varepsilon)u = u\}),$$

and there is a sequence $\{u_k\}$ of vectors such that $\|u_k\| = 1$, $E(\lambda; 1/k)u_k = u_k$ and $\lim_{k\to\infty} \langle u_k, [H, \mathrm{i}A]u_k\rangle = \rho_H(\lambda)$.

For the development of the theory, for example in order to show that the limiting absorption principle holds, one has to require more regularity than $C^1(A)$ (see [1] and references there). Let S be a bounded operator on $\mathcal{H}$. We say that S is of class $\mathcal{C}^{1,1}(A)$ if $\int_0^1 \|(\mathcal{W}_t - 1)^2 S\| \frac{\mathrm{d}t}{t^2} < \infty$; where $\mathcal{W}_t(S) = \mathrm{e}^{-\mathrm{i}At} S \mathrm{e}^{\mathrm{i}At}$. Let H be a self-adjoint operator in $\mathcal{H}$. We shall say that H is of class $\mathcal{C}^{1,1}(A)$ if $(H - z)^{-1}$ is of class $\mathcal{C}^{1,1}(A)$ for some (then for all) $z \in \mathbb{C} \setminus \sigma(H)$.

2.2. Let $\mathcal{H} = L^2(\mathbb{R}^n)$. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ be the space of tempered test functions and $\mathcal{S}^* = \mathcal{S}^*(\mathbb{R}^n)$ the space of tempered distributions. The usual identifications $\mathcal{S} \subset \mathcal{H} \cong \mathcal{H}^* \subset \mathcal{S}^*$ are made. We denote by $\mathcal{F}$ the Fourier transformation acting in $\mathcal{S}^*$ defined by:

$$(\mathcal{F}f)(k) = \widehat{f}(k) = (2\pi)^{-n/2} \int \mathrm{e}^{-\mathrm{i}kx} f(x)\mathrm{d}x.$$

We denote by $f(Q)$ the operator of multiplication by a function f in $\mathcal{H}$ and by $f(P) = \mathcal{F}^* f(Q)\mathcal{F}$ the associated convolution operator in $\mathcal{H}$. We denote $\langle x\rangle = (1 + | x |^2)^{1/2}$. Then we define the Sobolev spaces

$$\mathcal{H}^s(\mathbb{R}^n) = \{f \in \mathcal{S}^*(\mathbb{R}^n) : \langle P\rangle^s f \in \mathcal{H}(\mathbb{R}^n)\}.$$

We introduce now a class of weighted Sobolev spaces. Let $\theta \in C_0^\infty(\mathbb{R}^n)$ be a real function such that $\theta(x) > 0$ if $2^{-1} < | x | < 2$ and $\theta(x) = 0$ otherwise. Choose one more real function $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) > 0$ if $| x | < 1$. Then for any $s, t \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$ let $\mathcal{H}^s_{t,p}$ be the space of distributions u that belong locally to $\mathcal{H}^s$ and such that

$$\|\eta(Q)u\|_{\mathcal{H}^s} + \Big[\int_1^\infty \|r^t \theta(r^{-1}Q)u\|^p_{\mathcal{H}^s} \frac{\mathrm{d}r}{r}\Big]^{1/p} < \infty.$$

If $p = \infty$ the second term is interpreted as $\sup_{r \geqslant 1} \|r^t \theta(r^{-1}Q)u\|_{\mathcal{H}^s}$. The left hand side above is a norm on $\mathcal{H}^s_{t,p}$ which provides this space with a Banach space structure. The spaces $\mathcal{H}^s_t := \mathcal{H}^s_{t,2}$ are the usual weighted Sobolev spaces defined by the norms $\|\langle P\rangle^s\langle Q\rangle^t u\|$. If $t_1 < t < t_2$, $t = (1-\lambda)t_1 + \lambda t_2$ and $p, p_1, p_2 \in [1,\infty], s \in \mathbb{R}$, then

$$\mathcal{H}^s_{t,p} = (\mathcal{H}^s_{t_1,p_1}, \mathcal{H}^s_{t_2,p_2})_{\lambda,p}.$$

Moreover, if $1 \leqslant p < \infty$ and $1/p + 1/p' = 1$, then $(\mathcal{H}^s_{t,p})^* = \mathcal{H}^{-s}_{-t,p'}$.

3. MOURRE ESTIMATE FOR OPERATORS OF THE FORM $H_1 \otimes I + I \otimes H_2$

In this section we shall explicitly calculate the function ρ of an operator of the form $H = H_1 \otimes I + I \otimes H_2$ in terms of those of H_j assuming that A is similarly decomposable.

We begin by recalling some results on tensor products of operators. We denote by $\mathcal{H}_1 \otimes \mathcal{H}_2$ the Hilbert tensor product of the two Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$. If S_1 and S_2 are densely defined closed linear operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, we denote by $S_1 \otimes S_2$ the closure of their algebraic tensor product (we use the conventions of [15], pp. 298–299). We write $S_1 \otimes I + I \otimes S_2$ for the closure of the sum of $S_1 \otimes I$ and $I \otimes S_2$

Assume that two self-adjoint operators H_1 and H_2 are given in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We write $D(H_j)$ for the domain of H_j provided with the graph-norm. Let $\mathcal{M}_j$ $(j = 1,2)$ be a core for H_j in $\mathcal{H}_j$ and let $\mathcal{M}$ the set of all finite linear combinations of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of the form $f_1 \otimes f_2$ with $f_j \in \mathcal{M}_j$. Let $H = H_1 \otimes I + I \otimes H_2$ (defined as indicated above). Let us make some remarks concerning the spectral properties of H in $\mathcal{H}$:

(i) H is self-adjoint and $\mathcal{M}$ is a core for it;

(ii) the spectrum of H is the closure of the sum of those of H_1 and H_2:

$$\sigma(H) = \overline{\sigma(H_1) + \sigma(H_2)};$$

(iii) one has for each $t \in \mathbb{R}$:

$$\mathrm{e}^{\mathrm{i}Ht} = \mathrm{e}^{\mathrm{i}H_1 t} \otimes \mathrm{e}^{\mathrm{i}H_2 t};$$

(iv) if one of the operators H_1 or H_2 has a purely absolutely continuous spectrum, then H has a purely absolutely continuous spectrum too;

(v) a real number λ is an eigenvalue of H if and only if it is of the form $\lambda = \lambda_1 + \lambda_2$, where λ_j is an eigenvalue of H_j $(j = 1,2)$.

These assertions are well known from Theorem VIII.33 in [15], Theorem 8.33 and Theorem 8.35 in [17].

DEFINITION 3.1. Let X be a metric space. We say that a function $g : X \to \mathbb{R}$ is *uniformly lower semi-continuous* (u.l.s.c.) if:

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } |\,\lambda - \mu\,| < \delta \ \Rightarrow\ g(\mu) \geqslant g(\lambda) - \varepsilon.$$

The next proposition contains a technical result that will be useful for the proof of the main theorem of this section.

PROPOSITION 3.2. *Let $\theta_1, \theta_2 : \mathbb{R} \to \mathbb{R}$ be two bounded from below* u.l.s.c. *functions. Let $\theta : \mathbb{R} \to \mathbb{R}$ be the function defined by:*

$$\theta(\lambda) = \inf_{\lambda=\lambda_1+\lambda_2} (\theta_1(\lambda_1) + \theta_2(\lambda_2)).$$

Then θ is bounded from below u.l.s.c. *function.*

Proof. We only have to show that θ is uniformly lower semi-continuous. Let $\varepsilon > 0$. Then $\theta(\lambda) - \varepsilon \leqslant \left(\theta_1(\lambda_1) - \frac{\varepsilon}{2}\right) + \left(\theta_2(\lambda_2) - \frac{\varepsilon}{2}\right)$ if $\lambda = \lambda_1 + \lambda_2$. Since θ_1 and θ_2 are uniformly lower semi-continuous, there is $\delta > 0$ such that $|\,\lambda_i - \mu_i\,| < \delta \Rightarrow \theta_i(\lambda_i) - \frac{\varepsilon}{2} \leqslant \theta_i(\mu_i)$ for $i = 1, 2$. Hence, if $\sup\limits_{i=1,2} |\,\lambda_i - \mu_i\,| < \delta$ for some (λ_1, λ_2) with $\lambda = \lambda_1 + \lambda_2$ then $\theta(\lambda) - \varepsilon \leqslant \theta_1(\mu_1) + \theta_2(\mu_2)$. In other terms, if $\triangle_\lambda$ is the straight line $\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda = \lambda_1 + \lambda_2\}$ and $\operatorname{dist}((\mu_1, \mu_2), \triangle_\lambda) < \delta$, then the last inequality holds. Now, if $|\,\lambda - \mu\,| < \frac{\delta}{\sqrt{2}}$ then $\mu = \mu_1 + \mu_2$ with $|\,\lambda_i - \mu_i\,| < \delta$. Hence $\theta(\lambda) - \varepsilon \leqslant \theta(\mu)$. This proves the proposition. ∎

We pass now to the main result of this section, namely the calculation of the function ρ for an operator of the form $H_1 \otimes I + I \otimes H_2$ when A admits a similar decomposition.

THEOREM 3.3. *Let H_1, H_2 be two self-adjoint operators in the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. Let A_j be self-adjoint operators in $\mathcal{H}_j$ such that $\mathrm{e}^{\mathrm{i}tA_j} D(H_j) \subset D(H_j)$ and $[H_j, \mathrm{i}A_j] \in B(\mathcal{H}_j)$. Assume that for each $\eta > 0$ there are functions $\theta_1, \theta_2 : \mathbb{R} \to \mathbb{R}$ bounded from below and* u.l.s.c. *and $\varepsilon > 0$ such that:*

(H1) $\quad E_j(\mu;\varepsilon)\,[H_j, \mathrm{i}A_j]\,E_j(\mu;\varepsilon) \geqslant \theta_j(\mu)\,E_j(\mu;\varepsilon)$ *for each $\mu \in \mathbb{R}$,*

(H2) $$\begin{cases} \theta_j(\mu) \geqslant \rho_{H_j}^{A_j}(\mu) - \eta & \text{if } \mu \in \sigma(H_j), \\ \theta_j(\mu) \geqslant N \ (N < \infty) & \text{if } \mu \notin \sigma(H_j). \end{cases}$$

Let $H = H_1 \otimes I + I \otimes H_2$ and $A = A_1 \otimes I + I \otimes A_2$ be the self-adjoint operators in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ defined as indicated above. Finally, assume that $\sigma(H_1) + \sigma(H_2)$ is closed. Then H is of class $C^1(A)$ and for all $\lambda \in \mathbb{R}$

$$\rho_H^A(\lambda) = \inf_{\lambda=\lambda_1+\lambda_2} [\rho_{H_1}^{A_1}(\lambda_1) + \rho_{H_2}^{A_2}(\lambda_2)]. \tag{3.1}$$

Proof. Step 1. Let us prove first that H is of class $C^1(A)$. It is clear that the core $\mathcal{M}$ of H is invariant under the action of the unitary group W_α in $\mathcal{H}$ generated by A. Then we get easily that

$$\langle v, W_{-\alpha} H W_\alpha u\rangle = \langle v, Hu\rangle + \int_0^\alpha \langle v, W_{-\tau}\{[H_1, \mathrm{i}A_1] \otimes I + I \otimes [H_2, \mathrm{i}A_2]\} W_\tau u\rangle \mathrm{d}\tau$$

for all $\alpha \in \mathbb{R}$ and $u, v \in \mathcal{M}$. Replacing v by $W_{-\alpha} v$ we get

$$\langle v, H W_\alpha u\rangle = \langle v, W_\alpha H u\rangle + \int_0^\alpha \langle v, W_{\alpha-\tau}\{[H_1, \mathrm{i}A_1] \otimes I + I \otimes [H_2, \mathrm{i}A_2]\} W_\tau u\rangle \mathrm{d}\tau.$$

Since $\mathcal{M}$ is dense in $\mathcal{H}$, we deduce from the last equality that $\|HW_\alpha u\| \leqslant \|Hu\| + c\alpha\|u\|$. Then $D(H)$ is invariant under W. By using the fact that $W_{-\alpha}HW_\alpha = \mathrm{e}^{-\mathrm{i}\alpha A_1}H_1\mathrm{e}^{\mathrm{i}\alpha A_1} \otimes I + I \otimes \mathrm{e}^{-\mathrm{i}\alpha A_2}H_2\mathrm{e}^{\mathrm{i}\alpha A_2}$ we deduce that for each $u \in D(H)$ the function $\alpha \mapsto \langle W_\alpha u, HW_\alpha u\rangle$ is of class C^1.

Step 2. Let us prove now formula (3.1). Set $\rho_j = \rho_{H_j}^{A_j}$. Since $\sigma(H_1) + \sigma(H_2)$ is closed, then $\sigma(H) = \sigma(H_1) + \sigma(H_2)$ and therefore (3.1) is obvious if $\lambda \notin \sigma(H)$ as both members are equal to $+\infty$.

Step 3. Let us fix some arbitrary $\lambda \in \sigma(H)$ and some numbers $\eta > 0$. Then by hypothesis there exists $\varepsilon > 0$ such that the hypothesis (H1) holds for all $\mu \in \mathbb{R}$. Set $T_1 = H_1 \otimes I$ and $T_2 = I \otimes H_2$. By working in a spectral representation of the operator H_j, we easily deduce from (H_1) that

$$\begin{aligned} E(\lambda;\varepsilon)[T_1, \mathrm{i}A]E(\lambda;\varepsilon) &\geqslant \theta_1(\lambda - T_2)E(\lambda;\varepsilon),\\ E(\lambda;\varepsilon)[T_2, \mathrm{i}A]E(\lambda;\varepsilon) &\geqslant \theta_2(\lambda - T_1)E(\lambda;\varepsilon), \end{aligned}$$

and therefore

$$E(\lambda;\varepsilon)[H, \mathrm{i}A]E(\lambda;\varepsilon) \geqslant [\theta_1(\lambda - T_2) + \theta_2(\lambda - T_1)]E(\lambda;\varepsilon). \tag{3.2}$$

By remarking that we can write $\lambda - T_2 = \lambda - H + T_1$ and $\lambda - T_1 = \lambda - H + T_2$ and by using the fact that $|\, H - \lambda \,| < \varepsilon$, we obtain

$$E(\lambda;\varepsilon)[H, \mathrm{i}A]E(\lambda;\varepsilon) \geqslant \inf_{|\,\tau\,|<\varepsilon}\{\theta_1(\tau + T_1) + \theta_2(\tau + T_2)\}E(\lambda;\varepsilon). \tag{3.3}$$

Here $T_1 + T_2 = H$; then $|\, T_1 + T_2 - \lambda \,| < \varepsilon$. Now, by working in a spectral representation for both H_1 and H_2, we deduce from (3.3):

$$\begin{aligned} E(\lambda;\varepsilon)[H, \mathrm{i}A]E(\lambda;\varepsilon) &\geqslant \inf_{|\,\tau\,|<\varepsilon, |\,\mu_1+\mu_2-\lambda\,|<\varepsilon}\{\theta_1(\tau + \mu_1) + \theta_2(\tau + \mu_2)\}E(\lambda;\varepsilon)\\ &\geqslant \inf_{|\,\lambda_1+\lambda_2-\lambda\,|<3\varepsilon}\{\theta_1(\lambda_1) + \theta_2(\lambda_2)\}E(\lambda;\varepsilon)\\ &= \inf_{|\,\nu-\lambda\,|<3\varepsilon}\ \inf_{\nu=\lambda_1+\lambda_2}\{\theta_1(\lambda_1) + \theta_2(\lambda_2)\}E(\lambda;\varepsilon)\\ &= \inf_{|\,\nu-\lambda\,|<3\varepsilon}\theta(\nu)E(\lambda;\varepsilon), \end{aligned}$$

where θ is the function defined in Proposition 3.2. By the same proposition we get that $\rho(\lambda) \geqslant \theta(\lambda)$. By using the hypothesis (H2) we obtain $\rho(\lambda) \geqslant \inf_{\lambda=\lambda_1+\lambda_2}(\rho_1(\lambda_1) + \rho_2(\lambda_2)) - 2\eta$ for all $\eta > 0$. So, we can conclude that: $\rho(\lambda) \geqslant \inf_{\lambda=\lambda_1+\lambda_2}(\rho_1(\lambda_1) + \rho_2(\lambda_2))$.

Step 4. To show the opposite inequality it is enough to prove that $\rho(\lambda) \leqslant \rho_1(\lambda_1) + \rho_2(\lambda_2)$ if $\lambda \in \sigma(H), \lambda_j \in \sigma(H_j)$ $(j = 1, 2)$ and $\lambda = \lambda_1 + \lambda_2$. Let $a < \rho(\lambda)$. Then there is $\varepsilon > 0$ such that $aE(\lambda;\varepsilon) \leqslant E(\lambda;\varepsilon)[H, \mathrm{i}A]E(\lambda;\varepsilon)$. We can show that

$$\begin{aligned} &aE_1(\lambda_1;\varepsilon/2) \otimes E_2(\lambda_2;\varepsilon/2) \leqslant\\ &\qquad \{E_1(\lambda_1;\varepsilon/2)[H_1, \mathrm{i}A_1]E_1(\lambda_1;\varepsilon/2)\} \otimes E_2(\lambda_2;\varepsilon/2)\\ &\qquad + E_1(\lambda_1;\varepsilon/2) \otimes \{E_2(\lambda_2;\varepsilon/2)[H_2, \mathrm{i}A_2]E_2(\lambda_2;\varepsilon/2)\}. \end{aligned}$$

By the definition of $\rho_j(\lambda_j)$ there is a sequence of vectors $\{f_n^j\}_{n\in\mathbb{N}}$ in $\mathcal{H}_j$ $(j=1,2)$ such that $\|f_n^j\|=1$, $f_n^j = E_j(\lambda_j;\varepsilon/2)f_n^j$ and $\langle f_n^j,[H_j,\mathrm{i}A_j]f_n^j\rangle \to \rho_j(\lambda_j)$ as $n\to\infty$. If $f_n = f_n^1\otimes f_n^2$, the last inequality implies:

$$a = \langle f_n, aE_1(\lambda_1;\varepsilon/2)\otimes E_2(\lambda_2;\varepsilon/2)f_n\rangle \leqslant \langle f_n^1,[H_1,\mathrm{i}A_1]f_n^1\rangle + \langle f_n^2,[H_2,\mathrm{i}A_2]f_n^2\rangle$$

for any $n\in\mathbb{N}$. Hence $a\leqslant \rho_1(\lambda_1)+\rho_2(\lambda_2)$, which completes the proof of the theorem. ∎

REMARK 3.4. (i) Our hypothesis on the regularity of H_j with respect to A_j is stronger than in the case where H_j are bounded from below (see Theorem 3.4 in [2]). Indeed, our conditions imply that $H_j\in C^1(A_j)$ because $B(\mathcal{H}_j)\subset B(D(H_j),D(H_j)^*)$. However, the following example shows that in the case where H_j is not bounded from below the hypothesis $H_j\in C^1(A_j)$ is not sufficient to obtain $H\in C^1(A)$.

Let $\mathcal{H}_1 = L^2(\mathbb{R})$. We consider in $\mathcal{H}_1$ the free Stark Hamiltonian $H_1 = P^2 + Q$. Let $A_1 = -P$. It follows that $[H_1,\mathrm{i}A_1] = 1$, which is obviously a bounded operator in $\mathcal{H}_1$. In particular $H_1\in C^1(A_1)$. Let $\mathcal{H}_2 = L^2(\mathbb{R}^n)$, in which we consider the Laplacian $H_2=\Delta$. Let $A_2 = D = \frac{1}{4}(PQ+QP)$ the generator of dilatations in $\mathcal{H}_2$. Since $[\Delta,\mathrm{i}D]=\Delta$, the operator H_2 is of class $C^1(A_2)$. Now we consider in $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ the self-adjoint operators $H = H_1\otimes 1 + 1\otimes H_2$ and $A = A_1\otimes 1 + 1\otimes A_2$. We denote by $D(H)$ the domain of H provided with the graph topology. We then have $[H,\mathrm{i}A] = 1 + 1\otimes H_2$. We shall show that $[H,\mathrm{i}A]\notin B(D(H),D(H)^*)$. Suppose that $[H,\mathrm{i}A]\in B(D(H),D(H)^*)$, then there exists a constant $c<\infty$ such that $1\otimes H_2 \leqslant c(1+H^2)$. Define the unitary operator $U = \mathrm{e}^{-\mathrm{i}P^3/3}$ in $\mathcal{H}_1$. We can show that $H_1 = U^{-1}QU$. Thus H_1 is equivalent to the operator of multiplication by the function $g:\mathbb{R}\to\mathbb{R}$ defined by $g(x)=x$. So we have $\sigma(H_1)=\mathbb{R}$ and H_1 is not bounded from below. In its spectral representation H_2 becomes the operator of multiplication by the function $h:\mathbb{R}^n\to\mathbb{R}$ defined by $h(y) = |\,y\,|^2$. In the spectral representation of H_1 and H_2 the inequality $1\otimes H_2\leqslant c(1+H^2)$ is equivalent to the statement: for each $x\in\mathbb{R}$ and each $y\in\mathbb{R}^n$ we have $h(y)\leqslant c(1+(g(x)+h(y))^2)$. Thus $h(y)\leqslant c$ for each $y\in\mathbb{R}^n$, which is obviously absurd. So $[H,\mathrm{i}A]\notin B(D(H),D(H)^*)$, which implies that $H\notin C^1(A)$.

(ii) The condition $\sigma(H_1)+\sigma(H_2)$ is closed, which is automaticaly verified in the case where H_j are bounded from below, is simply technical and is not very restrictive in the applications.

APPLICATION 3.5. As an example, let us show how the theorem should be used for the case of time periodic relativistic Hamiltonians.

Let $H_1 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}$ in $\mathcal{H}_1 = L^2(\mathbb{T})$ where $\mathbb{T}$ is the torus $\mathbb{R}/2\pi\mathbb{Z}$. Let $H_2 = h(P) = \sqrt{1+P^2}$ in $\mathcal{H}_2 = L^2(\mathbb{R}^n)$. Let $H = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\otimes 1 + 1\otimes\sqrt{P^2+1}$ be the self-adjoint operator defined as above.

Since H_1 is an operator with purely discrete spectrum by simple arguments then the operator $A_1 = 0$ is conjugate to it and one has

$$\rho_1(\lambda) = 0 \text{ if } \lambda \in \mathbb{Z} \quad \text{and} \quad \rho_1(\lambda) = +\infty \text{ if } \lambda \notin \mathbb{Z}.$$

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field defined by $F(\xi) = \frac{h'(\xi)}{1+|h'(\xi)|^2}$. Then $F \in BC^1(\mathbb{R}^n)$ and the operator $A_2 = \frac{1}{2}[F(P)Q + QF(P)]$ is essentially self-adjoint on $\mathcal{H}_2$. The group $\{e^{iA_2 t}\}$ generated by A_2 leaves invariant the domain of H_2 and we have $[H_2, iA_2] = \frac{|h'(P)|^2}{1+|h'(P)|^2} = \frac{|P|^2}{1+|P|^2}$. Therefore $[H_2, iA_2] \in B(\mathcal{H}_2)$. Furthermore one has

$$\rho_2(\lambda) = \inf_{h(x)=\lambda} \frac{x^2}{1+x^2} = \frac{\lambda^2-1}{2\lambda^2-1} \text{ if } \lambda \geqslant 1 \quad \text{and} \quad \rho_2(\lambda) = +\infty \text{ if } \lambda < 1.$$

Now we construct the functions θ_1, θ_2 of the Theorem 3.3. Let us fix some arbitrary $\eta > 0$ as well as numbers $\varepsilon, \alpha, \beta, \delta > 0$ such that $\eta > \beta, \eta > \delta$. Let us also choose a number $N < \infty$ large enough. Then we define θ_1 and θ_2 as follows:

$$\theta_1(\mu) = \begin{cases} N & \text{if } \mu \notin]k-\varepsilon, k+\varepsilon[\ \forall\, k \in \mathbb{Z}, \\ -\beta & \text{if } \mu \in]k-\varepsilon, k+\varepsilon[\ \forall\, k \in \mathbb{Z}, \end{cases}$$

and

$$\theta_2(\mu) = \begin{cases} N & \text{if } \mu < 1-\alpha, \\ -\delta & \text{if } 1-\alpha \leqslant \mu \leqslant 1+\alpha, \\ \rho_2(\mu) - \eta & \text{if } \mu > 1+\alpha. \end{cases}$$

We verify easily that all the hypotheses of the Theorem 3.3 are satisfied. Therefore, $H \in C^1(A)$ and

$$\begin{aligned} \rho_H^A(\lambda) &= \inf_{\lambda=\lambda_1+\lambda_2} [\rho_1(\lambda_1) + \rho_2(\lambda_2)] = \inf_{\lambda=k+\lambda_2,\, k\in\mathbb{Z}} \rho_2(\lambda_2) \\ &= \inf_{\lambda-k\geqslant 1,\, k\in\mathbb{Z}} \rho_2(\lambda-k) = \inf_{\lambda-k\geqslant 1,\, k\in\mathbb{Z}} \frac{(\lambda-k)^2-1}{2(\lambda-k)^2-1}. \end{aligned}$$

The result is that the time periodic relativistic Hamiltonians satisfy a strict Mourre estimate on $\mu^A(H) = \mathbb{R} \setminus \mathbb{Z}$.

4. SPECTRAL THEORY OF DISPERSIVE TIME-PERIODIC SYSTEMS

In this section we study spectral properties of operators of the form $H = H_0 + V$, where $H_0 = H_1 \otimes 1 + 1 \otimes H_2$ is a self-adjoint operator and V is, in some sense, a small symmetric perturbation of H_0. The perturbation theory we use is described in [1].

4.1. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $H_1 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau}$ the momentum operator in the Hilbert space $\mathcal{H}_1 = L^2(\mathbb{T})$. Let $X = \mathbb{R}^n$, $n \geqslant 2$ and $h : X \to \mathbb{R}$ be a real function of class C^2. We denote by H_2 the self-adjoint operator $h(P) = \mathcal{F}^* h(Q)\mathcal{F}$ in the Hilbert space $\mathcal{H}_2 = L^2(X)$ with domain $D(H_2) = \{u \in \mathcal{H}_2 : h(P)u \in \mathcal{H}_2\}$.

We denote by $H_0 = H_1 \otimes 1 + 1 \otimes H_2$ the self-adjoint operator in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 = L^2(\mathbb{T} \times X)$ defined as indicated in Section 3. The operator H_0 will be called *dispersive time periodic Hamiltonian*.

We denote by $\kappa(h)$ the set of critical values of h:

$$\kappa(h) = \{\lambda \in \mathbb{R} : \exists k \in X \text{ such that } h(k) = \lambda \text{ and } h'(k) = 0\}. \tag{4.1}$$

Note that $\kappa(h)$ is a closed subset of $\mathbb{R}$. The set $\kappa(h)$ plays an important role in spectral theory. One may give the following physical interpretation of $\kappa(h)$: if the particle has kinetic energy $\lambda = h(k) \in \kappa(h)$, then its corresponding velocity $h'(k)$ is equal to zero. At these energies the particle has bad propagation properties.

We assume that the critical set $\kappa(h)$ is compact and we denote by $\tau(h)$ the closed set $\tau(h) = \kappa(h) + \mathbb{Z}$.

Since the conjugate operator plays an important role in the Mourre theory, we shall begin by constructing a suitable conjugate operator for the free Hamiltonian H_0.

Since the operator H_1 has a purely discrete spectrum, it is conjugate to the operator $A_1 = 0$ (see Application 3.5). In order to find a conjugate operator A_2 for H_2 we shall follow the construction made in [1], which is motivated by the following argument.

Let $F : X \to X$ be a vector field and set

$$A_2 = \frac{1}{2}[F(P) \cdot Q + Q \cdot F(P)] = QF(P) - \frac{\mathrm{i}}{2} f(P), \tag{4.2}$$

where $f = \mathrm{div} F = \sum_{j=1}^{n} \partial_j F_j$. Since $[h(P), \mathrm{i}\, Q_j] = (\partial_j h)(P)$, then $[h(P), \mathrm{i}A_2] = (Fh')(P)$, where $h' = \nabla h$ and $Fh' = \sum_{j=1}^{n} F_j \partial_j h$. As we shall see in the proof of the Proposition 4.1, in order to get local positivity of the first order commutator, it is appropriate to use the vector field F defined by:

$$F(k) = \frac{h'(k)}{1 + |\, h'(k)\, |^2}. \tag{4.3}$$

Since F belongs to $BC^1(X)$ if $|\, h''(k)\, | \leqslant c(1 + |\, h'(k)\, |^2)$, it follows that the operator A_2 defined in (4.2) is essentially self-adjoint in $\mathcal{H}_2$. Finally, for the free hamiltonian H_0 we adopt the following conjugate operator:

$$A = 1 \otimes A_2. \tag{4.4}$$

Suitable conditions on V will make this operator conjugate to H too.

We now state the main result of this section.

PROPOSITION 4.1. *Let $h : X \to \mathbb{R}$ be a real function of class C^2 such that $|h''(k)| \leqslant c(1+|h'(k)|^2)$. We suppose also that the function h satisfies the following strict positivity global condition:*

$$\forall\, \delta > 0\ \exists\, a > 0 \text{ such that if } \operatorname{dist}(h(k),\kappa(h)) \geqslant \delta \text{ then } |h'(k)| \geqslant a. \tag{4.5}$$

Let A and H_0 be defined as above. Then

(i) *A is strictly conjugate to H_0 on $\mathbb{R}\setminus\tau(h)$, i.e.*

$$\mu^A(H_0) = \mathbb{R}\setminus\tau(h).$$

In particular, H_0 has no eigenvalues in $\mathbb{R}\setminus\tau(h)$.

(ii) *The limits $\lim_{\mu\to\pm 0}(H-\lambda-\mathrm{i}\mu)^{-1} \equiv (H-\lambda\mp \mathrm{i}0)^{-1}$ exist locally uniformly on each compact subset of $\lambda \in \mathbb{R}\setminus\tau(h)$ in the weak*-topology of $B(\mathcal{H}_{1/2,1},\mathcal{H}_{-1/2,\infty})$.*

In particular, H_0 has no singular continuous spectrum outside $\tau(h)$.

Proof. Step 1. It is clear that the domain $D(H_2)$ of H_2 is invariant under the group $\mathrm{e}^{-\mathrm{i}tA_2}$. Furthermore, we have:

$$[H_2,\mathrm{i}A_2] = (Fh')(P) = \frac{|h'(P)|^2}{1+|h'(P)|^2}$$

which is a bounded operator in $\mathcal{H}_2$. It follows from Theorem 3.3 that H is of class $C^1(A)$.

Let $\lambda \in \mathbb{R}\setminus\tau(h)$. Denote $d(\lambda,\tau(h))$ the distance from λ to $\tau(h)$. Let $0 < \eta < d(\lambda,\tau(h))$ and $\varphi \in C_0^\infty([\lambda-\eta/3,\lambda+\eta/3])$. We remark that the spectrum of H_1 is discrete and constituted by eigenvalues $k\in\mathbb{Z}$. Let P_k be the eigenprojection associated to the eigenvalue k. Then H_0 can be decomposed as

$$H_0 = \sum_{k\in\mathbb{Z}} (k+H_2)\otimes P_k.$$

Consequently, we have

$$\varphi(H_0)[H_0,\mathrm{i}A]\varphi(H_0) = \sum_{k\in\mathbb{Z}} \frac{|h'(P)|^2}{1+|h'(P)|^2}\varphi(k+H_2)^2\otimes P_k.$$

Since $\operatorname{supp}\varphi(k+\cdot) \subset [\lambda-\eta/3-k,\lambda+\eta/3-k]$ we have that if there is $k\in\mathbb{Z}$ such that $h'(p) \in [\lambda-\eta/3-k,\lambda+\eta/3-k]$ then $d(h(p),\kappa(h)) \geqslant \frac{\eta}{3}$. From the assumption (see (4.5)) there is $a > 0$ such that $|h'(p)| \geqslant a$ and therefore there exists a constant $c > 0$ such that:

$$\varphi(H_0)[H_0,\mathrm{i}A]\varphi(H_0) \geqslant c\varphi(H_0)^2$$

which proves that A is strictly conjugate to H_0 in λ.

Step 2. It is easy to see that $[[H_2,\mathrm{i}A_2],\mathrm{i}A_2] = ((F\nabla)^2 h)(P) \in B(\mathcal{H}_2)$ and consequently, H_0 is of class $C^2(A)$. Now, (ii) follows from (i) by the well-known arguments. ∎

REMARK 4.2. The condition (4.5) which we impose in order to make a simple spectral analysis of H_0 can be stated in physical terms as follows: If the kinetic energy belongs to a set located at a positive distance from the critical set $\kappa(h)$, then the velocity is bounded from below by a strictly positive constant.

4.2. We shall describe now a class of potentials V for which we can extend the results of the preceding proposition to the perturbed operators of the form $H = H_0 + V$.

We first introduce some notation. For each multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ we set $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let $\mathcal{G} = D(|H_0|^{1/2})$ be the form domain of H_0 and $\mathcal{G}_1 = \mathcal{H}_1 \otimes \mathcal{G}$. Let $\mathcal{G}^* = D(|H_0|^{-1/2})$ and $\mathcal{G}_1^* = \mathcal{H}_1 \otimes \mathcal{G}^*$ be the corresponding adjoint spaces.

We say that h is an elliptic symbol of degree $2r > 0$ if $h \in C^\infty(X)$, $|h^{(\alpha)}(x)| \leqslant c_\alpha \langle x \rangle^{2r-|\alpha|}$ for each multi-index α and $|h(x)| \geqslant c\langle x \rangle^{2r}$, for some $c > 0$, outside a compact set. In this case $\mathcal{G} = (\mathcal{H}_2)^r$ the usual Sobolev space associated to the momentum operator P in the Hilbert space $\mathcal{H}_2$, and $\mathcal{G}^* = (\mathcal{H}_2)^{-r}$ its adjoint.

If V is a symmetric operator in $\mathcal{H}$ then to each $t \in \mathbb{T}$ we associate a symmetric operator $V(t)$ in $\mathcal{H}_2$. Therefore, if V can be decomposed into a sum $V = V_{\mathrm{S}} + V_{\mathrm{L}}$ where V_{S} and V_{L} are symmetric operators in $\mathcal{H}$, then to each $t \in \mathbb{T}$ we can associate two symmetric operators $V_{\mathrm{S}}(t)$ and $V_{\mathrm{L}}(t)$ in $\mathcal{H}_2$. The component $V_{\mathrm{S}}(t)$ is called the short range part of the interaction which can be very singular but it must decay quickly at infinity. The component $V_{\mathrm{L}}(t)$, called the long range part of the interaction, may be a non local operator, quite strong locally singular and decays very slowly at infinity but it must be regular.

THEOREM 4.3. *Let A and H_0 be as above. Suppose that h is an elliptic symbol and let $V : \mathcal{G}_1 \to \mathcal{G}_1^*$ be a symmetric operator such that V is relatively bounded with respect to H_0 with relative bound strictly less than* 1. *Denote by H the self-adjoint operator in $\mathcal{H}$ induced by $H_0 + V$ and assume that $(H + \mathrm{i})^{-1} - (H_0 + \mathrm{i})^{-1}$ is compact in $\mathcal{H}$. Assume that $V = V_{\mathrm{S}} + V_{\mathrm{L}}$, where $V_{\mathrm{S}}(t) : \mathcal{G} \to \mathcal{G}^*$ and $V_{\mathrm{L}}(t) : \mathcal{G} \to \mathcal{G}^*$ are symmetric operators satisfying:*

(S) *There is $\theta \in C_0^\infty(X)$ with $\theta(x) > 0$ in an annulus $0 < a < |x| < b < \infty$ and $\theta(x) = 0$ otherwise, such that:*

$$\int_1^\infty \sup_{t \in \mathbb{T}} \|\theta(r^{-1}Q)V_{\mathrm{S}}(t)\|_{B(\mathcal{G},\mathcal{G}^*)} \mathrm{d}r < \infty.$$

(L) *There is $\zeta \in C^\infty(X)$ with $\zeta(x) = 0$ near zero and $\zeta(x) = 1$ near infinity such that:*

$$\sum_{j=1}^n \int_1^\infty \sup_{t \in \mathbb{T}} \{\|\zeta(r^{-1}Q)[Q_j, V_{\mathrm{L}}(t)]\|_{B(\mathcal{G},\mathcal{G}^*)} + \|\zeta(r^{-1}Q)|Q|[P_j, V_{\mathrm{L}}(t)]\|_{B(\mathcal{G},\mathcal{G}^*)}\} \frac{\mathrm{d}r}{r} < \infty.$$

Then the operator H has the following properties:

(i) *A is locally conjuguate to H on $\mathbb{R} \setminus \tau(h)$, i.e.*

$$\widetilde{\mu}_A(H) = \mathbb{R} \setminus \tau(h).$$

(ii) *The eigenvalues of H which do not belong to* $\tau(h)$ *are of finite multiplicity and do not have accumulation points outside* $\tau(h)$.

(iii) *For each* $\lambda \in \mathbb{R} \setminus [\tau(h) \cup \sigma_{\mathrm{p}}(H)]$, *the limits* $\lim_{\mu \to +0}(H - \lambda \mp \mathrm{i}\mu)^{-1}$ *exist in the weak*-*topology of* $B(\mathcal{H}_{1/2,1}, \mathcal{H}_{-1/2,\infty})$, *and this holds uniformly in* λ *in each compact set of* $\mathbb{R} \setminus [\tau(h) \cup \sigma_{\mathrm{p}}(H)]$.

(iv) *The singular continuous spectrum of H is included in* $\tau(h)$. *In particular, if* $\tau(h)$ *is countable, then H has no singular continuous spectrum.*

Proof. (i) As h is an elliptic symbol and $[[h(P), \mathrm{i}A_2], \mathrm{i}A_2] = ((F\nabla)^2 h)(P)$ is a bounded operator in $\mathcal{H}_2$, then it is easy to see that H_0 is of class $C^2(A)$. Since $(H+\mathrm{i})^{-1} - (H_0+\mathrm{i})^{-1}$ is a compact operator in $\mathcal{H}$, it follows from Theorem 7.2.9 in [1] that if H is of class $\mathcal{C}^{1,1}(A)$ (in fact we need only $H \in C^1_u(A)$), then $\widetilde{\rho_H} = \widetilde{\rho_{H_0}}$. In particular,

$$\widetilde{\mu}^A(H) = \widetilde{\mu}^A(H_0) = \mu^A(H_0) = \mathbb{R} \setminus \tau(h).$$

Clearly the conditions (i), (ii) and (iii) of Proposition 7.5.6 in [1] are satisfied. Therefore, all assertions follow immediately as soon as we show that V is of class $\mathcal{C}^{1,1}(A; \mathcal{G}_1, \mathcal{G}_1^*)$.

(ii) Now we show that $V_{\mathrm{S}}, V_{\mathrm{L}} \in \mathcal{C}^{1,1}(A; \mathcal{G}_1, \mathcal{G}_1^*)$. For this we shall use Proposition 7.5.7 and Theorem 7.5.8 in [1], with $\Lambda = 1 \otimes \langle Q \rangle$ where Q is the position operator on $\mathcal{H}_2$. By using Lemma 7.6.7 in [1] we have that the group $\{\mathrm{e}^{\mathrm{i}\langle Q \rangle \tau}\}$ leaves invariant $\mathcal{G}$ and induces a C_0-group of polynomial growth in $\mathcal{G}$ and $\mathcal{G}^*$. Besides, we have $D(\langle Q \rangle, \mathcal{G}^*) \subset D(A_2, \mathcal{G}^*)$ and $\langle Q \rangle^{-2} A_2^2 \in B(\mathcal{G})$, so the properties $\mathrm{e}^{\mathrm{i}\Lambda\tau}\mathcal{G}_1 \subset \mathcal{G}_1$ for each $\tau \in \mathbb{R}$, $\|\mathrm{e}^{\mathrm{i}\Lambda\tau}\|_{B(\mathcal{G}_1)} \leqslant c\langle \tau \rangle^N$ and $D(\Lambda, \mathcal{G}_1^*) \subset D(\Lambda, \mathcal{G}_1^*)$ and $\Lambda^{-2}A^2 \in B(\mathcal{G}_1)$ required are easy to verify. Also, we have

$$\int_1^\infty \|\theta(r^{-1}\Lambda)V_{\mathrm{S}}\|_{B(\mathcal{G}_1, \mathcal{G}_1^*)} \mathrm{d}r = \int_1^\infty \sup_{t \in \mathbb{T}} \|\theta(r^{-1}Q)V_{\mathrm{S}}(t)\|_{B(\mathcal{G}, \mathcal{G}^*)} \mathrm{d}r < \infty,$$

so (7.5.29) in [1] is true and V_{S} is of class $\mathcal{C}^{1,1}(A; \mathcal{G}_1, \mathcal{G}_1^*)$. Finally, we consider the operator V_{L} and we propose to verify the relation (7.5.26) in [1]. From the proof of Theorem 7.6.8 in [1] we deduce:

$$\|\zeta(r^{-1}Q)[A_2, \mathrm{i}V_{\mathrm{L}}(t)]\|_{B(\mathcal{G}, \mathcal{G}^*)} \leqslant \varphi(r, t)$$

where φ is an integrable function on $(1, \infty)$ with respect to the measure $\frac{\mathrm{d}r}{r}$. Then

$$\int_1^\infty \|\zeta(r^{-1}\Lambda)[A, \mathrm{i}V_{\mathrm{L}}]\|_{B(\mathcal{G}_1, \mathcal{G}_1^*)} = \int_1^\infty \sup_{t \in \mathbb{T}} \|\zeta(r^{-1}Q)[A_2, \mathrm{i}V_{\mathrm{L}}(t)]\|_{B(\mathcal{G}, \mathcal{G}^*)} \frac{\mathrm{d}r}{r} < \infty,$$

and therefore $V_{\mathrm{L}} \in \mathcal{C}^{1,1}(A; \mathcal{G}_1, \mathcal{G}_1^*)$. ∎

EXAMPLES 4.4. We consider two physically important situations, namely the non-relativistic and relativistic Schrödinger operator periodic in time.

(i) *Non-relativistic Schrödinger operator periodic in time:* A trivial example of elliptic symbol is the quadratic function $h(k) = |k|^2$, which is of degree 2 (i.e.

$r = 1$). In this case $\mathcal{G} = (\mathcal{H}_2)^1$ and $\mathcal{G}^* = (\mathcal{H}_2)^{-1}$ and the hamiltonian is the non-relativistic Schrödinger operator periodic in time $H_0 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} \otimes 1 + 1 \otimes \Delta$. In this case, we have $\kappa(h) = \{0\}$ and $\tau(h) = \mathbb{Z}$. We can compare our result with those of Yokoyama (see [18]). However, the class of interactions that we consider is sensibly more general.

(ii) *Relativistic Schrödinger operator periodic in time:* Another physically important situation is obtained when $h(k) = (|\, k\,|^2 + 1)^{1/2}$, which is an elliptic symbol of degree 1 (i.e. $r = \frac{1}{2}$). In this case $\mathcal{G} = (\mathcal{H}_2)^{1/2}$ and $\mathcal{G}^* = (\mathcal{H}_2)^{-1/2}$ and the hamiltonian is the relativistic Schrödinger operator periodic in time $H = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} \otimes 1 + 1 \otimes \langle \Delta \rangle$. In this case, we have $\kappa(h) = \{1\}$ and $\tau(h) = \mathbb{Z}$.

In the particular case where $V = V_{\mathrm{S}} + V_{\mathrm{L}}$ is the operator of multiplication in $\mathcal{H}$ by a function $V(t, Q) = V_{\mathrm{S}}(t, Q) + V_{\mathrm{L}}(t, Q)$ our hypotheses are simple and natural: we only require that $V(t, Q) = V_{\mathrm{S}}(t, Q) + V_{\mathrm{L}}(t, Q)$ with

$$|\, V_{\mathrm{S}}(t)(x)\,| \leqslant c\langle x \rangle^{-1-\varepsilon}$$
$$|\, \partial_x^\alpha V_{\mathrm{L}}(t)(x)\,| \leqslant C\langle x \rangle^{-|\,\alpha\,|-\varepsilon}$$

for some $\varepsilon > 0$ and some constant C independent of t.

Now, we shall describe another class of potentials V for which all conclusions of the preceding theorem are true. We are interested in the case $V = V_{\mathrm{S}} + V_{\mathrm{L}}$ where V_{S} is a bounded operator having a decay of short range type at infinity and $V_{\mathrm{L}} = a(Q, P)$ is a bounded pseudo-differential operator of long range type. In order to state the long range assumptions we introduce the following definition:

A function $b : \mathbb{R}^{2n} \to \mathbb{C}$ is called an *admissible symbol* if b is the inverse Fourier transform of a measure $\widehat{b}$ having the following property:

$$\iint\limits_{\mathbb{R}^{2n}} \Big[|\,\widehat{b}(x,y)\,| \ln(2 + |\,y\,|) + \int\limits_{|\,z\,|<1} |\,\widehat{b}(x+z,y) - \widehat{b}(x,y)\,||\,z\,|^{-n}\mathrm{d}z\Big]\mathrm{d}x\mathrm{d}y < \infty.$$

THEOREM 4.5. *Let A and H_0 be defined as in Proposition* 4.1. *Let V_{S} and V_{L} be bounded symmetric operators on $\mathcal{H}$. Let H be a self-adjoint operator in $\mathcal{H}$ associated to the sum $H_0 + V_{\mathrm{S}} + V_{\mathrm{L}}$. We assume that $(H + \mathrm{i})^{-1} - (H_0 + \mathrm{i})^{-1}$ is a compact operator, and that V_{S}, V_{L} satisfy the following conditions:*

(S) *There is $\theta \in C_0^\infty(X)$ with $\theta(x) > 0$ if $0 < a < |\,x\,| < b < \infty$ and $\theta(x) = 0$ otherwise, such that:*

$$\int_1^\infty \sup_{t \in \mathbb{T}} \|\theta(r^{-1}Q)V_{\mathrm{S}}(t)\|_{B(\mathcal{H}_2)}\mathrm{d}r < \infty.$$

(L) *$V_{\mathrm{L}}(t)$ is a pseudodifferential operator having as Weyl symbol a distribution $a : \mathbb{R}^{2n} \to \mathbb{R}$ such that $a_j(x,y) \equiv \partial_{x_j} a(x,y)$ and $a_{j,k}(x,y) \equiv (\partial_{y_k} - \mathrm{i}x_k)a_j(x,y)$ are admissible symbols for all $j, k = 1, \ldots, n$. Furthermore, assume that the operators $V_{\mathrm{L}}(t) \equiv a(Q,P)$ and $[V_{\mathrm{L}}(t), \mathrm{i}Q_j]$ decay at infinity in the following sense: if $\zeta \in C^\infty(X)$*

with $\zeta(x) = 0$ *near* 0 *and* $\zeta(x) = 1$ *near infinity, then*

$$\int_1^\infty \sup_{t\in\mathbb{T}}\{\|\zeta(r^{-1}Q)V_{\mathrm{L}}(t)\|_{B(\mathcal{H}_2)} + \|\zeta(r^{-1}Q)[V_{\mathrm{L}}(t), \mathrm{i}Q_j]\|_{B(\mathcal{H}_2)}\}\frac{\mathrm{d}r}{r} < \infty.$$

Then all conclusions of the preceding theorem are true.

Proof. As in the proof of the previous theorem and by the same arguments, it suffices to show that $V_{\mathrm{S}}, V_{\mathrm{L}} \in \mathcal{C}^{1,1}(A)$. This assertion follows by mimicking the proof of [4]. For the short range part,

$$\int_1^\infty \|\theta(r^{-1}\Lambda)V_{\mathrm{S}}\|_{B(\mathcal{H})}\mathrm{d}r = \int_1^\infty \sup_{t\in\mathbb{T}}\|\theta(r^{-1}Q)V_{\mathrm{S}}(t)\|_{B(\mathcal{H}_2)}\mathrm{d}r < \infty.$$

For the long range part, from the proof of Theorem 2.5 in [3] it is easy to see that:

$$\int_1^\infty \|\zeta(r^{-1}\Lambda)[A, \mathrm{i}V_{\mathrm{L}}]\|_{B(\mathcal{H})} = \int_1^\infty \sup_{t\in\mathbb{T}}\|\zeta(r^{-1}Q)[A_2, \mathrm{i}V_{\mathrm{L}}]\|_{B(\mathcal{H}_2)}\frac{\mathrm{d}r}{r} < \infty,$$

and therefore $V_{\mathrm{L}} \in \mathcal{C}^{1,1}(A)$. ∎

EXAMPLE 4.6. In the particular case where $V = V_{\mathrm{S}} + V_{\mathrm{L}}$ is the operator of multiplication in $\mathcal{H}$ by a function $V(t,Q) = V_{\mathrm{S}}(t,Q) + V_{\mathrm{L}}(t,Q)$ our hypotheses are simple: $V_{\mathrm{S}}(t)$, $V_{\mathrm{L}}(t)$ are two bounded real functions such that

$$\int_1^\infty \sup_{t\in\mathbb{T}}\sup_{\tau\leqslant|x|\leqslant 2\tau}|V_{\mathrm{S}}(t)(x)|\mathrm{d}\tau < \infty,$$

$V_{\mathrm{L}}(t)$ is of class C^{m+1} (with $m = [n/2] + 1$) and

$$|\partial_x^\alpha V_{\mathrm{L}}(t)(x)| \leqslant C\langle x\rangle^{-|\alpha|-\varepsilon}$$

for some $\varepsilon > 0$, C some constant independent of t and $|\alpha| \leqslant m+1$.

Acknowledgements. I take this opportunity to express my gratitude to Vladimir Georgescu for suggesting me the subject of this work and for many helpful discussions. I am also indebted to Marius Măntoiu for his assistance to proof reading the paper.

REFERENCES

[1] W. AMREIN, A. BOUTET DE MONVEL-BERTHIER, V. GEORGESCU, C_0-*Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Progr. Math., Birkhauser, 1996.

[2] A. BOUTET DE MONVE-BERTHIERL, V. GEORGESCU, Graded C^*-Algebras and Many-Body Perturbation Theory: II. The Mourre estimate, in *Méthodes Semi-Classiques, Colloque international, (Nantes, juin 1991)*, vol. 2, Astérisque, vol. 210, Soc. Math. France, town 1992, pp. 75–96.

[3] A. BOUTET DE MONVEL-BERTHIER, V. GEORGESCU, Limiting absorption principle for long-range perturbations of pseudo-differential operators, in *Advances in Dynamical Systems and Quantum Physics (Capri 1993)*, World Sci. Publ., River Edge, NJ, 1995, pp. 38–74.

[4] A. BOUTET DE MONVEL-BERTHIER, V. GEORGESCU, Spectral theory and scattering theory by the conjugate operator method, *Algebra i Analiz* **4**(1992), 73–116; *St. Petersburg Math. J.* **4**(1993), 469–501.

[5] A. BOUTET DE MONVEL-BERTHIER, V. GEORGESCU, M. MĂNTOIU, Locally smooth operators and the limiting absorption principle for N-body Hamiltonians, *Rev. Math. Phys.* **5**(1993), 105–189.

[6] H.L. CYCON , R.G. FROESE, W. KIRSCH, B. SIMON, *Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry*, Springer, Berlin 1987.

[7] M. DAMAK, On the spectral theory of dispersive N-body Hamiltonians, *J. Math. Phys.* **40**(1999), 35–48.

[8] R.G. FROESE, I. HERBST, A new proof of the Mourre estimate, *Duke Math. J.* **49**(1982), 1075–1085.

[9] V. GEORGESCU, C. GÉRARD, J.S. MOLLER, Commutators, C_0-semigroups and resolvent estimates, preprint mp-arc, university, town 2003.

[10] C. GÉRARD, F. NIER, The Mourre theory for analytically fibered operators, *J. Funct. Anal.* **152**(1998), 202–219.

[11] T. HOSHIRO, Mourre's method and smoothing properties of dispersive equations, *Comm. Math. Phys.* **202**(1999), 255–265.

[12] E. MOURRE, Absence of singular continuous spectrum for certain self-adjoint operators, *Comm. Math. Phys.* **78**(1981), 391–408.

[13] E. MOURRE, Opérateurs conjugués et propriétés de propagation, *Comm. Math. Phys.* **91**(1983), 279–300.

[14] P. PERRY , I.M. SIGAL, B. SIMON, Spectral analysis of N-body Schrödinger operators, *Ann. Math.* **114**(1981), 519–567.

[15] M. REED, B. SIMON, *Methods of Modern Mathematical Physics*, vol. 1, Academic Press, London 1972.

[16] J. SAHBANI, The conjugate operator method for locally regular Hamiltonians, *J. Operator Theory* **38**(1997), 297–322.

[17] J. WEIDMANN, *Linear Operators in Hilbert Spaces*, Springer, New York 1980.

[18] K. YOKOYAMA, Mourre theory for time-periodic systems, *Nagoya Math. J.* **149**(1998), 193–210.

MONDHER DAMAK, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF SFAX, B.P. 802, SFAX, 3018, TUNISIA

E-mail address: Mondher.Damak@fss.rnu.tn

Received May 17, 2004; revised February 1, 2005.