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TRANSLATION INVARIANT ASYMPTOTIC HOMOMORPHISMS: EQUIVALENCE OF TWO APPROACHES IN INDEX THEORY

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ABSTRACT. The algebra $\Psi(M)$ of order zero pseudodifferential operators on a compact manifold M defines a well-known C^*-extension of the algebra $C(S^*M)$ of continuous functions on the cospherical bundle $S^*M \subset T^*M$ by the algebra $\mathbb{K}$ of compact operators. In his proof of the index theorem, Higson defined and used an asymptotic homomorphism T from $C_0(T^*M)$ to $\mathbb{K}$, which plays the role of a deformation for the commutative algebra $C_0(T^*M)$. Similar constructions exist also for operators and symbols with coefficients in a C^*-algebra. We show that the image of the above extension under the Connes–Higson construction is T and that this extension can be reconstructed out of T. This explains, why the classical approach to index theory coincides with the one based on asymptotic homomorphisms.

KEYWORDS: *Index, pseudodifferential operator, C^*-algebra extension, asymptotic homomorphism, Connes–Higson construction.*

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1. TWO WAYS TO DEFINE INDEX

The standard way to define the index of a pseudodifferential elliptic operator on a compact manifold M comes from the short exact sequence of C^*-algebras

$$0 \longrightarrow \mathbb{K} \longrightarrow \Psi(M) \longrightarrow C(S^*M) \longrightarrow 0, \tag{1.1}$$

where $\mathbb{K}$ is the algebra of compact operators on $L^2(M)$, $\Psi(M)$ denotes the norm closure of the algebra of order zero pseudodifferential operators in the algebra of bounded operators on $L^2(M)$ and S^*M denotes the cospherical bundle, $S^*M = \{(x,\xi) \in T^*M : |\xi| = 1\}$, in the cotangent bundle T^*M. If one deals with operators having coefficients in a C^*-algebra A then one has to tensor the short exact sequence (1.1) by A:

$$0 \longrightarrow \mathbb{K} \otimes A \longrightarrow \Psi_A(M) \longrightarrow C(S^*M; A) \longrightarrow 0, \tag{1.2}$$

where $C(X;A)$ denotes the C^*-algebra of continuous functions on X taking values in A. The (main) symbol of a pseudodifferential elliptic operator of order zero is an invertible element in $C(S^*M;A)$ and the K-theory boundary map $\partial : K_1(C(S^*M;A)) \to K_0(\mathbb{K} \otimes A)$ maps the symbol to a class in $K_0(A)$, which is called the index of the operator.

Another approach, suggested by Higson in [6], is based on the notion of an asymptotic homomorphism [4]. Here one starts with a symbol σ of a pseudodifferential operator of order one and constructs a *symbol class* $[a_\sigma] \in K_0(C_0(T^*M))$ (see details in [6]). Then one constructs an asymptotic homomorphism from $C_0(T^*M)$ to $\mathbb{K}$ as follows. In the local coordinates (x,ξ) in $U \times \mathbb{R}^n \subset TM$ take a smooth function $a(x,\xi)$ with a compact support, $a \in C_c^\infty(U \otimes \mathbb{R}^n)$. Then define a continuous family of operators $T_{a,t}$, $t \in \mathbb{R}_+ = (0,\infty)$, on $L^2(U)$ by

$$T_{a,t}f(x) = \int a(x,t^{-1}\xi)e^{ix\xi}\widehat{f}(\xi)d\xi, \tag{1.3}$$

where $\widehat{f}$ is the Fourier transform for f. Fix an atlas $\{U_k\}$ of charts on a compact manifold M and let $\{\varphi_k\}$ be a smooth partition of unity subordinate to that atlas. Take also smooth functions ψ_k on M such that $\operatorname{supp}\psi_k \subset U_k$ and $\psi_k\varphi_k = \varphi_k$ for all k. For $f \in L^2(M)$ put

$$T_t(a)f = \sum_k T_{\psi_k a,t}(\varphi_k f). \tag{1.4}$$

It is shown in Lemma 8.4 of [6], that this family of operators defines an asymptotic homomorphism $T = (T_t)_{t\in\mathbb{R}_+}$ from $C_c^\infty(T^*M)$ to $\mathbb{K}$ (it is also shown in Lemma 8.7 of [6], that if we take another atlas or other functions φ_k and ψ_k then the resulting asymptotic homomorphism is asymptotically equal to this one). Therefore this asymptotic homomorphism defines a $*$-homomorphism $\overline{T}$ from $C_c^\infty(T^*M)$ to the asymptotic C^*-algebra $C_b(\mathbb{R}_+;\mathbb{K})/C_0(\mathbb{R}_+;\mathbb{K})$, where $C_b(\mathbb{R}_+;\mathbb{K})$ denotes the algebra of bounded continuous $\mathbb{K}$-valued functions on $\mathbb{R}_+$ and due to automatic continuity of C^*-algebra $*$-homomorphisms one can extend $\overline{T}$ to a $*$-homomorphism $\widehat{T} : C_0(T^*M) \to C_b(\mathbb{R}_+;\mathbb{K})/C_0(\mathbb{R}_+;\mathbb{K})$. Applying the Bartle–Graves selection theorem [7], we obtain an asymptotic homomorphism $\widetilde{T} = (\widetilde{T}_t)_{t\in\mathbb{R}_+} : C_0(T^*M) \to \mathbb{K}$, which is uniformly continuous and asymptotically equal to T on smooth functions (i.e. $\lim_{t\to\infty} T_t(a) - \widetilde{T}_t(a) = 0$ for any $a \in C_c^\infty(T^*M)$). Finally the index of the operator with a symbol σ is defined as the class of the image of $[a_\sigma]$ under the map $K_0(C_0(T^*M)) \to K_0(\mathbb{K})$ induced by $\widetilde{T}$. Once more, one can tensor everything by A and construct an asymptotic homomorphism

$$T = (T_t)_{t\in\mathbb{R}_+} : C_c^\infty(T^*M;A) \to \mathbb{K} \otimes A$$

and then change it by a uniformly (with respect to t) continuous asymptotic homomorphism extended to $C_0(T^*M;A)$,

$$\widetilde{T} = (\widetilde{T}_t)_{t\in\mathbb{R}_+} : C_0(T^*M;A) \to \mathbb{K} \otimes A \tag{1.5}$$

(we keep the same notation T and $\widetilde{T}$ for the case of A-valued symbols). Remark that the asymptotic homomorphism T is translation invariant, i.e. $T_{ts}(a) = T_t(a_s)$, where $a_s(x,\xi) = a(x, s^{-1}\xi)$, $a \in C_c^\infty(T^*M; A)$. The asymptotic homomorphism $\widetilde{T}$ enjoys the property of asymptotic translation invariance, i.e. $\lim_{t\to\infty} \widetilde{T}_{ts}(a) - \widetilde{T}_t(a_s) = 0$ for any $a \in C_0(T^*M; A)$.

The purpose of the present paper is to explain why these two approaches to define the index of elliptic pseudodifferential operators are equivalent.

2. THE CONNES–HIGSON CONSTRUCTION AND ITS INVERSE

In the pioneering paper on asymptotic homomorphisms [4] a construction was given, which transforms C^*-algebra extensions into asymptotic homomorphisms. Given a C^*-extension $0 \to B \to E \to D \to 0$, one obtains an asymptotic homomorphism from the suspension $SD = C_0((0,1); D)$ into B. One of the main results of [10] was that the asymptotic homomorphisms obtained via this Connes–Higson construction possess an additional important property — translation invariance. In order to make its description easier we identify the suspension SA with $C_0(\mathbb{R}_+; D)$ instead of using $(0,1)$. There is a natural action τ of $\mathbb{R}_+$ on itself by multiplication, $\tau_s(x) = xs$, $s, x \in \mathbb{R}_+$, which extends to an action on SD by $\tau_s(f)(x) = f(sx)$, where $f \in SD = C_0(\mathbb{R}_+; D)$ (in [10] the additive structure on $\mathbb{R}$ was used instead of the multiplicative structure on $\mathbb{R}_+$).

DEFINITION 2.1. An asymptotic homomorphism $\varphi = (\varphi_t)_{t\in\mathbb{R}_+} : SD \to B$ is *translation invariant* if $\varphi_t(\tau_s(f)) = \varphi_{ts}(f)$ for any $f \in SD$ and for any $t, s \in \mathbb{R}_+$ and if $\lim_{t\to 0} \varphi_t(f) = 0$ for any $f \in SD$. It is *asymptotically translation invariant* if $\lim_{t\to\infty} \varphi_t(\tau_s(f)) - \varphi_{ts}(f) = 0$ for any $f \in SA$ and for any $t, s \in \mathbb{R}_+$ and if $\lim_{t\to 0} \varphi_t(f) = 0$ for any $f \in SD$. Two (asymptotically) translation invariant asymptotic homomorphisms $\varphi^{(0)}, \varphi^{(1)} : SD \to B$ are homotopic if there is an (asymptotically) translation invariant asymptotic homomorphism $\Phi : SD \to C[0,1] \otimes B$, whose restrictions onto the endpoints of $[0,1]$ coincide with $\varphi^{(0)}$ and $\varphi^{(1)}$ respectively.

Note that, by passing to spherical coordinates in the fibers, the suspension $SC(S^*M; A)$ can be identified with the algebra $C_{00}(T^*M; A)$ of continuous functions on T^*M vanishing both at infinity and at the zero section and the asymptotic homomorphism $\widetilde{T}$ (1.5) can be restricted onto $C_{00}(T^*M; A)$.

LEMMA 2.2. *The asymptotic homomorphism* $\widetilde{T} : C_{00}(T^*M; A) \to \mathbb{K} \otimes A$ *is asymptotically translation invariant.*

Proof. The family of maps (1.5) is obviously asymptotically invariant under the action of $\mathbb{R}_+$ and one easily checks that

$$\lim_{t\to 0} \widetilde{T}_t(a) = 0 \quad \text{for any} \quad a \in C_{00}(T^*M; A). \qquad ▮$$

From now on we assume that A and D are separable and B is stable and σ-unital. Let $\mathrm{Ext}_h(D,B)$ denote the semigroup of homotopy classes of C^*-extensions of D by B. In the case when D is nuclear, this functor was defined and studied in [2]. Let $[[SD,B]]_{a,\tau}$ denote the semigroup of asymptotically translation invariant asymptotic homomorphisms from SD to B. Note that there is a forgetful map

$$[[SD,B]]_{a,\tau} \longrightarrow [[SD,B]] \tag{2.1}$$

to the group of homotopy classes of all asymptotic homomorphisms from SD to B, which is the E-theory group $E(SD,B)$. The Connes–Higson construction [4] defines a map

$$CH : \mathrm{Ext}_h(D,B) \longrightarrow [[SD,B]].$$

It was shown in [10] that this map factorizes through the map (2.1) and the modified Connes–Higson construction

$$\widetilde{CH} : \mathrm{Ext}_h(D,B) \longrightarrow [[SD,B]]_{a,\tau}. \tag{2.2}$$

The main result of [10] is that the map (2.2) is an isomorphism. This was proved by constructing an inverse map

$$I : [[SD,B]]_{a,\tau} \longrightarrow \mathrm{Ext}_h(D,B).$$

The map I is constructed as follows (see details in [10]). Let $\varphi = (\varphi_t)_{t\in\mathbb{R}_+} : SD \to B$ be an asymptotically translation invariant asymptotic homomorphism. Then, by the Bartle–Graves continuous selection theorem [7], there is an asymptotically translation invariant asymptotic homomorphism $\widetilde{\varphi}$, which is asymptotically equal to φ and such that the family of maps $\widetilde{\varphi}_t : SD \to B$ is uniformly continuous.

Let $\gamma_0 \in C_0(\mathbb{R}_+)$ be a (smooth) function with support in $[1/2, 2]$ such that $\sum_{i\in\mathbb{Z}} \gamma_i^2 = 1$, where $\gamma_i = \tau_{2^i}(\gamma_0)$. Note that $\gamma_i\gamma_j = 0$ when $|i-j| \geqslant 2$. Let e_{ij} denote the standard elementary operators on the standard Hilbert C^*-module $H_B = l^2(\mathbb{Z}) \otimes B$. We identify the algebra of compact (respectively adjointable) operators on H_B with the C^*-algebra $B \otimes \mathbb{K}$ (respectively the multiplier C^*-algebra $M(B\otimes\mathbb{K})$) and let

$$q : M(B\otimes\mathbb{K}) \longrightarrow Q(B\otimes\mathbb{K}) = M(B\otimes\mathbb{K})/B\otimes\mathbb{K}$$

be the quotient $*$-homomorphism.

Put, for $a \in D$,

$$I_0(\varphi)(a) = \sum_{i,j\in\mathbb{Z}} \widetilde{\varphi}_{2^i}(\tau_{2^{-i}}(\gamma_i\gamma_j) \otimes a) \otimes e_{ij} \in M(B\otimes\mathbb{K})$$

and $I(\varphi)(a) = q(I_0(\varphi)(a))$. Then the map $I : D \to Q(B \otimes \mathbb{K})$ is a $*$-homomorphism, so it defines an extension of D by $B \otimes \mathbb{K}$, being its Busby invariant [3].

3. MAIN RESULT

Denote by $[\Psi_A(M)] \in \mathrm{Ext}_h(C(S^*M; A), \mathbb{K} \otimes A)$ the homotopy class of the extension (1.2).

THEOREM 3.1. *The image of* $[\Psi_A(M)]$ *under the Connes–Higson construction coincides with the homotopy class of the asymptotic homomorphism* $\widetilde{T}$ *if* A *is separable.*

Proof. Due to [10] we do not need to prove that $CH([\Psi_A(M)])$ is homotopic to $\widetilde{T}$. It is sufficient to prove instead that $I(\widetilde{T})$ is homotopic to the Busby invariant of the extension (1.2), which is easier.

In order to construct the Busby invariant for the extension (1.2) one can use the same atlas of charts and the same functions φ_k and ψ_k as in the construction of the asymptotic homomorphism T_t (1.4). Let θ be a smooth cutting function on $[0, \infty)$, which equals 1 outside a compact set and vanishes at the origin and let $U \subset M$ be a subset diffeomorphic to a domain in a Euclidean space. In the local coordinates (x, ξ) in $U \times \mathbb{R}^n \subset T^*M$ take a smooth function $a(x, \xi)$ with a compact support with respect to the first coordinate and order zero homogeneous with respect to the second coordinate. Let f be an element of the Hilbert C^*-module $L^2(U) \otimes A$ over A. Define an operator $\mathrm{Op}(a)$ on this Hilbert C^*-module by

$$\mathrm{Op}(a)f(x) = \int a(x, \xi)\theta(|\xi|)\mathrm{e}^{\mathrm{i}x\xi}\widehat{f}(\xi)\,\mathrm{d}\xi,$$

where $\widehat{f}$ is the Fourier transform for f. Then, for a main symbol $a(x, \xi) \in C^\infty(S^*M; A)$ defined on the whole M, one can construct an operator $\mathrm{Op}(a)$ on the Hilbert C^*-module $L^2(M) \otimes A$ by

$$\mathrm{Op}(a)(f) = \sum_k \mathrm{Op}(\psi_k a)(\varphi_k f), \quad \mathrm{Op}(a) \in M(\mathbb{K} \otimes A).$$

The map $q \circ \mathrm{Op} : C^\infty(S^*M; A) \to Q(\mathbb{K} \otimes A)$ is a $*$-homomorphism (cf. [11], [8]), so, due to automatic continuity, it extends to a $*$-homomorphism $\underline{\mathrm{Op}} : C(S^*M; A) \to Q(\mathbb{K} \otimes A)$. Using the Bartle–Graves selection theorem one can obtain a continuous homogeneous lifting $\widetilde{\mathrm{Op}} : C(S^*M; A) \to M(\mathbb{K} \otimes A)$ for $\underline{\mathrm{Op}}$.

Let γ_0^s and $\gamma_{\pm 1}^s$, $s \in (0, 1]$, be smooth functions in $C_0(\mathbb{R}_+)$ with support in $[2^{-1/s}, 2^{1/s}]$ and in $[2^{\pm 1/s - 1}, 2^{\pm 1/s + 1}]$ respectively, such that $\sum_{i \in \mathbb{Z}} \gamma_i^2 = 1$, where $\gamma_{\pm i}^s = \tau_{2^{\pm(i-1)}}(\gamma_{\pm 1}^s)$ for $i > 1$.

Let at first $a \in C^\infty(S^*M; A)$. Define a map from $C^\infty(S^*M; A)$ to $M(\mathbb{K} \otimes A)$ by

$$\Psi_s(a) = \sum_{i,j \in \mathbb{Z}} T_1(\gamma_i^s \gamma_j^s \theta) \otimes a) \otimes e_{ij}$$

for $s \in (0,1]$ and
$$\Psi_0(a) = \mathrm{Op}(a) \otimes e_{00}.$$

Strict continuity of the family $\Psi_s(a)$ at any $s \in (0,1]$ is obvious, so we have to check it at $s = 0$. By construction, $\gamma_0^s(x) = 1$ for $x \in [2^{-1/s+1}, 2^{1/s-1}]$, hence γ_i^s strictly converges to zero as $s \to 0$ for any $i \neq 0$, so for any $f \in L^2(U) \otimes A$ in local coordinates one has
$$\lim_{s\to 0} \int a(x,\xi)\gamma_i^s(|\xi|)\gamma_j^s(|\xi|)\theta(|\xi|)\mathrm{e}^{\mathrm{i}x\xi}\widehat{f}(\xi)\,\mathrm{d}\xi = \begin{cases} 1 & \text{if } i=j=0, \\ 0 & \text{otherwise,} \end{cases}$$
because $\widehat{f} \in L^2(\mathbb{R}^n) \otimes A$, hence

$$\lim_{s\to 0} T_1((\gamma_0^s)^2\theta \otimes a)(f) - \mathrm{Op}(a)(f) = 0 \tag{3.1}$$

and

$$\lim_{s\to 0} T_1(\gamma_i^s\gamma_j^s\theta \otimes a)(f) = 0 \tag{3.2}$$

whenever either i or j differs from zero. Since the set $\Psi_s(a)$ is uniformly bounded for any $a \in C^\infty(S^*M; A)$, it follows from (3.1) and (3.2) that the family of maps Ψ_s, $s \in [0,1]$, is strictly continuous with respect to s, hence this family defines a map
$$\Psi : C^\infty(S^*M; A) \longrightarrow M(C([0,1]; \mathbb{K} \otimes A)),$$
which is obviously a $*$-homomorphism modulo the ideal $C([0,1]; \mathbb{K} \otimes A)$. The $*$-homomorphism $q \circ \Psi : C^\infty(S^*M; A) \to Q(C([0,1]; \mathbb{K} \otimes A))$ extends by continuity to a $*$-homomorphism $\widetilde{\Psi} : C(S^*M; A) \to Q(C([0,1]; \mathbb{K} \otimes A))$.

It remains to show that $\widetilde{\Psi}$ is the required homotopy. One easily sees that $\widetilde{\Psi}_0 = q \circ \widetilde{\mathrm{Op}} \oplus 0$, so one has to check that $\widetilde{\Psi}_1 = I(\widetilde{T})$ and it is sufficient to check the latter equality on $C^\infty(S^*M; A)$. Since, for big enough positive i, $\gamma_i\gamma_j\theta = \gamma_i\gamma_j$ and since
$$\lim_{i\to -\infty} T_1(\gamma_i\gamma_j \otimes a) = \lim_{i\to -\infty} T_1(\gamma_i\gamma_j\theta \otimes a) = 0$$
for any $a \in C^\infty(S^*M; A)$ (for all j, because the only non-trivial values for j are i and $i \pm 1$), so $q \circ \Psi_1 = q \circ \Psi'$, where
$$\Psi'(a) = \sum_{i,j\in\mathbb{Z}} T_1((\gamma_i\gamma_j) \otimes a) \otimes e_{ij}.$$

By properties of asymptotic homomorphisms T and $\widetilde{T}$ one has
$$\lim_{i\to -\infty} T_1(\gamma_i\gamma_j \otimes a) = 0,$$
$$\lim_{i\to -\infty} \widetilde{T}_{2^i}(\tau_{2^{-i}}(\gamma_i\gamma_j) \otimes a) = \lim_{i\to -\infty} \widetilde{T}_{2^i}(\gamma_0\gamma_{j-i}) \otimes a) = 0,$$
$$\lim_{i\to \infty} \widetilde{T}_{2^i}(\tau_{2^{-i}}(\gamma_i\gamma_j) \otimes a) - T_1(\gamma_i\gamma_j \otimes a) = \lim_{i\to \infty} \widetilde{T}_{2^i}(\gamma_0\gamma_{j-i} \otimes a) - T_{2^i}(\gamma_0\gamma_{j-i} \otimes a = 0,$$
for any $a \in C^\infty(S^*M; A)$. Therefore, $\widetilde{\Psi}_1 = q \circ \Psi' = I(\widetilde{T})$ on $C^\infty(S^*M; A)$. ∎

4. CONCLUDING REMARKS

So we have proved that the extension (1.2) and the restriction of the asymptotic homomorphism (1.5) to $C_{00}(S^*M;A)$ define each other. This is the reason beneath the fact that the two definitions of the index give the same result. To complete the picture we have to show this well known fact. For that purpose consider the extensions

$$0 \longrightarrow C_0(T^*(M);A) \longrightarrow C(D^*M;A) \longrightarrow C(S^*M;A) \longrightarrow 0 \tag{4.1}$$

and

$$0 \longrightarrow C_{00}(T^*(M);A) \longrightarrow C_0(D^*M;A) \longrightarrow C(S^*M;A) \longrightarrow 0, \tag{4.2}$$

where D^*M denotes the ball bundle obtained from T^*M by compactifying each fiber by a sphere and $C_0(D^*M;A) \subset C(D^*M;A)$ denotes the subset of functions vanishing on the zero section of D^*M.

Consider the diagram

$$\begin{array}{ccccc} K_1(C(S^*M;A)) & & \xrightarrow{\quad\partial\quad} & & K_0(\mathbb{K}\otimes A) \\ \downarrow j & \searrow^{\partial'} & & \nearrow_{\widetilde{T}_*} & \\ K_0(C_{00}(T^*M;A)) & \xrightarrow{i} & K_0(C_0(T^*M;A)) & & \end{array} \tag{4.3}$$

where the map j is the standard isomorphism $K_1(B) = K_0(SB)$, the map ∂ is the K-theory boundary map induced by the extension (1.2), the map ∂' is the K-theory boundary map induced by the extension (4.1), the map $\widetilde{T}_*$ is induced by the asymptotic homomorphism $\widetilde{T}$ (1.5) and the map i is induced by the inclusion $C_{00}(T^*M) \subset C_0(T^*M)$.

PROPOSITION 4.1. *The diagram* (4.3) *commutes.*

Proof. It obviously follows from the properties of E-theory [4] (more details can be found in [5], cf. Exercise 25.7(d) of [1]) that for any extension $0 \to B \to E \to A \to 0$ with the Busby invariant $\psi : SA \to Q(B)$ the map $K_0(SA) \to K_0(B)$ induced by the asymptotic homomorphism $CH(\psi)$ coincides with the K-theory boundary map $K_1(A) \to K_0(B)$ after the standard identification $K_1(A) = K_0(SA)$. Thus $\partial = \widetilde{T}_* \circ i \circ j$. Since the extension (4.2) is the restriction of the extension (4.1), we have $\partial' = i \circ j$. Hence the whole diagram commutes. ∎

It may seem that the asymptotic homomorphism $\widetilde{T}$ (1.5) contains more information than the extension (1.2) since it is defined not only on $C_{00}(T^*M;A)$, but on the bigger C^*-algebra $C_0(T^*M;A)$. In fact, the extension (1.2) also possesses an additional property, which is equivalent to that additional property of the asymptotic homomorphism $\widetilde{T}$. Namely, there is a subalgebra $C(M;A) \subset C(S^*M;A)$ consisting of functions that are constants on the fibers and the Busby invariant of the extension (1.2) restricted onto $C(M;A)$ can be lifted to $M(\mathbb{K}\otimes A)$. Indeed, the multiplication $\pi(a)f = af$ for $a \in C(M;A)$ and $f \in L^2(M)\otimes A$ defines such a lifting, i.e. a $*$-homomorphism $\pi : C(M;A) \to M(\mathbb{K}\otimes A)$. Using a relative version of the Bartle–Graves theorem [7], one can construct a continuous section $\overline{\mathrm{Op}} : C(S^*M;A) \to M(\mathbb{K}\otimes A)$ such that its restriction onto $C(M;A)$ coincides with π. So we now describe how one can extend the Connes–Higson construction to the case when an extension of a C^*-algebra D restricted to a C^*-subalgebra $C \subset D$ is liftable. Denote the Busby invariant of such an extension by $\chi : D \to Q(B)$ and let $\overline{\chi} : D \to M(B)$ be a continuous homogeneous lifting for χ such that $\overline{\chi}|_C$ is a $*$-homomorphism. Consider the C^*-subalgebra

$$C_0([0,\infty);C) \cup C_0(\mathbb{R}_+;D) \tag{4.4}$$

in $C_0([0,\infty);D)$. The Connes–Higson construction on $SD = C_0(\mathbb{R}_+;D)$ can be defined on elementary tensors of the form $f\otimes d$, $f \in C_0(\mathbb{R}_+)$, $d \in D$, by the formula

$$CH(\chi)_t(f\otimes d) = \overline{\chi}(d)(f\circ\kappa)(u_t), \tag{4.5}$$

where $(u_t)_{t\in\mathbb{R}_+} \subset B$ is a quasicentral (with respect to $\overline{\chi}(D)$) approximate unit, $0 \leqslant u_t \leqslant 1$, and $\kappa : (0,1] \to [0,\infty)$ is a homeomorphism (cf. [4]). In order to extend this construction to the C^*-algebra (4.4) we have to define the asymptotic homomorphism $CH(\chi)$ on $C_0([0,\infty);C)$ compatible with (4.5). Let $g\otimes c \in C_0([0,\infty);C)$ be an elementary tensor, $g \in C_0[0,\infty)$, $c \in C$. Then apply the same formula,

$$CH(\chi)_t(g\otimes c) = \overline{\chi}(c)(g\circ\kappa)(u_t).$$

In the case when $D = C(S^*M;A)$ and $C = C(M;A)$, the C^*-algebra (4.4) obviously coincides with the C^*-algebra $C_0(T^*M;A)$ and the extended Connes–Higson construction gives us the asymptotic homomorphism $\widetilde{T}$ defined on the whole $C_0(T^*M;A)$.

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