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TESTING SCHATTEN CLASS HANKEL OPERATORS, CARLESON EMBEDDINGS AND WEIGHTED COMPOSITION OPERATORS ON REPRODUCING KERNELS

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ABSTRACT. Given an operator A on a Hilbert space $\mathcal{H}$ and $c \in \mathcal{H}$, we consider operators $\Lambda_{A,c}$ defined on analytic functions f by $\Lambda_{A,c}f = f(A)c$. Special cases of $\Lambda_{A,c}$ include vectorial Hankel operators, Carleson embeddings and weighted composition operators. For certain A, we determine conditions under which $\Lambda_{A,c}$ extends to an operator of Schatten von-Neumann class on the Hardy or Bergman space of the disc. These conditions involve only the action of $\Lambda_{A,c}$ on reproducing kernels and their derivatives. We also give corresponding results for operators on the Hardy space of the half-plane.

KEYWORDS: *Schatten classes, reproducing kernels, Hardy space, Bergman space, Hankel operators, Carleson embeddings, composition operators.*

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1. INTRODUCTION AND NOTATION

Establishing whether a given operator on a function space is bounded, compact or belongs to a Schatten–von Neumann class is an important problem in functional analysis. A fruitful approach to this problem has been to employ a "small" set of test functions such that properties of the operator are determined solely by its action on these functions. When the space is a reproducing kernel Hilbert space, it is natural to use the kernels themselves as test functions.

The "Reproducing Kernel Thesis" asserts that for many classes of operators on reproducing kernel Hilbert spaces, boundedness of a particular operator is equivalent to boundedness on the kernels, see [11]. Two important examples of this phenomenon are Hankel operators and Carleson embeddings on the Hardy space; for these two classes of operators the Reproducing Kernel Thesis is equivalent to fundamental results from harmonic analysis — namely C. Fefferman's duality theorem $(H^1)^* = BMOA$ and the Carleson measure theorem, respectively.

In [7], a general class of operators was considered in a Hardy space and Bergman space setting. This class was motivated by linear systems theory and included Hankel operators, Carleson embeddings and weighted composition operators as special cases. Boundedness and compactness criteria for this class of operators were obtained in terms of reproducing kernels.

In [8], the problem of determining whether a Hankel operator on the Hardy space belonged to a particular Schatten–von Neumann class in terms of the operator's behaviour on the kernels was first considered and some partial results were obtained. A complete characterisation, both for Hankel operators and Carleson embeddings, was achieved in [16].

The aim of this paper is to classify Schatten–von Neumann class membership for the general operators considered in [7] in terms of their action on kernels.

1.1. REPRODUCING KERNEL HILBERT SPACES. A Hilbert space $\mathcal{H}$ of functions on a set Ω is a reproducing kernel Hilbert space if for any point $z \in \Omega$, there exists a function $k_z \in \mathcal{H}$ (the associated reproducing kernel) such that for all $f \in \mathcal{H}$, $f(z) = \langle f, k_z \rangle$, i.e. point evaluations are bounded linear functionals.

The Hardy space H^2 consists of those functions f which are analytic on the open unit disc $\mathbb{D}$ and such that

$$\|f\|_{H^2}^2 = \sup_{0\leqslant r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^2 \mathrm{d}\theta < \infty.$$

The Bergman space L^2_{a} consists of those functions f which are analytic on $\mathbb{D}$ and such that

$$\|f\|_{L^2_{\mathrm{a}}}^2 = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \mathrm{d}A(z) < \infty,$$

where $\mathrm{d}A$ denotes Lebesgue area measure on $\mathbb{D}$. Both H^2 and L^2_{a} are reproducing kernel Hilbert spaces over $\mathbb{D}$. The reproducing kernels for H^2 are given by

$$k_z(w) = \frac{1}{1-\overline{z}w}; \quad \text{thus} \quad \widetilde{k}_z(w) = \frac{(1-|z|^2)^{1/2}}{1-\overline{z}w},$$

where $\widetilde{f}$ denotes $f/\|f\|$. The reproducing kernels for L^2_{a} are given by

$$h_z(w) = \frac{1}{(1-\overline{z}w)^2}, \quad \text{so that} \quad \widetilde{h}_z(w) = \frac{1-|z|^2}{(1-\overline{z}w)^2}.$$

We shall also require the derivatives of the kernels. Let

$$\dot{k}_z(w) = \frac{\partial}{\partial \overline{z}} k_z(w) = \frac{w}{(1-\overline{z}w)^2}, \quad \dot{h}_z(w) = \frac{\partial}{\partial \overline{z}} h_z(w) = \frac{2w}{(1-\overline{z}w)^3}.$$

For $f \in H^2$, we have $f'(z) = \langle f, \dot{k}_z \rangle_{H^2}$, with a corresponding result for L^2_{a}. Note that

$$\|\dot{k}_z\|_{H^2}^2 = \dot{k}_z'(z) = \frac{1+|z|^2}{(1-|z|^2)^3} \approx (1-|z|^2)^{-3}, \tag{1.1}$$

where $\approx$ denotes equivalence up to constants. Similarly,

$$\|\dot{h}_z\|^2_{L^2_{\mathrm{a}}} = \dot{h}'_z(z) = \frac{2+4|z|^2}{(1-|z|^2)^4} \approx (1-|z|^2)^{-4}. \tag{1.2}$$

We will need to use the vectors $\widetilde{k}_z = \dot{k}_z / \|\dot{k}_z\|$ and $\widetilde{h}_z = \dot{h}_z / \|\dot{h}_z\|$ later.

1.2. OPERATORS ARISING FROM LINEAR SYSTEMS THEORY. Let $\mathcal{H}$ be a Hilbert space (always assumed separable). Suppose that $A \in \mathcal{B}(\mathcal{H})$ (the collection of bounded linear operators on $\mathcal{H}$) with spectral radius $r(A) \leqslant 1$. This means that A has spectrum $\mathrm{Sp}(A) \subseteq \overline{\mathbb{D}}$. In particular, this will hold if A is a contraction, so that $\|A\| \leqslant 1$.

Let $\mathcal{O}(\,\overline{\mathbb{D}}\,)$ denote the collection of all functions which are holomorphic in some neighbourhood of $\overline{\mathbb{D}}$, the closed unit disc. Note that $\mathcal{O}(\,\overline{\mathbb{D}}\,)$ contains all of the reproducing kernels of both H^2 and L^2_{a} and their derivatives, so that $\mathcal{O}(\,\overline{\mathbb{D}}\,)$ is a dense subspace of both H^2 and L^2_{a}. Then $f(A) \in \mathcal{B}(\mathcal{H})$ is well-defined whenever $f \in \mathcal{O}(\,\overline{\mathbb{D}}\,)$ (because the associated power series in A converges absolutely).

DEFINITION 1.1. For $\mathcal{H}$ and A as above with $r(A) \leqslant 1$, we define for each $c \in \mathcal{H}$ the operator $\Lambda_{A,c}$ with symbols A and c as follows:

$$\Lambda_{A,c} : \mathcal{O}(\,\overline{\mathbb{D}}\,) \to \mathcal{H}; \quad \Lambda_{A,c} f = f(A)c.$$

We will be interested in criteria which characterise when such operators extend to bounded, compact or Schatten class operators on H^2 or L^2_{a}. Such operators occur in linear systems theory; indeed the boundedness of such an operator on H^2 is equivalent to the infinite-time admissibility of the observation operator for a discrete-time linear system (see [7] for details). However certain special cases of these operators have been studied before without reference to linear systems. In addition to Carleson embeddings and Hankel operators which are related to C. Fefferman's important duality theorem $(H^1)^* = BMOA$, we mention also the following example:

EXAMPLE 1.2. Weighted composition operators may be considered as particular examples of the $\Lambda_{A,c}$ operators introduced above. If μ is a positive, finite Borel measure on $\overline{\mathbb{D}}$ and $\phi : \mathrm{supp}(\mu) \to \overline{\mathbb{D}}$ is a measurable function then if $\mathcal{H} = L^2(\mu)$ and we let $A = M_\phi$ be the operator of pointwise multiplication by ϕ on $\mathcal{H}$ we obtain

$$\Lambda_{A,c}(f) = f(M_\phi)c = M_{f\circ\phi}(c) = c \cdot (f \circ \phi)$$

for each $c \in \mathcal{H}$.

In [7], necessary and sufficient conditions for $\Lambda_{A,c}$ to extend to a bounded or compact operator on H^2 or L^2_{a} are derived, in terms of the action of $\Lambda_{A,c}$ on the reproducing kernels of those spaces. We have the following results:

THEOREM 1.3. *Suppose that $\mathcal{H}$ is a (separable) Hilbert space and that $A \in \mathcal{B}(\mathcal{H})$ is a subnormal operator with $r(A) \leqslant 1$. Let $c \in \mathcal{H}$. Then $\Lambda_{A,c}$ extends to a bounded operator from L^2_{a} to $\mathcal{H}$ if and only if*

$$\sup_{z \in \mathbb{D}} \|\Lambda_{A,c}(\widetilde{h}_z)\| < \infty.$$

Moreover, the above supremum is equivalent to $\|\Lambda_{A,c}\|$ (with constants of equivalence independent of $\mathcal{H}$, A and c). $\Lambda_{A,c}$ extends to a compact operator from L^2_{a} to $\mathcal{H}$ if and only if

$$\lim_{r \to 1^-} \sup_{|z|=r} \|\Lambda_{A,c}(\widetilde{h}_z)\| = 0.$$

The proof of this result relies upon the spectral theorem for normal operators and corresponding results for Carleson embedding operators on L^2_{a}; see Theorem 2.5 later (which is quoted from [7]). If $\Lambda_{A,c}$ is considered to be acting on H^2, we can obtain corresponding results for A in a larger class of operators.

THEOREM 1.4. *Suppose that $\mathcal{H}$ is a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ is a contraction, so that $\|A\| \leqslant 1$. Let $c \in \mathcal{H}$. Then $\Lambda_{A,c}$ extends to a bounded operator from H^2 to $\mathcal{H}$ if and only if*

$$\sup_{z \in \mathbb{D}} \|\Lambda_{A,c}(\widetilde{k}_z)\| < \infty.$$

Moreover, the above supremum is equivalent to $\|\Lambda_{A,c}\|$ (with constants of equivalence independent of $\mathcal{H}$, A and c). $\Lambda_{A,c}$ extends to a compact operator from H^2 to $\mathcal{H}$ if and only if

$$\lim_{r \to 1^-} \sup_{|z|=r} \|\Lambda_{A,c}(\widetilde{k}_z)\| = 0.$$

The proof of this result relies upon the Sz.-Nagy–Foias functional model for contraction operators and corresponding results for Hankel operators on H^2 — see Theorem 3.6 later (which is quoted from [7]).

The aim of this paper is to classify Schatten–von Neumann class membership for certain $\Lambda_{A,c}$ operators in terms of their action on reproducing kernels.

1.3. SCHATTEN–VON NEUMANN CLASSES. Suppose that T is a compact operator between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then T has a Schmidt decomposition, so that there are orthonormal bases $\{e_n\}$ and $\{\sigma_n\}$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and a sequence $\{\lambda_n\}$ with $\lambda_n \geqslant 0$ and $\lambda_n \to 0$ such that for all $f \in \mathcal{H}_1$,

$$Tf = \sum_{n=0}^{\infty} \lambda_n \langle f, e_n \rangle \sigma_n.$$

For $1 \leqslant p < \infty$, such a compact operator T belongs to the Schatten–von Neumann p-class $S_p = S_p(\mathcal{H}_1, \mathcal{H}_2)$ if and only if

$$\|T\|_{S_p}^p = \sum_{n=0}^{\infty} \lambda_n^p < \infty.$$

For $T \in S_1(\mathcal{H}_1, \mathcal{H}_1)$, we may also define the trace of T as follows:

$$\mathrm{Tr}(T) = \sum_{n=0}^{\infty} \langle Te_n, e_n \rangle,$$

where $\{e_n\}$ is an arbitrary orthonormal basis. For more details on the Schatten classes, we refer to the books [15] and [18].

Throughout this paper, C and C_p stand for various absolute constants which may vary from line to line, with C_p dependent only upon p.

2. OPERATORS ON THE BERGMAN SPACE

2.1. CRITERIA FOR GENERAL OPERATORS. If one is trying to use reproducing kernels to classify the boundedness or compactness of operators on the Bergman space then there are obvious necessary conditions: if T is a bounded operator on L^2_{a} then

$$\sup_{z \in \mathbb{D}} \|T\widetilde{h}_z\| < \infty,$$

and if T is a compact operator on L^2_{a} then

$$\lim_{r \to 1^-} \sup_{|z|=r} \|T\widetilde{h}_z\| = 0,$$

because $\widetilde{h}_z \to 0$ weakly as $|z| \to 1$. We have seen that the converses to these statements hold for those $\Lambda_{A,c}$ operators from L^2_{a} to $\mathcal{H}$ with $A \in \mathcal{B}(\mathcal{H})$ a subnormal contraction operator. The aim here is to find corresponding conditions which classify Schatten class membership for operators on L^2_{a} in terms of their action on the reproducing kernels. Recall that $\mathrm{d}\lambda$ is the measure on $\mathbb{D}$ given by

$$\mathrm{d}\lambda(z) = \frac{\mathrm{d}A(z)}{(1-|z|^2)^2}.$$

PROPOSITION 2.1. *Let $T \in \mathcal{B}(L^2_{\mathrm{a}}, \mathcal{H})$ for a Hilbert space $\mathcal{H}$. Then:*

(i) $\|T\|^2_{S_2} = \frac{1}{\pi} \int\limits_{\mathbb{D}} \|Th_z\|^2 \mathrm{d}A(z) = \frac{1}{\pi} \int\limits_{\mathbb{D}} \|T\widetilde{h}_z\|^2 \mathrm{d}\lambda(z)$.

(ii) *If $p > 2$ and $T \in S_p(L^2_{\mathrm{a}}, \mathcal{H})$ then*

$$\int\limits_{\mathbb{D}} \|T\widetilde{h}_z\|^p \mathrm{d}\lambda(z) \leqslant \pi \|T\|^p_{S_p}.$$

(iii) *If $1 \leqslant p < 2$ and*

$$\int\limits_{\mathbb{D}} \|T\widetilde{h}_z\|^p \mathrm{d}\lambda(z) < \infty,$$

then $T \in S_p(L^2_{\mathrm{a}}, \mathcal{H})$. Moreover,

$$\|T\|^p_{S_p} \leqslant \frac{1}{\pi} \int\limits_{\mathbb{D}} \|T\widetilde{h}_z\|^p \mathrm{d}\lambda(z).$$

Proof. We recall the following from pages 115 and 117 in [18]: if A is a positive operator on L^2_{a} then

$$\mathrm{Tr}(A) = \frac{1}{\pi}\int_{\mathbb{D}} \langle A\widetilde{h}_z, \widetilde{h}_z\rangle \mathrm{d}\lambda(z);$$

if $f \in L^2_{\mathrm{a}}$ has unit norm then

$$\langle A^q f, f\rangle \geqslant \langle Af, f\rangle^q, \quad \text{for } q \geqslant 1, \quad \langle A^q f, f\rangle \leqslant \langle Af, f\rangle^q, \quad \text{for } 0 < q \leqslant 1. \tag{2.1}$$

But

$$\|T\|^p_{S_p} = \mathrm{Tr}((T^*T)^{p/2}) = \frac{1}{\pi}\int_{\mathbb{D}} \langle (T^*T)^{p/2}\widetilde{h}_z, \widetilde{h}_z\rangle \mathrm{d}\lambda(z).$$

Since by (2.1) $\langle (T^*T)^{p/2}\widetilde{h}_z, \widetilde{h}_z\rangle \geqslant \|T\widetilde{h}_z\|^p$ for $p \geqslant 2$ and the reverse inequality holds if $p \leqslant 2$, the result follows. ∎

We will show that the converses to the above statements hold for many $\Lambda_{A,c}$ operators. We will need an extra condition in terms of $\widetilde{\dot{h}}_z$, the normalised derivatives of the kernels, which characterises S_1 operators.

PROPOSITION 2.2. *Let $T \in \mathcal{B}(L^2_{\mathrm{a}}, \mathcal{H})$ for a Hilbert space $\mathcal{H}$. If*

$$\int_{\mathbb{D}} \|T\widetilde{\dot{h}}_z\| \mathrm{d}\lambda(z) < \infty,$$

then $T \in S_1(L^2_{\mathrm{a}}, \mathcal{H})$. Moreover,

$$\|T\|_{S_1} \leqslant C\Big(\int_{\mathbb{D}} \|T\widetilde{\dot{h}}_z\| \mathrm{d}\lambda(z) + \|Th_0\|\Big).$$

Proof. First note that, for $f \in L^2_{\mathrm{a}}$ with power series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, elementary calculations show that

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^2 \mathrm{d}A(z) + |f(0)|^2 &= |a_0|^2 + \sum_{n=1}^{\infty} \frac{2n}{(n+1)(n+2)}|a_n|^2 \\ &\approx \sum_{n=0}^{\infty} \frac{1}{n+1}|a_n|^2 = \|f\|^2_{L^2_{\mathrm{a}}}.\end{aligned} \tag{2.2}$$

Thus, if $A \in S_1(L^2_{\mathrm{a}}, L^2_{\mathrm{a}})$ is positive and with Schmidt decomposition

$$Af = \sum_{n=0}^{\infty} \lambda_n \langle f, e_n\rangle e_n$$

then

$$\langle A\dot{h}_z, \dot{h}_z\rangle = \sum_{n=0}^{\infty} \lambda_n |e_n'(z)|^2 \text{ and } \langle Ah_0, h_0\rangle = \sum_{n=0}^{\infty} \lambda_n |e_n(0)|^2.$$

Therefore, using (2.2), we see that

$$\int_{\mathbb{D}} \langle A\dot{h}_z, \dot{h}_z\rangle (1-|z|^2)^2 \mathrm{d}A(z) + \langle Ah_0, h_0\rangle \approx \sum_{n=0}^{\infty} \lambda_n \|e_n\|^2 = \mathrm{Tr}(A).$$

Thus if we set $A = (T^*T)^{1/2}$ and substitute, we get

$$\begin{aligned}\|T\|_{S_1} = \mathrm{Tr}(A) &\approx \int_{\mathbb{D}} \langle (T^*T)^{1/2}\widetilde{h}_z, \widetilde{h}_z\rangle \mathrm{d}\lambda(z) + \langle (T^*T)^{1/2}h_0, h_0\rangle \\ &\leqslant \int_{\mathbb{D}} \|T\widetilde{h}_z\| \mathrm{d}\lambda(z) + \|Th_0\|\end{aligned}$$

by using equation (1.2) and equation (2.1) in the proof of Proposition 2.1. ∎

2.2. SCHATTEN CLASS CARLESON EMBEDDINGS ON THE BERGMAN SPACE. In [7], it is shown that the operators $\Lambda_{A,c}$, where A is a normal contraction operator, are equivalent to Carleson embeddings. We shall therefore characterise precisely S_p membership for Carleson embeddings on L^2_{a} in terms of reproducing kernels.

For a finite, positive Borel measure μ on $\mathbb{D}$, let $I_\mu : L^2_{\mathrm{a}} \to L^2(\mu)$ denote the embedding operator. In [10], Luecking classified S_p membership for such operators in terms of a dyadic partition of the unit disc. For $k = 0, 1, 2, \ldots$ and $l = -2^k + 1, \ldots, 2^k$, let

$$B_{k,l} = \{re^{\mathrm{i}\theta} : 2^{-k}(l-1) \leqslant \theta/\pi < 2^{-k}l \text{ and } 2^{-k-1} \leqslant 1 - r < 2^{-k}\}.$$

THEOREM 2.3. *For $1 \leqslant p < \infty$, $I_\mu \in S_p$ if and only if*

$$\sum_{k=0}^{\infty} \sum_{l=-2^k+1}^{2^k} (\mu(B_{k,l})2^{2k})^{p/2} < \infty,$$

and $\|I_\mu\|^p_{S_p}$ is equivalent to the above expression with constants of equivalence depending only on p.

See Corollary 1 of [10]. We shall show that this condition may be interpreted in terms of reproducing kernels. The analogous results for embedding operators on H^2 are included in Theorem 3.8 later as a special case and are also given in [16]. We will find it convenient later to consider measures supported on the whole closed disc $\overline{\mathbb{D}}$ instead; this is possible because functions in $\mathcal{O}(\overline{\mathbb{D}})$ are certainly defined on $\overline{\mathbb{D}}$.

PROPOSITION 2.4. *Let μ be supported now on $\overline{\mathbb{D}}$.*

(i) *For $1 < p < \infty$, $I_\mu \in S_p(L^2_{\mathrm{a}}, L^2(\mu))$ if and only if*

$$\int_{\mathbb{D}} \|I_\mu \widetilde{h}_z\|^p \mathrm{d}\lambda(z) < \infty \tag{2.3}$$

and moreover $\|I_\mu\|^p_{S_p}$ is equivalent to the above expression.

(ii) $I_\mu \in S_1(L^2_{\mathrm{a}}, L^2(\mu))$ *if and only if*

$$\int_{\mathbb{D}} \|I_\mu \widetilde{h}_z\| \mathrm{d}\lambda(z) < \infty \tag{2.4}$$

and moreover $\|I_\mu\|_{S_1}$ *is equivalent to*

$$\int_{\mathbb{D}} \|I_\mu \widetilde{h}_z\| \mathrm{d}\lambda(z) + \|I_\mu h_0\|.$$

Proof. By Propositions 2.1 and 2.2, we need only prove the sufficiency of these conditions when $2 < p < \infty$ and the necessity when $1 \leqslant p < 2$. Let us suppose for the moment that μ is supported on $\mathbb{D}$ as in Theorem 2.3; we will deal with the part supported on $\mathbb{T}$, the unit circle, later.

First suppose that $2 < p < \infty$ and that (2.3) holds. Then

$$\begin{aligned}
\int_{\mathbb{D}} \|I_\mu \widetilde{h}_z\|^p \mathrm{d}\lambda(z) &= \sum_{k=0}^{\infty} \sum_{l=-2^k+1}^{2^k} \int_{B_{k,l}} \Big(\int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\overline{z}w|^4} \mathrm{d}\mu(w) \Big)^{p/2} \mathrm{d}\lambda(z) \\
&\geqslant \sum_{k=0}^{\infty} \sum_{l=-2^k+1}^{2^k} \int_{B_{k,l}} \Big(\int_{B_{k,l}} \frac{(1-|z|^2)^2}{|1-\overline{z}w|^4} \mathrm{d}\mu(w) \Big)^{p/2} \mathrm{d}\lambda(z) \\
&\geqslant C_p \sum_{k=0}^{\infty} \sum_{l=-2^k+1}^{2^k} \Big(\mu(B_{k,l}) 2^{2k} \Big)^{p/2} \lambda(B_{k,l}),
\end{aligned}$$

since $1-|z|^2 \approx 2^{-k}$ and $|1-\overline{z}w| \approx 2^{-k}$ for $w, z \in B_{k,l}$ — see for instance p. 122 in [1]. Since $\lambda(B_{k,l}) \approx 1$, it follows by Theorem 2.3 that $I_\mu \in S_p(L^2_{\mathrm{a}}, L^2(\mu))$.

Now let us show the necessity of (2.4); the proof of the necessity of (2.3) when $1 < p < 2$ is similar. So, suppose that $I_\mu \in S_1(L^2_{\mathrm{a}}, L^2(\mu))$ and that $c_{k,l} \in B_{k,l}$. Then we have

$$\int_{\mathbb{D}} \|I_\mu \widetilde{h}_z\| \mathrm{d}\lambda(z) \approx \int_{\mathbb{D}} \Big(\sum_{k=0}^{\infty} \sum_{l=-2^k+1}^{2^k} \int_{B_{k,l}} \frac{|w|^2(1-|z|^2)^4}{|1-\overline{z}w|^6} \mathrm{d}\mu(w) \Big)^{1/2} \mathrm{d}\lambda(z)$$

$$\begin{aligned}
&\leqslant C \int_{\mathbb{D}} \Big(\sum_{k=0}^{\infty} \sum_{l=-2^k+1}^{2^k} \frac{(1-|z|^2)^4}{|1-\overline{z}c_{k,l}|^6} \mu(B_{k,l}) \Big)^{1/2} \mathrm{d}\lambda(z) \\
&\leqslant C \sum_{k=0}^{\infty} \sum_{l=-2^k+1}^{2^k} \mu(B_{k,l})^{1/2} \int_{\mathbb{D}} \frac{1}{|1-\overline{z}c_{k,l}|^3} \mathrm{d}A(z),
\end{aligned}$$

using $(\sum a_n)^{1/2} \leqslant \sum a_n^{1/2}$ for $a_n \geqslant 0$ and changing the order of summation and integration. The main inequality we used was the following:

$$\int_{B_{k,l}} \frac{\mathrm{d}\mu(w)}{|1-\bar{z}w|^6} \leqslant C\,\frac{\mu(B_{k,l})}{|1-\bar{z}c|^6} \quad (z \in \mathbb{D},\, c \in B_{k,l})$$

with C independent of z, k, l and the point c, which is true simply because for any $\eta \in \mathbb{D}$ the function $|1-\eta w|$ as a function of w does not change much as w varies over $B_{k,l}$. More precisely, there is a fixed $\delta > 0$ independent of k, l such that

$$\delta \cdot |1-\eta c\,| \leqslant \inf_{w \in B_{k,l}} |1-\eta w| \leqslant |1-\eta c\,| \quad \text{for all } \eta \in \mathbb{D}. \tag{2.5}$$

We can see that this is true as follows: it is easy to see that the set $B_{k,l}$ has diameter about 2^{-k}. Every point of $B_{k,l}$ has modulus at most $1-2^{-(k+1)}$ and $|\eta^{-1}| > 1$, so that

$$\operatorname{dist}(\eta^{-1}, B_{k,l}) = \inf_{w \in B_{k,l}} |\eta^{-1} - w| \geqslant 2^{-(k+1)} \approx \operatorname{diam}(B_{k,l}).$$

Thus we get

$$|\eta^{-1} - c| \leqslant \operatorname{dist}(\eta^{-1}, B_{k,l}) + \operatorname{diam}(B_{k,l}) \leqslant C \cdot \operatorname{dist}(\eta^{-1}, B_{k,l})$$

so that multiplying throughout by $|\eta|$ gives the inequality (2.5) needed.

Now we use the estimate:

$$\int_{\mathbb{D}} \frac{1}{|1-\bar{z}c_{k,l}|^3}\mathrm{d}A(z) \approx (1-|c_{k,l}|^2)^{-1} \approx 2^k$$

which can be found on p. 53 of [18] to obtain

$$\int_{\mathbb{D}} \|I_\mu \widetilde{h}_z\| \mathrm{d}\lambda(z) \leqslant C \sum_{k=0}^{\infty} \sum_{l=-2^k+1}^{2^k} (\,\mu(B_{k,l}) 2^{2k})^{1/2} \leqslant C \|I_\mu\|_{S_1},$$

by Theorem 2.3. Trivially, $\|I_\mu h_0\| \leqslant \|I_\mu\| \leqslant \|I_\mu\|_{S_1}$, so we are finished in the case of $\operatorname{supp}(\mu) \subseteq \mathbb{D}$.

Now we deal with the general case. If μ is supported on $\overline{\mathbb{D}}$ then we can write $\mu = \mu_0 + \nu$ where μ_0 is supported on $\mathbb{D}$ and ν is supported on $\mathbb{T}$. We have $\|I_\mu f\|^2 = \|I_{\mu_0} f\|^2 + \|I_\nu f\|^2$ and the results are already proved for μ_0, so we will now consider ν; it turns out that ν has to be zero anyway.

Let $J \subset \mathbb{T}$ be a subarc (which we call a subinterval) of $\mathbb{T}$ given by

$$J = \{\mathrm{e}^{\mathrm{i}\theta} : \theta_0 - \varepsilon < \theta \leqslant \theta_0 + \varepsilon\}$$

for some $\theta_0 \in \mathbb{R}$, where $|J| = 2\varepsilon$ is the length of J. Define the associated region (which is similar to the Carleson square of J in the standard Carleson measure theorem for H^2)

$$B(J) = \{r\mathrm{e}^{\mathrm{i}\theta} : 1-2\varepsilon < r < 1-\varepsilon, \quad \theta_0 - \varepsilon < \theta \leqslant \theta_0 + \varepsilon\} \subset \mathbb{D}$$

which is a nondyadic version of the regions $B_{j,k}$ considered earlier (we could prove our results by taking only dyadic partitions of $\mathbb{T}$ and using $B_{j,k}$). We recall the following familiar Hardy space estimates which can be proved in a similar way to (2.5) above:

$$|\widetilde{k}_z(\mathrm{e}^{\mathrm{i}\theta})|^2 = P_z(\mathrm{e}^{\mathrm{i}\theta}) \geqslant \delta \cdot \frac{\chi_J(\mathrm{e}^{\mathrm{i}\theta})}{|J|} \quad \text{for all } z \in B(J) \text{ and } \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{T},$$

where P_z is the Poisson kernel for z. Here the functions are considered as functions on $\mathbb{T}$, χ_J equals 1 on J and 0 otherwise and $\delta > 0$ is independent of J.

But $|\widetilde{h}_z| = |\widetilde{k}_z|^2$ so that $|\widetilde{h}_z|^2 \geqslant \delta^2 \cdot \chi_J/|J|^2$; now let us partition $\mathbb{T}$ into n subintervals J_j of length $2\pi/n$ and observe that

$$\|I_\nu \widetilde{h}_z\|^2 = \int_{\mathbb{T}} |\widetilde{h}_z|^2 \,\mathrm{d}\nu \geqslant \delta^2 \cdot \frac{\nu(J_j)}{|J_j|^2} \quad \text{for all } z \in B(J_j). \tag{2.6}$$

Suppose first that I_ν is merely bounded from L^2_{a} to $L^2(\mathbb{T}, \nu)$, which certainly holds if I_ν lies in S_p. For each j pick $z \in B(J_j)$. Then

$$\frac{\nu(J_j)}{|J_j|^2} \leqslant \delta^{-2} \|I_\nu \widetilde{h}_z\|^2 \leqslant M$$

for some M independent of j and n, so that

$$\nu(\mathbb{T}) = \sum_{j=1}^{n} \nu(J_j) \leqslant M \sum_{j=1}^{n} |J_j|^2 = \frac{4M\pi^2}{n}.$$

Thus letting $n \to \infty$ gives $\nu(\mathbb{T}) = 0$ so that $\nu = 0$ as required.

Now suppose, conversely, that (2.3) above holds for $p > 2$ (which is the only remaining case we need consider because of Propositions 2.1 and 2.2 as remarked before). It is clear that the regions $B(J_j) \subset \mathbb{D}$ are pairwise disjoint and that $\lambda(B(J)) \approx 1$. Hence by equation (2.6) above

$$\sum_{j=1}^{n} \Big(\frac{\nu(J_j)}{|J_j|^2} \Big)^{p/2} \leqslant M \sum_{j=1}^{n} \int_{B(J_j)} \|I_\nu \widetilde{h}_z\|^p \,\mathrm{d}\lambda(z) \leqslant M'$$

so that (using Hölder's inequality with exponent $p/2 > 1$)

$$\frac{n^2}{4\pi^2} \sum_{j=1}^{n} \nu(J_j) = \sum_{j=1}^{n} \frac{\nu(J_j)}{|J_j|^2} \leqslant (M')^{2/p} \cdot n^{1-2/p}$$

because $|J_j| = 2\pi/n$ for each j, so that letting $n \to \infty$ as before again gives $\nu = 0$ as required. ∎

2.3. SCHATTEN CLASS $\Lambda_{A,c}$ OPERATORS ON THE BERGMAN SPACE. The following theorem may be found in [7]; it is the discrete version of an earlier result for normal semigroups given in [17]. See Section 4 later for a discussion of the continuous semigroup results.

THEOREM 2.5. *Let A be a subnormal operator on a Hilbert space $\mathcal{H}$ with $r(A) \leqslant 1$. Let $c \in \mathcal{H}$. Then there exists a finite, positive Borel measure μ on $\overline{\mathbb{D}}$ such that*

$$\|\Lambda_{A,c} f\|_{\mathcal{H}}^2 = \int_{\overline{\mathbb{D}}} |f(z)|^2 \, \mathrm{d}\mu(z) = \|I_\mu f\|^2,$$

for all $f \in \mathcal{O}(\overline{\mathbb{D}})$.

Here I_μ means the inclusion operator mapping $\mathcal{O}(\overline{\mathbb{D}})$ into $L^2(\mu)$ as above. We can apply this result to our $\Lambda_{A,c}$ operators with A subnormal.

THEOREM 2.6. *Let A be a subnormal operator on a Hilbert space $\mathcal{H}$ with $r(A) \leqslant 1$. Let $c \in \mathcal{H}$.*

(i) *For $1 < p < \infty$, $\Lambda_{A,c} \in S_p(L^2_{\mathrm{a}}, \mathcal{H})$ if and only if*

$$\int_{\mathbb{D}} \|\Lambda_{A,c} \widetilde{h}_z\|^p \mathrm{d}\lambda(z) < \infty, \tag{2.7}$$

and moreover $\|\Lambda_{A,c}\|_{S_p}^p$ is equivalent to the above expression.

(ii) *$\Lambda_{A,c} \in S_1(L^2_{\mathrm{a}}, \mathcal{H})$ if and only if*

$$\int_{\mathbb{D}} \|\Lambda_{A,c} \widetilde{\dot{h}}_z\| \mathrm{d}\lambda(z) < \infty, \tag{2.8}$$

and moreover $\|\Lambda_{A,c}\|_{S_1}$ is equivalent to

$$\int_{\mathbb{D}} \|\Lambda_{A,c} \widetilde{\dot{h}}_z\| \mathrm{d}\lambda(z) + \|c\|.$$

Proof. By Theorem 2.5, there exists a positive, finite Borel measure μ supported on $\overline{\mathbb{D}}$ such that $\|\Lambda_{A,c} f\| = \|I_\mu f\|$ for all $f \in \mathcal{O}(\overline{\mathbb{D}})$. Because $\Lambda_{A,c}(h_0) = c$ we see that the results follow immediately from Proposition 2.4. ∎

3. OPERATORS ON THE HARDY SPACE

3.1. CRITERIA FOR GENERAL OPERATORS. As in the case of the Bergman space, we require criteria for operators on the Hardy space which characterise Schatten class membership in terms of an operator's action on the reproducing kernels. The following was proved in [16]; see Lemmas 1 and 2 and Theorems 1 and 2 of [16].

THEOREM 3.1. *Let $T \in \mathcal{B}(H^2, \mathcal{H})$ for any Hilbert space $\mathcal{H}$. Then:*

(i) $\|T\|_{S_2}^2 = \sup\limits_{0 \leqslant r < 1} \frac{1}{2\pi} \int\limits_0^{2\pi} \|T k_{r\mathrm{e}^{\mathrm{i}\theta}}\|^2 \mathrm{d}\theta \approx \int\limits_{\mathbb{D}} \|T \dot{k}_z\|^2 (1 - |z|^2) \mathrm{d}A(z) + \|T k_0\|^2.$

(ii) *If $p > 2$ and $T \in S_p(H^2, \mathcal{H})$ then*

$$\int_{\mathbb{D}} \|T\widetilde{k}_z\|^p \mathrm{d}\lambda(z) \leqslant C_p \|T\|_{S_p}^p.$$

(iii) *If $1 \leqslant p < 2$ and*

$$\int_{\mathbb{D}} \|T\widetilde{\widetilde{k}}_z\|^p \mathrm{d}\lambda(z) < \infty,$$

then $T \in S_p(H^2, \mathcal{H})$. Moreover,

$$\|T\|_{S_p}^p \leqslant C_p \Big(\int_{\mathbb{D}} \|T\widetilde{\widetilde{k}}_z\|^p \mathrm{d}\lambda(z) + \|Tk_0\|^p \Big).$$

It is shown in [16] that the converses to the above statements hold for T a *scalar* Hankel operator or T a Carleson embedding on H^2.

3.2. SCHATTEN CLASS VECTORIAL HANKEL OPERATORS. In [7], it is shown that operators $\Lambda_{A,c}$, where A is a contraction operator, are isometrically equivalent to a direct sum of a vectorial Hankel operator and a Carleson embedding operator — see Theorem 3.6 later. We shall therefore characterise precisely S_p membership for vectorial Hankel operators on H^2 in terms of reproducing kernels.

For a separable Hilbert space F, let $L^2(\mathbb{T}, F)$ denote the corresponding Hilbert space of F-valued measurable functions on $\mathbb{T}$ and $H^2(F)$ the Hardy space of F-valued holomorphic functions on $\mathbb{D}$, which may be considered as a closed subspace of $L^2(\mathbb{T}, F)$. Let P_+ denote the orthogonal projection from $L^2(\mathbb{T}, F)$ onto $H^2(F)$.

Given $h \in H^2(F)$, we define the (anti-linear) vectorial Hankel operator

$$\Gamma_h : H^2 \to H^2(F); \quad \Gamma_h f = P_+(h\overline{f}\,).$$

Here $h\overline{f}$ denotes the pointwise multiplication of the two functions h and $\overline{f}$ defined almost everywhere on $\mathbb{T}$. Note that Γ_h is certainly defined at least for $f \in \mathcal{O}(\overline{\mathbb{D}}\,)$ and indeed for $f \in H^\infty(\mathbb{D})$. There are obvious definitions for boundedness, compactness and Schatten–von Neumann classes of anti-linear operators. It is well known that the boundedness and compactness of such operators is determined by their action on the reproducing kernels of H^2 — see pp. 81–83 of [15]. The aim here is to produce analogous conditions for S_p membership — we shall see that the converses of the statements in Theorem 3.1 hold. Since this was done for the special case $F = \mathbb{C}$ in [16], we shall just provide sketches of the proofs.

For $g \in L^2(\mathbb{T})$ and $x \in F$, let $g \otimes x$ denote the member of $L^2(\mathbb{T}, F)$ defined by

$$(g \otimes x)(z) = g(z)x \quad \text{for each } z \in \mathbb{T}.$$

Let S denote the shift on $H^2(F)$, so that $Sf(z) = zf(z)$, for $f \in H^2(F)$ and $z \in \overline{\mathbb{D}}$.

PROPOSITION 3.2. *Let $2 < p < \infty$. Then $\Gamma_h \in S_p(H^2, H^2(F))$ if and only if*

$$\int_{\mathbb{D}} \|\Gamma_h \widetilde{k}_z\|^p \mathrm{d}\lambda(z) < \infty, \tag{3.1}$$

with $\|\Gamma_h\|_{S_p}^p$ equivalent to the above expression.

Proof. By Theorem 3.1, we need only show the sufficiency of (3.1). Peller [14] characterised S_p vectorial Hankel operators in terms of Besov space properties of their symbols, so we have $\Gamma_h \in S_p(H^2, H^2(F))$ if and only if

$$\int_{\mathbb{D}} \|(Sh)'(z)\|^p (1 - |z|^2)^p \mathrm{d}\lambda(z) < \infty, \tag{3.2}$$

with $\|\Gamma_h\|_{S_p}^p$ equivalent to the above expression; see p. 293 of [15] for details.

First note that, for any $x \in F$ and $z \in \mathbb{D}$,

$$(1 - |z|^2)\langle (Sh)'(z), x\rangle_F = \langle \Gamma_h \widetilde{k}_z, \widetilde{k}_z \otimes x\rangle_{H^2(F)}.$$

This is a simple vectorial analogue of Proposition 4 in [4]. Now we get

$$(1 - |z|^2) \cdot |\langle (Sh)'(z), x\rangle_F| \leqslant \|\Gamma_h \widetilde{k}_z\|_{H^2(F)} \cdot \|x\|_F$$

for all $x \in F$ because $\|\widetilde{k}_z \otimes x\|_{H^2(F)} = \|x\|_F$, so that

$$(1 - |z|^2) \cdot \|(Sh)'(z)\|_F \leqslant \|\Gamma_h \widetilde{k}_z\|_{H^2(F)}.$$

Thus (3.1) implies that (3.2) holds, so that $\Gamma_h \in S_p(H^2, H^2(F))$ by Peller's result. ∎

To obtain a necessary condition for S_1 membership of vectorial Hankel operators, we shall require an atomic decomposition theorem for the corresponding vectorial Besov space. This is well known in the scalar case, but does not seem to appear in the literature in the vectorial case. We shall therefore state this as a separate result, as it may be of independent interest.

For a separable Hilbert space F, let $B_1(F)$ denote the vectorial Besov space of exponent 1, i.e. the space of all functions Φ which are F-valued and analytic on $\mathbb{D}$, vanish at 0 and satisfy

$$\|\Phi\|_{B_1(F)} := \|\Phi'(0)\|_F + \int_{\mathbb{D}} \|\Phi''(z)\|_F \, \mathrm{d}A(z) < \infty.$$

Let $B_\infty(F)$ denote the space of all functions Ψ which are F-valued and analytic on $\mathbb{D}$, vanish at 0 and satisfy

$$\|\Psi\|_{B_\infty(F)} := \sup_{z\in\mathbb{D}} (1 - |z|^2)\|\Psi(z)\|_F < \infty.$$

It is well known that the dual space of $B_1(F)$ is $B_\infty(F)$ under the pairing

$$\langle \Phi, \Psi \rangle = \sum_{j=1}^{\infty} \langle \widehat{\Phi}(j), \widehat{\Psi}(j) \rangle_F,$$

where $\widehat{\Phi}(j)$ denotes the jth Fourier coefficient of Φ; see e.g. pp. 292–293 of [15].

Let

$$\phi_w = \frac{k_w}{\|k_w\|^2}, \qquad \text{so that} \qquad \phi_w(z) = \frac{1-|w|^2}{1-\overline{w}z},$$

and let $\psi_w(z) = z\phi_w(z)$.

THEOREM 3.3. *$\Phi \in B_1(F)$ if and only if there exist sequences $(w_n) \subseteq \mathbb{D}$ and $(x_n) \subseteq F$ such that*

$$\Phi = \sum_{n=1}^{\infty} \psi_{w_n} \otimes x_n \quad \textit{and} \quad \sum_{n=1}^{\infty} \|x_n\|_F < \infty. \tag{3.3}$$

Moreover,

$$\|\Phi\|_{B_1(F)} \approx \inf \Big\{ \sum_{n=1}^{\infty} \|x_n\|_F : (3.3) \text{ holds} \Big\}.$$

Proof. By Bonsall's general atomic decomposition theorem in [5], it is sufficient to show that there exist constants $m, M > 0$ such that for all $\Psi \in B_\infty(F)$,

$$m\|\Psi\|_{B_\infty(F)} \leqslant \sup_{w\in\mathbb{D},\, \|x\|_F=1} |\langle \psi_w \otimes x, \Psi \rangle| \leqslant M\|\Psi\|_{B_\infty(F)}. \tag{3.4}$$

Fix $x \in F$ and let $\Psi_x \in B_\infty(\mathbb{C})$ denote the scalar function defined by $\Psi_x(z) = \langle \Psi(z), x \rangle_F$. Then

$$\langle \psi_w \otimes x, \Psi \rangle = \sum_{j=1}^{\infty} \widehat{\psi}_w(j)\, \overline{\widehat{\Psi}_x(j)} = \langle \psi_w, \Psi_x \rangle.$$

It may be easily shown that

$$\sup_{\|x\|=1} \|\Psi_x\|_{B_\infty(\mathbb{C})} = \|\Psi\|_{B_\infty(F)}.$$

It is well known that there exists an atomic decomposition of the scalar Besov space $B_1(\mathbb{C})$ in terms of the functions $\{\psi_w\}$; see e.g. p. 70 of [12] or p. 89 of [18]. Thus, by [5] again, there exist constants $m, M > 0$ such that for all $\psi \in B_\infty(\mathbb{C})$,

$$m\|\psi\|_{B_\infty(\mathbb{C})} \leqslant \sup_{w\in\mathbb{D}} |\langle \psi_w, \psi \rangle| \leqslant M\|\psi\|_{B_\infty(\mathbb{C})}.$$

In particular this holds for each $\psi = \Psi_x$. Taking the supremum over all x with $\|x\| = 1$ gives (3.4). ∎

LEMMA 3.4. *There is a constant C such that if $\Gamma_h \in S_1(H^2, H^2(F))$ then*

$$\int_{\mathbb{D}} \|\Gamma_h \widetilde{k}_z\| \mathrm{d}\lambda(z) \leqslant C\|\Gamma_h\|_{S_1}.$$

Proof. Let us first suppose that $h = \phi_w \otimes x$ for $w \in \mathbb{D}$ and $x \in F$. It is easy to show that $\Gamma_{\phi_w \otimes x} f = \overline{f(w)} \phi_w \otimes x$. Then

$$\|\Gamma_{\phi_w \otimes x} \widetilde{\bar{k}}_z\| = |\widetilde{\bar{k}}_z(w)| \cdot \|\phi_w\| \cdot \|x\| \approx (1-|z|^2)^{3/2} \frac{|w|(1-|w|^2)^{1/2}}{|1-\bar{z}w|^2} \|x\|,$$

so we get

$$\begin{aligned} \int_{\mathbb{D}} \|\Gamma_{\phi_w \otimes x} \widetilde{\bar{k}}_z\| \mathrm{d}\lambda(z) &\leqslant C_0 \|x\| (1-|w|^2)^{1/2} \int_{\mathbb{D}} \frac{(1-|z|^2)^{-1/2}}{|1-\bar{z}w|^2} \mathrm{d}A(z) \\ &\leqslant C_1 \|x\| \end{aligned}$$

by a result on p. 53 of [18], where C_1 does not depend on the point $w \in \mathbb{D}$.

For a general h, Peller's results imply that $\Gamma_h \in S_1(H^2, H^2(F))$ if and only if $Sh \in B_1(F)$ and so, using Theorem 3.3,

$$h = \sum_{n=0}^{\infty} \phi_{w_n} \otimes x_n, \tag{3.5}$$

for some $\{w_n\} \subseteq \mathbb{D}$ and $\{x_n\} \subseteq F$ with $\sum \|x_n\| < \infty$. Moreover,

$$\|\Gamma_h\|_{S_1} \approx \inf \Big\{ \sum_{n=0}^{\infty} \|x_n\| : \text{(3.5) holds} \Big\}. \tag{3.6}$$

Therefore,

$$\int_{\mathbb{D}} \|\Gamma_h \widetilde{\bar{k}}_z\| \mathrm{d}\lambda(z) \leqslant C_1 \sum_{n=0}^{\infty} \|x_n\| \leqslant C \|\Gamma_h\|_{S_1}$$

by using the estimate (3.6). ∎

THEOREM 3.5. *Let* $1 \leqslant p \leqslant 2$. *Then* $\Gamma_h \in S_p(H^2, H^2(F))$ *if and only if*

$$\int_{\mathbb{D}} \|\Gamma_h \widetilde{\bar{k}}_z\|^p \mathrm{d}\lambda(z) < \infty, \tag{3.7}$$

and moreover

$$\|\Gamma_h\|_{S_p}^p \approx \int_{\mathbb{D}} \|\Gamma_h \widetilde{\bar{k}}_z\|^p \mathrm{d}\lambda(z) + \|h\|^p.$$

Proof. The sufficiency of (3.7) follows from Theorem 3.1, noting that $\Gamma_h k_0 = h$. We will use an interpolation argument to prove necessity; the book [3] discusses interpolation theory in detail.

Let $\Gamma S_p(F)$ denote the space of all S_p class vectorial Hankel operators from H^2 to $H^2(F)$ and let $\Gamma S_\infty(F)$ denote the space of all compact vectorial Hankel operators from H^2 to $H^2(F)$. Let $L^p(F, \mathrm{d}\lambda)$ be the space of all strongly measurable F-valued functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} \|f(z)\|_F^p \mathrm{d}\lambda(z) < \infty,$$

with the usual modification for $p = \infty$. Consider

$$\Phi : \Gamma S_\infty(F) \to L^\infty(F, \mathrm{d}\lambda); \quad \Phi(\Gamma_h)(z) = \Gamma_h \widetilde{k}_z.$$

Φ is clearly a bounded map; we need to show that it is also bounded from $\Gamma S_p(F)$ to $L^p(F, \mathrm{d}\lambda)$ for $1 \leqslant p \leqslant 2$. We will show by interpolation that this actually holds for all $1 \leqslant p < \infty$. By Lemma 3.4, Φ is bounded from $\Gamma S_1(F)$ to $L^1(F, \mathrm{d}\lambda)$. Let $(X_0, X_1)_{\theta,p}$ denote the corresponding real interpolation space of X_0 and X_1. Then

$$(L^1(F, \mathrm{d}\lambda), L^\infty(F, \mathrm{d}\lambda))_{\theta,p} = L^p(F, \mathrm{d}\lambda)$$

if $p = 1/(1-\theta)$ — see p. 109 of [3]. Also,

$$(\Gamma S_1(F), \Gamma S_\infty(F))_{\theta,p} = \Gamma S_p(F)$$

if $p = 1/(1-\theta)$. The scalar valued version of this result using the AAK theorem may be found on p. 254 of [15], and the vectorial version is proved in the same manner. Thus, by interpolation, Φ is bounded from $\Gamma S_p(F)$ to $L^p(F, \mathrm{d}\lambda)$, which gives the required result. ∎

3.3. SCHATTEN CLASS $\Lambda_{A,c}$ OPERATORS ON THE HARDY SPACE. The following may be found in [7], as a consequence of the Sz.-Nagy–Foias functional model for contractions on a Hilbert space.

THEOREM 3.6. *Let A be a contraction on a separable Hilbert space $\mathcal{H}$ and let $C \in \mathcal{H}^*$. Then there exists a finite, positive Borel measure μ on $\mathbb{T}$, a separable Hilbert space F and a vector $h \in H^2(F)$ such that*

$$\|Cf(A)\|^2_{\mathcal{H}^*} = \int_{\mathbb{T}} |f(z)|^2 \mathrm{d}\mu(z) + \|\Gamma_h f\|^2$$

for all $f \in \mathcal{O}(\overline{\mathbb{D}})$.

In order to transform this result into one about $\Lambda_{A,c}$ operators we introduce the involution

$$f \mapsto f^* \in \mathcal{O}(\overline{\mathbb{D}}); \quad f^*(z) = \overline{f(\overline{z})} = \sum_{n=0}^{\infty} \overline{a}_n z^n$$

for f with Taylor series $f(z) = \sum a_n z^n$. Since $\mathcal{H}^*$ may be identified with $\mathcal{H}$, let $c \in \mathcal{H}$ be such that $Cx = \langle x, c\rangle$. Then we get

$$\|Cf(A)\|_{\mathcal{H}^*} = \|f^*(A^*)c\|_{\mathcal{H}}$$

which gives the following immediate corollary by replacing A by A^*:

COROLLARY 3.7. *Let A be a contraction on $\mathcal{H}$ and let $c \in \mathcal{H}$. Then there exist μ, F and h as in Theorem 3.6 such that*

$$\|\Lambda_{A,c}(f^*)\|^2 = \int_{\mathbb{T}} |f(z)|^2 \mathrm{d}\mu(z) + \|\Gamma_h f\|^2.$$

Now we can give our main result for $\Lambda_{A,c}$ operators; recall that Theorem 1.4 earlier gives similar characterisations for boundedness and compactness of the $\Lambda_{A,c}$ operators when A is a contraction.

THEOREM 3.8. *Let A be a contraction on a separable Hilbert space $\mathcal{H}$ and let $c \in \mathcal{H}$.*

(i) *If $p > 2$ then $\Lambda_{A,c} \in S_p(H^2, \mathcal{H})$ if and only if*

$$\int_{\mathbb{D}} \|\Lambda_{A,c}\widetilde{k}_z\|^p \mathrm{d}\lambda(z) < \infty. \tag{3.8}$$

Moreover, $\|\Lambda_{A,c}\|_{S_p}^p$ is equivalent to the above expression.

(ii) *If $1 \leqslant p < 2$ then $\Lambda_{A,c} \in S_p(H^2, \mathcal{H})$ if and only if*

$$\int_{\mathbb{D}} \|\Lambda_{A,c}\tilde{\tilde{k}}_z\|^p \mathrm{d}\lambda(z) < \infty. \tag{3.9}$$

Moreover,

$$\|\Lambda_{A,c}\|_{S_p}^p \approx \int_{\mathbb{D}} \|\Lambda_{A,c}\tilde{\tilde{k}}_z\|^p \mathrm{d}\lambda(z) + \|c\|^p.$$

Proof. By Corollary 3.7, there exists $h \in H^2(F)$ and a finite, positive Borel measure μ on $\mathbb{T}$ such that $\|\Lambda_{A,c}(f^*)\|^2 = \|I_\mu f\|^2 + \|\Gamma_h f\|^2$. The involution $*$ applied to $\tilde{\tilde{k}}_z$ or $\widetilde{k}_z$ merely changes z into $\overline{z}$ and $\mathrm{d}\lambda(z)$ is invariant under this reflection, so the results follow from Theorems 3.2 and 3.5 once we have shown that $\mu = 0$.

The proof that $\mu = 0$ for $\mathrm{supp}(\mu) \subseteq \mathbb{T}$ is very similar to the Bergman space version in the proof of Theorem 2.4 earlier and so is omitted. We remark that any Borel measure μ on $\mathbb{T}$ for which I_μ is *compact* from $H^2(\mathbb{D})$ to $L^2(\mu)$ must be zero because $\|I_\mu \widetilde{k}_z\| \to 0$ as $|z| \to 1$, although clearly there are many nonzero *bounded* I_μ operators — for example, the inclusion operator into $L^2(\mathbb{T})$. ∎

REMARK. Recalling Example 1.2 earlier, given any finite, positive Borel measure μ on $\overline{\mathbb{D}}$ we can take $\mathcal{H} = L^2(\mu)$, $c = 1$ and $A = M_z$. A is then a *contraction* and $\|\Lambda_{A,c}f\| = \|I_\mu f\|$ so that Carleson embedding operators can be dealt with as special cases of the theorem, thus giving alternative proofs of Theorem 5 in [16].

4. HARDY SPACE RESULTS FOR THE HALF PLANE

This section is essentially a sequel to Section 5 of [7]. There, boundedness and compactness results for certain operators were given. Here, we obtain similar results for S_p membership; our main result is Corollary 4.9 below.

Our $\Lambda_{A,c}$ operators on $H^2(\mathbb{D})$ in fact were originally motivated by similar operators on $H^2(\mathbb{C}_+)$, where $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and $H^2(\mathbb{C}_+)$ is the space of analytic functions on $\mathbb{C}_+$ such that

$$\|F\|^2_{H^2(\mathbb{C}_+)} = \sup_{x>0} \int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^2 \,\frac{\mathrm{d}y}{2\pi} < \infty.$$

The paper [7] gives a detailed discussion of a class of operators on $H^2(\mathbb{C}_+)$ arising from linear systems theory and how they may be related to our $\Lambda_{A,c}$ operators. The approach taken there uses semigroups and Laplace transform methods. We shall adopt a functional calculus approach instead which is more direct but does not make clear the connection with linear systems theory and differential equations.

Let us consider a separable Hilbert space $\mathcal{H}$ and a closed, possibly unbounded operator A with dense domain $\mathcal{D}(A) \subset \mathcal{H}$. Suppose that $\mathrm{Sp}(A) \subseteq \{z : \mathrm{Re}(z) \leqslant 0\}$, so that $(\lambda I - A)^{-1} \in \mathcal{B}(\mathcal{H})$, for all $\lambda \in \mathbb{C}_+$. Then $A(\lambda I - A)^{-1} = I + \lambda(\lambda I - A)^{-1}$ is also a *bounded* operator on $\mathcal{H}$.

PROPOSITION 4.1. *Let $RH^2(\mathbb{C}_+) = \mathrm{Rat}(\,\overline{\mathbb{C}}_+\,) \cap H^2(\mathbb{C}_+)$ denote the set of rational functions $F(z)$ which only have poles in $\{z : \mathrm{Re}(z) < 0\}$ and satisfy $F(\infty) = 0$. Then for each $F \in RH^2(\mathbb{C}_+)$ we can define $F(-A)$ and $A\,F(-A)$ in $\mathcal{B}(\mathcal{H})$.*

Proof. It is clear that $RH^2(\mathbb{C}_+)$ contains precisely the $H^2(\mathbb{C}_+)$ rational functions and is a dense subspace of $H^2(\mathbb{C}_+)$. Note that any $F \in RH^2(\mathbb{C}_+)$ always has a finite factorisation of the form

$$F(z) = \frac{\alpha}{z+\nu} \prod_i \Big(\frac{1}{z+\nu_i}\Big) \prod_j \Big(\frac{z+\alpha_j}{z+\lambda_j}\Big) \tag{4.1}$$

where $\alpha, \alpha_j \in \mathbb{C}$, $\nu, \nu_i, \lambda_j \in \mathbb{C}_+$ and the products $\prod_i$ and $\prod_j$ may be empty. We can thus define $F(-A)$ and $A\,F(-A)$ formally by substituting $-A$ for z in (4.1).

The operators $(\lambda I - A)^{-1}$ and $(\alpha I - A)(\lambda I - A)^{-1}$ for $\alpha \in \mathbb{C}$ and $\lambda \in \mathbb{C}_+$ are bounded operators on $\mathcal{H}$. Thus it is clear that $F(-A)$ will be bounded. Because (4.1) always contains at least one unpaired $(\nu + z)^{-1}$ term and $A(\nu I - A)^{-1}$ is bounded, it follows that $A\,F(-A)$ is bounded also. ∎

DEFINITION 4.2. Let c_1, $c_2 \in \mathcal{H}$ and consider the operators

$$\Lambda^{\mathrm{con}}_{A,C} : RH^2(\mathbb{C}_+) \to \mathcal{H}; \quad \Lambda^{\mathrm{con}}_{A,C}(F) = F(-A)c_1 + A\,F(-A)c_2$$

where $C = (c_1, c_2)$ stands for a pair of vectors.

Often in applications we can take $c_2 = 0$, but notice that $A\,F(-A)c_2$ makes sense even when $c_2 \notin \mathcal{D}(A)$. $\Lambda^{\mathrm{con}}_{A,C}$ is the $H^2(\mathbb{C}_+)$ analogue of the $\Lambda_{A,c}$ operators considered before (with the “con” standing for continuous) — but note that $\Lambda^{\mathrm{con}}_{A,C}$ is *not* necessarily defined on $\mathcal{O}(\,\overline{\mathbb{C}}_+\,)$.

Previous papers considering these operators have taken a semigroup approach. That is, a bounded C_0-semigroup $(T_t)_{t\geqslant 0}$ is considered for which A is the infinitesimal generator of (T_t), i.e.

$$Ax = \frac{\mathrm{d}}{\mathrm{d}t}(T_t x)\Big|_{t=0} = \lim_{t\to 0^+} \frac{(T_t - I)x}{t}$$

for all x for which the limit exists. There are many books dealing with semigroups; for example [6] gives a basic introduction to the theory.

In the semigroup case we can define the $\Lambda^{\mathrm{con}}_{A,C}$ operators via $F(-A)$ in another way using the Laplace transform $\mathcal{L}$. If $g \in L^1 \cap L^2(0,\infty)$ and

$$F = \mathcal{L}g, \quad \text{so that} \quad F(z) = \int_0^\infty \exp(-zt)\, g(t)\,\mathrm{d}t \quad \text{for } z \in \mathbb{C}_+,$$

then we can define $F(-A)$ by using the operator valued integral

$$F(-A) = \int_0^\infty T_t\, g(t)\,\mathrm{d}t \in \mathcal{B}(\mathcal{H}).$$

For such g, $A\,F(-A)$ will not be bounded in general.

Our operators $\Lambda^{\mathrm{con}}_{A,C}$ make sense even when A does not generate a semigroup; however in most cases of interest A is the infinitesimal generator of a *contraction* semigroup (T_t), so that $\|T_t\| \leqslant 1$ for all $t \geqslant 0$. This holds in each of the following examples so that Corollary 4.9 below applies to all of them, characterising boundedness, compactness and S_p membership for each $\Lambda^{\mathrm{con}}_{A,C}$ operator.

EXAMPLE 4.3. Let $\mathcal{H} = H^2(\mathbb{C}_+)$ and let $\phi : \mathbb{C}_+ \to \mathbb{C}_+$ be a holomorphic map. Let $(T_t x)(z) = \exp(-t\phi(z))x(z)$, so that $Ax(z) = -\phi(z)x(z)$. Then for $c_2 = 0$ we have

$$\Lambda^{\mathrm{con}}_{A,C}(F) = F(-A)c_1 = c_1 \cdot (F \circ \phi)$$

which is a weighted composition operator on $H^2(\mathbb{C}_+)$.

EXAMPLE 4.4. Let $\mathcal{H} = H^2(\mathbb{C}_+)$ and $T_t x = P_+(\mathrm{e}^{-\overline{z}t}x(z))$, where $P_+ : L^2(\mathrm{i}\mathbb{R}) \to H^2(\mathbb{C}_+)$ is the standard projection. Then if $c_2 = 0$ we have

$$\Lambda^{\mathrm{con}}_{A,C}(F)(z) = P_+(c_1(z)F(\overline{z})) \quad \text{for } z \in \mathrm{i}\mathbb{R}$$

which is a Hankel operator on $H^2(\mathbb{C}_+)$.

EXAMPLE 4.5. Let $\mathcal{H} = L^2(\mathbb{C}_+, \mu)$ for μ a positive Borel measure on $\mathbb{C}_+$ and $(T_t x)(z) = \exp(-tz)x(z)$ for $x \in \mathcal{H}$ so that $Ax(z) = -z\,x(z)$. Then

$$\|\Lambda^{\mathrm{con}}_{A,C}(F)\|^2 = \int_{\mathbb{C}_+} |F(z)|^2 |c_1(z) - z\,c_2(z)|^2\,\mathrm{d}\mu(z) = \|F\|^2_{L^2(w\,\mathrm{d}\mu)}$$

with $w(z) = |c_1 - zc_2|^2 \geqslant 0$ so that $\Lambda^{\mathrm{con}}_{A,C}$ is a Carleson embedding operator.

In this example, we see why the $A\,F(-A)$ term should be included: define a new measure $\mathrm{d}\nu(z) = |c_1 - zc_2|^2 \mathrm{d}\mu(z)$. If $c_2 = 0$ then ν has to satisfy $\nu(\mathbb{C}_+) = \|c_1\|^2 < \infty$, which is an unnecessary restriction for the Carleson embedding theorem on $H^2(\mathbb{C}_+)$ (because $1 \notin H^2(\mathbb{C}_+)$, unlike the case of the disc).

We are interested in determining whether $\Lambda^{\mathrm{con}}_{A,C}$ initially defined on $RH^2(\mathbb{C}_+)$ extends to a bounded, compact or S_p-class operator from $H^2(\mathbb{C}_+)$ to $\mathcal{H}$. The paper [7] considers operators constructed with semigroups using functionals $C : \mathcal{D}(A) \to \mathbb{C}$ instead of vectors $c_1, c_2 \in \mathcal{H}$, which is equivalent to our $\Lambda^{\mathrm{con}}_{A,C}$ operators for the adjoint semigroup (T_t^*). The $\Lambda^{\mathrm{con}}_{A,C}$ operators can be reduced to $\Lambda_{A,c}$ operators using a conformal equivalence $\mathcal{M}$ between $\mathbb{C}_+$ and $\mathbb{D}$ which produces a corresponding transformation between $H^2(\mathbb{C}_+)$ and $H^2(\mathbb{D})$.

DEFINITION 4.6. Define $\mathcal{M} : \mathbb{C}_+ \to \mathbb{D}$ and $\mathcal{W} : \mathrm{Hol}(\mathbb{C}_+) \to \mathrm{Hol}(\mathbb{D})$ by

$$\mathcal{W}F(z) = \frac{F(\mathcal{M}z)}{1+z} \quad (z \in \mathbb{D}), \quad \mathcal{M}(z) = \mathcal{M}^{-1}(z) = \frac{1-z}{1+z}.$$

The inverse $\mathcal{W}^{-1} : \mathrm{Hol}(\mathbb{D}) \to \mathrm{Hol}(\mathbb{C}_+)$ is given by

$$(\mathcal{W}^{-1}f)(z) = (1 + \mathcal{M}z) f(\mathcal{M}z) = \frac{2}{1+z} f(\mathcal{M}z) \quad (z \in \mathbb{C}_+).$$

It is a standard fact that restricting $\sqrt{2}\,\mathcal{W}$ to $H^2(\mathbb{C}_+)$ gives a unitary operator mapping $H^2(\mathbb{C}_+)$ onto $H^2(\mathbb{D})$ — see [13]. We note also that $\mathcal{W}$ maps $RH^2(\mathbb{C}_+)$ into $RH^2(\mathbb{D}) \subset \mathcal{O}(\overline{\mathbb{D}})$ but $\mathcal{W}$ does *not* map $\mathcal{O}(\overline{\mathbb{C}_+})$ into $\mathcal{O}(\overline{\mathbb{D}})$.

PROPOSITION 4.7. *Let $B = (I + A)(I - A)^{-1} \in \mathcal{B}(\mathcal{H})$. Then $r(B) \leqslant 1$ and there exists $c \in \mathcal{H}$ such that*

$$\Lambda^{\mathrm{con}}_{A,C}(F) = (\mathcal{W}F)(B)c = \Lambda_{B,c}(\mathcal{W}F) \quad \textit{for } F \in RH^2(\mathbb{C}_+). \tag{4.2}$$

If A is the infinitesimal generator of a contraction semigroup then B is a contraction operator.

Proof. The bounded operator $B = \mathcal{M}(-A)$ satisfies $r(B) \leqslant 1$ because for each $\alpha \notin \overline{\mathbb{D}}$ we can check that $(\alpha I - B)^{-1}$ exists and is bounded. The fact that for A the infinitesimal generator of a contraction *semigroup*, B is (minus) the cogenerator operator of the semigroup and thus a contraction *operator* is standard — see pp. 92–96 of [2].

For each $F \in RH^2(\mathbb{C}_+)$ we have $F(z) = 2(\mathcal{W}F \circ \mathcal{M})(z)/(1+z)$ and so

$$\Lambda^{\mathrm{con}}_{A,C}(F) = F(-A)c_1 + A\,F(-A)c_2 = (\mathcal{W}F)(B)c$$

where $c = 2((I - A)^{-1}c_1 + A(I - A)^{-1}c_2)$. We have used the formula $(f \circ \mathcal{M})(-A) = f(B)$ which is easy to prove directly from the factorisation (4.1) of F. ∎

The equation (4.2) shows that the operators $\Lambda^{\mathrm{con}}_{A,C}$ densely defined on $H^2(\mathbb{C}_+)$ and $\Lambda_{B,c}$ on $H^2(\mathbb{D})$ have the same boundedness, compactness or S_p membership properties. Thus we can apply our earlier Theorem 3.8 about $\Lambda_{B,c}$ operators for B a contraction operator to get an $H^2(\mathbb{C}_+)$ version. First we must calculate:

PROPOSITION 4.8. *If $a \in \mathbb{C}_+$, consider the following functions in $RH^2(\mathbb{C}_+)$ defined for $w \in \mathbb{C}_+$ by:*

$$K_a(w) = (w+\bar{a})^{-1}, \qquad \widetilde{K}_a = \operatorname{Re}(a)^{1/2} K_a;$$
$$\dot{K}_a(w) = (1+w)(w+\bar{a})^{-2}, \qquad \widetilde{\dot{K}}_a = \operatorname{Re}(a)^{3/2}|1+a|^{-1}\dot{K}_a.$$

Then K_a is the reproducing kernel for $H^2(\mathbb{C}_+)$ at the point $a \in \mathbb{C}_+$. Furthermore for $z = \mathcal{M}a \in \mathbb{D}$ we have

$$\mathcal{W}^{-1}(\widetilde{k}_z) = 2\frac{1+\bar{a}}{|1+a|}\widetilde{K}_a, \quad \mathcal{W}^{-1}(\widetilde{\dot{k}}_z) = \eta(a)\widetilde{\dot{K}}_a \quad \text{with } |\eta(a)| \approx 1, \tag{4.3}$$

so that $\|\widetilde{K}_a\|_{H^2(\mathbb{C}_+)} \approx \|\widetilde{\dot{K}}_a\|_{H^2(\mathbb{C}_+)} \approx 1$.

Proof. Equation (4.3) is a straightforward calculation if we use the formulae

$$1 - |\mathcal{M}z|^2 = 4\operatorname{Re}(z)|1+z|^{-2} \quad \text{and} \quad \|\dot{k}_z\| \approx (1-|z|^2)^{-3/2}.$$

Because $\widetilde{k}_z$ and $\widetilde{\dot{k}}_z$ have norm 1 in $H^2(\mathbb{D})$ and $\mathcal{W}$ is a multiple of a unitary operator, it follows that $\|\widetilde{K}_a\|$ is constant and that $\|\widetilde{\dot{K}}_a\|$ is bounded above and below independently of a. The fact that K_a is the reproducing kernel for $H^2(\mathbb{C}_+)$ at the point a is standard — see [13], for example. ∎

Finally we can apply our results to get a version of Theorem 3.8 for the $\Lambda^{\mathrm{con}}_{A,C}$ operators:

COROLLARY 4.9. *Let A be the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$. If $\mathrm{d}A$ is the standard Lebesgue area measure on $\mathbb{C}_+$ and $\widetilde{K}_a$, $\widetilde{\dot{K}}_a$ are as in Proposition 4.8 above then we have the following conditions:*

(i) *$\Lambda^{\mathrm{con}}_{A,C}$ extends to a bounded operator on $H^2(\mathbb{C}_+)$ if and only if $\sup_{a\in\mathbb{C}_+} \|\Lambda^{\mathrm{con}}_{A,C}(\widetilde{K}_a)\|_{\mathcal{H}} < \infty$.*

(ii) *$\Lambda^{\mathrm{con}}_{A,C}$ extends to a compact operator on $H^2(\mathbb{C}_+)$ if and only if $\|\Lambda^{\mathrm{con}}_{A,C}(\widetilde{K}_a)\|_{\mathcal{H}} \to 0$ whenever $\operatorname{Re}(a) \to 0$ and whenever $|a| \to \infty$ in $\mathbb{C}_+$.*

(iii) *If $p > 2$ then $\Lambda^{\mathrm{con}}_{A,C} \in S_p(H^2(\mathbb{C}_+), \mathcal{H})$ if and only if*

$$\int_{\mathbb{C}_+} \|\Lambda^{\mathrm{con}}_{A,C}(\widetilde{K}_a)\|^p_{\mathcal{H}} \frac{\mathrm{d}A(a)}{\operatorname{Re}(a)^2} < \infty \tag{4.4}$$

with $\|\Lambda^{\mathrm{con}}_{A,C}\|^p_{S_p}$ being equivalent to the above expression.

(iv) *If $1 \leqslant p < 2$ then $\Lambda^{\mathrm{con}}_{A,C} \in S_p(H^2(\mathbb{C}_+), \mathcal{H})$ if and only if*

$$\int_{\mathbb{C}_+} \|\Lambda^{\mathrm{con}}_{A,C}(\widetilde{\dot{K}}_a)\|^p_{\mathcal{H}} \frac{\mathrm{d}A(a)}{\operatorname{Re}(a)^2} < \infty. \tag{4.5}$$

Moreover with $c \in \mathcal{H}$ as in Proposition 4.7, $\|\Lambda^{\mathrm{con}}_{A,C}\|^p_{S_p} \approx \|c\|^p + \int\limits_{\mathbb{C}_+} \|\Lambda^{\mathrm{con}}_{A,C}(\widetilde{\dot{K}}_a)\|^p_{\mathcal{H}} \frac{\mathrm{d}A(a)}{\mathrm{Re}(a)^2}$.

Proof. We use Proposition 4.7 and Proposition 4.8 — letting $a = \mathcal{M}z \in \mathbb{C}_+$ gives

$$\|\Lambda^{\mathrm{con}}_{A,C}(\widetilde{K}_a)\| \approx \|\Lambda_{B,c}(\widetilde{k}_z)\|, \quad \|\Lambda^{\mathrm{con}}_{A,C}(\widetilde{\dot{K}}_a)\| \approx \|\Lambda_{B,c}(\widetilde{\dot{k}}_z)\|.$$

The boundedness and compactness results easily follow from the corresponding results given in Theorem 1.4 earlier for $\Lambda_{B,c}$ — details can be found in [7]. The boundedness result was first proved in [9] directly by applying the Sz.-Nagy–Foias functional model for contraction semigroups instead of operators. Noting that $\mathrm{d}\lambda(z) = \mathrm{d}A(a)/4\,\mathrm{Re}(a)^2$ and changing variables in (3.8) and (3.9) from Theorem 3.8 gives the required results for S_p membership. ∎

5. REMARKS

It would be interesting to know whether it is possible to characterise boundedness, compactness or Schatten class membership for the $\Lambda_{A,c}$ operators on the Bergman space L^2_{a} with A any contraction, thus generalising Theorems 1.3 and 2.6. By Theorem 3.6, this would be possible if we knew that these classes of vectorial Hankel operators from L^2_{a} to $H^2(F)$ were characterised by the corresponding expressions. However, we have not been able to do this thus far.

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