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SPECTRAL TRIPLES FOR AF C^*-ALGEBRAS AND METRICS ON THE CANTOR SET

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ABSTRACT. An AF C^*-algebra has a natural filtration as an increasing sequence of finite dimensional C^*-algebras. We show that it is possible to construct a Dirac operator which relates to this filtration in a natural way and which will induce a metric for the weak*-topology on the state space of the algebra. It turns out that for AF C^*-algebras, there is no limit to the growth of the eigenvalues of such a Dirac operator. We have obtained a kind of an inverse to this result, by showing that a phenomenon like this can only occur for AF C^*-algebras. The results are then applied to a study of the classical Cantor set.

KEYWORDS: *AF C^*-algebras, metrics, non commutative compact spaces, spectral triples, Dirac operator, Cantor set, dimension.*

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1. INTRODUCTION

Alain Connes has extended the notion of a compact metric space to the non commutative setting of C^*-algebras and unbounded operators on Hilbert spaces, [5], [6]. For a compact, spin, Riemannian manifold $\mathcal{M}$, Connes has shown that the geodesic distance can be expressed in terms of an unbounded Fredholm module over the C^*-algebra $\mathrm{C}(\mathcal{M})$, such that the distance between two points p, q in $\mathcal{M}$ is obtained via the Dirac operator D by the formula

$$d(p,q) = \sup\{|a(p) - a(q)| \;:\; a \in \mathrm{C}(\mathcal{M}), \|[D,a]\| \leqslant 1\}.$$

For a commutative unital C^*-algebra $\mathrm{C}(X)$, the compact space X embeds naturally into the set of regular Borel probability measures on the space X. By Riesz' representation theorem the latter space is the weak*-compact subset of the dual of $\mathrm{C}(X)$, named the state space of the algebra. For the case $X = \mathcal{M}$, the notion $d(p,q)$ can then be extended to states on $\mathrm{C}(\mathcal{M})$ right away, using the same formula, and in this way it is possible to obtain a metric for the weak*-topology on the state space of $\mathrm{C}(\mathcal{M})$. This construction does not rely on the commutativity of the algebra $C(\mathcal{M})$ but on the C^*-algebra structure of $C(\mathcal{M})$ and the existence of an operator like D, named a *Dirac operator*. Given a C^*-algebra $\mathcal{A}$, the natural question is then which properties an operator D should have in order to deserve such a name? According to Connes, a candidate for a *Dirac operator* must as a minimum have properties which he has defined in the terms *unbounded Fredholm module* and *spectral triple*.

DEFINITION 1.1. Let $\mathcal{A}$ be a unital C^*-algebra. An *unbounded Fredholm module* (H, D) over $\mathcal{A}$ is:

(i) a Hilbert space H which is a left $\mathcal{A}$-module, that is, a Hilbert space H and a $*$-representation of $\mathcal{A}$ on H;

(ii) an unbounded, selfadjoint operator D on H such that the set

$$\{a \in \mathcal{A} : [D, a] \text{ is densely defined and extends to a bounded operator on } H\}$$

is norm dense in $\mathcal{A}$;

(iii) $(I + D^2)^{-1}$ is a compact operator.

The triple $(\mathcal{A}, H, D)$ with the above description is called *a spectral triple*.

Condition (iii) is quite often strengthened in the way that D is said to be finitely summable or *p-summable*, [5], [6], if for some $p > 0$

$$\mathrm{trace}((I + D^2)^{-p/2}) < \infty.$$

Given a spectral triple $(\mathcal{A}, H, D)$, one can then introduce a pseudo-metric on the state space $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ by the formula

$$\forall \varphi, \psi \in \mathcal{S}(\mathcal{A}) : \; d(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathcal{A}, \|[D, a]\| \leqslant 1\}.$$

We use the term *pseudo-metric* because it is not clear that $d(\varphi, \psi) < \infty$ for all pairs, but the other axioms of a metric are fulfilled.

Marc A. Rieffel has studied several aspects of this extension of the concept of a compact metric space to the framework of C^*-algebras, and he has obtained a lot of results [19], [20], [21], [22], [23]. Among the questions he has dealt with, we have been most attracted by the one which asks whether a spectral triple will induce a metric for the weak*-topology on the state space. If the metric topology coincides with the weak*-topology on the state space, then the metric topology should give the state space a finite diameter, since the state space is compact for the weak*-topology. A nice characterization of when the metric is bounded on the state space and furthermore when it induces the weak*-topology on this space was given by Rieffel, [19], and Pavlović, [17]. This characterization reads:

THEOREM 1.2. *Let (H, D) be an unbounded Fredholm module over a unital C^*-algebra $\mathcal{A}$, and let the pseudo-metric d on $\mathcal{S}(\mathcal{A})$ be defined by the formula:*

$$d(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathcal{A}, \|[D, a]\| \leqslant 1\}$$

for $\varphi, \psi \in \mathcal{S}(\mathcal{A})$. Then:

(i) *d is a bounded metric on $\mathcal{S}(\mathcal{A})$ if and only if*

$$\{a \in \mathcal{A} : \|[D, a]\| \leqslant 1\}$$

has a bounded image in the quotient space $\mathcal{A}/\mathbb{C}1$, equipped with the quotient norm.

(ii) *The metric topology coincides with the weak*-topology if and only if the set*

$$\{a \in \mathcal{A} : \|[D, a]\| \leqslant 1\}$$

has a precompact image in the quotient space $\mathcal{A}/\mathbb{C}1$, equipped with the quotient norm.

We will recall some classical results, which serve as motivation for the definitions given in the non commutative case. The compact manifold $\mathbb{T}$, i.e. the unit circle, has a canonical differential operator and is also a compact group. This group is the dual of the integers and the integers has a canonical length function, which, via the Fourier transform, has close relations to the differentiation operator on the circle. While respecting all these classical structures, Connes considered in [5] a discrete group G endowed with a *length function* $\ell : G \to \mathbb{R}_+$. In analogy with the situation for $C(\mathbb{T})$, or the 2π-periodic functions on $\mathbb{R}$ where $\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}nt} = n\mathrm{e}^{\mathrm{i}nt}$, Connes then defined a Dirac operator D on $\ell^2(G)$ by $(D\xi)(g) = \ell(g)\xi(g)$ and he proved, that if the length function ℓ is a *proper length function*, i.e. $\ell^{-1}([0, c])$ is finite for each $c \in \mathbb{R}_+$, then $(\ell^2(G), D)$ is an unbounded Fredholm module for $C^*_r(G)$. It is a remarkable result of [5], that only amenable discrete groups can have a p-summable Fredholm module. This result is extended by Voiculescu in [24], Sections 4 and 5. The common features of these results are that certain boundedness properties of a spectral triple $(\mathcal{A}, H, D)$ imply that $\mathcal{A}$ has a faithful trace state which extends to an $\mathcal{A}$ invariant state on $B(H)$. This in turn implies ([4]) that the representation of $\mathcal{A}$ on H must be hyperfinite, and in case of a reduced group C^*-algebra of a discrete group, it therefore follows that the group must be amenable. In particular these results tell that for certain spectral triples $(\mathcal{A}, H, D)$ involving a non nuclear C^*-algebras it is not possible to perturb D very much without destroying the properties of a spectral triple. It is one of the main insights in the present investigation that for some of the spectral triples $(\mathcal{A}, H, D)$ one can construct for an approximately finite dimensional C^*-algebras there is an abundance of possible perturbations of D which will still give a spectral triple.

In [5], [6], [24], [16] and many more places the concept *filtration of a C^*-algebra* plays an important role in the investigation of spectral triples. The reason is of course that the filtration quite often induces a natural candidate for a spectral triple. The present authors were inspired by this and then wanted to see what this line of investigation can yield for approximately finite dimensional, or AF, C^*-algebras.

An AF C^*-algebra $\mathcal{A}$ has a natural filtration since $\mathcal{A}$, by definition, is the norm closure of an increasing sequence $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ of finite dimensional C^*-algebras. These algebras were studied first by Bratteli, [2]. In this paper we will only consider unital AF C^*-algebras which have a faithful state, and we will therefore always assume that $\mathcal{A}_0 = \mathbb{C}I_{\mathcal{A}}$. Based on the GNS-representation coming from a faithful state and the given increasing sequence of subalgebras, we show that it is possible to construct an unbounded Fredholm module over this C^*-algebra in much the same way as it was done by Connes for discrete groups. To verify the agreement between the induced metric topology and the weak*-topology on the state space we follow ideas of the same type as was used in the group C^*-algebra case, [1], [5], [6], [16], [21]. This means that we try to obtain estimates of the norm of an element a by careful estimation of the norm of some parts of the commutator $[D, a]$. The whole point in these computations is to show that there exists a D such that the set $\mathcal{D} = \{a \in \mathcal{A} \, : \, \|[D, a]\| \leqslant 1\}$ is so big that it separates the states of $\mathcal{A}$ and so small that it has precompact image in $\mathcal{A}/\mathbb{C}I$. It is not difficult to see that such a Dirac operator must exist, and further that there is a lot of freedom in the choice of the eigenvalues for such a D. Especially for the Dirac operators obtained, it turns out that a certain minimal growth in the eigenvalues is needed in order to get the right properties, but once this level is attained, it is possible to increase the numerical values of eigenvalues arbitrarily, without destroying the topological properties of the spectral triple. This implies that for any $p > 0$ it is possible to construct a p-summable Fredholm module, so it is not possible to assign a dimension other than 0 to a unital AF C^*-algebra via spectral triples.

As mentioned above this is very much in contrast to results by Connes and Voiculescu [5], [24]. On the other hand it was suggested to us by Connes that the possibility to increase the numerical values of the eigenvalues of a Dirac operator can only happen for AF C^*-algebras. We have included a theorem which confirms that conjecture if the eigenspaces of the Dirac operator is fixed.

In the special case of a UHF C^*-algebra, the Bratteli diagram is easy to analyze and the unique trace state is faithful, so it is quite easy to give a *natural* description of a spectral triple with the right properties.

Another special type of AF C^*-algebras are the commutative ones. The approximately finite dimensionality implies here, that such an algebra consists of the continuous complex functions on a totally disconnected compact space. It is well known that the algebra of continuous functions on the standard Cantor subset of the unit interval is an approximately finite dimensional C^*-algebra, so we have tried to see what the spectral triples could look like in this case. We are not the first ones who try to apply the non commutative tools on this commutative algebra. We have had the opportunity to see some notes by Connes [7], where he constructs a spectral triple for the algebra of continuous functions on the Cantor set. His construction is different from ours and very accurate in its reflection of the geometrical properties of the usual Cantor set which is obtained by successive cuttings of open intervals from the unit interval. In particular Connes spectral triple makes it possible to recover the metric inherited from $\mathbb{R}$ exactly and he can find the differentiation operator too. Our emphasis is to see how a general construction which works for any AF C^*-algebra will work in this special case. This has the effect that the module we propose is quite different from the one Connes has constructed. We try, later, shortly to describe the major difference between the 2 types of modules. Our spectral triple will not give exact geometric data for the middle third Cantor set, but it will in this case induce a metric equivalent to the one coming from the standard embedding of the Cantor set in $\mathbb{R}$. On the other hand we can see that there are several other possible choices of spectral triples and we have found a *"natural"* family $(D_\gamma)_{\gamma \in]0,1[}$ of Dirac operators, which all act on the Hilbert space coming from the standard representation of the Cantor algebra. It turns out that for any γ in $]0,1[$ the corresponding Dirac operator will yield a metric on the Cantor set, such that this compact metric space, say $\mathcal{C}_\gamma$, will have Hausdorff dimension $\frac{\log 2}{-\log \gamma}$. Having this, we searched the literature for a unified representation theory for Cantor sets of any positive dimension. We have not found a general theory, but we found some examples [9], [15]. These examples have only a small connection to our spectral triples, so we have constructed a family of compact subsets of $\ell^1(\mathbb{N}, \mathbb{R})$ which serves our purpose. Each of these spaces is bi-Lipschitz equivalent to *a generalized Cantor set* in some space $\mathbb{R}^e$. Following a suggestion by Marc Rieffel we have computed an upper bound for the Gromov–Hausdorff distance between two such compact metric spaces $\mathcal{C}_\gamma$ and $\mathcal{C}_\mu$. Finally this investigation showed us how to construct a compact metric space which for any $\gamma \in]0,1[$ contains a subset which is bi-Lipschitz equivalent to $\mathcal{C}_\gamma$.

A full matrix algebra $\mathcal{M}_n(\mathbb{C})$ is a special UHF C^*-algebra and it has a very special compact metric on the state space, namely the one induced by the norm. We show that for $\mathcal{M}_n(\mathbb{C})$ acting on itself with respect to the trace state and the Dirac operator given by transposition on $\mathcal{M}_n(\mathbb{C})$, we get a spectral triple such that the norm distance on the state space is the metric induced by the Fredholm module.

Some more investigations into this construction show that it is possible to extend the properties of the transposition operator to the setting of a general C^*-algebra $\mathcal{A}$. Having this, it turns out that the metric induced by the norm on $\mathcal{S}(\mathcal{A})$ can always be obtained via a Dirac operator which is a selfadjoint unitary.

2. AF C^*-ALGEBRAS

We consider now the case of AF C^*-algebras. Let $\mathcal{A}$ be a unital AF C^*-algebra, such that $\mathcal{A}$ is the norm closure of an increasing sequence $(\mathcal{A}_n)_{n \in \mathbb{N}_0}$ of finite dimensional C^*-algebras. As mentioned above we will always stick to the case of a unital C^*-algebra $\mathcal{A}$ and assume that $\mathcal{A}_0 = \mathbb{C}I_\mathcal{A}$. This restriction is not really serious since we may always add a unit to a non unital AF C^*-algebra and in this way we get a unital AF C^*-algebra for which we can construct spectral triples. It is not difficult to see that such a spectral triple will also be a spectral triple for the original non unital algebra. Further we mention that an AF C^*-algebra always has a faithful state and we will fix such a faithful state in the arguments to come. Usually such a state is denoted τ. For a natural number k we will let $\mathcal{M}_k$ denote the algebra of complex $k \times k$ matrices. Then for $n \geqslant 1$ each $\mathcal{A}_n$ is isomorphic to a sum of full matrix algebras $\mathcal{M}_{m_1} \oplus \cdots \oplus \mathcal{M}_{m_k}$ and it embeds into $\mathcal{A}_{n+1}$ such that the unit of $\mathcal{A}$ always is the unit of each algebra in the sequence. The GNS Hilbert space H of $\mathcal{A}$ is just the completion of the prehilbert space $\mathcal{A}$ equipped with the inner product $(a, b) = \tau(b^*a)$, and the GNS representation π is just given by left multiplication. In order to avoid the writing of too many $\pi's$ we will assume that $\mathcal{A}$ is a subalgebra of $B(H)$ and that ξ is a unit vector in H which is separating and cyclic for $\mathcal{A}$ and further has the property that the vector state ω_ξ equals τ. Since ξ is separating, the mapping $\eta : \mathcal{A} \to H$ given by $\eta(a) = a\xi$ induces, for each n in $\mathbb{N}$, a bijective linear homeomorphism of the algebra $\mathcal{A}_n$ onto a finite dimensional subspace, say H_n of H. The corresponding growing sequence of orthogonal projections from H onto H_n is denoted (P_n). The basic idea in the construction of a Dirac operator is, to let it have its eigenspaces equal to the sequence of differences $H_n \ominus H_{n-1}$. We do therefore define a sequence $(F_n)_{n\in\mathbb{N}}$ of pairwise orthogonal finite dimensional subspaces of H by $F_0 = H_0$ and $F_n = H_n \ominus H_{n-1}$ for $n \geqslant 1$. The corresponding sequence of pairwise orthogonal projections is denoted $(Q_n)_{n\in\mathbb{N}}$, so $Q_0 = P_0$ and $Q_n = P_n - P_{n-1}$ for $n \geqslant 1$. We will, in the following theorem, show that it is possible to find a sequence of positive reals $(\alpha_n)_{n\in\mathbb{N}_0}$ such that the operator D given by

$$D = \sum_{n=1}^{\infty} \alpha_n Q_n$$

can serve as Dirac operator for a reasonable spectral triple. Remark that α_0 is not needed in the description above so we will therefore fix $\alpha_0 = 0$. In particular this means that $D\xi = 0$. Based on all this notation we can now formulate the main result of this section which in combination with Theorem 1.2 shows that the metric induced by D on the state space of $\mathcal{A}$ will be a metric for the weak*-topology on the state space.

THEOREM 2.1. *Let $\mathcal{A}$ be an infinite dimensional unital AF C^*-algebra acting on a Hilbert space H and let ξ be a separating and cyclic, unit vector for $\mathcal{A}$.*

(i) *There exists a sequence of real numbers $(\alpha_n)_{n\in\mathbb{N}_0}$ such that $\alpha_0 = 0$ and with the notation introduced above, the unbounded selfadjoint operator $D = \sum_{n=1}^{\infty} \alpha_n Q_n$ on H has the property that the set*

$$\mathcal{D} = \{a = a^* \in \mathcal{A} \, : \, \| \, [D, a] \, \| \leqslant 1\}$$

separates the states and has a precompact image in the quotient space $\mathcal{A}/\mathbb{C}1$, equipped with the quotient norm. Further the metric which $\mathcal{D}$ induces on the state space generates the weak-topology.*

(ii) *Given any $p > 0$ it is possible to choose the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$, such that the Fredholm module is p-summable.*

Proof. We remind the reader of the fact that the algebras $\mathcal{A}_n$ are finite dimensional and as such, they are all complemented subspaces of $\mathcal{A}$. In the special case where ξ is a trace vector it is well known how to construct a completely positive projection of $\mathcal{A}$ onto $\mathcal{A}_n$, [3]. In the general case with ξ just separating and cyclic, the method can be mimicked and we can define a continuous projection, say π_n of $\mathcal{A}$ onto $\mathcal{A}_n$ by

$$\forall\, a \ \in \mathcal{A} :\ \pi_n(a) \ := \ \eta^{-1}(P_n \eta(a)).$$

This definition has one fundamental consequence upon which we shall build our arguments, namely, that when we consider both the operator algebra structure and the Hilbert space structure on $\mathcal{A}$ simultaneously we get

$$\forall a \in \mathcal{A} :\ \pi_n(a)\xi = P_n a \xi.$$

In order to prove that the set $\mathcal{D}$ has the properties stated we start by studying the domain of definition for the unbounded derivation $\delta(a) = [D, a]$. Hence we think that a sequence $(\alpha_n)_{n \in \mathbb{N}_0}$ with $\alpha_0 = 0$ is given and we will show that for any $n \in \mathbb{N}$ and any $a \in \mathcal{A}_n$ the commutator $[D, a]$ is densely defined and bounded. This shows that the union of multiples of $\mathcal{D}$ given as $\bigcup_{n \in \mathbb{N}} n\mathcal{D}$ is dense in the selfadjoint part of $\mathcal{A}$ and consequently must separate the states on $\mathcal{A}$. So let us fix an n in $\mathbb{N}$, a selfadjoint a in $\mathcal{A}_n$ and let m in $\mathbb{N}$ be chosen such that $m > n$. Since $a\mathcal{A}_m \subset \mathcal{A}_m$ and $a\mathcal{A}_{m-1} \subset \mathcal{A}_{m-1}$ we find that both of the projections P_m and P_{m-1} commute with a and consequently a commutes with $Q_m = P_m - P_{m-1}$. This means that for the closure of the commutator $[D, a]$ we get

$$\forall n \in \mathbb{N},\, \forall a \in \mathcal{A}_n :\ \text{closure}(\,[D, a]\,) = \sum_{i=1}^{n} \alpha_i [Q_i, a].$$

Since the sum above is finite, some positive multiple of a must be in $\mathcal{D}$. It is important to notice that this is true for any sequence (α_n), and the statement that $\mathcal{D}$ separates the states will be independent of the actual size of the eigenvalues α_n.

The main ingredient of our construction is that we show that there exists a sequence of positive reals $(c_n)_{n \in \mathbb{N}}$ which only depends on the sequence $(\mathcal{A}_n)$ and the vector ξ such that for any $a \in \mathcal{D}$, where now $\mathcal{D}$ depends on the actual values of α_n, we have

$$\forall n \in \mathbb{N}_0,\, \forall a \in \mathcal{D} :\ \|\pi_{n+1}(a) - \pi_n(a)\| \ \leqslant \ \frac{c_{n+1}}{\alpha_{n+1}}.$$

When this relation is established it follows easily that we can get very nice convergence estimates by choosing the eigenvalues α_n sufficiently big.

We will now describe how the sequence $(c_n)_{n \in \mathbb{N}}$ is obtained. For any $n \in \mathbb{N}_0$ the seminorms $a \to \|\pi_{n+1}(a) - \pi_n(a)\|_{\mathcal{A}}$ and $a \to \|Q_{n+1} a \xi\|_H$ are equivalent, since Q_{n+1} is of finite dimension and ξ is a separating vector. Consequently there exists a positive real $c_{n+1} \geqslant 1$ such that

$$\forall n \in \mathbb{N}_0,\, \forall a \in \mathcal{A} :\ \|\pi_{n+1}(a) - \pi_n(a)\| \ \leqslant \ c_{n+1} \|Q_{n+1} a \xi\|.$$

We will now again suppose that a sequence (α_n) is given such that $\alpha_0 = 0$, fix an arbitrary $n \in \mathbb{N}_0$ and an a in $\mathcal{D}$, then since $D\xi = 0$ we get a series of estimates

$$1 \geqslant \|[D,a]\| \geqslant \|Q_{n+1}[D,a]Q_0\| = \|\alpha_{n+1}Q_{n+1}a\xi\| \geqslant \frac{|\alpha_{n+1}|}{c_{n+1}}\|\pi_{n+1}(a) - \pi_n(a)\|.$$

In conclusion for any $n \in \mathbb{N}_0$ and any $a \in \mathcal{D}$ we have

$$\|\pi_{n+1}(a) - \pi_n(a)\| \leqslant \frac{c_{n+1}}{|\alpha_{n+1}|}.$$

We will now make the choice of the elements in the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ such that the sequence of fractions $\left(\frac{c_{n+1}}{|\alpha_{n+1}|}\right)_{n\in\mathbb{N}_0}$ is summable. Let us then consider an absolutely convergent series $\sum_{\mathbb{N}} \beta_n$ of positive reals and for $n \in \mathbb{N}$ we define $\alpha_n = \beta_n^{-1}c_n$, then $\forall a \in \mathcal{D}, \forall n \in \mathbb{N}, \forall k \in \mathbb{N}$

$$\|\pi_{n+k}(a) - \pi_n(a)\| \leqslant \sum_{j=1}^{k} \|\pi_{(n+j)}(a) - \pi_{n+j-1}(a)\| \leqslant \sum_{j=1}^{\infty} \frac{c_{n+j}}{\alpha_{n+j}} = \sum_{j=1}^{\infty} \beta_{n+j}.$$

From this inequality we first deduce that the sequence $(\pi_n(a))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}$ and hence convergent. Let b denote the limit point for this sequence, then

$$b\xi = \lim \pi_n(a)\xi = \lim P_n a\xi = a\xi,$$

and since ξ is separating $b = a$ and we get from above $\forall a \in \mathcal{D}, \forall n \in \mathbb{N} :\ \|a - \pi_n(a)\| \leqslant \sum_{j=1}^{\infty} \beta_{n+j}$. For $n = 0$ the inequality above gives

$$\forall a \in \mathcal{D} :\ \|a - \omega_\xi(a)I\| \ = \ \|a - \pi_0(a)\| \ \leqslant \ \sum_{j=1}^{\infty} \beta_j < \infty.$$

In particular this shows that for any a in $\mathcal{D}$, the norm of $a + \mathbb{C}I$ in the quotient space is at most $\sum \beta_k$, so $\mathcal{D}/\mathbb{C}I$ is bounded.

The inequality also yields the precompactness of $\mathcal{D}/\mathbb{C}I$ right away. Let $\varepsilon > 0$ be given and let n in $\mathbb{N}$ be chosen such that $\sum_{j\geqslant n} \beta_{j+1} < \frac{\varepsilon}{2}$, then

$$\forall a \in \mathcal{D} :\ \|a - \pi_n(a)\| \ \leqslant \ \frac{\varepsilon}{2} \quad \text{and} \quad \|\pi_n(a) - \omega_\xi(a)I\| \ \leqslant \ \sum \beta_j.$$

Since $\mathcal{A}_n$ is finite dimensional, a closed ball in this space of radius $\sum \beta_k$ is norm compact and can be covered by finite number of balls of radius $\frac{\varepsilon}{2}$. This shows that the set

$$\{a - \omega_\xi(a)I \ : \ a \in \mathcal{D}\}$$

can be covered by a finite number of balls of radius ε and hence this set is precompact in $\mathcal{A}$ and consequently $\mathcal{D}/\mathbb{C}I$ is precompact in $\mathcal{A}/\mathbb{C}I$.

Let $p > 0$ be given. With respect to the p-summability of the above Fredholm module we may assume that the sequence of algebras satisfies $\mathcal{A}_{n+1} \neq \mathcal{A}_n$. If this was not so, then $Q_{n+1} = 0$ and the repetition of $\mathcal{A}_n$ would not have any impact on the Dirac operator. We will therefore assume that for all n in $\mathbb{N}_0$ we have $\dim \mathcal{A}_{n+1} > \dim \mathcal{A}_n$. Since $\mathcal{A}_0$ is one dimensional we get then the rough estimates

$$\forall n \in \mathbb{N}_0 : \dim \mathcal{A}_n \geqslant n+1 \quad \text{and} \quad \dim \mathcal{A}_{n+1} > \dim \mathcal{A}_n.$$

For the given p we define $t = \max\{ 2, \frac{3}{p} \}$, then for $\beta_n = (\dim \mathcal{A}_n)^{-t}$ we have $\sum \beta_n < \infty$ since

$$\sum_{n=1}^{\infty} \beta_n \leqslant \sum_{n=1}^{\infty} (\dim \mathcal{A}_n)^{-2} \leqslant \sum_{n=1}^{\infty} (n+1)^{-2} \leqslant 1.$$

The p-summability follows in the same way since we know for all n we have $c_n \geqslant 1$:

$$\begin{aligned} \operatorname{tr}((I+D^2)^{-(p/2)}) &= 1 + \sum_{n=1}^{\infty} (1 + (\dim \mathcal{A}_n)^{2t} c_n^2)^{-(p/2)} (\dim \mathcal{A}_n - \dim \mathcal{A}_{n-1}) \\ &\leqslant 1 + \sum_{n=1}^{\infty} (\dim \mathcal{A}_n)^{1-pt} \leqslant 1 + \sum_{n=1}^{\infty} (n+1)^{-2} \leqslant 2. \quad \blacksquare \end{aligned}$$

REMARK 2.2. It may, in the first place, seem plausible that the construction given just above should be applicable in a wider setting than just the one of AF C^*-algebras. We have, of course, tried to follow such a path, and realized that, at least for us, it is not easy to get much further along this road. Suppose for a moment that the elements in the filtration $(\mathcal{A}_n)$ are no longer algebras but just finite dimensional subspaces such that $\mathcal{A}_n \mathcal{A}_m \subset \mathcal{A}_{m+n}$, then the spaces $H_m = \mathcal{A}_m \xi$ can not be expected to be invariant for operators a in $\mathcal{A}_n$ when $n \leqslant m$. In this case, the boundedness of the commutator $[D, a]$, for $a \in \mathcal{A}_n$ can not be established easily unless the sequence $(\alpha_n)_{n \in \mathbb{N}_0}$ is the very special one given as $\alpha_n = n$. To realize that it is, in general, not possible just to increase the eigenvalues arbitrarily we refer, again, to Connes' [5] and Voiculescu's results [24]. In particular they show that the reduced group C^*-algebra of a non amenable discrete group can not have a finitely summable Fredholm module. On the other hand for AF C^*-algebras there is no upper limit to the growth of the eigenvalues of $|D|$. This is not only remarkable when compared to the just mentioned results of Connes and Voiculescu, but also quite opposite to classical results for commutative C^*-algebras associated to compact smooth manifolds. In the commutative world an AF C^*-algebra is the continuous functions on a totally disconnected compact space and the present general C^*-algebraic result fits well with this fact. Yet another aspect is discussed in [24], namely the possibility of having a slow growth of the dimensions $\dim \mathcal{A}_n$ of the elements in the filtration. We have not been able to obtain results for spectral triples related to a filtration with slow growth, but it seems likely that such an assumption might have non commutative geometrical consequences.

In a presentation of Theorem 2.1 to an audience containing A. Connes, he suggested that this freedom in the choice of unbounded Fredholm modules might be a characteristic property for AF C^*-algebras. We have tried to prove this conjecture, but in vain for this sort of generality. On the other hand we have found a more restrictive property, which of course always holds for an AF C^*-algebra, and we can show that algebras which have this property must be AF C^*-algebras.

Our difficulty in proving a general result lies in the problems involved in comparing Dirac-operators connected to different representations and/or with different spectral projections. If we fix the representation, and the spectral projections of D and further impose an extra condition on some dense subset of the algebra, then we can obtain a result of the type conjectured by Connes.

THEOREM 2.3. *Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space H, ξ be a separating unit vector for $\mathcal{A}$ and $(Q_n)_{n\in\mathbb{N}_0}$ a sequence of pairwise orthogonal finite-dimensional projections with sum I such that $Q_0\xi = \xi$. For any sequence of real numbers $(\lambda_n)_{n\in\mathbb{N}_0}$ such that $|\lambda_n| \to \infty$ for $n \to \infty$ the symbol $D_{(\lambda_n)}$ shall denote the closed selfadjoint operator which formally can be written as $\sum \lambda_n Q_n$. The common domain of definition,* span $\big(\bigcup Q_n H\big)$ *for all the operators $D_{(\lambda_n)}$ is denoted $\mathcal{D}_0$.*

If $\mathcal{A}$ contains a dense subset $\mathcal{S}$ such that for any s in $\mathcal{S}$ and any $D_{(\lambda_n)}$ the commutator $[D_{(\lambda_n)}, s]$ is defined and bounded on $\mathcal{D}_0$ then $\mathcal{A}$ is an AF C^*-algebra.

Proof. The sequence $(Q_n)_{n\in\mathbb{N}_0}$ induces a matrix description of the operators on H so we will define $H_n = Q_n H$ and for an operator x in $B(H)$ we will write $x = (x_{ij})$ such that x_{ij} is an operator in $B(H_j, H_i)$ given by $x_{ij} = Q_i x | H_j$. For any bounded operator x and any sequence (λ_n) we can then formally describe the commutator $[D_{(\lambda_n)}, x]$ by

$$\forall i, j \in \mathbb{N}_0 : \ [D_{(\lambda_n)}, x]_{ij} = (\lambda_i - \lambda_j) x_{ij}.$$

This may be nothing but a formal infinite matrix, but if we know that the commutator is defined and bounded on $\mathcal{D}_0$ then the closure of the commutator will be a bounded operator whose matrix is the one just described.

Given the assumptions on the elements in $\mathcal{S}$ we may then work on the matrix representations of their commutators with various $D_{(\lambda_n)}$'s without worrying about the domain of definition for the commutators. Our first aim is to prove that for elements in $\mathcal{S}$ all, but finitely many, of the matrix entries outside the main diagonal vanish. Let us then assume that there is an element s in $\mathcal{S}$ which has infinitely many non zero matrix entries outside the main diagonal, and let $(s_{i_k j_k})_{k\in\mathbb{N}}$ be an infinite sequence of non vanishing entries such that for all k, $i_k \neq j_k$. Without loss of generality we may assume that the sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers given by $n_0 = 0$ and $n_k := \max\{i_k, j_k\}$ for $k \geqslant 1$ is strictly increasing. We will then inductively over $k \in \mathbb{N}$ define a sequence $(\lambda_n)_{n\in\mathbb{N}_0}$, which will yield the contradiction, by first defining $n_0 := 0$, $\lambda_0 := 0$ and then for k a natural number:

$$\lambda_n = \lambda_{n_{k-1}} \quad \text{if } n_{k-1} < n < n_k,$$
$$\lambda_{n_k} = \lambda_{n_{k-1}} + k + \frac{k}{\|s_{i_k j_k}\|}.$$

For this sequence (λ_n) and for k in $\mathbb{N}$ we want to estimate the norm of the matrix entry $[D_{(\lambda_n)}, s]_{i_k j_k} = (\lambda_{i_k} - \lambda_{j_k}) s_{i_k j_k}$. Since $i_k \neq j_k$ the norm will satisfy $\| [D_{(\lambda_n)}, s]_{i_k j_k} \| \geqslant k$, so $[D_{(\lambda_n)}, s]$ is not bounded on $\mathcal{D}_0$ and any element s in $\mathcal{S}$ has only finitely many non zero entries outside the main diagonal in its matrix (s_{ij}). In particular this means that for each s in $\mathcal{S}$ there exists a natural number n_s such that for any natural number $n > n_s$ the projection Q_n will commute with s. In order to get our skeleton of finite dimensional C^*-subalgebras of $\mathcal{A}$ we then define an increasing sequence of finite dimensional projections $(P_n)_{n \in \mathbb{N}_0}$ in $B(H)$ by

$$\forall n \in \mathbb{N}_0 : \quad P_n := \sum_{i=0}^{n} Q_i.$$

For each $n \in \mathbb{N}_0$ we can then define an increasing sequence of unital C^*-subalgebras $\mathcal{A}_n$ of $\mathcal{A}$ by

$$\mathcal{A}_n := \{ a \in \mathcal{A} : \forall k \geqslant n, [P_k, a] = 0 \}.$$

Since each s in the set $\mathcal{S}$ eventually commutes with the elements in the sequence (Q_n), the same is true for the sequence (P_n) and we get that

$$\forall s \in \mathcal{S}, \exists n_s \in \mathbb{N}_0 : s \in \mathcal{A}_{n_s}.$$

In particular $\mathcal{S}$ is contained in the union of the algebras $\mathcal{A}_n$, so this union is dense in $\mathcal{A}$ and we only have to prove that each $\mathcal{A}_n$ is finite dimensional. Let us then fix a natural number n. By the definition of $\mathcal{A}_n$ we get a $*$-homomorphism $\rho_n : \mathcal{A}_n \to B(P_n H)$ by $\rho_n(a) := P_n a | P_n H = a | P_n H$, so in order to prove that $\mathcal{A}_n$ is finite dimensional it suffices to prove that ρ_n is faithful on $\mathcal{A}_n$. Suppose now that a in $\mathcal{A}_n$ satisfies $\rho_n(a) = 0$, then since the separating vector ξ is in $P_n H$ we get $0 = \rho_n(a)\xi = a\xi$ and $a = 0$. ∎

3. UHF C^*-ALGEBRAS

A UHF C^*-algebra is a special sort of AF C^*-algebra where at each stage the algebra $\mathcal{A}_n$ is a full matrix algebra. In this section we will then consider a UHF C^*-algebra $\mathcal{A}$ which is the norm closure of an increasing sequence of finite dimensional full matrix algebras $(\mathcal{A}_n)_{n \in \mathbb{N}_0}$ such that $\mathcal{A}_0 = \mathbb{C} I_{\mathcal{A}}$. Since each $\mathcal{A}_n$ is a full matrix algebra there is an increasing sequence of natural numbers $(m_n)_{n \in \mathbb{N}}$ such that $\mathcal{A}_n$ is isomorphic to the full matrix algebra $\mathcal{M}_{m_n}$. The assumption that the unit of $\mathcal{A}$ is the unit in all the algebras $\mathcal{A}_n$ implies that there must be natural numbers d_n such that

$$\forall n \in \mathbb{N} : \ \mathcal{A}_n = \mathcal{A}_{n-1} \otimes \mathcal{M}_{d_n}.$$

In order to avoid trivial complications we will, as in the previous section, assume that the sequence $(m_n)_{n \in \mathbb{N}}$ is strictly increasing, i.e. all $d_n \geqslant 2$. Based on this we then have $d_1 = m_1, m_n = d_1, \dots, d_n$ and

$$\mathcal{A}_{n-1} = \mathcal{M}_{d_1} \otimes \cdots \otimes \mathcal{M}_{d_{n-1}} \otimes \mathbb{C}I_{\mathcal{M}_{d_n}} \subset \mathcal{M}_{d_1} \otimes \cdots \otimes \mathcal{M}_{d_n} = \mathcal{A}_n.$$

Going back to the notation from the section on AF C^*-algebras we can now determine the space denoted $\mathcal{Q}_n$ which is determined via the unique trace state by

$$\forall n \in \mathbb{N}: \ \mathcal{Q}_n \ := \ \{\pi_n(a) - \pi_{n-1}(a) \ : \ a \in \mathcal{A}\}.$$

Since a full matrix algebra only has one trace state, the normalized trace, it follows that the UHF C^*-algebra $\mathcal{A}$ also only has one trace state, the restriction of which to $\mathcal{A}_{n+1}$ is the tensor product of the trace states from each of the factors in the tensor product decomposition of $\mathcal{A}_{n+1}$. Based on this, it is for n in $\mathbb{N}_0$ possible to describe $\mathcal{Q}_{n+1}$ in terms of tensor products. Let $\mathcal{M}^{\circ}_{d_{n+1}}$ denote the elements in $\mathcal{M}_{d_{n+1}}$ of trace zero, then the product description of the trace state shows that

$$\forall n \in \mathbb{N}_0: \ \mathcal{Q}_{n+1} \ = \ \{\pi_{n+1}(a) - \pi_n(a) \ : \ a \in \mathcal{A}\} = \mathcal{M}_{d_1} \otimes \cdots \otimes \mathcal{M}_{d_n} \otimes \mathcal{M}^{\circ}_{d_{n+1}}.$$

In the proof of Theorem 2.1 we introduced the constants c_{n+1}. In the present case of a UHF algebra we can compute the numbers c_{n+1} exactly. Going back to the previous section we find that c_{n+1} is the maximal ratio between the operator-norm and the 2-norm, with respect to the trace state, for elements in $\mathcal{Q}_{n+1}$. It is not difficult to see that this maximum will be attained for an operator which is a rank one operator in $\mathcal{A}_{n+1}$ provided that $\mathcal{Q}_{n+1}$ contains a rank one operator. In order to see, that this is so, one can consider an operator f in $\mathcal{A}_{n+1}$ which is a product $f = e_1 \otimes \cdots \otimes e_{n+1}$ where each e_j is a matrix unit in $\mathcal{M}_{d_j}$ and $\mathrm{trace}(e_{n+1}) = 0$. Such an operator f must belong to $\mathcal{Q}_{n+1}$ and be a rank one operator in $\mathcal{A}_{n+1}$. Then for this f we get

$$c_{n+1} \ = \ \frac{\|f\|}{\|f\|_2} \ = \frac{1}{(m_{n+1})^{-(1/2)}} \ = \ \sqrt{m_{n+1}}.$$

We will now go back to the start Section 2 and take the notation introduced there and apply it to the present situation. Having the concrete values of the elements in the sequence $(c_n)_{n\in\mathbb{N}}$ we can then modify the proof of Theorem 2.1 in accordance with the extra notation introduced just above, such that we can obtain the following result.

THEOREM 3.1. *Let $\mathcal{A}$ be a* UHF *C^*-algebra with an increasing sequence $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ of full matrix algebras such that $\mathcal{A}_0 = \mathbb{C}I_{\mathcal{A}}$ and the union of the sequence is dense in $\mathcal{A}$. With the notation introduced we have:*

(i) *For an absolutely convergent series $\sum \beta_n$ of non zero reals, the selfadjoint Dirac operator given by*

$$D = \sum_{n=1}^{\infty} (\beta_n)^{-1} \sqrt{m_n} Q_n$$

will induce a metric on the state space for the weak-topology. This Fredholm module is* 4*-summable.*

(ii) *Given a positive $p < 2$ for $s > \frac{2}{p} > 1$, the operator D given as*

$$D = \sum_{n=1}^{\infty} (m_n)^s Q_n$$

will induce a metric on the state space for the weak-topology. This Fredholm module is p-summable.*

Proof. The first statement, in (i), is a direct consequence of Theorem 2.1. For the second statement one just has to remark that $\dim(Q_n H) \leqslant \dim(\mathcal{A}_n) = m_n^2$. We can now do the estimation which will show the 4-summability:

$$\mathrm{tr}((I + D^2)^{-(4/2)}) \leqslant 1 + \sum_{n=1}^{\infty} |\beta_n|^4 m_n^{-2} \dim(Q_n H) \leqslant 1 + \sum_{n=1}^{\infty} |\beta_n|^4 < \infty.$$

The p-summability works by a similar sequence of estimates, one just has to remember that the assumption $\mathcal{A}_n \neq \mathcal{A}_{n+1}$ implies that $m_{n+1} \geqslant 2m_n$, so $m_n \geqslant 2^n$. As above we have that the multiplicities of the eigenvalues are given by $(\dim\mathcal{A}_n - \dim\mathcal{A}_{n-1})$ which is dominated by $(m_n)^2$, so we get

$$\mathrm{tr}((I + D^2)^{-(p/2)}) \leqslant 1 + \sum_{n=1}^{\infty} (m_n)^{(2-ps)} \leqslant 1 + \sum_{n=1}^{\infty} (2^{(2-ps)})^n < \infty. \quad \blacksquare$$

4. UNBOUNDED FREDHOLM MODULES FOR THE CANTOR SET

The usual Cantor set is a subset of the unit interval in $\mathbb{R}$ which is obtained by successive cuttings of 2^{n-1} open sub-intervals of length $(\frac{1}{3})^n$. This space can also be considered as the compact topological space $\Pi\mathbb{Z}_2$ which is an infinite product of the compact two element group $\{0, 1\}$. When viewed as an infinite product space it is quite easy to see that it can equipped with several inequivalent metrics, and it turns out that the Hausdorff dimension for these metric spaces can attain any value in the interval $]0, \infty[$. When we want to emphasize that we are considering the classical Cantor set we will call it the middle third Cantor set and denote it $\mathfrak{C}_{1/3}$, the Hausdorff dimension of this space is $\frac{\log 2}{\log 3}$. To give the reader, who may not be familiar with other representations of the Cantor set, an idea of how to construct such a set, we just mention that for a positive real number $\gamma \in]0, \frac{1}{2}[$ it is possible to construct a homeomorphic copy of the middle third Cantor set inside the unit interval by successive cuttings of 2^{n-1} intervals of length $(1 - 2\gamma)\gamma^{n-1}$. This space is denoted $\mathfrak{C}_\gamma$ and its Hausdorff dimension is $\frac{\log 2}{-\log \gamma}$.

For a C^*-algebraist it is well known that the algebra of continuous functions on the Cantor set is an AF C^*-algebra, but the C^*-algebra carries only the topological information, so all the geometry which comes from a particular metric on the space has to be obtained from other sources. The non commutative geometry as developed by Connes offers the tools to describe geometric data for C^*-algebras in general, and Connes has tried to see how his theory can be applied to the Cantor set $\mathfrak{C}_{1/3}$. This is done to some extent in the book [6] (IV 3.ε), and in an unpublished note [7] we have had access to. In the note Connes constructs an unbounded Fredholm module over the algebra $C(\mathfrak{C}_{1/3})$ which carries the exact geometrical structure of the compact subset $\mathfrak{C}_{1/3}$ of $\mathbb{R}$.

We can not get so much information from the spectral triples we associate to AF C^*-algebras. It seems as the spectral triples we study for AF C^*-algebras can reflect the structures which are invariant under bi-Lipschitz mappings, and not much more. What we do in this section is to take the results from Section 2 and use the extra structure to see what we can get for the Cantor C^*-algebra. This means that the representation of the algebra will be a fixed standard representation and the eigenspaces of the Dirac operator will be determined by the increasing sequence of finite dimensional subalgebras. The spectral triples we can obtain this way are quite different from the one Connes constructs in [6], [7], so we will just shortly describe the major difference between Connes' module over $C(\mathfrak{C}_{1/3})$ and the one we get from Section 2.

In his book [6], (IV 3.ϵ) Connes describes a Fredholm module over a general Cantor set which is a subset of the interval $[0, 1]$. In the note [7] Connes concentrates on the middle third Cantor set and constructs a spectral triple for the continuous functions on this set. The module is an infinite sum of one dimensional modules which are associated with the points in the unit interval which are either 0, 1 or an end point of some cut-out-interval. The Dirac operator has eigenvalues which reflects the distance between points which are either end points of a cut out interval or endpoints of an interval which is left back after a certain number of cuts has been made. This coding contains a surprising lot of the geometrical data for the space $\mathfrak{C}_{1/3}$. In this construction, the module H is a Hilbert space, which is an infinite sum of one dimensional modules over $C(\mathfrak{C}_{1/3})$. In particular H has an orthonormal basis of vectors, all of which are joint eigenvectors for all the elements in $C(\mathfrak{C}_{1/3})$. In a language which makes sense for non commutative C^*-algebras too, one can say that Connes' module is a subrepresentation of the reduced atomic representation of $C(\mathcal{C})$.

The module, we have proposed as part of a spectral triple for a general AF C^*-algebras with a faithful state, yields for $C(\mathfrak{C}_{1/3})$ a module which has no non trivial common eigenvectors but instead a separating and cyclic trace vector.

The special thing, for the C^*-algebra consisting of the continuous functions on the Cantor set, is that it has a concrete description, and this makes it possible to perform detailed analysis. From Section 2 we know that there is a lot of freedom in the choice of the eigenvalues for the Dirac operator. This suggests that it would be interesting to see if it is possible to describe the geometrical significance of some of the possible choices of the eigenvalues. We show below that for any real γ such that $0 < \gamma < 1$ we can use the sequence $(\gamma^{-n+1})_{n \in \mathbb{N}}$ as a sequence of eigenvalues, such that the corresponding metric on the state space generates the weak*-topology. Having this it seems natural to look for geometrical consequences of this result and it turns out that if one equips the usual Cantor set with the metric induced by a Dirac operator with the sequence $(\gamma^{-n+1})_{n\in\mathbb{N}}$ as eigenvalues, then this compact metric space will be of Hausdorff dimension $\frac{\log 2}{-\log\gamma}$. We have found a concrete realization of these fractals as a continuous family of subsets of $\ell_1(\mathbb{N},\mathbb{R})$ and as subsets or $\mathbb{R}^e$ for $e\in\mathbb{N}$ and $e>\frac{\log 2}{-\log\gamma}$. All of this is of course based on the description of the topological representation of the Cantor set as the compact Abelian group $\Pi\mathbb{Z}_2$. According to this description of the Cantor set we would like to mention that the discrete Abelian group which is the dual of this infinite product group is the infinite sum group which we denote $\bigoplus\mathbb{Z}_2$, and the continuous functions on the Cantor set is just the group C^*-algebra for this discrete group. If one now equips this group with the length function given by

$$\forall g=(g_n)_{n\in\mathbb{N}}\in\bigoplus\mathbb{Z}_2\ :\quad \ell(g):=\begin{cases}\max\{n\ :\ g_n=1\} & \text{if } \mathrm{g}\neq 0,\\ 0 & \text{if } \mathrm{g}=0,\end{cases}$$

then the spectral triple as constructed by Connes for the reduced C^*-algebra of a discrete group is, except for the size of the eigenvalues, exactly the one we get from the AF-construction.

Before we state the theorem we would like to set up the frame, inside which we will work. First we will let $\mathcal{A}$ denote the C^*-algebra $\mathrm{C}(\Pi\mathbb{Z}_2,\mathbb{C})$ and for n in $\mathbb{N}$ we will let $e_n:\Pi\mathbb{Z}_2\to\mathbb{Z}_2$ denote the coordinate mapping $e_n((x_i))=x_n$. Then e_n is a selfadjoint projection in $\mathcal{A}$ and we will define a sequence of symmetries or self adjoint unitaries $(s_n)_{n\in\mathbb{N}_0}$ in $\mathcal{A}$ by $s_0=I$ and for $n>0,\ s_n=2e_n-I$. The sequence of finite dimensional subalgebras $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ is then defined such that $\mathcal{A}_n$ is the least selfadjoint complex algebra containing the set $\{s_0,s_1,\dots,s_n\}$. Then $\mathcal{A}_n$ is isomorphic to $\mathbb{C}^{2^n}$, the continuous functions on 2^n points. The union of these algebras is denoted $\mathcal{A}_\infty$ and is a selfadjoint unital subalgebra of $\mathrm{C}(\Pi\mathbb{Z}_2)$ which clearly separates the points in $\Pi\mathbb{Z}_2$, so by Stone–Weierstrass' Theorem $\mathcal{A}_\infty$ is dense in $\mathcal{A}$ and $\mathcal{A}$ is a unital AF C^*-algebra. In order to apply Theorem 2.1 we have to fix a faithful state, or here a Borel probability measure on $\Pi\mathbb{Z}_2$ with support equal to $\Pi\mathbb{Z}_2$. The natural choice is the measure which is determined in such a way that the coordinate functions are treated symmetrically. This means that the state, say τ is defined on $\mathcal{A}_\infty$ by

$$\tau(I)=1\quad\text{and}\quad\tau(e_{i_1}e_{i_2}\cdots e_{i_m})=2^{-|\{i_1,i_2,\dots,i_m\}|}.$$

Then τ is extended to all of $\mathcal{A}$ by continuity and we will let the corresponding regular Borel probability measure on $\Pi\mathbb{Z}_2$ be denoted by μ. It should be noted that the symmetry s_n corresponds to the generator of the n'th summand in $\bigoplus\mathbb{Z}_2$ and that this identification can be pursued to an isomorphism which shows that the GNS-representation of $\mathcal{A}$ on $L^2(\mathcal{A},\tau)$ is nothing but the left regular representation of $\mathrm{C}^*_r(\bigoplus\mathbb{Z}_2)$.

We can now formulate our theorem on Dirac operators for the Cantor set using this notation and the definitions used in the set up for Theorem 2.1.

THEOREM 4.1. *Let $\mathcal{A}$ denote the AF C*-algebra* $C(\Pi\mathbb{Z}_2)$, $\mathcal{A}_0 = \mathbb{C}I_{\mathcal{A}}$, *and for* $n \in \mathbb{N}$, $\mathcal{A}_n$ *the subalgebra of* $\mathcal{A}$ *generated by the n first coordinate functions.*

(i) *If* $(\alpha_n)_{n\in\mathbb{N}_0}$ *is a sequence of real numbers such that*

$$\alpha_0 = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \sup\{|\alpha_n - \alpha_i|^{-1} \,:\, 0 \leqslant i \leqslant n-1\} \,<\, \infty,$$

then the Dirac operator D given by

$$D = \sum_{n=1}^{\infty} \alpha_n Q_n$$

will generate a metric for the weak-topology on the state space of* $C(\Pi\mathbb{Z}_2)$ *and in particular a metric denoted* $d_{(\alpha_n)}$ *for the compact space* $\Pi\mathbb{Z}_2$.

(ii) *In the special case where there exists a real* γ *such that* $0 < \gamma < 1$ *and for* $n \in \mathbb{N}$, $\alpha_n = \gamma^{-n+1}$ *the conditions under* (i) *are fulfilled and the module will be p-summable for* $p > \frac{\log 2}{-\log\gamma}$ *and not for* $p = \frac{\log 2}{-\log\gamma}$. *The metric induced by D on* $\Pi\mathbb{Z}_2$ *will in this case be denoted* d_γ *and it will satisfy the following inequalities:*

$$\forall x \neq y \in \Pi\mathbb{Z}_2 \text{ let } m(x,y) = \min\{n \in \mathbb{N} \,:\, x(n) \neq y(n)\}, \text{ then}$$
$$2\gamma^{m(x,y)-1} \leqslant d_\gamma(x,y) \leqslant 2\frac{\gamma^{m(x,y)-1}}{(1-\gamma)^2}.$$

Proof. In the proof of Theorem 2.1 the important step is to get an estimate of $\|\pi_{n+1}(a) - \pi_n(a)\|$ for an $a \in \mathcal{A}$ such that the norm of the commutator of $[D,a]$ is at most 1. Since the present situation is much easier to deal with than the general one, we can get some rather exact estimates of this type. So assume that we have an operator a in $\mathcal{D}$, which means that $\|[D,a]\| \leqslant 1$. We start by examining the expression $\|\pi_n(a) - \pi_{n-1}(a)\|$ in more details. First we remark that the algebra $\mathcal{A}_n$ is represented faithfully on the subspace H_n, which is invariant for $\mathcal{A}_n$, so we have $\|\pi_n(a) - \pi_{n-1}(a)\| = \|(\pi_n(a) - \pi_{n-1}(a))P_n\|$. Moreover $P_{n-1}(\pi_n(a) - \pi_{n-1}(a))P_{n-1} = 0$ since

$$\forall b,c \in \mathcal{A}_{n-1} :\, (\pi_n(a)b\xi, c\xi) \,=\, (P_n a\xi, cb^*\xi) = (P_{n-1}a\xi, cb^*\xi) \,=\, (\pi_{n-1}(a)b\xi, c\xi).$$

A closer examination also shows, as we shall see, that for this particular algebra also $Q_n(\pi_n(a) - \pi_{n-1}(a))Q_n = 0$. This last statement, follows from the facts that the space $F_n = Q_nH$ equals $\mathcal{A}_{n-1}s_n\xi$ and then for any a in $\mathcal{A}$ we have $\pi_n(a) - \pi_{n-1}(a) \in \mathcal{A}_{n-1}s_n$. Since $s_n^2 = I$ and $\mathcal{A}$ is commutative we get $(\pi_n(a) - \pi_{n-1}(a))Q_nH \subset P_{n-1}H$. This all means that

$$P_n(\pi_n(a) - \pi_{n-1}(a))P_n = P_{n-1}(\pi_n(a) - \pi_{n-1}(a))Q_n + Q_n(\pi_n(a) - \pi_{n-1}(a))P_{n-1}.$$

For operators x in $\mathcal{A}$ which satisfy the relation

$$x = P_{n-1}xQ_n + Q_nxP_{n-1}$$

we have $\|x\| = \max\{\|P_{n-1}xQ_n\|, \|Q_nxP_{n-1}\|\}$; this follows from the C^*-relation $\|x^*x\| = \|x\|^2$. We can therefore obtain

$$\begin{aligned}
&\|\pi_n(a) - \pi_{n-1}(a)\| \\
&\qquad = \|(\pi_n(a) - \pi_{n-1}(a))P_n\| \\
&\qquad = \max\{\|P_{n-1}(\pi_n(a) - \pi_{n-1}(a))Q_n\|, \|Q_n(\pi_n(a) - \pi_{n-1}(a))P_{n-1}\|\}.
\end{aligned}$$

We are now going to relate the expression

$$\max\{\|P_{n-1}(\pi_n(a) - \pi_{n-1}(a))Q_n\|, \|Q_n(\pi_n(a) - \pi_{n-1}(a))P_{n-1}\|\}$$

to the commutator $[D, a]$ and its norm. Since D commutes with the projections P_n and any operator in $\mathcal{A}_n$ commutes with all the Q_m for $m \geqslant n$ we get

$$\forall a \in \mathcal{D} : \; P_n[D, a]P_n \;=\; [D, \pi_n(a)]P_n \;=\; P_n[D, \pi_n(a)] \;=\; [D, \pi_n(a)].$$

Based on this we get for an a in $\mathcal{D}$ and an n in $\mathbb{N}$

$$\begin{aligned}
P_{n-1}[D, a]Q_n &= (DP_{n-1} - \alpha_n P_{n-1})aQ_n = (DP_{n-1} - \alpha_n P_{n-1})\pi_n(a)Q_n \\
&= (DP_{n-1} - \alpha_n P_{n-1})(\pi_n(a) - \pi_{n-1}(a))Q_n
\end{aligned}$$

and similarly

$$\begin{aligned}
Q_n[D, a]P_{n-1} &= Q_n a(\alpha_n P_{n-1} - DP_{n-1}) = Q_n\pi_n(a)(\alpha_n P_{n-1} - DP_{n-1}) \\
&= Q_n(\pi_n(a) - \pi_{n-1}(a))(\alpha_n P_{n-1} - DP_{n-1}).
\end{aligned}$$

The assumptions made on the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ say that the series $\sum_{n\in\mathbb{N}} \beta_n$ of positive reals defined by $\beta_n = \max\{|\alpha_n - \alpha_i|^{-1} \,:\, 0 \leqslant i \leqslant n-1\}$ is summable. When examining the expression $(\alpha_n P_{n-1} - DP_{n-1})$ one finds that the properties of the sequence (α_n) implies that this operator is invertible on $P_{n-1}H$ with an inverse, say B_n, on this space such that $\|B_n\| = \beta_n$. We can now make estimates of $\max\{\|P_{n-1}(\pi_n(a) - \pi_{n-1}(a))Q_n\|, \|Q_n(\pi_n(a) - \pi_{n-1}(a))P_{n-1}\|\}$ by combining the previous computations. Since it all depends on a max operation we may just as well assume that $\|\pi_n(a) - \pi_{n-1}(a)\| = \|P_{n-1}(\pi_n(a) - \pi_{n-1}(a))Q_n\|$ so

$$\begin{aligned}
\|\pi_n(a) - \pi_{n-1}(a)\| &= \|P_{n-1}(\pi_n(a) - \pi_{n-1}(a))Q_n\| \\
&= \|B_n(DP_{n-1} - \alpha_n P_{n-1})(\pi_n(a) - \pi_{n-1}(a))Q_n\| \\
&= \|B_nP_{n-1}[D, \pi_n(a) - \pi_{n-1}(a)]Q_n\| \\
&= \|B_nP_{n-1}[D, a]Q_n\| \leqslant \|B_n\|\|[D, a]\| \leqslant \beta_n.
\end{aligned} \tag{4.1}$$

Now the arguments run as in the proof of Theorem 2.1 and we get that the metric generated by the operator D induces a metric for the weak*-topology on the state space. For later use we remark that

$$\forall a \in \mathcal{D}: \quad \|a - \tau(a)I\| = \|a - \pi_0(a)\| \leqslant \sum_{i=1}^{\infty} \beta_i; \tag{4.2}$$

$$\forall a \in \mathcal{D}, \forall n \in \mathbb{N}: \quad \|a - \pi_n(a)\| \leqslant \sum_{i=n+1}^{\infty} \beta_i. \tag{4.3}$$

We will now turn to the proof of item (ii), so we will assume that for some $\gamma \in]0,1[$ we have the eigenvalues of D given by $\alpha_0 = 0$ and for $n \in \mathbb{N} : \alpha_n = \gamma^{-n+1}$. In this setting the numbers β_n are given by

$$\beta_n = (\gamma^{-n+1} - \gamma^{-n+2})^{-1} = \frac{\gamma^{n-1}}{1-\gamma}, \quad \text{and} \quad \sum \beta_n = (1-\gamma)^{-2}.$$

Having this we can for any positive real s compute the trace $\mathrm{tr}(D^{-s})$, where as usual we disregard the kernel of the selfadjoint positive operator D. The way the algebras $\mathcal{A}_n$ are defined shows that for $n \geqslant 1$ we have $\mathcal{A}_n = \mathcal{A}_{n-1} \oplus (\mathcal{A}_{n-1}s_n)$, hence $\dim Q_n H = \dim \mathcal{A}_{n-1}s_n = \dim \mathcal{A}_{n-1} = 2^{n-1}$ and we may compute the trace,

$$\mathrm{tr}(D^{-s}) = \lim_{k\to\infty} \sum_{n=1}^{k} \gamma^{(n-1)s} 2^{n-1} = \lim_{k\to\infty} \frac{1-(2\gamma^s)^k}{(1-2\gamma^s)} \tag{4.4}$$

and it follows that this module is summable if and only if $s > \frac{\log 2}{-\log \gamma}$.

In order to determine the restriction to $\Pi\mathbb{Z}_2$, of the metric which D induces on the set of regular Borel probability measures, we first remark that for each $n \in \mathbb{N}$ we have $\pi_n(s_n) - \pi_{n-1}(s_n) = s_n$, so from the computations done under item (i) we have

$$\begin{aligned} [D, s_n] &= P_{n-1}[D, s_n]Q_n + Q_n[D, s_n]P_{n-1} \\ &= (DP_{n-1} - \gamma^{-n+1}P_{n-1})s_nQ_n - Q_ns_n(DP_{n-1} - \gamma^{-n+1}P_{n-1}). \end{aligned}$$

Since this expression again is of the form $Q_nxP_{n-1} + P_{n-1}xQ_n$ the norm is the maximum of the norms of the summands. On the other hand, and as described above, s_nQ_n is a partial isometry with range projection P_{n-1} and support projection Q_n. Hence each of the summands above have norm equal to $\|DP_{n-1} - \gamma^{-n+1}P_{n-1}\| = |\gamma^{-n+1} - 0| = \gamma^{-n+1}$ and we have got

$$\forall n \in \mathbb{N}: \ \|[D, s_n]\| = \gamma^{-n+1}, \tag{4.5}$$

which means that

$$\forall n \in \mathbb{N}: \ \gamma^{n-1}s_n \in \mathcal{D}. \tag{4.6}$$

We will now determine the quantitative consequence of (4.6) for the metric d_γ. For $x \neq y \in \Pi\mathbb{Z}_2$ we recall that $m(x,y)$ is the least natural number for which the coordinates satisfy $x(n) \neq y(n)$. Let then χ_x and χ_y denote the states on $\mathcal{A}$

or probability measures on $\Pi\mathbb{Z}_2$ which are the point evaluations at x and y. We then get

$$d_\gamma(x,y) \geqslant |\chi_x(\gamma^{m(x,y)-1}s_{m(x,y)}) - \chi_y(\gamma^{m(x,y)-1}s_{m(x,y)})| = 2\gamma^{m(x,y)-1}. \tag{4.7}$$

In order to obtain an inequality in the opposite direction we have to return to the series of inequalities (4.3). Given $x \neq y$ in $\Pi\mathbb{Z}_2$, an $\varepsilon > 0$ and an a in $\mathcal{D}$ such that $|\chi_x(a) - \chi_y(a)| \geqslant d_\gamma(x,y) - \varepsilon$, then we get for $n \in \mathbb{N}_0$ and $n < m(x,y)$ that $\chi_x(\pi_n(a)) = \chi_y(\pi_n(a))$, so by (4.3)

$$\begin{aligned} d_\gamma(x,y) &\leqslant |\chi_x(a) - \chi_y(a)| + \varepsilon \\ &= |\chi_x(a - \pi_{(m(x,y)-1)}(a)) - \chi_y(a - \pi_{(m(x,y)-1)}(a))| + \varepsilon \\ &\leqslant 2\|a - \pi_{(m(x,y)-1)}(a)\| + \varepsilon \quad (\text{then by (4.3) and } \beta_i = \frac{\gamma^{i-1}}{1-\gamma}) \qquad (4.8) \\ &\leqslant 2\sum_{i=m(x,y)}^{\infty} \frac{\gamma^{i-1}}{1-\gamma} + \varepsilon = 2\frac{\gamma^{m(x,y)-1}}{(1-\gamma)^2} + \varepsilon. \end{aligned}$$

We can then conclude that

$$\forall x,y \in \Pi\mathbb{Z}_2 : \quad 2\gamma^{m(x,y)-1} \leqslant d_\gamma(x,y) \leqslant 2\frac{\gamma^{m(x,y)-1}}{(1-\gamma)^2}. \quad ∎ \tag{4.9}$$

The natural question raised by Theorem 4.1, is whether there exist *"natural"* geometrical representations of the Cantor sets corresponding to all the values of $\gamma \in]0,1[$. At a first sight it is not even clear what the question means, but for the type of metric spaces we consider the relevant equivalence concept seems to be that of *bi-Lipschitz*. We will recall this concept [8] in the definition below.

DEFINITION 4.2. Let (S,μ) and (T,ν) denote two metric spaces and $F : S \to T$ a map. The map F is said to be *bi-Lipschitz* if it is bijective and there exist strictly positive constants k, K such that

$$\forall x,y \in S : k\mu(x,y) \leqslant \nu(F(x), \quad F(y)) \leqslant K\mu(x,y).$$

If $S = T$ and the identity mapping on S is bi-Lipschitz then we say that μ and ν are equivalent metrics on S.

Having this definition we will first remark that it is quite clear from the last inequalities in the Theorem 4.1 that for a pair of different values of γ the corresponding metrics are inequivalent. A closer look at the mentioned inequalities makes it apparent that for a given $\gamma \in]0,1[$ there may be a representative for the metrics in the equivalence class containing d_γ which may look more attractive than the rest. The metric which we have chosen to like the most is given by

DEFINITION 4.3. Let $\Pi\mathbb{Z}_2$ denote the Cantor set. For any $\gamma \in]0,1[$ we let δ_γ denote the metric on $\Pi\mathbb{Z}_2$ given by

$$\forall x,y \in \Pi\mathbb{Z}_2 : \ \delta_\gamma(x,y) = \sum_{n=1}^{\infty} |x(n) - y(n)|\gamma^{n-1}(1-\gamma).$$

We will now compute the bounds for the equivalence between δ_γ and d_γ.

PROPOSITION 4.4. *For any $\gamma \in]0,1[$ and any $x, y \in \Pi\mathbb{Z}_2$ we have:*

(i) $\gamma^{(m(x,y)-1)}(1-\gamma) \leqslant \delta_\gamma(x,y) \leqslant \gamma^{(m(x,y)-1)}$.

(ii) $2\delta_\gamma(x,y) \leqslant d_\gamma(x,y) \leqslant \frac{2}{(1-\gamma)^3}\delta_\gamma(x,y)$.

(iii) *For any two different values of γ the corresponding metrics are inequivalent.*

Proof. The sum formula for geometric series yields (i) right away.

For (ii) we see that a combination of (i) with the results of Theorem 4.1 (ii) yields the equivalence as stated. The very same reference shows that different γ's give inequivalent metrics. ∎

5. CONTINUOUS FAMILIES OF CANTOR SETS OF DIFFERENT DIMENSIONS

Based on the construction of the metric δ_γ on the Cantor set, it is possible to construct an *isometric embedding* of the metric space $(\Pi\mathbb{Z}_2, \delta_\gamma)$ onto a compact subset of the Banach space $\ell^1(\mathbb{N}, \mathbb{R})$ of absolutely summable sequences of real numbers. It is quite easy to compute the dimension of this space as $\frac{\log 2}{-\log\gamma}$, but it was less obvious to us that for the natural number d defined as

$$d = \left\lfloor \frac{\log 2}{-\log\gamma} \right\rfloor + 1,$$

it is possible to construct a bi-Lipschitz map of $(\Pi\mathbb{Z}_2, \delta_\gamma)$ onto a compact subset of $\mathbb{R}^d$ equipped with the usual metric. If $0 < \gamma < \frac{1}{2}$, the image is the usual Cantor set $\mathfrak{C}_\gamma$, defined above, which is obtained by cutting intervals of the unit interval, such that at step n we cut out 2^{n-1} intervals each of length $(1-2\gamma)\gamma^{n-1}$. If $d > 1$ then the image is a product in $\mathbb{R}^d$ of d copies of the Cantor set $\mathfrak{C}_{\gamma^d}$ where the cutting length in the n'th step is $(1-2\gamma^d)\gamma^{d(n-1)}$.

We will now define the embedding of $(\Pi\mathbb{Z}_2, \delta_\gamma)$ into $\ell^1(\mathbb{N}, \mathbb{R})$.

DEFINITION 5.1. Let $\gamma \in]0,1[$; then $f_\gamma : (\Pi\mathbb{Z}_2, \delta_\gamma) \to \ell^1(\mathbb{N}, \mathbb{R})$ is defined by

$$\forall x \in \Pi\mathbb{Z}_2, \forall n \in \mathbb{N} : \ f_\gamma(x)(n) := \gamma^{n-1}(1-\gamma)x(n).$$

The image in ℓ^1 of the metric set $(\Pi\mathbb{Z}_2, \delta_\gamma)$ by f_γ is denoted $\mathcal{C}_\gamma$.

PROPOSITION 5.2. *For any $\gamma \in]0,1[$ the map f_γ is an isometry of $(\Pi\mathbb{Z}_2, \delta_\gamma)$ onto $(\mathcal{C}_\gamma, \|\cdot\|_1)$.*

Proof. We have:

$$\forall x, y \in \Pi\mathbb{Z}_2 : \|f_\gamma(x) - f_\gamma(y)\|_1 = \sum_{n=1}^{\infty} |x(n) - y(n)|\gamma^{n-1}(1-\gamma) = \delta_\gamma(x,y). \quad ∎$$

REMARK 5.3. We have chosen to let the embedding take place in the space $\ell^1(\mathbb{N}, \mathbb{R})$ because of the isometric embedding. On the other hand, as sets, we have

$\ell^1(\mathbb{N},\mathbb{R}) \subset \ell^p(\mathbb{N},\mathbb{R}) \subset \ell^\infty(\mathbb{N},\mathbb{R})$ for any $p \in]1,\infty[$, so it might be reasonable to look at the other embeddings too. It turns out that the metrics coming from the other norms are all equivalent on the set $\mathcal{C}_\gamma$. An elementary computation shows these equivalences:

$$\forall x,y \in \Pi\mathbb{Z}_2,\ \forall p \geqslant 1:\ \gamma^{(m(x,y)-1)}(1-\gamma) = \|f_\gamma(x) - f_\gamma(y)\|_\infty \leqslant \|f_\gamma(x) - f_\gamma(y)\|_p \leqslant \|f_\gamma(x) - f_\gamma(y)\|_1 \leqslant \gamma^{(m(x,y)-1)}.$$

The Hausdorff dimension of the metric spaces $\mathcal{C}_\gamma$ can be computed in the same way as it is done for the usual Cantor set contained in the unit interval. We will base our computations on the one K. Falconer offers in [8] on the pages 31–32. We will use the same notation as Falconer, so $\mathscr{H}^s(\mathcal{C}_\gamma)$ denotes the s dimensional Hausdorff measure of $\mathcal{C}_\gamma$. The Hausdorff dimension, say t, is characterized by the fact that $\mathscr{H}^s(\mathcal{C}_\gamma) = 0$, if $s > t$ and the value is infinite if $s < t$. Let us then fix a $\gamma \in]0,1[$ and define $t := \frac{\log 2}{-\log\gamma}$. We will then show that the t dimensional Hausdorff measure of $\mathcal{C}_\gamma$ is a strictly positive real number, so the Hausdorff dimension of $\mathcal{C}_\gamma$ is $\frac{\log 2}{-\log\gamma}$.

In order to ease the translation of the proof from [8] we will give a short description of the *standard intervals* in $\mathcal{C}_\gamma$ and describe some of their properties in a lemma. For $n \in \mathbb{N}$ we will let $\mathcal{S}_n$ denote all the points in $\Pi\mathbb{Z}_2$ whose coordinates are all zero from coordinate number $n+1$ and onwards, or formally $\mathcal{S}_n = \bigoplus_{i=1}^{n} \mathbb{Z}_2 \subset \Pi\mathbb{Z}_2$. The union of the sets $\mathcal{S}_n$ is denoted $\mathcal{S}$. It seems to be convenient to introduce the projection mappings $\pi_n : \Pi\mathbb{Z}_2 \to \mathcal{S}_n$ which for an $x \in \Pi\mathbb{Z}_2$ replaces all the coordinates of x from the number $n+1$ and onwards by 0. For an s in $\mathcal{S}$ and an n in $\mathbb{N}$ we then define the standard interval $V(s,n)$ by

$$\forall s \in \mathcal{S}\,\forall n \in \mathbb{N}: \quad V(s,n) = \begin{cases} \emptyset & \text{if } s \notin \mathcal{S}_n, \\ \{x \in \Pi\mathbb{Z}_2 \ :\ \pi_n(x) = s\} & \text{if } s \in \mathcal{S}_n. \end{cases}$$

This description of $V(s,n)$ works independently of the chosen γ, but in the following lemma, where we list some properties of the standard intervals, we also state some properties which relate to the present metric, so we will now consider the intervals as subsets of $\mathcal{C}_\gamma$.

LEMMA 5.4. *The standard intervals $V(s,n)$ of $\mathcal{C}_\gamma$ have the following properties:*

(i) *$\forall n \in \mathbb{N}$ there are exactly 2^n nonempty standard intervals of the form $V(s,n)$, and they are indexed by the points in $\mathcal{S}_n$ by $\{V(s,n) \ :\ s \in \mathcal{S}_n\}$. Further the sets are pairwise disjoint and their union equals $\Pi\mathbb{Z}_2$.*

(ii) *The 2^n sets $\{V(s,n) \ :\ s \in \mathcal{S}_n\}$ are open and closed and each one has diameter equal to γ^n in $\mathcal{C}_\gamma$.*

(iii) *For any subset U of $\mathcal{C}_\gamma$ of diameter $|U| < (1-\gamma)$ there exists a standard interval $V(s,n)$ which contains U and such that its diameter satisfies $|V(s,n)| \leqslant \frac{|U|}{1-\gamma}$.*

(iv) *For* $t = \frac{\log 2}{-\log \gamma}$, *for any* $n, m \in \mathbb{N}$ *and for any* s_0 *in* $\mathcal{S}_n$

$$2^{-n} = |V(s_0, n)|^t = \sum_{s \in \{x \in \mathcal{S}_{m+n} \,:\, \pi_n(x) = s_0\}} |V(s, n+m)|^t.$$

Proof. The content of (i) follows from the fact that $\mathcal{S}_n$ has exactly 2^n points. With respect to (ii) the topological content is obvious since the coordinate mappings map into a two-point space. The diameter estimate follows from the inequality below:

$$\forall n \in \mathbb{N},\ \forall s \in \mathcal{S}_n,\ \forall x, y \in V(s,n) :\ \delta_\gamma(x,y) = \sum_{i=n+1}^{\infty} |x_i - y_i| \gamma^{i-1}(1-\gamma) \leqslant \gamma^n.$$

It is evident that for any x in $V(s,n)$ there is a y in $V(s,n)$ such that $\delta_\gamma(x,y) = \gamma^n$, so the maximum distance is not only attained, but it can be attained from any point in the set ! The result of (iii) is not so obvious and demands some more computations. Let then U be given such that $|U| < 1 - \gamma$. We will fix a point u in U, then for any x in U we have

$$|U| \geqslant \delta_\gamma(u,x) = \sum_{i=m(u,x)}^{\infty} |u_i - x_i| \gamma^{i-1}(1-\gamma) \geqslant \gamma^{(m(u,x)-1)}(1-\gamma).$$

This shows that

$$\forall x \in U :\ m(u,x) - 1 \geqslant \left\lceil \log\left(\frac{|U|}{1-\gamma}\right) \frac{1}{\log \gamma} \right\rceil.$$

Hence for

$$n = \left\lceil \log\left(\frac{|U|}{1-\gamma}\right) \frac{1}{\log \gamma} \right\rceil \quad \text{and} \quad s = \pi_n(u) :\ U \subset V(s,n)$$

and since the diameter of $V(s,n) = \gamma^n$ we get that

$$|V(s,n)| = \gamma^{\left\lceil \log\left(\frac{|U|}{1-\gamma}\right) \frac{1}{\log \gamma} \right\rceil} \leqslant \gamma^{\log\left(\frac{|U|}{1-\gamma}\right) \frac{1}{\log \gamma}} = \frac{|U|}{1-\gamma},$$

so (iii) follows. With respect to (iv) we remark that t is defined such that $\gamma^t = \frac{1}{2}$. In order to prove (iv) it is by the induction principle enough to consider the case where $m = 1$. So let n in $\mathbb{N}$ and s in $\mathcal{S}_n$ be given then $|V(s,n)|^t = (\gamma^n)^t = 2^{-n}$. Each such interval is divided into two intervals $V(s_0, n+1)$ and $V(s_1, n+1)$ at the level $n+1$, and by the computations just performed we have

$$|V(s_0, n+1)|^t + |V(s_1, n+1)|^t = 2^{-n-1} + 2^{-n-1} = 2^{-n} = |V(s,n)|^t. \quad \blacksquare$$

The result on the Hausdorff dimension for $\mathcal{C}_\gamma$ can now be proved, and it is done in much the same way as Falconer in [8] computes the Hausdorff measure for the usual Cantor set $\mathfrak{C}_{1/3}$. We will also use the notation from [8] in the following theorem.

PROPOSITION 5.5. *For any* γ *in* $]0,1[$, *and for* $t = \frac{\log 2}{-\log \gamma}$ *the* t*-dimensional Hausdorff measure* $\mathscr{H}^t(\mathcal{C}_\gamma)$ *of* $\mathcal{C}_\gamma$ *satisfies* $(1-\gamma)^t \leqslant \mathscr{H}^t(\mathcal{C}_\gamma) \leqslant 1$.

Proof. With respect to the upper estimate, we take an n in $\mathbb{N}$ and consider the collection of intervals $\{V(s,n) \,:\, s \in \mathcal{S}_n\}$. This constitutes a covering of $\mathcal{C}_\gamma$ by 2^n sets of diameter γ^n. By Lemma 5.4 (iv) or by the definition of t, we have $\sum_{s\in\mathcal{S}_n} |V(s,n)|^t = 1$, so $\mathscr{H}^t(\mathcal{C}_\gamma) \leqslant 1$.

The inequality the other way is determined after a couple of reduction steps. As Falconer mentions one can without loss of generality restrict the attention to covering families (U_i), where each set U_i is an open set. Since $\mathcal{C}_\gamma$ is compact this means, in turn, that we can restrict to finite families (U_i). Let then $0 < \delta < 1-\gamma$ and a finite δ-cover (U_i) be given. For each i we can then by Lemma 5.4 (iii) find a natural number n_i and an s_i in $\mathcal{S}_{n_i}$ such that $U_i \subset V(s_i,n_i)$ and $(1-\gamma)|V(s_i,n_i)| \leqslant |U_i|$. Further since there are only finitely many n_i we can define $m = \max\{(n_i)\}$ and by Lemma 5.4 (iv) we can start estimating:

$$\begin{aligned}\sum |U_i|^t &\geqslant (1-\gamma)^t \sum |V(s_i,n_i)|^t = (1-\gamma)^t \sum_i \sum_{s\in\mathcal{S}_m \text{ and } \pi_{n_i}(s)=s_i} |V(s,m)|^t \\ &\geqslant (1-\gamma)^t \sum_{s\in\mathcal{S}_m} |V(s,m)|^t = (1-\gamma)^t,\end{aligned}$$

so $\mathscr{H}^t(\mathcal{C}_\gamma) \geqslant (1-\gamma)^t$. ∎

For each γ in $]0,1[$ we will find a suitable natural number e_γ such that we can embed $\mathcal{C}_\gamma$ into $\mathbb{R}^{e_\gamma}$ via a bi-Lipschitz mapping. Such an embedding preserves the dimension.

THEOREM 5.6. *Let γ be in $]0,1[$, $e_\gamma := \lfloor \frac{\log 2}{-\log\gamma} \rfloor + 1$ and let $F_\gamma : \mathcal{C}_\gamma \to \mathbb{R}^{e_\gamma}$ be defined via its i'th coordinate function, F^i_γ, as*

$$\forall i \in \{1,\dots,e_\gamma\},\ \forall x \in \Pi\mathbb{Z}_2 :\ F^i_\gamma(x) = \sum_{p=1}^{\infty} x(i+(p-1)e_\gamma)(\gamma^{e_\gamma})^{p-1}(1-\gamma^{e_\gamma}).$$

Then F_γ is a bi-Lipschitz continuous mapping of $\mathcal{C}_\gamma$ onto its image as subset of $\mathbb{R}^{e_\gamma}$.

If $\gamma < \frac{1}{2}$ then $e_\gamma = 1$ and the image is the usual γ-Cantor subset of the unit interval, $\mathfrak{C}_\gamma$. If $\gamma \geqslant \frac{1}{2}$ then the image is the product of e_γ copies of the one dimensional $\mathfrak{C}_{\gamma^{e_\gamma}}$-Cantor subset of the unit interval.

Proof. It is obvious that F_γ is continuous, but the bi-Lipschitz property can only be seen after a few computations, but before we start the computations we mention that the symbol $\|z\|$ means the usual Euclidian norm in $\mathbb{R}^{e_\gamma}$. Other norms will be indicated by a subscript. The continuity and the Lipschitz property for F_γ follow from the following inequalities:

$$\begin{aligned}\forall x,y \in \mathcal{C}_\gamma : &\|F_\gamma(x) - F_\gamma(y)\| \\ &\leqslant \|F_\gamma(x) - F_\gamma(y)\|_1 \\ &\leqslant \sum_{i=1}^{e_\gamma}\sum_{p=1}^{\infty} |x(i+(p-1)e_\gamma) - y(i+(p-1)e_\gamma)|(\gamma^{e_\gamma})^{p-1}(1-\gamma^{e_\gamma})\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{e_\gamma} \sum_{p=1}^{\infty} \gamma^{1-i}(1-\gamma^{e_\gamma})|x(i+(p-1)e_\gamma) - y(i+(p-1)e_\gamma)|\gamma^{(i+(p-1)e_\gamma-1)} \\
&\leqslant \sum_{i=1}^{e_\gamma} \sum_{p=1}^{\infty} \gamma^{(1-e_\gamma)}|x(i+(p-1)e_\gamma) - y(i+(p-1)e_\gamma)|\gamma^{(i+(p-1)e_\gamma-1)} \\
&= \frac{\gamma^{(1-e_\gamma)}}{1-\gamma}\delta_\gamma(x,y).
\end{aligned}
$$

Before we start to prove that F_γ has a Lipschitz inverse we remind the reader that the choice of e_γ as consequence has, that $\gamma^{e_\gamma} < \gamma^{\log 2/-\log\gamma} = \frac{1}{2}$. Let us fix a pair x, y in $\mathcal{C}_\gamma$ and let i in $\{1, \dots, e_\gamma\}$ be chosen such that $i \equiv m(x,y)$ mod e_γ then we will first determine p such that $i + (p-1)e_\gamma = m(x,y)$. Since $i = e_\gamma$ is possible, it turns out that $p = \left\lceil \frac{m(x,y)}{e_\gamma} \right\rceil$. We will base the estimates below on the i'th coordinate function, still the same i. We will start the estimation on $|F^i_\gamma(x) - F^i_\gamma(y)|$, by using the knowledge on p just obtained:

$$
\begin{aligned}
|F^i_\gamma(x) - F^i_\gamma(y)| &= \Big| \sum_{q=p}^{\infty} (x(i+(q-1)e_\gamma) - y(i+(q-1)e_\gamma))(\gamma^{e_\gamma})^{(q-1)}(1-\gamma^{e_\gamma}) \Big| \\
&\geqslant (\gamma^{e_\gamma})^{(p-1)}(1-\gamma^{e_\gamma}) - \sum_{q=p+1}^{\infty} (\gamma^{e_\gamma})^{(q-1)}(1-\gamma^{e_\gamma}) \\
&= (\gamma^{e_\gamma})^{(p-1)}(1-\gamma^{e_\gamma}) - (\gamma^{e_\gamma})^{p} = (\gamma^{e_\gamma})^{(p-1)}(1-2\gamma^{e_\gamma}) \\
&= (\gamma^{(m(x,y)-i)}(1-2\gamma^{e_\gamma}) \geqslant \gamma^{m(x,y)}(1-2\gamma^{e_\gamma}).
\end{aligned}
$$

The estimates on F_γ can now be performed using Proposition 4.4:

$$
\|F_\gamma(x) - F_\gamma(y)\| \geqslant |F^i_\gamma(x) - F^i_\gamma(y)| \geqslant \gamma^{m(x,y)}(1-2\gamma^{e_\gamma}) \geqslant (1-2\gamma^{e_\gamma})\gamma\delta_\gamma(x,y).
$$

We have now proved that each map F_γ is bi-Lipschitz and we only have to study the image $F_\gamma(\mathcal{C}_\gamma)$. We will stick to the case where $\gamma = \frac{1}{3}$ and consider an x in $\Pi\mathbb{Z}_2$

$$
F_\gamma(x) = \sum_{n=1}^{\infty} x(n)\Big(\frac{1}{3}\Big)^{n-1}\Big(1-\frac{1}{3}\Big) = \sum_{n=1}^{\infty} x(n)2\Big(\frac{1}{3}\Big)^{n}.
$$

Since $x(n)$ is in the set $\{0,1\}$ we see that the image of F_γ is exactly the set of points in the unit interval which in the base 3, can written using only digits from the set $\{0,2\}$. This is a well known characterization of the standard Cantor set, $\mathfrak{C}_{1/3}$. ∎

6. ON THE GROMOV–HAUSDORFF DISTANCE BETWEEN $\mathcal{C}_\gamma$ AND $\mathcal{C}_\mu$

Hausdorff has defined a metric on the closed subsets of a compact metric space and in this way obtained a new compact metric space. In the book [11]

M. Gromov extends this idea and defines a distance between any pair of compact metric spaces. This metric on compact metric spaces is denoted the *Gromov–Hausdorff distance*. In order to provide a frame, inside which it makes sense to speak about convergence of a family of non commutative compact metric spaces, Marc A. Rieffel has studied many aspects of the Gromov–Hausdorff distance between non commutative compact metric spaces, and published a memoir in the series of the American Mathematical Society on these matters [23]. After the first presentation of our application of the C^*-algebraic approach to the Cantor set, Marc A. Rieffel asked us, if we could compute the Gromov–Hausdorff distance between the different metric spaces $\mathcal{C}_\gamma$ obtained from the Cantor set. We can not answer this question completely but below we can give an upper bound on the distance between two such spaces $\mathcal{C}_\gamma$ and $\mathcal{C}_\mu$. The estimate is based on the isometric embedding of each $\mathcal{C}_\gamma$ in $\ell^1(\mathbb{N})$ as constructed in the previous subsection.

In order to make this computation we first recall the definition of the Hausdorff distance between closed subsets of a metric space and then the definition of the Gromov–Hausdorff distance between metric spaces.

DEFINITION 6.1. Let X and Y be closed subsets of a compact metric space (Z, d) then the Hausdorff distance between X and Y is given by

$$\begin{aligned}&\mathrm{dist}^d_{\mathrm{H}}(X, Y)\\ &\quad = \inf\{r > 0 : \forall x \in X, \exists y \in Y, d(x, y) \leqslant r \text{ and } \forall y \in Y, \exists x \in X, d(y, x) \leqslant r\}.\end{aligned}$$

For compact metric spaces (X, d_x) and (Y, d_y) let $X \dot\cup Y$ denote the disjoint union of the sets and let $\mathcal{M}(d_x, d_y)$ denote the set of all metrics on this space such that the restriction of any of these metrics to each of the subsets X and Y agrees with the given metric on that space. The Gromov–Hausdorff distance between X and Y is then given by

$$\mathrm{dist}_{\mathrm{GH}}((X, d_x), (Y, d_y)) = \inf\{\mathrm{dist}^d_H(X, Y) \,:\, d \in \mathcal{M}(d_x, d_y)\,\}.$$

PROPOSITION 6.2. *Let* $0 < \mu < \gamma < 1$ *then*

$$\mathrm{dist}_{\mathrm{GH}}(\mathcal{C}_\gamma, \mathcal{C}_\mu) \leqslant 2\frac{\gamma - \mu}{1 - \gamma}.$$

Proof. The spaces $\mathcal{C}_\gamma$ are all subsets of $\ell^1(\mathbb{N})$, but they are not disjoint inside this metric space since for instance 0 belongs to all of them. This deficiency can easily be repaired by considering the normed space $E = \mathbb{R} \times \ell^1(\mathbb{N}, \mathbb{R})$ which is equipped with the norm $\|(t, x)\| := \max\{|t|, \|x\|_1\}$ and by the construction of isometric copies $\mathcal{D}_\gamma$ of $\mathcal{C}_\gamma$ in E. This is done by defining

$$\mathcal{D}_\gamma := \{(\gamma, (x_n\gamma^{n-1}(1 - \gamma))_{n\in\mathbb{N}}) \in E \,:\, x_n \in \{0, 1\}\,\}.$$

We first remark that since $\mu < \gamma$ we must have $\gamma^{n-1}(1 - \gamma) \leqslant \mu^{n-1}(1 - \mu)$ for at least $n = 1$ and we define m as the largest natural number such that the inequality above is satisfied. Then $\gamma^{n-1}(1 - \gamma) > \mu^{n-1}(1 - \mu)$ for $n > m$. We can now make an estimate of the Hausdorff distance between $\mathcal{D}_\mu$ and $\mathcal{D}_\gamma$.

Let $u = (\mu, (x_n\mu^{n-1}(1-\mu))_{n\in\mathbb{N}})$ be a point in $\mathcal{D}_\mu$ then for $v = (\gamma, (x_n\gamma^{n-1}(1-\gamma))_{n\in\mathbb{N}})$ in $\mathcal{D}_\gamma$, with the same sequence (x_n) we can perform the estimates $\|u - v\|_E = \max\left\{\gamma-\mu, \sum_{n=1}^{\infty} x_n|\gamma^{n-1}(1-\gamma)-\mu^{n-1}(1-\mu)|\right\}$. Since $x_n \in \{0,1\}$ and m is chosen as it is we can get the following upper estimate for the sum just above:

$$\begin{aligned}
&\sum_{n=1}^{\infty} x_n|\gamma^{n-1}(1-\gamma)-\mu^{n-1}(1-\mu)| \\
&\quad \leqslant \sum_{n=1}^{m}(\mu^{n-1}(1-\mu)-\gamma^{n-1}(1-\gamma)) + \sum_{n=m+1}^{\infty}(\gamma^{n-1}(1-\gamma)-\mu^{n-1}(1-\mu)) \\
&\quad = 2(\gamma^m - \mu^m)
\end{aligned}$$

and finally we get $\|u-v\|_E \leqslant \max\{\gamma-\mu, 2(\gamma^m-\mu^m)\}$.

By definition of m we have $\gamma^{m-1}(1-\gamma) \leqslant \mu^{m-1}(1-\mu)$ so

$$\gamma^{m-1} \leqslant \mu^{m-1}\frac{1-\mu}{1-\gamma} \quad \text{and}$$

$$\gamma^m - \mu^m \leqslant \frac{\gamma\mu^{m-1}(1-\mu)-\mu^m(1-\gamma)}{1-\gamma} = \frac{\mu^{m-1}(\gamma-\mu)}{1-\gamma} \leqslant \frac{\gamma-\mu}{1-\gamma}.$$

If instead of starting with a point $u \in \mathcal{D}_\mu$ we had started with a point $v \in \mathcal{D}_\gamma$ we could have chosen $u \in \mathcal{D}_\mu$ and made exactly the same computations as above. This symmetry in the choice of u and v shows we that the Hausdorff distance between $\mathcal{D}_\mu$ and $\mathcal{D}_\gamma$ is at most $2\frac{\gamma-\mu}{1-\gamma}$, so the Gromov–Hausdorff distance between $\mathcal{C}_\mu$ and $\mathcal{C}_\gamma$ is at most this number too. ∎

7. A COMPACT METRIC SPACE WHICH CONTAINS CANTOR SETS OF ANY DIMENSION

The title for this subsection indicates its content. By the simple definition $\mathcal{E}_\gamma := (1-\gamma)\mathcal{C}_\gamma$ we define a compact subset of $\ell^1(\mathbb{N}, \mathbb{R})$ which is bi-Lipschitz equivalent to $\mathcal{C}_\gamma$. The closure of the union of all these spaces, denoted $\mathcal{E}$, is a compact space, which contains Cantor sets of any dimension. In order to state the result a bit more precise we define e_1 to be the unit vector in $\ell^1(\mathbb{N}, \mathbb{R})$ which is the first basis vector, i.e. the coordinates of e_1 are given by $e_1(n) = \delta_{1n}$. Further we define a subset $\mathcal{F}$ of $\ell^1(\mathbb{N}, \mathbb{R})$ by $\mathcal{F} = \left\{x \in \ell^1(\mathbb{N}, \mathbb{R}) \,:\, 0 \leqslant x(n) \leqslant \frac{4}{(n+1)^2}\right\}$. The set $\mathcal{F}$ is a compact subset of $\ell^1(\mathbb{N}, \mathbb{R})$ since for elements in $\mathcal{F}$ we will have the following uniform estimates of the norm of the tails of the elements

$$\forall x \in \mathcal{F}, \forall k \in \mathbb{N} : \sum_{n=k}^{\infty} |x(n)| \leqslant \frac{4}{k}.$$

THEOREM 7.1. *The space $\mathcal{E}$ is a compact subset of $\mathcal{F}$ of infinite Hausdorff dimension. It contains closed Cantor sets of any positive Hausdorff dimension and it can be* *described as the union*

$$\mathcal{E} = e_1 \cup \Big\{ \bigcup_{\gamma \in]0,1[} (1-\gamma)\mathcal{C}_\gamma \Big\} = \text{closure}\Big(\bigcup_{\gamma \in]0,1[} \mathcal{E}_\gamma \Big).$$

Proof. In order to prove that $\mathcal{E}$ is contained in $\mathcal{F}$ we fix a γ in $]0,1[$ and look at an element x in $(1-\gamma)\mathcal{C}_\gamma$. For each n in $\mathbb{N}$ we have $x(n)$ is in the two-point set $\{0, (1-\gamma)^2\gamma^{n-1}\}$, so we must investigate the function $g_n(t) := (1-t)^2 t^{n-1}$ on the unit interval in order to find an upper estimate for its maximum on this interval. For $n = 1$ the maximal value of g_1 is 1. For $n > 1$, elementary calculus yields that g_n has maximal value in the point $\frac{n-1}{n+1}$ and at this point

$$g_n\Big(\frac{n-1}{n+1}\Big) = \Big(\frac{n-1}{n+1}\Big)^{n-1}\Big(1 - \frac{n-1}{n+1}\Big)^2 < \Big(\frac{2}{n+1}\Big)^2,$$

and $\mathcal{E}$ is contained in $\mathcal{F}$.

In order to show that e_1 belongs to $\mathcal{E}$ and that it is the only point which has to be added to the union of the sets $(1-\gamma)\mathcal{C}_\gamma$, when closing up, we first remark that for any γ in $]0,1[$, the vector $(1-\gamma)^2 e_1$ belongs to $(1-\gamma)\mathcal{C}_\gamma$, so e_1 must belong to $\mathcal{E}$. Suppose now that x is a point in $\mathcal{E}$. Then there exist sequences (γ_i) and (x^i) such that for each i, $0 < \gamma_i < 1$, x^i is in $(1-\gamma_i)\mathcal{C}_{\gamma_i}$ and (x^i) converges to x. The sequence (γ_i) is bounded so it has a convergent subsequence and we may as well assume that the sequence (γ_i) is convergent with limit say ξ in $[0,1]$. If $\xi = 1$ then $x = 0$, but 0 is in all the sets $\mathcal{C}_\gamma$, so we may assume that $\xi < 1$. If $\xi = 0$ then $x = e_1$ and we have dealt with this case already. We can then assume that $0 < \xi < 1$. It is now quite elementary to check that for a coordinate, say $x(n)$ the convergence of (x^i) to x implies that

$$\forall n \in \mathbb{N} : x(n) = 0 \quad \text{or} \quad x(n) = (1-\xi)^2\xi^{n-1}.$$

This means that x is an element of $(1-\xi)\mathcal{C}_\xi$. ∎

8. THE UNIFORM METRIC ON THE STATE SPACE

Let $\mathcal{A}$ be a unital C^*-algebra. In this section we are studying the metric on the state space $\mathcal{S}(\mathcal{A})$ which is induced by the norm on the dual space $\mathcal{A}^*$ of $\mathcal{A}$. For a finite dimensional C^*-algebras the norm topology and the weak*-topology agree, so we looked for a spectral triple for the algebra $\mathcal{M}_n$ of complex $n \times n$ matrices such that the metric induced on the state space would be that of the norm. As we mentioned in the introduction we considered a *standard spectral triple* given by $\mathcal{A} = \mathcal{M}_n$, $H = L^2(\mathcal{M}_n, \frac{1}{n}\text{tr})$ and the operators in $\mathcal{A}$ acting on H by left multiplication. The Dirac operator is then $D = T$ the selfadjoint unitary operator on H which consists of transposing a matrix. It turned out that one can extend the idea behind the above spectral triple such that it is possible, for any C^*-algebra $\mathcal{A}$, to construct a representation π of $\mathcal{A}$ on a Hilbert space H such that there exists a projection $P \in B(H)$ which has the property that the norm distance on the state space is recovered exactly if this projection P plays the role of the Dirac operator. This is the main result of this section. The proof is builded upon the following lemmas. The first lemma is well known ([19]) and easy to prove.

LEMMA 8.1. *Let A be a unital C^*-algebra then for any two states φ, ψ on A*

$$\| \varphi - \psi \| \; = \; \sup \Big\{ | \, (\varphi - \psi)(a) \, | : \; a = a^* \in \mathcal{A} \text{ and } \inf_{\alpha \in \mathbb{R}} \| \, a - \alpha I \, \| \; \leqslant 1 \Big\}.$$

Further we state:

LEMMA 8.2. *Let $\mathcal{A}$ be a unital C^*-algebra and let ρ denote a faithful representation of $\mathcal{A}$ on a Hilbert space H. Let H_1 denote the Hilbert space tensor product $H_1 = H \otimes H$, S the flip on H_1 given by $S(\xi \otimes \eta) = \eta \otimes \xi$ and P the projection $P = \frac{I+S}{2}$. Then the representation π of $\mathcal{A}$ on H_1 given by the amplification $\pi(a) = \rho(a) \otimes I$ satisfies:*

$$\forall a = a^* \in \mathcal{A} : \quad \inf_{\gamma \in \mathbb{R}} \| \, a - \gamma I \, \| \; = \; \| \, [P, \pi(a)] \, \|.$$

Proof. We will first transform the commutator slightly in order to ease the computations. For $\forall \gamma \in \mathbb{R}$, $\forall a = a^* \in \mathcal{A}$ we have:

$$\begin{aligned} \|[P, \pi(a)]\| &= \frac{1}{2}\|[S, \pi(a)]\| = \frac{1}{2}\|S[S, \pi(a)]\| = \frac{1}{2}\|\pi(a) - S\pi(a)S\| \\ &= \frac{1}{2}\|\rho(a) \otimes I - I \otimes \rho(a)\| = \frac{1}{2}\|\rho(a - \gamma I) \otimes I - I \otimes \rho(a - \gamma I)\|. \end{aligned}$$

From this series of identities it follows immediately that $\forall a = a^* \in \mathcal{A}$, $\forall \gamma \in \mathbb{R} : \;\; \| \, a - \gamma I \, \| \geqslant \; \| \, [P, \pi(a)] \, \|$. In order to show the inequality in the opposite direction, for a certain γ, we use the series of identities again. Remark that by spectral theory it follows that for $a = a^* \in \mathcal{A}$ with spectrum contained in the smallest possible interval $[\alpha, \beta] \subseteq \mathbb{R}$ one has $\inf_{\gamma \in \mathbb{R}} \| \, a - \gamma I \, \| \; = \; \left\| \, a - \frac{\alpha+\beta}{2} \, \right\| \; = \frac{\beta - \alpha}{2}$. Let $\varepsilon > 0$ and choose unit vectors $\xi, \eta \in H$ such that $(\rho(a)\xi, \xi) \geqslant \beta - \varepsilon$ and $(\rho(a)\eta, \eta) \leqslant \alpha + \varepsilon$. Then, $\xi \otimes \eta$ is a unit vector in H_1 and

$$\begin{aligned} \|[P, \pi(a)]\| &= \frac{1}{2}\|\rho(a) \otimes I - I \otimes \rho(a)\| \; \geqslant \; \frac{1}{2}((\rho(a) \otimes I - I \otimes \rho(a))\xi \otimes \eta, \xi \otimes \eta) \\ &\geqslant \; \frac{\beta - \alpha}{2} - \varepsilon \; = \; \inf_{\gamma \in \mathbb{R}} \| \, a - \gamma I \, \| \; - \varepsilon. \quad \blacksquare \end{aligned}$$

We are now ready to give the main result of this section.

THEOREM 8.3. *Let $\mathcal{A}$ be a C^*-algebra and ρ a faithful non-degenerate representation of $\mathcal{A}$ on a Hilbert space H. Then there exists a representation π of $\mathcal{A}$ on a Hilbert space H_1 which is an amplification of ρ, and a projection P in $B(H_1)$ such that for any pair of states φ, ψ on $\mathcal{A}$*

$$\| \; \varphi - \psi \; \| \; = \; \sup\{ | \, (\varphi - \psi)(a) \, | : \; a = a^* \in \mathcal{A} \text{ and } \|[P, \pi(a)]\| \; \leqslant 1 \; \}.$$

If H is separable and the commutant of $\rho(\mathcal{A})$ is a properly infinite von Neumann algebra then $\pi = \rho$ is possible. If $\mathcal{A} = M_n$ and ρ is the standard representation of $\mathcal{A}$ on

$L^2\big(M_n, \frac{1}{n}\mathrm{tr}\big)$ then $\pi = \rho$ is possible and the projection $P = \frac{1}{2}(I+T)$ where T is the transposition on M_n can be used.

Proof. If $\mathcal{A}$ has no unit then we add a unit in order to obtain a unital C^*-algebra $\tilde{\mathcal{A}}$. It is well known that the state space of $\mathcal{A}$ embeds isometrically into the state space of $\tilde{\mathcal{A}}$. We can then deduce the result for the non-unital case from the unital one by remarking that both of the expressions

$$|\,(\varphi - \psi)(a)\,| \quad \text{and} \quad \|[P, \pi(a)]\|$$

are left unchanged if a is replaced by $(a - \alpha I)$.

Let us then assume that $\mathcal{A}$ is unital. Then, by Lemma 8.2 we can choose to amplify ρ by the Hilbert dimension of H, but less might do just as well. It all depends on the multiplicity of the representation ρ, or rather whether the commutant of $\rho(\mathcal{A})$ contains a subfactor isomorphic to $B(H)$. In particular, this situation occurs if H is separable and the commutant is properly infinite.

If $\mathcal{A} = M_n$ and ρ is the "left regular representation" of $\mathcal{A}$ on $L^2(M_n, \mathrm{tr})$, then this Hilbert space is naturally identified with $\mathbb{C}^n \otimes \mathbb{C}^n$ via the mapping

$$M_n(\mathbb{C}) \ni a \to \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} e_j \otimes e_i,$$

where the elements e_i denote the elements of the standard basis for $\mathbb{C}^n$. From here it is easy to see that the flip on the Hilbert space tensor product is nothing but the transposition operator on M_n. The Lemma 8.2 now applies directly for $\pi = \rho$ and the projection $P = \frac{1}{2}(I+T)$, where T is the transposition operator on M_n. In the arguments above we have used the trace rather than the trace-state as stated in the formulation of the theorem, the reason being that the identification of M_n with $\mathbb{C}^n \otimes \mathbb{C}^n$ fits naturally with the trace. ∎

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