J. OPERATOR THEORY
56:1(2006), 199–217

ACTIONS AND COACTIONS OF MEASURED GROUPOIDS ON W^*-ALGEBRAS

KARLA J. OTY and ARLAN RAMSAY

Dedicated to the memory of George W. Mackey

Communicated by William B. Arveson

ABSTRACT. This paper answers some questions involved in extending from groups to groupoids the theory of actions and coactions on W^*-algebras. In particular, we explain the connection between actions of a measured groupoid, G, on a bundle of W^*-algebras, and Hopf actions of the Hopf algebroid $L^\infty(G)$ on the direct integral of the bundle of W^*-algebras. The Hopf algebroid structure on $L^\infty(G)$ is determined by G and can be used to construct G up to a set of measure zero.

KEYWORDS: *Groupoids, W^*-algebras, actions, coactions, duality.*

MSC (2000): 46L55, 22A22, 22D35.

INTRODUCTION

In this introduction, we recall some material from [9]. We have chosen to use left Haar measures rather than the right Haar measures of [9]. Because we use measure theoretic tools, we impose a global separability hypothesis: with only a few necessary exceptions, all Hilbert spaces are separable, all topological spaces such as groups are second countable, all W^*-algebras have separable preduals, and all measures are σ-finite. Unless it causes confusion, all identity mappings are denoted by i. Operator algebra and groupoid terms and facts are explained in the next section. The main issue that is new for measured groupoids relative to locally compact groups is the fact that sets of measure 0 in the space of units of a groupoid must be handled.

Let G be a locally compact group and let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Let α be an action of G on M, i.e., a homomorphism from G to $\mathrm{Aut}(M)$ such that $s \mapsto \alpha_s(x)$ is continuous from G to M for every $x \in M$, using the weak operator topology on M. If we think of M as an abstract W^*-algebra, the σ-weak topology can be used. For actions that are unitarily implemented, as ours will be, the fact that several operator topologies agree on the group of unitaries simplifies the situation. The action α gives rise to a von Neumann algebra called the *crossed product of M by G* and denoted by $M \times_\alpha G$.

One construction of $M \times_\alpha G$ begins with a convolution algebra of integrable functions from G to M, and we use that approach later for groupoids, but the first construction in [9] can be described as follows. Let $L^2(G;\mathcal{H})$ be the Hilbert space of square integrable functions from G to $\mathcal{H}$, relative to left Haar measure. For $x \in M$, define a bounded operator, $\pi_\alpha(x)$ on the space $L^2(G;\mathcal{H})$ by $(\pi_\alpha(x)\xi)(s) = \alpha_s(x)(\xi(s))$. Then π_α is a normal, injective, $*$-homomorphism of M into the von Neumann algebra of bounded operators on $L^2(G;\mathcal{H})$, which we denote by $\mathcal{L}(L^2(G;\mathcal{H}))$. There is also a unitary representation L of G on $L^2(G;\mathcal{H})$ defined by $(L(s)\xi)(t) = \xi(s^{-1}t)$. The crossed product $M \times_\alpha G$ is the von Neumann algebra $(\pi_\alpha(M) \cup L(G))''$ in $\mathcal{L}(L^2(G;\mathcal{H}))$. We may also regard $M \times_\alpha G$ as a von Neumann algebra on $\mathcal{H} \otimes L^2(G)$. If the representation L is integrated to produce a representation of the convolution $*$-algebra $L^1(G)$, the possibility of expressing $M \times_\alpha G$ as the completion of a convolution algebra comes into view.

If G is abelian, and $\widehat{G}$ is its dual, then there is a unitary representation V of $\widehat{G}$ on $L^2(G;\mathcal{H})$ defined by $(V(p)\xi)(s) = p(s)\xi(s)$. It is easy to verify that $V(p)\pi_\alpha(x)V(p)^* = \pi_\alpha(x)$ for $x \in M$, and that $V(p)L(s)V(p)^* = p(s)L(s)$ for $s \in G$. Thus conjugating by $V(p)$ gives an action of $\widehat{G}$ on $M \times_\alpha G$. That action of $\widehat{G}$, denoted by $\widehat{\alpha}$ and called a *coaction of* G, is said to be *dual* to α. This situation is completely symmetrical between G and $\widehat{G}$ and their actions and coactions, but only for abelian groups.

In reformulating the theory for nonabelian groups, Nakagami and Takesaki used the fact that if G is a locally compact group, then $L^\infty(G)$ has a Hopf algebra structure ([15], Section 1; [9], Chapter I, Section 1; [2], Section 1.3). The comultiplication on $L^\infty(G)$ can be obtained by first observing that the natural isomorphism of $L^2(G) \otimes L^2(G)$ with $L^2(G \times G)$ carries $L^\infty(G) \otimes L^\infty(G)$ to $L^\infty(G \times G)$. Thus the mapping, α_G, defined by $(\alpha_G f)(s,t) = f(st)$ for $f \in L^\infty(G)$ and $(s,t) \in G \times G$, can be regarded as a mapping of $L^\infty(G)$ into $L^\infty(G) \otimes L^\infty(G)$, and hence is a candidate to be a comultiplication.

Associativity of multiplication in G is equivalent to a property of α_G called *coassociativity*, namely $(\alpha_G \otimes \mathrm{i}) \circ \alpha_G = (\mathrm{i} \otimes \alpha_G) \circ \alpha_G$, which is represented pictorially by the following commutative diagram:

$$\begin{array}{ccc} L^\infty(G) & \xrightarrow{\alpha_G} & L^\infty(G) \otimes L^\infty(G) \\ {\scriptstyle \alpha_G}\Big\downarrow & & {\scriptstyle \alpha_G \otimes \mathrm{i}}\Big\downarrow \\ L^\infty(G) \otimes L^\infty(G) & \xrightarrow{\mathrm{i} \otimes \alpha_G} & L^\infty(G) \otimes L^\infty(G) \otimes L^\infty(G) \end{array} .$$

Thus $L^\infty(G)$ is a Hopf algebra, and the imbedding π_α of M into $L^\infty(G;M)$ can be seen as an action of the Hopf algebra $L^\infty(G)$ by recognizing that $L^\infty(G;M)$ is naturally isomorphic to $M \otimes L^\infty(G)$ and using the fact that α is a homomorphism. Such a mapping π_α can actually be defined for any *measurable* mapping

α of G into $\mathrm{Aut}(M)$. Then the homomorphism property of α is equivalent to the equation $(\alpha_G \otimes \mathrm{i}) \circ \pi_\alpha = (\mathrm{i} \otimes \pi_\alpha) \circ \pi_\alpha$ which defines coassociativity. This equation is illustrated by the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\pi_\alpha} & M \otimes L^\infty(G) \\ {\scriptstyle \pi_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \pi_\alpha \otimes \mathrm{i}} \\ M \otimes L^\infty(G) & \xrightarrow[\mathrm{i}\otimes\alpha_G]{} & M \otimes L^\infty(G) \otimes L^\infty(G) \end{array} .$$

In Hopf algebra terms, π_α and α_G are actions of $L^\infty(G)$ on M and $L^\infty(G)$ respectively.

It is easy to prove that π_α is a Hopf action of $L^\infty(G)$ on M, but to have a one-one correspondence between actions of G on M and actions of $L^\infty(G)$ on M it is necessary to prove that every action of $L^\infty(G)$ on a von Neumann algebra M is of the form π_α for a unique action α, as is done in Proposition 2.1 of Chapter 1 of [9].

1. PRELIMINARIES

In this section we establish notation and state a few known results in forms convenient for this paper.

1.1. GROUPOIDS. Suppose that G is a groupoid with unit space X (or G^0). We denote the range and source maps by $r, s : G \to X$, respectively. Let G^2 denote the collection of composable/multipliable pairs, namely $\{(\gamma, \gamma') \in G \times G : s(\gamma) = r(\gamma')\}$, and write G^x for $r^{-1}(x)$ and G_x for $s^{-1}(x)$, if $x \in X$. Additional information on groupoids can be found in [10], [11], [12] or [13].

A groupoid G is *an analytic (a standard) Borel groupoid* if G has an analytic (a standard) Borel structure, $X = G^0$ is a Borel set when regarded as a subset of G, and r, s, the inverse map, and the multiplication map are Borel functions. For G to be topological, we require that r, s, the inverse map, and multiplication be continuous and that r, s be open. We also require that G be at least locally Hausdorff.

Many groupoids that one encounters are topological, but Borel structure is all that is used in this paper. If G is an analytic or standard Borel groupoid and Y is a Borel set of units we can form the *reduction to* Y, namely $G|Y = \{\gamma \in G : r(\gamma) \in Y, s(\gamma) \in Y\}$. Then $G|Y$ is also a Borel groupoid of the same type as G. For topological groupoids, Y must be chosen more carefully to obtain another topological groupoid; it is always sufficient for Y to be open. In the presence of a measure on X, $G|Y$ is called an *inessential reduction* provided that Y has full measure, i.e., its complement has measure 0.

If G is an analytic Borel groupoid, a *compatible* measure on G is one possessing appropriate (quasi)invariance properties. Since σ-finite measures are always equivalent to finite measures, we describe what is needed first for a finite (Borel) measure ν on G. We say that ν is *quasisymmetric* if it has the same null sets as its image ν^{-1} under $(\cdot)^{-1}$. Translation quasiinvariance for ν must be stated in terms of a decomposition of ν relative to r and $\widetilde{\nu} = r_*(\nu)$. Write

$$\nu = \int \nu^x \mathrm{d}\widetilde{\nu}(x)$$

for such a decomposition. (For a discussion and a proof of existence and uniqueness a.e. of decompositions of measures, see Lemma 4.4 of [1].) We will use the Borel property of the family of measures $\{\nu^x : x \in X\}$, namely, if $f \geqslant 0$ is Borel on G, then $x \mapsto \int f \mathrm{d}\nu^x$ is a Borel function on X. (Sometimes we regard ν^x as a measure on G such that $\nu^x(G \backslash r^{-1}(x)) = 0$ and sometimes (e.g., the next paragraph) as a measure on G^x. There is a natural isomorphism between the two sets of measures.)

For $\gamma \in G$, the mapping $\gamma' \mapsto \gamma\gamma'$ translates $G^{s(\gamma)}$ Borel isomorphically onto $G^{r(\gamma)}$ and carries $\nu^{s(\gamma)}$ to a measure that we denote by $\gamma\nu^{s(\gamma)}$. If there exist a decomposition of ν and a Borel set X_0 of full measure in X such that if $\gamma \in G|X_0$ then $\gamma\nu^{s(\gamma)} \sim \nu^{r(\gamma)}$, we say that ν is *left quasiinvariant.* The measure ν is called *quasiinvariant* if it is both quasisymmetric and left quasiinvariant. A pair (G, ν) is a *measured groupoid* if G is an analytic groupoid and ν is a quasiinvariant Borel measure on G. We allow ν to be σ-finite instead of finite, but can always choose an equivalent finite measure if it is convenient.

Suppose that ν is quasiinvariant on G, and X_0 is as above. Then there is a Borel set of full measure, X_1, contained in X_0 such that $(G|X_1, \nu|(G|X_1))$ is a measured groupoid. To see this, set $N_0 = X \backslash X_0$. For $x \in X_0$, $\nu^x(r^{-1}(N_0)) = 0$ because $r^{-1}(x) \cap r^{-1}(N_0) = \varnothing$. Also, the set $s^{-1}(N_0)$ has measure 0, so it has measure 0 for almost all ν^x. The set, X_1, of x in X_0 for which $\nu^x(s^{-1}(N_0)) = 0$ is invariant for $G|X_0$ because X_0 serves to verify that ν is left quasiinvariant. It is also a Borel set because $x \mapsto \nu^x$ is Borel. Hence, restricting all measures to $G|X_1$ gives a measured groupoid in which $\gamma\nu^{s(\gamma)} \sim \nu^{r(\gamma)}$ holds for every γ.

Since an inessential reduction preserves what is measure theoretically important, we may as well assume that the left quasiinvariance condition holds for all γ.

According to Theorem 3.7 in [4], if (G, ν) is a measured groupoid then G has a *compatible left Haar system*, denoted by $\{\lambda^x : x \in X\}$. This means that $\lambda^x \sim \nu^x$ for every x and $\gamma\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$ for all γ. Note that a Haar system is also required to have the Borel property. The associated right Haar system is denoted by $\{\lambda_x : x \in X\}$, where λ_x is defined to be $(\lambda^x)^{-1}$. On the other hand, given a left Haar system, if μ is any finite or even σ-finite measure on X, we define a measure λ^μ on G by

$$\lambda^\mu = \int_X \lambda^x \,\mathrm{d}\mu(x),$$

giving the decomposition of λ^μ by its definition. Then we say that μ is *quasi-invariant* provided that λ^μ is quasisymmetric, which guarantees that λ^μ itself is quasiinvariant.

In this case, we set $\nu = \lambda^\mu$. According to Theorem 3.8 of [4] the (Borel) Radon-Nikodym derivative of ν relative to ν^{-1}, denoted by Δ, can be taken always positive and automatically satisfies the homomorphism identity a.e. on G^2, as a function taking values in $\mathbb{R}_+^*$. By Theorem 5.1 of [11], we may change Δ on a Borel set of measure 0 and pass to an inessential reduction to arrange that Δ is a homomorphism. This homomorphism is called the *modular function of G relative to* μ, and sometimes denoted by Δ_μ. It appears in the structure of a left Hilbert algebra for G constructed in [5]. This all makes sense only in the presence of a fixed Haar system and uses the measure ν_1 defined next.

For the set G^2 of composable pairs, there is a measure which we will call ν_1 defined by

$$\int_{G^2} f(\gamma_1,\gamma_2)\mathrm{d}\nu_1(\gamma_1,\gamma_2) = \int_X \int_{G_x} \int_{G^{s(\gamma_1)}} f(\gamma_1,\gamma_2)\mathrm{d}\lambda^{s(\gamma_1)}(\gamma_2)\mathrm{d}\lambda_x(\gamma_1)\mathrm{d}\mu(x).$$

We will also need the set $G^2_{r,r} = \{(\gamma_1,\gamma_2) : r(\gamma_1) = r(\gamma_2)\}$. The appropriate measure on this set will be denoted by ν_2 and is defined by

$$\int_{G^2_{r,r}} f(\gamma_1,\gamma_2)\mathrm{d}\nu_2(\gamma_1,\gamma_2) = \int_X \int_{G^x} \int_{G^{r(\gamma_1)}} f(\gamma_1,\gamma_2)\mathrm{d}\lambda^{r(\gamma_1)}(\gamma_2)\mathrm{d}\lambda^x(\gamma_1)\mathrm{d}\mu(x).$$

1.2. W^*-ALGEBRAS AND VON NEUMANN ALGEBRAS. While the title of the paper is stated in terms of W^*-algebras, we work with their faithful representations as von Neumann algebras. By definition, a W^*-algebra is isomorphic to a von Neumann algebra. Later in this section, we recall why that von Neumann algebra can be taken to be in standard form, in an essentially unique way. Thus we are able to establish results for W^*-algebras by working with von Neumann algebras in standard form.

Following [15], page 72, if $\mathcal{H}$ is a Hilbert space and N is a $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ equal to its own second commutant, we call the pair $(N,\mathcal{H})$ a *von Neumann algebra.* If A is a C^*-algebra that possesses a representation π on a Hilbert space $\mathcal{H}$ such that $(\pi(A),\mathcal{H})$ is a von Neumann algebra, we say that A is a W^**-algebra* ([15], page 130). It is a theorem that a C^*-algebra A is a W^*-algebra if and only if A is isometric to the dual space of some Banach space ([15], page 133). There can be at most one such Banach space ([15], page 135). It is called the *predual of A* and is denoted by A_*.

The main reason for requiring every W^*-algebra, N, to have a separable predual is that it is equivalent to requiring N to have a faithful representation on a separable Hilbert space, so that direct integral theory works properly. The faithful representation on a separable Hilbert space comes about as follows. Having a separable predual implies that N is σ-finite and has a faithful normal state, φ

([15], page 78). To say that a state or representation is *normal* means that it is completely additive on orthogonal families of projections, or equivalently, preserves least upper bounds of monotone nets of selfadjoint elements. A state on N is normal if and only if it belongs to N_*. Denote the representation determined by φ by π_φ and the Hilbert space by $\mathcal{H}_\varphi$. Then π_φ is also normal and the pair $(\pi_\varphi(N), \mathcal{H}_\varphi)$ has a modular conjugation J_φ and a self-dual cone $\mathcal{P}_\varphi$ contained in $\mathcal{H}_\varphi$, completing the ingredients of a *standard form* ([16], page 151).

By page 152, Theorem IX.1.14 of [16], if $(M_1, \mathcal{H}_1, J_1, \mathcal{P}_1)$ and $(M_2, \mathcal{H}_2, J_2, \mathcal{P}_2)$ are standard forms and π is an isomorphism of M_1 onto M_2 then there is exactly one unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that for $x \in M_1$ $\pi(x) = UxU^*$, $J_2 = UJ_1U^*$, and $U(\mathcal{P}_1) = \mathcal{P}_2$. This result is fundamental for our purposes. One important fact is that every von Neumann algebra that has a cyclic and separating vector "is" in standard form, i.e., the necessary J and $\mathcal{P}$ exist.

Direct integral decompositions of von Neumann algebras are used in this paper, primarily in order to give explicit constructions of relative tensor products and fiber products. For more information on direct integrals we refer to Section 4 of [1] and to Section IV.8 of [15]. We recall some of the essential structure and facts.

The basic operation is the (direct integral) decomposition of a Hilbert space, $\mathcal{H}$, with respect to a normal representation of an algebra $L^\infty(S, \mu)$, where (S, μ) is a standard (or possibly only analytic) Borel probability space.

The normality requirement makes sense for a representation of $L^\infty(S, \mu)$, because we take $L^\infty(S, \mu)$ to be a von Neumann algebra of multiplication operators on $L^2(S, \mu)$, as follows. For $f \in L^\infty(S, \mu)$ and $\xi \in L^2(S, \mu)$, write $m(f)\xi$ for the equivalence class of the pointwise product of representatives of the two classes of functions. The operator norm of $m(f)$ is equal to $\|f\|_\infty$. It is a standard fact that $m(L^\infty(S, \mu))$ is a von Neumann algebra in $\mathcal{L}(L^2(S, \mu))$, and is in fact equal to its own commutant.

If $\mathcal{H}$ is a Hilbert space equipped with a normal representation of $L^\infty(S, \mu)$, we call $\mathcal{H}$ an *$L^\infty(S, \mu)$-module*, or more briefly an *S-module*. The direct integral decomposition of $\mathcal{H}$ relative to $L^\infty(S, \mu)$ amounts to a structure theorem for S-modules. In other words, there is a standard family of examples of S-modules, and every S-module is isomorphic to exactly one of the examples.

After $L^2(S, \mu)$ the simplest example of an S-module is a vector valued form of $L^2(S, \mu)$. If $\mathcal{K}$ is a Hilbert space, the space $L^2(S, \mu; \mathcal{K})$ of $\mathcal{K}$-valued square integrable Borel functions on S is a Hilbert space, $\mathcal{H}$, on which $L^\infty(S, \mu)$ can act by multiplication as before. This multiplication representation is a faithful normal representation.

The general case is essentially a vector valued L^2-space, but we must allow different Hilbert spaces at different points of S, as it is done for a direct sum. We bypass the general analysis of variable Hilbert spaces in favor of describing examples, one in each isomorphism class.

To produce the examples, begin with $\mathbb{N}^* = \mathbb{N} \cup \{0, \infty\}$, let $\mathcal{S} = \{S_n : n \in \mathbb{N}^*\}$ be a Borel partition of S, and let $\{\mathcal{K}_n : n \in \mathbb{N}^*\}$ be a collection of Hilbert spaces such that $\dim(\mathcal{K}_n) = n$ for all n. Write $\widetilde{\mathcal{H}}$ for $\bigcup_{n \in \mathbb{N}^*} (S_n \times \mathcal{K}_n)$.

In a Borel sense, $\widetilde{\mathcal{H}}$ has a "local" product structure, so we refer to any space constructed in the same way as $\widetilde{\mathcal{H}}$ as a *Hilbert bundle over S*. Starting with another collection $\{\mathcal{K}'_n : n \in \mathbb{N}^*\}$ clearly gives an isomorphic bundle. Also, starting with any Borel partition of S into countably many subsets and any function on the partition assigning a dimension to each element of the partition, it is easy to produce a Hilbert bundle compatible with that information. Hence direct sums of Hilbert bundles and tensor products of Hilbert bundles are again Hilbert bundles.

The function p assigning its first component to each element of $\widetilde{\mathcal{H}}$ is a Borel quotient map. A Borel function $f : S \to \widetilde{\mathcal{H}}$ for which $p \circ f = \mathrm{i}$ is called a *section* of $\widetilde{\mathcal{H}}$ over S.

There is an easy isomorphism between sections of $\widetilde{\mathcal{H}}$ over S and Borel functions f from S to $\bigcup_n \mathcal{K}_n$ such that for every n, $f|S_n$ takes values in $\mathcal{K}_n$. From this it is clear that the space $L^2(S, \mu; \widetilde{\mathcal{H}})$ of square integrable sections of $\widetilde{\mathcal{H}}$ is a Hilbert space, isomorphic to the direct sum of the spaces $L^2(S_n, \mu|S_n; \mathcal{K}_n)$. As before, there is a multiplication representation of $L^\infty(S, \mu)$ on $L^2(S, \mu; \widetilde{\mathcal{H}})$. That representation is faithful if and only if $\mu(S_0) = 0$.

The structure theorem for a general normal homomorphism, φ, of $L^\infty(S, \mu)$ into $\mathcal{L}(\mathcal{H})$ [1], [15] asserts that φ is unitarily equivalent to a multiplication representation on a space $L^2(S, \mu; \widetilde{\mathcal{H}})$ and that $S_0, S_1, S_2, \ldots, S_\infty$ are determined up to measure 0 by φ. Hence we always assume that φ is a multiplication representation on a space $L^2(S, \mu; \widetilde{\mathcal{H}})$.

From a Hilbert bundle $\widetilde{\mathcal{H}}$ two other useful bundles can be constructed. The first is $\bigcup_n (S_n \times \mathcal{L}(\mathcal{K}_n))$, which we denote by $\mathcal{L}(\widetilde{\mathcal{H}})$. The second is $\bigcup_n (S_n \times \mathcal{U}(\mathcal{K}_n) \times S_n)$, which we denote by $\mathcal{U}(\widetilde{\mathcal{H}})$.

Write $L^\infty(S, \mu; \mathcal{L}(\widetilde{\mathcal{H}}))$ for the algebra of bounded Borel sections of $\mathcal{L}(\widetilde{\mathcal{H}})$. If $A \in L^\infty(S, \mu; \mathcal{L}(\widetilde{\mathcal{H}}))$, then "multiplication by" A gives a bounded operator, $m(A)$: if $\xi \in L^2(S, \mu; \widetilde{\mathcal{H}})$ then $m(A)\xi$ is represented by the function in $L^2(S, \mu; \widetilde{\mathcal{H}})$ whose value at each $s \in S$ is $A(s)(\xi(s))$. The set $\{m(A) : A \in L^\infty(S, \mu; \mathcal{L}(\widetilde{\mathcal{H}}))\}$ is the commutant of $L^\infty(S, \mu)$, i.e., of $m(L^\infty(S, \mu))$. An operator that can be obtained in the form $m(A)$ is called *decomposable*. The mapping m is a normal isomorphism of $L^\infty(S, \mu; \mathcal{L}(\widetilde{\mathcal{H}}))$ into the bounded operators on $L^2(S, \mu; \widetilde{\mathcal{H}})$.

Subalgebras of $L^\infty(S, \mu; \mathcal{L}(\widetilde{\mathcal{H}}))$ are of interest to us, and each one is obtained by restricting the operator valued functions to take values in a subbundle of $\mathcal{L}(\widetilde{\mathcal{H}})$. In this setting, a bundle of von Neumann algebras is a Borel subset, $\widetilde{N}$, of some $\mathcal{L}(\widetilde{\mathcal{H}})$ such that the fiber of $\widetilde{N}$ over a point $s \in S_n$ is a von Neumann algebra on $\mathcal{K}_n$. We write $L^\infty(S, \mu; \widetilde{N})$ for the von Neumann algebra of operators, $m(A)$, on $L^2(S, \mu; \widetilde{\mathcal{H}})$ obtained from bounded sections, A, of $\widetilde{N}$.

The set $\mathcal{U}(\widetilde{\mathcal{H}})$ is an analytic Borel space and can be given the structure of a Borel groupoid. The unit space is S and the projections r and s map a triple onto its first and third components. The product is defined by

$$(s_1, U_1, s_2)(s_2, U_2, s_3) = (s_1, U_1 U_2, s_3),$$

while the units are of the form (s, i, s) and the inverse of (s_1, U, s_2) is (s_2, U^{-1}, s_1).

If a von Neumann algebra N on an S-module is contained in the commutant of $m(L^\infty(S,\mu))$, *to decompose N with respect to $\varphi(L^\infty(S,\mu))$* is to express it in the form $L^\infty(S,\mu;\widetilde{N})$. This can be done in essentially one way, so we may pass freely between bundles of von Neumann algebras and von Neumann algebras that commute with $L^\infty(S,\mu)$ ([15], Section V.8).

One result we need about such decompositions concerns quotients of the space S. Suppose that T is an analytic Borel space and that $p : S \to T$ is a quotient map. Decompose μ relative to p: set $\widetilde{\mu} = p_*(\mu)$ and get the decomposition $\mu = \int \mu_t \mathrm{d}\widetilde{\mu}(t)$. Lemma 4.5 of [1] says that the Hilbert spaces $\widetilde{\mathcal{H}}^p_t = L^2(S,\mu_t;\widetilde{\mathcal{H}})$ form a bundle over T as do the algebras $\widetilde{N}^p_t = L^\infty(S,\mu_t;\widetilde{N})$, and that $L^\infty(T,\widetilde{\mu};\widetilde{N}^p)$ is naturally isomorphic to $L^\infty(S,\mu;\widetilde{N})$. These bundles are first found in another form, but are isomorphic to special ones as described above. This result could be abbreviated by saying *direct integrals can be computed in stages*.

We will use specific information about decompositions of von Neumann algebras in standard form. The hypotheses are the von Neumann equivalent of a W^*-algebra having a separable predual and taking the representation obtained from a faithful normal state. The only part of this theorem not already proved in Section IV.8, Theorem 8.21, of [15] is the assertion about cyclic and separating vectors. That fact can be proved easily using the selection lemma techniques in Section IV.8 of [15].

THEOREM 1.1. *Suppose that $(N,\mathcal{H})$ is a von Neumann algebra that has a cyclic and separating vector, ξ_0. Suppose that S is a standard Borel space and μ is a σ-finite Borel measure on S. If $L^\infty(S,\mu)$ has a normal imbedding into the commutant of N, then $(N,\mathcal{H})$ has a direct integral decomposition over S, i.e., there is an isomorphism*

$$(N,\mathcal{H}) \simeq (L^\infty(S,\mu;\widetilde{N}), L^2(S,\mu;\widetilde{\mathcal{H}})).$$

If ξ_0 is represented in $L^2(S,\mu;\widetilde{\mathcal{H}})$ by a section $\widetilde{\xi}_0$ of $\widetilde{\mathcal{H}}$, then for almost every $s \in S$ the vector $\widetilde{\xi}_0(s)$ is cyclic and separating for $\widetilde{N}_s$.

1.3. RELATIVE TENSOR PRODUCTS AND FIBER PRODUCTS. For an action α of a group G on a W^*-algebra M, the corresponding action of the Hopf von Neumann algebra $L^\infty(G)$ is an embedding π_α of M into the von Neumann algebra tensor product $M \otimes L^\infty(G)$ that has the coassociative property. For groupoids a relative tensor product over $L^\infty(X,\mu)$ is the proper analog.

The notion of tensor product relative to $L^\infty(S,\mu)$ was defined by Sauvageot, for example in [14]. The ingredients begin with two S-modules $\mathcal{H}_1$ and $\mathcal{H}_2$ with representations φ_1 and φ_2, respectively. For $i = 1, 2$, let N_i be a von Neumann algebra on $\mathcal{H}_i$ that is contained in the commutant of $\varphi_i(L^\infty(S, \mu))$. Sauvageot shows how to express the relative tensor product $N_1 \otimes_S N_2$ by way of direct integral decompositions, in our context ([14], Section 1.1). To do this, take a decomposition of each N_i in the form $L^\infty(S, \mu; \widetilde{N}_i)$, with $\widetilde{N}_i$ a subbundle of $\mathcal{L}(\widetilde{\mathcal{H}}_i)$, and the fiber of $\widetilde{N}_i$ at the point s denoted by $(\widetilde{N}_i)_s$. Form the bundle $\widetilde{N}_1 \otimes \widetilde{N}_2$, whose fiber at s is $(\widetilde{N}_1)_s \otimes (\widetilde{N}_2)_s$, a von Neumann algebra on $(\widetilde{\mathcal{H}}_1)_s \otimes (\widetilde{\mathcal{H}}_2)_s$. Denote the Hilbert bundle whose fiber is $(\widetilde{\mathcal{H}}_1)_s \otimes (\widetilde{\mathcal{H}}_2)_s$ by $\widetilde{\mathcal{H}}_1 \otimes \widetilde{\mathcal{H}}_2$. Then

$$N_1 \otimes_S N_2 \simeq L^\infty(S, \mu; \widetilde{N}_1 \otimes \widetilde{N}_2),$$

which is a von Neumann algebra on $L^2(S, \mu; \widetilde{\mathcal{H}}_1 \otimes \widetilde{\mathcal{H}}_2)$.

If $\widetilde{M}$ is a bundle of W^*-algebras over X, then let $L^\infty(X, \widetilde{M})$ denote the algebra of bounded, operator valued Borel functions m such that if $x \in X$ then $m(x) \in \widetilde{M}_x$, i.e., the direct integral of the bundle. The technicalities can be treated by passing to von Neumann algebras. Note that $L^\infty(X, \widetilde{M})$ contains a copy of $L^\infty(X, \mu)$ in its center, and every W^*-algebra containing a copy of $L^\infty(X, \mu)$ in its center is of that form. A tensor product over $L^\infty(X, \mu)$ will be written as a tensor product over X. The rationale for this is that $L^\infty(X, \mu)$ is the only von Neumann algebra we associate with X.

The algebra $L^\infty(G)$ is a central module over X in two ways: composing elements of $L^\infty(X, \mu)$ with r and with s both embed $L^\infty(X, \mu)$ into $L^\infty(G)$. We denote these module structures by $L^\infty(G)_r$ and $L^\infty(G)_s$. It will be convenient for us to find other formulations of $M \otimes_X L^\infty(G)_r$ and $M \otimes_X L^\infty(G)_s$. These are special cases of the following lemma.

LEMMA 1.2. *Suppose that Z and W are standard Borel spaces and that $p : W \to Z$ is a surjection such that $B \subseteq Z$ is a Borel set if and only if $p^{-1}(B)$ is Borel in W. Suppose that μ and ν are Borel probability measures on Z and W such that $p_*(\nu)$ has the same null sets as μ. Let $\nu = \int \nu_z \, \mathrm{d}\mu(z)$ be a decomposition of ν over μ with respect to p. Let $\widetilde{\mathcal{H}}$ be a Hilbert bundle over Z and let $\widetilde{M}$ be a subbundle of the bundle whose fiber at z is $\mathcal{L}(\widetilde{\mathcal{H}}_z)$. If $M = L^\infty(Z, \mu; \widetilde{M})$ acts on $\mathcal{H} = L^2(Z, \mu; \widetilde{H})$, then the relative tensor product $M \otimes_Z L^\infty(W, \nu)$ is naturally isomorphic to $L^\infty(W, \nu; p^*(\widetilde{M}))$. In particular, the algebra of sections of $p^*(\widetilde{M})$ that are constant on p-fibers and $L^\infty(W, \nu)$, together generate $L^\infty(W, \nu; p^*(\widetilde{M}))$.*

Proof. It is proved in Section 1.1 of [14] that a relative tensor product of Hilbert spaces, in our context, can be computed by expressing each factor as a direct integral over the underlying space and taking the direct integral of the bundle whose fibers are the tensor products of the fibers for the two factors. In the present case, this means that $\mathcal{H} \otimes_Z L^2(W, \nu)$ is the space of L^2 sections of the bundle whose fiber at z is $\widetilde{\mathcal{H}}_z \otimes L^2(W, \nu_z)$. The latter tensor product is naturally isomorphic to the vector valued L^2-space, $L^2(W, \nu_z; \widetilde{\mathcal{H}}_z)$. That unitary isomorphism carries $\widetilde{M}_z \otimes L^\infty(\nu_z)$ to $L^\infty(W, \nu_z; \widetilde{M}_z)$, the direct integral of a constant bundle.

The direct integral of the vector valued Hilbert spaces is naturally isomorphic to $L^2(W, \nu; p^*(\widetilde{\mathcal{H}}))$. Finally, the unitary operators establishing the Hilbert space isomorphisms provide the desired isomorphisms between algebras of operators. ∎

In addition to relative tensor products, we will use the fiber products of von Neumann algebras over $L^\infty(X, \mu)$. For more information on fiber products see [14], [17]. We include the following definition for completeness.

DEFINITION 1.3. Let M_1 and M_2 be two von Neumann algebras over $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Suppose that M_1 and M_2 both contain copies of $L^\infty(X, \mu)$ as a von Neumann subalgebra. Then the *fiber product* $M_1 * M_2$ of M_1 and M_2 over $L^\infty(X, \mu)$ is defined to be the commutant of $\mathcal{L}_{M_1}(\mathcal{H}_1) \otimes_X \mathcal{L}_{M_2}(\mathcal{H}_2)$ in $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

The latter relative tensor product exists because each of the factor algebras is in the commutant of an image of $L^\infty(X, \mu)$.

1.4. FELL BUNDLES AND ACTIONS. The notion of Fell bundle over a Borel groupoid G in the C^*-algebra case is defined in [6]. An analogous definition for the W^*-algebra case is given in [3]. We use the following definition:

DEFINITION 1.4. A C^* *Fell bundle* over an analytic Borel groupoid G is a Borel bundle E of Banach spaces with projection $p : E \to G$ and associated set

$$E^2 = \{(e_1, e_2) \in E \times E : (p(e_1), p(e_2)) \in G^2\}$$

along with the following conditions:

(i) There is a Borel map from E^2 to E called the *multiplication map* which satisfies:

(a) $p(e_1e_2) = p(e_1)p(e_2)$ for all $(e_1, e_2) \in E^2$;

(b) the induced map, $E_{\gamma_1} \times E_{\gamma_2} \to E_{\gamma_1\gamma_2}$, is bilinear for every pair $(\gamma_1, \gamma_2) \in G^2$;

(c) $(e_1e_2)e_3 = e_1(e_2e_3)$ whenever the multiplication is defined;

(d) $\|e_1e_2\| \leqslant \|e_1\|\|e_2\|$ for all $(e_1, e_2) \in E^2$.

(ii) There is a Borel map from E to E denoted by $e \mapsto e^*$ called the *involution map* which satisfies:

(a) $p(e^*) = p(e)^{-1}$ for all $e \in E$;

(b) the induced map, $E_\gamma \to E_{\gamma^{-1}}$, is conjugate linear for all $\gamma \in G$;

(c) $e^{**} = e$ for all $e \in E$;

(iii) With the above two maps, these additional conditions are satisfied:

(a) $(e_1e_2)^* = e_2^*e_1^*$ for all $(e_1, e_2) \in E^2$;

(b) $\|e^*e\| = \|e\|^2$ for all $e \in E$;

(c) $e^*e \geqslant 0$ for all $e \in E$.

If, in addition, each E_γ has a predual $(E_\gamma)_*$, then we call the bundle a W^* *Fell bundle.*

We define the notion of action of the groupoid G on the bundle $\widetilde{M}$ as in [17], where he also uses the notion of functor, or homomorphism, into the isomorphism groupoid of $\widetilde{M}$.

DEFINITION 1.5. An action α of G on $\widetilde{M}$ is a Borel homomorphism of G into the isomorphism groupoid of the bundle $\widetilde{M}$. That is, α is a Borel map such that:

(i) $\forall \gamma \in G, \alpha_\gamma : \widetilde{M}_{s(\gamma)} \to \widetilde{M}_{r(\gamma)}$ is a $*$-isomorphism;
(ii) $\forall (\gamma_1, \gamma_2) \in G^2, \alpha_{\gamma_1\gamma_2} = \alpha_{\gamma_1} \circ \alpha_{\gamma_2}$;
(iii) $\forall x \in X, \alpha_{i_x} : \widetilde{M}_x \to \widetilde{M}_x$ is the identity map.

We use the following three W^* Fell bundles in this paper. For each of these, it is straightforward to check that the properties of a W^* Fell bundle are satisfied.

(i) Let $\widetilde{M}$ be a decomposition of a W^*-algebra, M, over X into fibers M_x which are W^*-algebras. Then $\{(x, m) : m \in \widetilde{M}_x\}$ is a W^* Fell bundle over the diagonal groupoid on X where p is the projection onto the first coordinate, multiplication is defined by $(x, m_1)(x, m_2) = (x, m_1 m_2)$, and involution is defined by $(x, m)^* = (x, m^*)$.

(ii) With an action α we can view the pull-back bundle over G, defined by $r^*(\widetilde{M}) = \{(\gamma, m) : m \in \widetilde{M}_{r(\gamma)}\}$, as a W^* Fell bundle as follows: The map $p : r^*(\widetilde{M}) \to G$ is the projection onto the first coordinate. Multiplication is defined by

$$(\gamma_1, m_1)(\gamma_2, m_2) = (\gamma_1\gamma_2, m_1\alpha_{\gamma_1}(m_2))$$

and involution is defined by $(\gamma, m)^* = (\gamma^{-1}, \alpha_{\gamma^{-1}}(m^*))$.

(iii) Let $p_1^*(r^*(\widetilde{M})) = \{((\gamma_1, \gamma_2), m) : m \in \widetilde{M}_{r(\gamma_1)}\}$. For this bundle, we define multiplication by

$$((\gamma, \gamma_1), m)((\gamma\gamma_1, \gamma_2), m_1) = ((\gamma, \gamma_1\gamma_2), mm_1)$$

and involution by $((\gamma, \gamma_1), m)^* = ((\gamma\gamma_1, \gamma_1^{-1}), m^*)$.

Instead of always writing elements of a W^* Fell bundle as ordered pairs, we occasionally suppress the first coordinate when the meaning is understood. Likewise, when we talk about sections of a bundle, we will usually write them as elements of $\prod_{s \in S} \widetilde{M}_s$ as long as this causes no confusion.

In [6], Kumjian constructs the associated C^*-algebra for a Fell bundle over an r-discrete groupoid, beginning with a convolution algebra of functions. If $\widetilde{M}$ is a Fell bundle of W^*-algebras over a measured groupoid, we construct a convolution algebra of sections of $\widetilde{M}$ whose completion is the W^*-algebra associated to the bundle. For the W^* Fell bundle constructed from an action, the associated W^*-algebra is the crossed product algebra.

We start by defining a norm for sections of $\widetilde{M}$ that mimics the I norm of [5] for scalar functions. Hahn begins with functions in $L^1(G, \nu)$, so we begin with the analog of that.

DEFINITION 1.6. If f is a Borel section of $r^*(\widetilde{M})$ such that $\gamma \mapsto \|f(\gamma)\|$ is integrable, then let $f^*(\gamma) = \Delta(\gamma)^{-1}\alpha_\gamma(f(\gamma^{-1})^*)$, and define

$$\|f\|_{I,r} = \sup_{x\in X} \int_{G^x} \|f(\gamma)\|_{\widetilde{M}_x} \mathrm{d}\lambda^x(\gamma),$$

and

$$\|f^*\|_{I,r} = \sup_{x\in X} \int_{G^x} \|f^*(\gamma)\|_{\widetilde{M}_x} \mathrm{d}\lambda^x(\gamma).$$

Finally, set

$$\|f\|_I = \max\{\|f\|_{I,r}, \|f^*\|_{I,r}\}.$$

On pages 37–38 of [5], Hahn proved that $\|\cdot\|_I$ for scalar valued functions was indeed a norm satisfying various useful inequalities. We denote the space of scalar function f such that $\|f\|_I < \infty$ by $L^I(G,\lambda,\mu)$. A minor variation of Hahn's proof shows that the above defines a norm for sections as well. We follow the reasoning of Yamanouchi, as summarized below.

From the unitary implementability result for isomorphisms of von Neumann algebras in standard form, there is always an associated Borel homomorphism u of G into the unitary groupoid $\mathcal{U}(\widetilde{\mathcal{H}})$ such that for $\gamma \in G$ and $a \in \widetilde{M}_{s(\gamma)}$, we have $\alpha_\gamma(a) = u_\gamma a u_\gamma^*$. This unitary representation is essential for the construction of the crossed product algebra. In [17], there was a different definition, but Yamanouchi proves in Lemma 4.2 that it gives the same result as the convolution construction we use here. The process takes place in a W^* Fell bundle.

Let $S(r^*(\widetilde{M}))$ denote the set of Borel sections of the bundle $r^*(\widetilde{M})$ that are integrable and finite in the I-norm. Then $S(r^*(\widetilde{M}))$ is a vector space under pointwise addition and scalar multiplication, and $\|\cdot\|_I$ is a norm on $S(r^*(\widetilde{M}))$. The elements of $S(r^*(\widetilde{M}))$ operate on the space $L^2(G,\nu;r^*(\widetilde{H}))$ of L^2 sections for the associated Hilbert bundle $r^*(\widetilde{H})$ by a convolution modified by α, defined as follows. For $\xi,\eta \in L^2(G,\nu;r^*(\widetilde{H}))$ and $f \in S(r^*(\widetilde{M}))$, the integral

$$\int\int \langle f(\gamma_1)u_\gamma(\xi(\gamma_1^{-1}\gamma)), \eta(\gamma)\rangle \mathrm{d}\lambda^{r(\gamma)}(\gamma_1)\mathrm{d}\nu(\gamma)$$

converges absolutely and has absolute value at most $\|f\|_I\|\xi\|_2\|\eta\|_2$, using the results on pages 37–44 of [5]. We write $\varphi_\alpha(f)$ for the operator defined by the Hermitian form whose value is the above integral. Then $\|\varphi_\alpha(f)\| \leqslant \|f\|_I$.

We can further equip $S(r^*(\widetilde{M}))$ with a $*$-algebra structure by defining involution as above and by defining convolution so that

$$\varphi_\alpha(f)\varphi_\alpha(g) = \varphi_\alpha(f * g).$$

Defining $f * g$ by the formula

$$(f * g)(\gamma) = \int f(\gamma_1)\alpha_\gamma(g(\gamma_1^{-1}\gamma))\,\mathrm{d}\lambda^{r(\gamma)}(\gamma_1),$$

it is possible to verify, by substituting into the integral formula, that for $\xi, \eta \in L^2(G, \nu; r^*(\widetilde{H}))$,

$$\langle \varphi_\alpha(f)\varphi_\alpha(g)\xi, \eta\rangle = (\varphi_\alpha(f * g)\xi, \eta),$$

as desired.

THEOREM 1.7. *Defining convolution and involution as above on the subspace $S(r^*(\widetilde{M}))$ makes it a $*$-subalgebra of $\mathcal{L}(L^2(G, \nu; r^*(\widetilde{H})))$.*

Proof. The remainder of the proof is a straightforward modification of arguments found in pages 41–42 of [5], pages 41–42 of [10], or pages 48–49 of [13]. See also pages 36–38 of [17]. ∎

DEFINITION 1.8. The weak closure of the $*$-algebra $S(r^*(\widetilde{M}))$ will be called the *crossed product of $\widetilde{M}$ by G with respect to α* and will be denoted by $\widetilde{M} \times_\alpha G$.

2. COMULTIPLICATION AND COACTIONS

Before defining a comultiplication on $\mathcal{R}(G)$ for groupoids, we discuss the comultiplication on $\mathcal{R}(G)$ for locally compact groups. Nakagami and Takesaki define the comultiplication, δ_G, in terms of the unitary operator W_G defined by $(W_G\xi)(s,t) = \xi(s,ts)$ for $\xi \in L^2(G \times G) \cong L^2(G) \otimes L^2(G)$ and $s, t \in G$. They define δ_G mapping from $\mathcal{R}(G)$ into $\mathcal{L}(L^2(G \times G))$ by

$$\delta_G(x) = W_G^*(x \otimes 1)W_G.$$

We wish to have a formula for δ_G in the groupoid case that does not involve direct reference to a unitary operator and that uses the left regular representation instead of the right regular representation. We will use π to denote both the left regular representation on G and, in its integrated form, the left regular representation on $L^1(G)$. The use should be clear based on the context.

For $f \in L^1(G)$ a calculation shows that the corresponding formula for δ_G using the left regular representation is

$$\delta_G(\pi(f))(h)(s,t) = \int f(r)h(rs, rs)\mathrm{d}r.$$

Using the tensor product $\pi \otimes \pi$, we can write the above integral as

$$\delta_G(\pi(f))(h)(s,t) = \int f(r)(\pi \otimes \pi)(r)h(s,t)\mathrm{d}r.$$

We will denote this integral by $(\pi \otimes \pi)(f)h(s,t)$ which gives $\delta_G(\pi(f)) = (\pi \otimes \pi)(f)$.

We now consider the case where G is a measured groupoid. Recall that the left regular representation for G is defined on the Hilbert Bundle $\widetilde{\mathcal{H}}$ whose fibers are $L^2(\lambda^x)$. For $\gamma \in G$ we define $\pi(\gamma) : L^2(G^{s(\gamma)}) \to L^2(G^{r(\gamma)})$ by

$$\pi(\gamma)(h)(\gamma_1) = h(\gamma^{-1}\gamma_1).$$

Combining Lemma 1.1 from [13] and Theorem 3.4 in [5], we have that the representation π integrates to give a representation π of $L^I(G,\lambda,\mu)$ on $L^2(X,\mu;\widetilde{H})$. For $f \in L^I(G,\lambda,\mu)$, and $\xi \in L^2(X,\widetilde{\mathcal{H}})$, the integrated representation is shown to be

$$\pi(f)\xi(x) = \int_{G^x} f(\gamma)\pi(\gamma)(\xi(s(\gamma)))\Delta^{-1/2}(\gamma)\mathrm{d}\lambda^x(\gamma).$$

DEFINITION 2.1. For $f \in L^I(G)$ and $\xi \in L^2(G^2_{r,r},\nu_2)$ we define δ_G by

$$(\delta_G(\pi(f))\xi)(\gamma_1,\gamma_2) = \int f(\gamma)\xi(\gamma^{-1}\gamma_1,\gamma^{-1}\gamma_2)\Delta^{-1/2}(\gamma)\mathrm{d}\lambda^{r(\gamma_1)}(\gamma).$$

Extend δ_G to all of $\mathcal{R}(G)$ by using the standard continuity arguments.

THEOREM 2.2. *The map δ_G as defined above is a normal $*$-isomorphism of $\mathcal{R}(G)$ into $\mathcal{R}(G) * \mathcal{R}(G)$ satisfying*

$$(\delta_G * \mathrm{i}) \circ \delta_G = (\mathrm{i} * \delta_G) \circ \delta_G\,.$$

We now define a coaction in general and a coaction dual to a given action.

DEFINITION 2.3. Suppose N is a von Neumann algebra and that N contains $L^\infty(X,\mu)$ as a von Neumann subalgebra. A *coaction* of G on N is a normal $*$-isomorphism δ from N into $N * \mathcal{R}(G)$ such that

$$(\delta * \mathrm{i}) \circ \delta = (\mathrm{i} * \delta_G) \circ \delta.$$

Note that δ_G as defined above is a coaction.

DEFINITION 2.4. Given an action α of G on the bundle $\widetilde{M}$ and the associated unitary representation u on $\widetilde{H}$, we define $\widehat{\alpha}$ on $\widetilde{M} \times_\alpha G$ as follows. For $f \in L^I(G)$, let $\widehat{\alpha}(f)$ be the bounded operator determined by the Hermitian form whose value on L^2 sections ξ and η of $\widetilde{H}$ is equal to

$$\int\int \langle \alpha_\gamma(f(\gamma^{-1}))u_\gamma(\xi(\gamma^{-1}\gamma_1,\gamma^{-1}\gamma_2)),\eta(\gamma_1,\gamma_2)\rangle \mathrm{d}\lambda^{r(\gamma_1)}(\gamma)\mathrm{d}\nu_2(\gamma_1,\gamma_2).$$

REMARK 2.5. Although Yamanouchi [17] defines $\widehat{\alpha}$ in terms of unitary maps, his Lemma 4.4 gives an equivalent formulation which is essentially our definition above.

THEOREM 2.6. *$\widehat{\alpha}$ as defined above is a coaction.*

Proof. See pages 42–46 of [17]. ∎

3. THE W^*-ALGEBRA FORMULATION

Suppose M is a W^*-algebra which is a central module over X and let $\widetilde{M}$ be the associated bundle. Let $L^\infty(X,\widetilde{M})$, $L^\infty(G,r^*(\widetilde{M}))$, and $L^\infty(G^2,p_1^*(r^*(\widetilde{M})))$ denote the appropriate algebras of bounded Borel sections. We will denote sections in the three algebras respectively by m, m^1, and m^2, and we will also use the same notation for the spaces of equivalence classes of bounded Borel sections. By Lemma 1.2, $L^\infty(G, r^*(\widetilde{M}))$ is generated by $M \circ r = \{f \circ r : f \in L^\infty(X, \widetilde{M})\}$ as a module over $L^\infty(G)$.

Notice that for groups this is the same situation that Nakagami and Takesaki consider in [9]. If G is a group, then X is a singleton, and $\widetilde{M}$ has only one fiber M, so $L^\infty(X, \widetilde{M}) = M$, $L^\infty(G, r^*(\widetilde{M})) = L^\infty(G, M) \cong M \otimes L^\infty(G)$ and $L^\infty(G^2, p_1^*(r^*(\widetilde{M}))) = L^\infty(G \times G, M) \cong M \otimes L^\infty(G) \otimes L^\infty(G)$. The mappings that appear in the definition of an action of the Hopf algebra $L^\infty(G)$ on a W^*-algebra M become more complicated for a groupoid.

First we define the following maps on sections of various bundles, and then we analyse them to see how much like the group case they are. The maps are the analogs of the mappings $\mathrm{i} \otimes \alpha_G$, π, and $\pi \otimes \mathrm{i}$ used in [9].

DEFINITION 3.1.

(i) Let $\alpha_G^2 : L^\infty(G, r^*(\widetilde{M})) \to L^\infty(G^2, p_1^*(r^*(\widetilde{M})))$ be defined by

$$\alpha_G^2(m^1)(\gamma_1, \gamma_2) = m^1(\gamma_1\gamma_2).$$

(ii) Let $\pi_\alpha^1 : L^\infty(X, \widetilde{M}) \to L^\infty(G, r^*(\widetilde{M}))$ be defined by

$$\pi_\alpha^1(m)(\gamma) = \alpha_\gamma(m(s(\gamma))).$$

(iii) Let $\pi_\alpha^2 : L^\infty(G, r^*(\widetilde{M})) \to L^\infty(G^2, p_1^*(r^*(\widetilde{M})))$ be defined by

$$\pi_\alpha^2(m^1)(\gamma_1, \gamma_2) = \alpha_{\gamma_1}(m^1(\gamma_2)).$$

To verify that α_G^2, π_α^1 and π_α^2 are the proper analogs of $\mathrm{i} \otimes \alpha_G$, π_α and $\pi_\alpha \otimes \mathrm{i}$, we must identify the analogs of their domains in the groupoid setting.

The mapping α_G is defined on $L^\infty(G)$ in both cases, and for $\mathrm{i} \otimes \alpha_G$ the domain is $M \otimes L^\infty(G)$, but $\mathrm{i} \otimes \alpha_G$ does not really depend on the first factor in the tensor product. The analog of $M \otimes L^\infty(G)$ for the groupoid case is $L^\infty(G, r^*(\widetilde{M}))$, which is also a relative tensor product of M and $L^\infty(G)$ over $L^\infty(X, \mu)$. Notice that the definition of α_G^2 likewise does not depend on the particular values of the section it is mapping.

It is clear that π_α^1 is defined in the same way as π_α, including the fact that it takes values in the analog of $M \otimes L^\infty(G)$, so it remains to see that π_α^2 is a good analog of $\pi_\alpha \otimes \mathrm{i}$. The domain of $\pi_\alpha \otimes \mathrm{i}$ is $M \otimes L^\infty(G)$, and its definition ignores the second factor in the tensor product, so it is determined by its values on $M \otimes \mathbb{C}$ together with the fact that it is a normal homomorphism on the tensor product algebra. The analog of $M \otimes \mathbb{C}$ is $M \circ r$ and π_α^2 is a normal homomorphism on $L^\infty(G, r^*(\widetilde{M}))$. To see how π_α^2 is determined by π_α^1 begin with a section $m \in L^\infty(X; \widetilde{M})$ and compute $\pi_\alpha^2(m \circ r)$. Its value at a pair $(\gamma_1, \gamma_2) \in G^2$ is

$$\alpha_{\gamma_1}(m(r(\gamma_1\gamma_2))) = \alpha_{\gamma_1}(m(r(\gamma_1))) = \pi_\alpha^1(m)(p_1(\gamma_1, \gamma_2)).$$

This suggests embedding $L^\infty(G, r^*(\widetilde{M}))$ into $L^\infty(G^2, p_1^*(r^*(\widetilde{M})))$ by composing with p_1, as we embedded M into $L^\infty(G, r^*(\widetilde{M}))$ by composing with r. Both of these correspond to the embeddings into tensor products that appear in the group case. Then we have $\pi_\alpha^2(m \circ r) = \pi_\alpha^1(m) \circ p_1$, so π_α^2 is indeed simply a promotion of π_α^1 to a map on $L^\infty(G, r^*(\widetilde{M}))$.

By the definition of action, we know that $\alpha_{\gamma_1} \circ \alpha_{\gamma_2} = \alpha_{\gamma_1\gamma_2}$. This combined with the fact that $s(\gamma_1\gamma_2) = s(\gamma_2)$ is enough to prove the following theorem.

THEOREM 3.2. *With the above maps, we have the following commutative diagram:*

$$\begin{array}{ccc} L^\infty(X, \widetilde{M}) & \xrightarrow{\pi_\alpha^1} & L^\infty(G, r^*(\widetilde{M})) \\ \pi_\alpha^1 \big\downarrow & & \big\downarrow \pi_\alpha^2 \\ L^\infty(G, r^*(\widetilde{M})) & \xrightarrow{\alpha_G^2} & L^\infty(G^2, p_1^*(r^*(\widetilde{M}))) \end{array} .$$

EXAMPLE 3.3. We now consider one specific map which expresses the multiplication of the groupoid. Define $T_\gamma : L^\infty(G_{s(\gamma)}, \lambda_{s(\gamma)}) \to L^\infty(G_{r(\gamma)}, \lambda_{r(\gamma)})$ by

$$(T_\gamma f)(\gamma_1) = f(\gamma_1\gamma).$$

Then T is an action on the bundle $\widetilde{M}$ over X whose fibers are $L^\infty(G_x, \lambda_x)$ and we can define maps π_G^1 and π_G^2 as above to get

$$\pi_G^2 \circ \pi_G^1 = \alpha_G^2 \circ \pi_G^1.$$

In particular, for $m \in L^\infty(X, \widetilde{M})$ we have

$$(\pi_G^2 \circ \pi_G^1)(m)(\gamma, \gamma_1, \gamma_2) = m((\gamma\gamma_1)\gamma_2)$$

and

$$(\alpha_G^2 \circ \pi_G^1)(m)(\gamma, \gamma_1, \gamma_2) = m(\gamma(\gamma_1\gamma_2))$$

which expresses associativity of multiplication.

We have shown that given a bundle and an action, we can define maps π_α^1 and π_α^2 that satisfy the appropriate commutative diagram, thereby defining an action of the Hopf algebroid $L^\infty(G)$. (The map α_G^2 is defined whether there is an α given or not.) We now prove the converse: Given appropriate maps that satisfy the correct commutative diagram, we have an action of the groupoid. To do this, however, involves some work with null sets and some more observations about actions.

Suppose first that α is an action of G as defined above on $\widetilde{M}$. For an element m^1 of $L^\infty(G, r^*(\widetilde{M}))$ and $\gamma \in G$, we define

$$\tau_\alpha(m^1)(\gamma) = \alpha_\gamma(m^1(\gamma^{-1})).$$

Then τ_α is an automorphism of the von Neumann algebra $L^\infty(G, r^*(\widetilde{M}))$ that has period 2 since $\alpha_\gamma \circ \alpha_{\gamma^{-1}}$ is the identity automorphism of $\widetilde{M}_{r(\gamma)}$. The fact that α is multiplicative appears in the commutative diagram satisfied by the maps α_G^2, π_α^1 and π_α^2.

For groups, it is easy to express all of this in terms of one map π, and Nakagami and Takesaki prove that such a map always is associated with an action. For groupoids, it is easier to include something about inverses in the von Neumann algebra formulation of the action. Note that the inverses are found in τ_α while the multiplicative properties are found in π^1_α.

We now look at the relationship between τ_α and π^1_α. Suppose that we are given τ_α, a section $m \in L^\infty(X, \widetilde{M})$, and $\gamma \in G$. If we define $\phi(m)(\gamma) = m(r(\gamma))$, then this definition is independent of α and we have that

$$\pi^1_\alpha(m)(\gamma) = \alpha_\gamma(m(s(\gamma))) = \tau_\alpha(\phi(m))(\gamma).$$

Next, we show that every triple of mappings τ, π^1, π^2 with the same behavior as τ_α, π^1_α and π^2_α must come from an action α of an inessential reduction of G. Since changing α on a set of measure 0 does not change τ_α, this is the best possible result.

THEOREM 3.4. *Let (G, ν) be a measured groupoid, and let M be a von Neumann algebra in standard form on a Hilbert space $\mathcal{H}$. Suppose that M is a central X-module and let*

$$(M, \mathcal{H}) = (L^\infty(X; \widetilde{M}), L^2(X, \mu; \widetilde{\mathcal{H}}))$$

be a direct integral decomposition of the pair. Suppose that

$$\pi^1 : M \to L^\infty(G, r^*(\widetilde{M}))$$

and

$$\pi^2 : L^\infty(G, r^*(\widetilde{M})) \to L^\infty(G^2, p_1^*(r^*(\widetilde{M})))$$

are normal embeddings such that the following diagram is commutative:

$$\begin{array}{ccc} L^\infty(X, \widetilde{M}) & \xrightarrow{\pi^1} & L^\infty(G, r^*(\widetilde{M})) \\ {\scriptstyle\pi^1}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi^2} \\ L^\infty(G, r^*(\widetilde{M})) & \xrightarrow{\alpha_G} & L^\infty(G^2, p_1^*(r^*(\widetilde{M}))) \end{array}.$$

Suppose also that there is an automorphism τ of period 2 of $L^\infty(G, r^*(\widetilde{M}))$ so that for $m \in M$, $\pi^1(m) = \tau(\phi(m))$, and for $f \in L^\infty(G) \subseteq L^\infty(G, r^*(\widetilde{M}))$, $\tau(f)(\gamma) = f(\gamma^{-1})$. Finally, suppose that for $m \in M$, $\pi^2(m \circ r) = \pi^1(m) \circ p_1$. Then there is an essentially unique action α of an inessential reduction of G on $\widetilde{M}$ such that $\tau = \tau_\alpha$, $\pi^1 = \pi^1_\alpha$, and $\pi^2 = \pi^2_\alpha$.

Proof. First notice that $\tau = \tau_\alpha$ implies that α is determined a.e. so the existence is the only substantial part of the proof. Let $\mathbb{Z}_2$ act on G, on the right, by having the non-identity element take γ to γ^{-1} and let $F = G \times \mathbb{Z}_2$ be the groupoid obtained from that action of $\mathbb{Z}_2$.

The pair $(L^\infty(G; r^*(\widetilde{M})), L^2(G; r^*(\widetilde{\mathcal{H}})))$ is in standard form because $(M, \mathcal{H})$ is in standard form. Hence the automorphism τ is implemented by an essentially unique unitary operator, U, on $L^2(G; r^*(\widetilde{\mathcal{H}}))$. Since $\tau^2 = \mathrm{i}$, $U^2 = \mathrm{i}$. Thus U

generates a unitary representation of the cyclic group of order 2, $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, by $n \mapsto U^n$. The formula for $\tau(f)$ implies that the multiplication representation of $L^\infty(G)$ on $L^2(G; r^*(\widetilde{\mathcal{H}}))$ is a system of imprimitivity for the representation U. By the Imprimitivity Theorem for non-transitive group actions ([11], Theorem 10.10), U is induced by a unitary cocycle on some inessential reduction of F. Since $\mathbb{Z}_2$ is finite, the reduction can be taken to be reduction to an invariant subset G_0 of G. In other words, after another inessential reduction, there is a Borel cocycle $L : G_0 \times \mathbb{Z}_2 \to \mathcal{U}(r^*(\widetilde{H}))$ and a multiplicative Borel cocycle $\rho : G_0 \times \mathbb{Z}_2 \to \mathbb{R}^+$ such that for an element $\xi \in L^2(G; r^*(\widetilde{\mathcal{H}}))$ we have

$$U(\xi)(\gamma) = \rho(\gamma, 1)L(\gamma, 1)(\xi(\gamma^{-1})).$$

In particular, $L(\gamma, 1)$ is unitary from $r^*(\widetilde{\mathcal{H}})_{\gamma^{-1}}$ to $r^*(\widetilde{H})_\gamma$. Since $U^2 = \mathrm{i}$, $L(\gamma^{-1}, 1)$ is almost always the inverse of $L(\gamma, 1)$. Hence we can use L to define α_γ for almost all γ by the formula

$$\alpha_\gamma(a) = L(\gamma, 1)aL(\gamma^{-1}, 1)$$

for $a \in r^*(\widetilde{M})_{\gamma^{-1}}$. If $L(\gamma^{-1}, 1) = L(\gamma, 1)^{-1}$, then α_γ is an isomorphism of $r^*(\widetilde{M})_{\gamma^{-1}}$ onto $r^*(\widetilde{M})_\gamma$ and $\alpha_{\gamma^{-1}}$ is its inverse. Thus α is defined a.e. Since U implements τ and has the given formula in terms of L, it follows that $\tau = \tau_\alpha$. Since $\pi^1 = \tau \circ \phi$ and $\pi^1_\alpha = \tau_\alpha \circ \phi$, it follows that $\pi^1 = \pi^1_\alpha$, and hence that $\pi^2 = \pi^2_\alpha$. Thus the commutative diagram for π^1 and π^2 ensures that $\alpha_{\gamma_1\gamma_2} = \alpha_{\gamma_1} \circ \alpha_{\gamma_2}$ a.e. By Theorem 5.1 of [11], we can modify α on a Borel set of measure 0 and then find an inessential reduction $G|X_0$ so that α is a homomorphism on $G|X_0$. ∎

It is natural to ask whether it is necessary to have the automorphism τ in order to recover α. Notice that the proof given here relies on the unitary implementation of isomorphisms of von Neumann algebras in standard form. Thus any proof that does not use a τ will require a different tool.

REFERENCES

[1] E.G. EFFROS, Global structure in von Neumann algebras, *Trans. Amer. Math. Soc.* **121**(1966), 434–454.

[2] M. ENOCK, J.-M. SCHWARTZ, *Kac Algebras and Duality of Locally Compact Groups*, Springer–Verlag, Heidelberg 1992.

[3] P. GHEZ, R. LIMA, J.E. ROBERTS, W^*-categories, *Pacific J. Math.* **120**(1985), 79–109.

[4] P. HAHN, Haar measure for measure groupoids, *Trans. Amer. Math. Soc.* **242**(1978), 1–33.

[5] P. HAHN, The regular representation of measure groupoids, *Trans. Amer. Math. Soc.* **242**(1978), 35–72.

[6] A. KUMJIAN, Fell bundles over groupoids, *Proc. Amer. Math. Soc.* **126**(1998), 1115–1125.

[7] G.W. MACKEY, *The Theory of Unitary Group Representations*, The University of Chicago Press, Chicago 1976.

[8] J. MRCUN, The Hopf algebroids on étale groupoids and their principal Morita equivalence, *J. Pure Appl. Algebra* **60**(2002), 249–262.

[9] Y. NAKAGAMI, M. TAKESAKI, *Duality for Crossed Products of von Neumann Algebras*, Lecture Notes in Math., vol. 731, Springer-Verlag, New York 1979.

[10] A.L.T. PATERSON, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Birkhauser, Boston 1999.

[11] A. RAMSAY, Virtual groups and group actions, *Adv. Math.* **6**(1971), 253–322.

[12] A. RAMSAY, M. WALTER, Fourier-Stieltjes algebras of locally compact groupoids, *J. Funct. Anal.* **148**(1997), 314–367.

[13] J. RENAULT, *A Groupoid Approach to C^*-Algebras*, Lecture Notes in Math., vol. 793, Springer Verlag, Berlin 1980.

[14] J. SAUVAGEOT, Produits tensoriels de Z-modules et applications, in *Operator Algebras and their Connections with Topology and Ergodic Theory (Buşteni, 1983)*, Lecture Notes in Math., vol. 1132, Springer Verlag, Berlin 1985, pp. 468–485.

[15] M. TAKESAKI, *Theory of Operator Algebras.* I, Springer Verlag, New York 1979.

[16] M. TAKESAKI, *Theory of Operator Algebras.* II, Springer Verlag, New York 2001.

[17] T. YAMANOUCHI, Duality for actions and coactions of measured groupoids on von Neumann algebras, *Mem. Amer. Math. Soc.* **484**(1993).

KARLA J. OTY, DEPARTMENT OF MATHEMATICAL SCIENCES, CAMERON UNIVERSITY, LAWTON, OK 73505, USA
E-mail address: koty@cameron.edu

ARLAN RAMSAY, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, CO 80309, USA
E-mail address: ramsay@euclid.colorado.edu

Received October 10, 2004.