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C^*-ALGEBRAS ASSOCIATED WITH SELF-SIMILAR SETS

TSUYOSHI KAJIWARA and YASUO WATATANI

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ABSTRACT. Let $\gamma = (\gamma_1, \dots, \gamma_N)$, $N \geqslant 2$, be a system of proper contractions on a complete metric space. Then there exists a unique self-similar non-empty compact subset K. We consider the union $\mathcal{G} = \bigcup\limits_{i=1}^{N} \{(x, y) \in K^2; x = \gamma_i(y)\}$ of the cographs of γ_i. Then $X = C(\mathcal{G})$ is a Hilbert bimodule over $A = C(K)$. We associate a C^*-algebra $\mathcal{O}_\gamma(K)$ with them as a Cuntz-Pimsner algebra $\mathcal{O}_X$. We show that if a system of proper contractions satisfies the open set condition in K, then the C^*-algebra $\mathcal{O}_\gamma(K)$ is simple, purely infinite and, in general, not isomorphic to a Cuntz algebra.

KEYWORDS: *Self-similar set, Hilbert bimodule, purely infinite C^*-algebra.*

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1. INTRODUCTION

The study of the self-similar set constructed from iterations of proper contractions has deep interactions with many areas of mathematics. The theory of C^*-algebras seems to be one of them. For example Bratelli-Jorgensen [3] considered relations among representations of the Cuntz algebra [5], wavelet theory and iterated function systems. See also [23]. In this paper we shall give a new construction of a C^*-algebra associated with a system of proper contractions on a self-similar set. The algebra is not a Cuntz algebra in general and its K-theory is closely related with the failure of the injectivity of the coding by the full shift. When the contractions are branches of the inverse of some map h, its K-theory is related to the structure of the branched points (critical points) of h.

Let $\gamma = (\gamma_1, \dots, \gamma_N)$, $N \geqslant 2$, be a system of proper contractions on a complete metric space Ω. Then there exists a unique compact non-empty subset $K \subset \Omega$ satisfying the self-similar condition such that $K = \bigcup\limits_{i} \gamma_i(K)$. In this paper, we often suppress reference to the ambient space Ω and regard each γ_i as a map on K. The subset $\{(x, y) \in K^2; x = \gamma_i(y)\}$ of K^2 is called the *cograph* of γ_i.

Define $\mathcal{G} = \bigcup_{i=1}^{N} \{(x,y) \in K^2; x = \gamma_i(y)\}$ the union of the cographs of γ_i. If the contractions are the continuous branches of the inverse of a certain map $h : K \to K$, then $\mathcal{G}$ is exactly the graph of h. Let $A = C(K)$ be the algebra of continuous functions on the self-similar set K. Define an endomorphism $\beta_i : A \to A$ by $(\beta_i(a))(y) = a(\gamma_i(y))$ for $a \in A$, $y \in K$. Let $C^*(A, \beta_1, \ldots, \beta_N)$ be the universal C^*-algebra generated by A and the Cuntz algebra $\mathcal{O}_N = C^*(S_1, \ldots, S_N)$ with the commutation relations $aS_i = S_i\beta_i(a)$ for $a \in A$ and $i = 1, \ldots, N$. Since each γ_i is a proper contraction, $C^*(A, \beta_1, \ldots, \beta_N)$ turns out to be isomorphic to the Cuntz algebra $\mathcal{O}_N$, as was shown in [28]. The problem with this construction is that we forgot to pay attention to the "branch points" and used the *disjoint union* of the cographs of the γ_i instead of the union of the cographs, $\mathcal{G}$. At the level of bimodules over $A = C(K)$, the disjoint union corresponds to the direct sum $\bigoplus_i {}_{\beta_i}A$, where ${}_{\beta_i}A$ denotes the bimodule over A induced by the endomorphism β_i, i.e., ${}_{\beta_i}A$ is A as a right A-Hilbert module and the left action is implemented by β_i. On the other hand, the bimodule associated with $\mathcal{G}$ is $X := C(\mathcal{G})$, which may be embedded as a submodule of $\bigoplus_i {}_{\beta_i}A$. The Cuntz-Pimsner algebra of X, which we denote $\mathcal{O}_\gamma(K)$, seems to reflect the dynamics of the iterated function system γ better than $C^*(A, \beta_1, \ldots, \beta_N)$. In particular, $\mathcal{O}_\gamma(K)$ is not always isomorphic to $\mathcal{O}_n$. In fact, its K_0-group can have a torsion free element.

In a recent paper [14], we introduced a C^*-algebra $\mathcal{O}_R(J_R)$ associated with a rational function R, viewed as a mapping on its Julia set, J_R. We were inspired by the pioneering work of Deaconu [6] and Deaconu and Muhly [7] who developed a groupoid approach for constructing C^*-algebras from branched coverings. See Renault [29] for groupoid C^*-algebras. If the inverse of R on J_R has continuous branches, $\gamma_1, \ldots, \gamma_N$, on J_R, then J_R may be viewed as the fractal coming from $\gamma_1, \ldots, \gamma_N$, and the algebra that we associate with them here, $\mathcal{O}_\gamma(J_R)$, turns out to be $\mathcal{O}_R(J_R)$, because the graph of R is the union of the cographs of the γ_i. We note that there exists an example of a rational function R whose Julia set J_R is homeomorphic to the Sierpinski gasket [15], [31] .

It is clear that if γ^1 and γ^2 are conjugate systems of contractions on self-similar sets K_1 and K_2, respectively, then the algebras $\mathcal{O}_{\gamma^1}(K_1)$ and $\mathcal{O}_{\gamma^2}(K_2)$ are isomorphic. Also, as we shall show, if the system of proper contractions γ satisfies the so-called open set condition on K, then the C^*-algebra $\mathcal{O}_\gamma(K)$ is simple and purely infinite. Thus, since the algebras are nuclear, separable and satisfy the UCT, when the open set condition is satisfied they are classified by their K-theory.

We have been inspired by the following analogy that derives from the study of simple, purely infinite C^*-algebras. In one direction, there is the crossed product C^*-algebra deriving from the action of a Kleinian group on its limit set. In another, there is the C^*-algebra $\mathcal{O}_R(J_R)$ coming from a rational function R acting on its Julia set J_R. And finally, there is the C^*-algebra $\mathcal{O}_\gamma(K)$ coming from a system γ of proper contractions acting on its self-similar set K. All share a number of features in common and the methods used in their analysis are similar in many respects. We would like to call attention to the works of Anatharaman-Delaroche [1], Laca and Spielberg [22] and Kumjian [20] for constructions of purely infinite, simple C^*-algebras that help to reinforce this analogy.

We also note that the C^*-algebras $\mathcal{O}_\gamma(K)$ are related to a number of other constructs that appear in the literature. First, there are graph C^*-algebras [21]. There are also the algebras associated to topological relations by Brenken [4]. And there are the C^*-algebras of topological graphs studied by Katsura [17], [16] as well as the C^*-algebras of topological quivers studied by Muhly and Solel [24] and by Muhly and Tomforde [25].

When this work was nearly complete, we learned of the preprint [25] of Muhly and Tomforde in which they derive conditions for simplicity of their topological quiver C^*-algebras. Their results include our simplicity conditions for $\mathcal{O}_\gamma(K)$. However, we give a more specialized proof of when $\mathcal{O}_\gamma(K)$ is simple, one which also identifies when $\mathcal{O}_\gamma(K)$ is purely infinite.

We have also learned that Nekrashevych has introduced interesting C^*-algebras associated with so-called graph-directed iterated function systems in a survey paper [2]. If the maps used are proper contractions, then Ionescu [11] has shown that the C^*-algebra is in fact isomorphic to the Cuntz-Krieger associated to the underlying finite graph. This work generalizes [28], where the graph consists of a bouquet of circles.

2. SELF-SIMILAR SETS AND HILBERT BIMODULES

Let (Ω, d) be a (separable) complete metric space Ω with a metric d. A map γ on Ω is called a contraction if its Lipschitz constant $Lip(\gamma) \leqslant 1$, that is,

$$Lip(\gamma) := \sup_{x \neq y} \frac{d(\gamma(x), \gamma(y))}{d(x, y)} \leqslant 1.$$

We say that contractions $\{\gamma_j : j = 1, 2, \ldots, N\}$ on Ω are *proper* if there exist positive constants $\{c_i\}$ and $\{c_i'\}$ with $0 < c_i \leqslant c_i' < 1$ satisfying the condition:

$$c_i d(x, y) \leqslant d(\gamma_i(x), \gamma_i(y)) \leqslant c_i' d(x, y)$$

for all $x, y \in \Omega$, $i = 1, 2, \ldots, N$.

We say that a non-empty compact set $K \subset \Omega$ is *self-similar* (in the weak sense) with respect to a system $\gamma = (\gamma_1, \ldots, \gamma_N)$ if K is the union of the images of the γ_i; that is, in case

$$K = \bigcup_{i=1}^{N} \gamma_i(K).$$

If the contractions are proper, then there is a unique self-similar set $K \subset \Omega$. If K is self-similar with respect to a system $\gamma = (\gamma_1, \ldots, \gamma_N)$, we shall write $K = K(\gamma)$ or $K = K(\gamma_1, \ldots, \gamma_N)$. See [9] and [18] for more on fractal sets.

Fix a natural number $N \geqslant 2$. For each natural number m, we write W_m for $\{1, \dots, N\}^m$ and call elements $w = (w_1, \dots, w_m)$ of W_m *words of length m* with symbols from $\{1, 2, \dots, N\}$. We set $W := \bigcup_{m \geqslant 1} W_m$ and we denote the length of a word w by $\ell(w)$.

The full N-shift space $\{1, 2, \dots, N\}^{\mathbb{N}}$ is the space of one-sided sequences $x = (x_n)_{n \in \mathbb{N}}$ of symbols $\{1, 2, \dots, N\}$. We define a metric d on $\{1, 2, \dots, N\}^{\mathbb{N}}$ by

$$d(x, y) = \sum_n \frac{1}{2^n}(1 - \delta_{x_n, y_n}).$$

Then $\{1, 2, \dots, N\}^{\mathbb{N}}$ is a compact metric space. Define a system $\{\sigma_j : j = 1, 2, \dots, N\}$ of N contractions on $\{1, 2, \dots, N\}^{\mathbb{N}}$ by

$$\sigma_j(x_1, x_2, \dots,) = (j, x_1, x_2, \dots,).$$

Then each σ_j is a proper contraction with the Lipschitz constant $Lip(\sigma_j) = \frac{1}{2}$. The self-similar set $K(\sigma_1, \sigma_2, \dots, \sigma_N) = \{1, 2, \dots, N\}^{\mathbb{N}}$.

Moreover for $w = (w_1, \dots, w_m) \in W_m$, let $\gamma_w = \gamma_{w_1} \circ \dots \circ \gamma_{w_m}$ and $K_w = \gamma_w(K)$. Then for any one-sided sequence $x = (x_n)_{n \in \mathbb{N}} \in \{1, 2, \dots, N\}^{\mathbb{N}}$, $\bigcap_{m \geqslant 1} K_{(x_1, \dots, x_m)}$ contains only one point $\pi(x)$. Therefore we can define a map $\pi : \{1, 2, \dots, N\}^{\mathbb{N}} \to K$ by $\{\pi(x)\} = \bigcap_{m \geqslant 1} K_{(x_1, \dots, x_m)}$. Since $\pi(\{1, 2, \dots, N\}^{\mathbb{N}})$ is also a self-similar set, we have $\pi(\{1, 2, \dots, N\}^{\mathbb{N}}) = K$. Thus π is a continuous onto map satisfying $\pi \circ \sigma_i = \gamma_i \circ \pi$ for $i = 1, \dots, N$. Moreover, for any $y \in K$ and any neighbourhood U_y of y there exists $n \in \mathbb{N}$ and $w \in W_n$ such that

$$y \in \gamma_w(K) \subset U_y.$$

In the note we usually forget an ambient space Ω and start with the following setting: Let (K, d) be a complete metric space and $\gamma = (\gamma_1, \dots, \gamma_N)$ be a system of proper contractions on K. We assume that K is self-similar, i.e. $K = \bigcup_{i=1}^{N} \gamma_i(K)$. We say that a system $\gamma = (\gamma_1, \dots, \gamma_N)$ satisfies the *open set condition in K* if there exists an non-empty open set $V \subset K$ such that

$$\bigcup_{i=1}^{N} \gamma_i(V) \subset V \quad \text{and} \quad \gamma_i(V) \cap \gamma_j(V) = \emptyset \quad \text{for } i \neq j.$$

It is easy to see that V is an open dense set of K. Moreover, for $n \in \mathbb{N}$ and $w, v \in W_n$, if $w \neq v$, then $\gamma_w(V) \cap \gamma_v(V) = \emptyset$.

We recall some basic facts about Cuntz-Pimsner algebras [27]. We follow the notation developed in [12], [13]. Let A be a C^*-algebra and X be a Hilbert right A-module. We denote by $L(X)$ be the algebra of the adjointable bounded operators on X. For $\xi, \eta \in X$, the "rank one" operator $\theta_{\xi,\eta}$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi(\eta|\zeta)$ for $\zeta \in X$. The closure of the linear span of rank one operators is denoted by $K(X)$. We say that X is a Hilbert bimodule over A if X is a Hilbert right A-module with a homomorphism $\phi : A \to L(X)$. We assume that X is full, meaning that the span of the inner products $(\eta|\zeta)$, $\eta, \zeta \in X$, is dense in A, and that ϕ is injective.

Let $F(X) = \bigoplus_{n=0}^{\infty} X^{\otimes n}$ be the full Fock module of X with the convention $X^{\otimes 0} = A$. For $\xi \in X$, the creation operator $T_\xi \in L(F(X))$ is defined by

$$T_\xi(a) = \xi a \quad \text{and} \quad T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

We define $i_{F(X)} : A \to L(F(X))$ by

$$i_{F(X)}(a)(b) = ab \quad \text{and} \quad i_{F(X)}(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \phi(a)\xi_1 \otimes \cdots \otimes \xi_n$$

for $a, b \in A$. The Cuntz-Toeplitz algebra $\mathcal{T}_X$ is the C^*-algebra on $F(X)$ generated by $i_{F(X)}(a)$ with $a \in A$ and T_ξ with $\xi \in X$. Let $j_K : K(X) \to \mathcal{T}_X$ be the homomorphism defined by $j_K(\theta_{\xi,\eta}) = T_\xi T_\eta^*$. We consider the ideal $I_X := \phi^{-1}(K(X))$ of A. Let $\mathcal{J}_X$ be the ideal of $\mathcal{T}_X$ generated by $\{i_{F(X)}(a) - (j_K \circ \phi)(a); a \in I_X\}$. Then the Cuntz-Pimsner algebra $\mathcal{O}_X$ is the the quotient $\mathcal{T}_X / \mathcal{J}_X$. Let $\pi : \mathcal{T}_X \to \mathcal{O}_X$ be the quotient map. Put $S_\xi = \pi(T_\xi)$ and $i(a) = \pi(i_{F(X)}(a))$. Let $i_K : K(X) \to \mathcal{O}_X$ be the homomorphism defined by $i_K(\theta_{\xi,\eta}) = S_\xi S_\eta^*$. Then $\pi((j_K \circ \phi)(a)) = (i_K \circ \phi)(a)$ for $a \in I_X$. We note that the Cuntz-Pimsner algebra $\mathcal{O}_X$ is the universal C^*-algebra generated by $i(a)$ with $a \in A$ and S_ξ with $\xi \in X$ satisfying that $i(a)S_\xi = S_{\phi(a)\xi}$, $S_\xi i(a) = S_{\xi a}$, $S_\xi^* S_\eta = i((\xi|\eta)_A)$ for $a \in A$, $\xi, \eta \in X$ and $i(a) = (i_K \circ \phi)(a)$ for $a \in I_X$. We usually identify $i(a)$ with a in A. We denote by $\mathcal{O}_X^{\mathrm{alg}}$ the $*$-algebra generated algebraically by A and S_ξ with $\xi \in X$. There exists an action $\alpha : \mathbb{R} \to \mathrm{Aut}\,\mathcal{O}_X$ with $\alpha_t(S_\xi) = \mathrm{e}^{it} S_\xi$, which is called the gauge action. Since we assume that $\phi : A \to L(X)$ is isometric, there is an embedding $\phi_n : L(X^{\otimes n}) \to L(X^{\otimes n+1})$ with $\phi_n(T) = T \otimes \mathrm{id}_X$ for $T \in L(X^{\otimes n})$, with the convention $\phi_0 = \phi : A \to L(X)$. We denote by $\mathcal{F}_X$ the C^*-algebra generated by all $K(X^{\otimes n})$, $n \geqslant 0$ in the inductive limit algebra $\varinjlim L(X^{\otimes n})$. Let $\mathcal{F}_n$ be the C^*-subalgebra of $\mathcal{F}_X$ generated by $K(X^{\otimes k})$, $k = 0, 1, \ldots, n$, with the convention $\mathcal{F}_0 = A = K(X^{\otimes 0})$. Then $\mathcal{F}_X = \varinjlim \mathcal{F}_n$.

We shall consider the union

$$\mathcal{G} = \mathcal{G}(\{\gamma_j : j = 1, 2, \ldots, N\}) := \bigcup_{i=1}^{N} \{(x, y) \in K^2; x = \gamma_i(y)\}$$

of the cographs of γ_i. For example, if $\{\gamma_j : j = 1, 2, \ldots, N\}$ are the continuous branches of the inverse of an expansive map $h : K \to K$, then $\mathcal{G}$ is exactly the graph of h. Consider a C^*-algebra $A = C(K)$ and let $X = C(\mathcal{G})$. Then X is an A-A-bimodule by

$$(a \cdot f \cdot b)(x, y) = a(x) f(x, y) b(y)$$

for $a, b \in A$ and $f \in X$. We introduce an A-valued inner product $(\cdot|\cdot)_A$ on X by

$$(f|g)_A(y) = \sum_{i=1}^{N} \overline{f(\gamma_i(y), y)} g(\gamma_i(y), y)$$

for $f, g \in X$ and $y \in K$. It is clear that the A-valued inner product $(\cdot|\cdot)_A$ is well defined, that is, $K \ni y \mapsto (f|g)_A(y) \in \mathbb{C}$ is continuous. Put $\|f\|_2 = \|(f|f)_A\|_\infty^{1/2}$. The left multiplication of A on X gives the left action $\phi : A \to L(X)$ such that $(\phi(a)f)(x,y) = a(x)f(x,y)$ for $a \in A$ and $f \in X$.

For any natural number n, we define $\mathcal{G}_n = \mathcal{G}(\{\gamma_w; w \in W_n\})$ and the Hilbert A-A-bimodule $X_n = C(\mathcal{G}_n)$ similarly. We also need to introduce a modified path space $\mathcal{P}_n$ of length n defined by

$$\mathcal{P}_n = \{(\gamma_{w_1,\dots,w_n}(y), \gamma_{w_2,\dots,w_n}(y), \gamma_{w_3,\dots,w_n}(y), \dots, \gamma_{w_n}(y), y) \in K^{n+1} : \\ w = (w_1, \dots, w_n) \in W_n, y \in K\}.$$

Then similarly $Y_n := C(\mathcal{P}_n)$ is a A-A-bimodule with an A-valued inner product defined by

$$(f|g)_A(y) = \sum_{w \in W_n} \overline{f(\gamma_{w_1,\dots,w_n}(y), \dots, \gamma_{w_n}(y), y)} g(\gamma_{w_1,\dots,w_n}(y), \dots, \gamma_{w_n}(y), y)$$

for $f, g \in Y_n$ and $y \in K$.

If there exists a continuous function $h : K \to K$ such that each contraction γ_i is a continuous branch of the inverse of h, then $\mathcal{P}_n$ can be identified with $\mathcal{G}_n$. Many examples in our paper have such functions h.

PROPOSITION 2.1. *Let $\gamma = (\gamma_1, \dots, \gamma_N)$ be a system of proper contractions on a compact metric space K. Let K be self-similar. Then $X = C(\mathcal{G})$ is a full Hilbert bimodule over $A = C(K)$ without completion. The left action $\phi : A \to L(X)$ is unital and faithful. Similar statements hold for $Y_n = C(\mathcal{P}_n)$.*

Proof. For any $f \in X = C(\mathcal{G})$, we have

$$\|f\|_\infty \leqslant \|f\|_2 = \Big(\sup_y \sum_{i=1}^N |f(\gamma_i(y), y)|^2\Big)^{1/2} \leqslant \sqrt{N}\|f\|_\infty.$$

Therefore two norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent. Since $C(\mathcal{G})$ is complete with respect to $\|\cdot\|_\infty$, it is also complete with respect to $\|\cdot\|_2$.

Since $(1_X|1_X)_A(y) = \sum_{i=1}^N 1_A = N$, $(X|X)_A$ contains the identity I_A of A. Therefore X is full. If $a \in A$ is not zero, then there exists $x_0 \in K$ with $a(x_0) \neq 0$. Since K is self-similar, there exists j and $y_0 \in K$ with $x_0 = \gamma_j(y_0)$. Choose $f \in X$ with $f(x_0, y_0) \neq 0$. Then $\phi(a)f \neq 0$. Thus ϕ is faithful. The statements for Y_n are similarly proved. ∎

DEFINITION 2.2. Let (K, d) be a compact metric space and $\gamma = (\gamma_1, \dots, \gamma_N)$ be a system of proper contractions on K. Assume that K is self-similar. We define the C^*-algebra $\mathcal{O}_\gamma(K)$ to be the Cuntz-Pimsner algebra $\mathcal{O}_X$ built from the Hilbert bimodule $X = C(\mathcal{G})$ over $A = C(K)$.

PROPOSITION 2.3. *Let (K, d) be a compact metric space and $\gamma = (\gamma_1, \dots, \gamma_N)$ be a system of proper contractions on K. Assume that K is self-similar. Then there is a*

Hilbert bimodule isomorphism $\varphi : X^{\otimes n} \to C(\mathcal{P}_n)$ *such that*

$$(\varphi(f_1 \otimes \cdots \otimes f_n))(\gamma_{w_1,\dots,w_n}(y), \gamma_{w_2,\dots,w_n}(y), \gamma_{w_3,\dots,w_n}(y), \dots, \gamma_{w_n}(y), y) \\ = f_1(\gamma_{w_1,\dots,w_n}(y), \gamma_{w_2,\dots,w_n}(y)) f_2(\gamma_{w_2,\dots,w_n}(y), \gamma_{w_3,\dots,w_n}(y)) \cdots f_n(\gamma_{w_n}(y), y)$$

for $f_1, \dots, f_n \in X$, $y \in K$ *and* $w = (w_1, \dots, w_n) \in W_n$. *Moreover, let* $\rho : \mathcal{P}_n \to \mathcal{G}_n$ *be the onto continuous map such that*

$$\rho(\gamma_{w_1,\dots,w_n}(y), \gamma_{w_2,\dots,w_n}(y), \gamma_{w_3,\dots,w_n}(y), \dots, \gamma_{w_n}(y), y) = (\gamma_{w_1,\dots,w_n}(y), y).$$

Then $\rho^* : C(\mathcal{G}_n) \ni f \mapsto f \circ \rho \in C(\mathcal{P}_n)$ *is an isometric Hilbert bimodule embedding of* $C(\mathcal{G}_n)$ *into* $C(\mathcal{P}_n)$.

Proof. It is easy to see that φ is well-defined and a bimodule homomorphism. We show that φ preserves inner products. Consider the case when $n = 2$ for simplicity of notation. Then

$$\begin{aligned}
&(f_1 \otimes f_2 | g_1 \otimes g_2)_A(y) \\
&\quad = (f_2 | (f_1|g_1)_A g_2)_A(y) \\
&\quad = \sum_i \overline{f_2(\gamma_i(y), y)} (f_1|g_1)_A(\gamma_i(y)) g_2(\gamma_i(y), y) \\
&\quad = \sum_i \overline{f_2(\gamma_i(y), y)} \Big(\sum_j \overline{f_1(\gamma_j\gamma_i(y), \gamma_i(y))} g_1(\gamma_j\gamma_i(y), \gamma_i(y)) g_2(\gamma_i(y), y) \Big) \\
&\quad = \sum_{i,j} \overline{f_1(\gamma_j\gamma_i(y), \gamma_i(y)) f_2(\gamma_i(y), y)} g_1(\gamma_j\gamma_i(y), \gamma_i(y)) g_2(\gamma_i(y), y) \\
&\quad = \sum_{i,j} \overline{(\varphi(f_1 \otimes f_2))(\gamma_j\gamma_i(y), \gamma_i(y), y)} (\varphi(g_1 \otimes g_2))(\gamma_j\gamma_i(y), \gamma_i(y), y) \\
&\quad = (\varphi(f_1 \otimes f_2) | \varphi(g_1 \otimes g_2))(y).
\end{aligned}$$

Since φ preserves inner products, φ is one to one. The non-trivial part of the argument is to show that φ is onto. Since $\varphi(1_X \otimes \cdots \otimes 1_X) = 1_X$ and

$$\varphi(f_1 \otimes \cdots \otimes f_n)\varphi(g_1 \otimes \cdots \otimes g_n) = \varphi(f_1 g_1 \otimes \cdots \otimes f_n g_n),$$

the image of φ is a unital $*$-subalgebra of $C(\mathcal{P}_n)$. If

$$(\gamma_{w_1,\dots,w_n}(y), \dots, \gamma_{w_n}(y), y) \neq (\gamma_{u_1,\dots,u_n}(z), \dots, \gamma_{u_n}(z), z)$$

for some $w, u \in W_n$ and $y, z \in K$, then there exists a certain i with $1 \leqslant i \leqslant n$ such that $\gamma_{w_i,\dots,w_n}(y) \neq \gamma_{u_i,\dots,u_n}(z)$, or $y \neq z$. Hence there exists $f_i \in X$ such that

$$f_i(\gamma_{(w_i,\dots,w_n)}(y), \gamma_{(w_{i+1},\dots,w_n)}(y)) \neq f_i(\gamma_{(u_i,\dots,u_n)}(z), \gamma_{(u_{i+1},\dots,u_n)}(z)),$$

where for $i = n$, this means that $f_n(\gamma_{w_n}(y), y) \neq f_n(\gamma_{u_n}(z), z)$. Then

$$\varphi(1_X \otimes \cdots f_i \cdots \otimes 1_X)(\gamma_{w_1,\dots,w_n}(y), \gamma_{w_2,\dots,w_n}(y), \dots, \gamma_{w_n}(y), y) \\ \neq \varphi(1_X \otimes \cdots f_i \cdots \otimes 1_X)(\gamma_{u_1,\dots,u_n}(z), \gamma_{u_2,\dots,u_n}(z), \dots, \gamma_{u_n}(z), z).$$

Thus the image of φ separates the two points. By the Stone-Weierstrass Theorem, the image of φ is dense in $C(\mathcal{P}_n)$ with respect to $\|\cdot\|_\infty$. Since the two norms $\|\cdot\|_2$

and $\|\cdot\|_\infty$ are equivalent and φ is isometric with respect to $\|\cdot\|_2$, φ is onto. The rest is clear. ∎

DEFINITION 2.4. Consider a (branched) covering map $\pi : \mathcal{G} \to K$ defined by $\pi(x,y) = y$ for $(x,y) \in \mathcal{G}$. Define the set

$$B(\gamma_1,\dots,\gamma_N) := \{x \in K; x = \gamma_i(y) = \gamma_j(y) \text{ for some } y \in K \text{ and } i \neq j\}.$$

Then $B := B(\gamma_1,\dots,\gamma_N)$ is a closed set, because

$$B = \bigcup_{i\neq j}\{x \in \gamma_i(K) \cap \gamma_j(K); \gamma_i^{-1}(x) = \gamma_j^{-1}(x)\}.$$

The set B is something like the branch set for a rational function and may be described by the ideal $I_X := \phi^{-1}(K(X))$ of A as in [14]. We define a branch index $e(x,y)$ at $(x,y) \in \mathcal{G}$ by

$$e(x,y) := {}^{\#}\{i \in \{1,\dots,N\}; \gamma_i(y) = x\}.$$

Hence $x \in B(\gamma_1,\dots,\gamma_N)$ if and only if there exists $y \in K$ with $e(x,y) \geqslant 2$. For $x \in K$ we define

$$I(x) := \{i \in \{1,\dots,N\}; \text{ there exists } y \in K \text{ such that } x = \gamma_i(y)\}.$$

LEMMA 2.5. *In the above situation, if $x \in K \setminus B(\gamma_1,\dots,\gamma_N)$, then there exists an open neighbourhood U_x of x satisfying the following:*

(i) $U_x \cap B = \emptyset$;
(ii) *if $i \in I(x)$, then $\gamma_j(\gamma_i^{-1}(U_x)) \cap U_x = \emptyset$ for $j \neq i$;*
(iii) *if $i \notin I(x)$, then $U_x \cap \gamma_i(K) = \emptyset$.*

Proof. Let $x \in K \setminus B$. Since B and $\bigcup_{i\notin I(x)} \gamma_i(K)$ are closed and x is not in either of them, there exists an open neighbourhood W_x of x such that

$$W_x \cap \Big(B \cup \Big(\bigcup_{i\notin I(x)} \gamma_i(K)\Big)\Big) = \emptyset.$$

For $i \in I(x)$ there exists a unique $y_i \in K$ with $x = \gamma_i(y_i)$, since $x \notin B$. For $j \in \{1,\dots,N\}$, if $j \neq i$, then $\gamma_j(y_i) \neq \gamma_i(y_i) = x$. Therefore there exists an open neighbourhood V_x^i of x such that $\gamma_j(\gamma_i^{-1}(V_x^i)) \cap V_x^i = \emptyset$ for $j \neq i$. Put $U_x := W_x \cap \Big(\bigcap_{i\in I(x)} V_x^i\Big)$. Then U_x is an open neighbourhood of x and satisfies all the requirement. ∎

PROPOSITION 2.6. *Let (K,d) be a compact metric space and $\gamma = (\gamma_1,\dots,\gamma_N)$ be a system of proper contractions on K. Assume that K is self-similar and the system $\gamma = (\gamma_1,\dots,\gamma_N)$ satisfies the open set condition in K. Then*

$$I_X = \{a \in A = C(K); a \text{ vanishes on } B(\gamma_1,\dots,\gamma_N)\}.$$

Proof. Let $B = B(\gamma_1, \dots, \gamma_N)$. Firstly, let us take $a \in A$ with a compact support $S = \mathrm{supp}(a) \subset K \setminus B$. For any $x \in S$, choose an open neighbourhood U_x of x as in Lemma 2.5. Since S is compact, there exists a finite subset $\{x_1, \dots, x_m\}$ such that $S \subset \bigcup_{i=1}^{m} U_{x_i} \subset K \setminus B$. By considering a partition of unity for the open covering $K = S^c \bigcup \Big(\bigcup_{i=1}^{m} U_{x_i} \Big)$, we can choose a finite family $(f_i)_i$ in $C(K)$ such that $0 \leqslant f_i \leqslant 1$, $\mathrm{supp}(f_i) \subset U_{x_i}$ for $i = 1, \dots, m$ and $\sum_{i=1}^{m} f_i(x) = 1$ for $x \in S$. Define $\xi_i, \eta_i \in C(\mathcal{G})$ by $\xi_i(x,y) = a(x)\sqrt{f_i(x)}$ and $\eta_i(x,y) = \sqrt{f_i(x)}$. Consider $T := \sum_{i=1}^{k} \theta_{\xi_i,\eta_i} \in K(X)$. We shall show that $T = \phi(a)$. For any $\zeta \in C(\mathcal{G})$, we have $(\phi(a)\zeta)(x,y) = a(x)\zeta(x,y)$ and

$$\begin{aligned}(T\zeta)(x,y) &= \sum_i \xi_i(x,y) \sum_j \overline{\eta_i(\gamma_j(y),y)}\zeta(\gamma_j(y),y) \\ &= \sum_i a(x)\sqrt{f_i(x)} \sum_j \sqrt{f_i(\gamma_j(y))}\zeta(\gamma_j(y),y).\end{aligned}$$

In the case when $a(x) = 0$, we have

$$(T\zeta)(x,y) = 0 = (\phi(a)\zeta)(x,y).$$

In the case when $a(x) \neq 0$, we have $x \in \mathrm{supp}(a) = S \subset \bigcup_{i=1}^{m} U_{x_i}$. Hence $x \in U_{x_i}$ for some i. Take any $y \in K$ with $(x,y) \in \mathcal{G}$. Since $x \notin B$, there exists a unique $k \in \{1, \dots, N\}$ with $x = \gamma_k(y)$. Then for any $j \neq k$ $f_i(\gamma_j(y)) = 0$, because $\gamma_j(y) \in \gamma_j(\gamma_k^{-1}(U_{x_i})) \subset U_{x_i}^c$. Therefore we have

$$\begin{aligned}(T\zeta)(x,y) &= \sum_i a(x)\sqrt{f_i(x)} \Big(\sum_j \sqrt{f_i(\gamma_j(y))}\zeta(\gamma_j(y),y) \Big) \\ &= \sum_i a(x)\sqrt{f_i(x)}\sqrt{f_i(\gamma_k(y))}\zeta(\gamma_k(y),y) \\ &= \sum_i a(x) f_i(x)\zeta(x,y) = a(x)\zeta(x,y) = (\phi(a)\zeta)(x,y).\end{aligned}$$

Thus $\phi(a) = T \in K(X)$. Now for a general $a \in A$ which vanishes on B, there exists a sequence $(a_n)_n$ in A with compact supports $\mathrm{supp}(a_n) \subset K \setminus B$ such that $\|a - a_n\|_\infty \to 0$. Hence $\phi(a) \in K(X)$, i.e., $a \in I_X$.

Conversely let $a \in A$ and $a(c) \neq 0$ for some $c \in B$. We may assume that $a(c) = 1$. Then $c = \gamma_k(d) = \gamma_r(d)$ for some $d \in K$ with $k \neq r \in \{1, \dots, N\}$. Thus the branch index $e(c,d) \geqslant 2$. We need to show that $\phi(a) \notin K(X)$. On the contrary suppose that $\phi(a) \in K(X)$. Then for $\varepsilon = \frac{1}{5\sqrt{N}}$, there exists a finite subset $\{\xi_i, \eta_i \in X; i = 1, \dots, M\}$ such that $\Big\| \phi(a) - \sum_{i=1}^{M} \theta_{\xi_i,\eta_i} \Big\| < \varepsilon$. Since the system satisfies the open set condition in K, there exists an open dense set $V \subset K$ such that $\bigcup_{i=1}^{N} \gamma(V) \subset V$ and $\gamma_i(V) \cap \gamma_j(V) = \emptyset$ for $i \neq j$. Thus $\gamma_i(V)$ is dense in $\gamma_i(K)$ and $\mathcal{G}_V := \bigcup_{i=1}^{N} \{(\gamma_i(y), y) \in \mathcal{G}; y \in V\}$ is dense in $\mathcal{G}$. We claim that for any open neighbourhood $U_{(c,d)}$ of (c,d) in $\mathcal{G}$, there exists $(x,y) \in U_{(c,d)}$ with $e(x,y) = 1$. On the contrary suppose that there were an open neighbourhood $U_{(c,d)}$ of (c,d) in $\mathcal{G}$ such that for any $(x,y) \in U_{(c,d)}$ we have $e(x,y) \geqslant 2$. Then there exists $(x,y) \in \mathcal{G}_V \cap U_{(c,d)}$ with $e(x,y) \geqslant 2$. Thus $y \in V$ and there exist i and j such that $i \neq j$ and $x = \gamma_i(y) = \gamma_j(y)$. Then $x \in \gamma_i(V) \cap \gamma_j(V)$. This is a contradiction and the claim is shown. Therefore there exists a sequence $(x_n, y_n)_n$ in $\mathcal{G}$ such that $e(x_n, y_n) = 1$ and $(x_n, y_n)_n$ converges to (c,d). Since $\mathcal{G}$ is the finite union of $\{(\gamma_i(y), y); y \in K\}$, $i = 1, \ldots, N$, we may assume that there exists a certain i_0 such that $\{(x_n, y_n); n \in \mathbb{N}\} \subset \{(\gamma_{i_0}(y), y); y \in K\}$, by taking a subsequence if necessary. Since $e(x_n, y_n) = 1$, as in the proof of Lemma 2.5, there exists an open neighbourhood $U_n = U_{x_n}$ of x_n such that $\gamma_j(\gamma_{i_0}^{-1}(U_n)) \cap U_n = \emptyset$ for $j \neq i_0$. We choose $\zeta_n \in X$ such that $\operatorname{supp} \zeta_n \subset \{(x,y) \in \mathcal{G}; x \in U_n \text{ and } x = \gamma_{i_0}(y)\}$, $\zeta_n(x_n, y_n) = 1$ and $0 \leqslant \zeta_n \leqslant 1$. Then $\|\zeta_n\|_2 \leqslant \sqrt{N}$. If $j \neq i_0$, then $\gamma_j(y_n) \notin U_n$ and $\zeta_n(\gamma_j(y_n), y_n) = 0$. If $j = i_0$, then $\zeta_n(\gamma_{i_0}(y_n), y_n) = \zeta_n(x_n, y_n) = 1$. Hence

$$
\begin{aligned}
&\Big| a(x_n) - \sum_{i=1}^{M} \xi_i(x_n, y_n) \overline{\eta_i(x_n, y_n)} \Big| \\
&\quad = \Big| a(x_n) - \sum_{i=1}^{M} \xi_i(x_n, y_n) \sum_{j=1}^{N} \overline{\eta_i(\gamma_j(y_n), y_n)} \zeta_n(\gamma_j(y_n), y_n) \Big| \\
&\quad = \Big| \Big(\Big(\phi(a) - \sum_{i=1}^{M} \theta_{\xi_i, \eta_i} \Big) \zeta_n \Big) (x_n, y_n) \Big| \\
&\quad \leqslant \Big\| \Big(\phi(a) - \sum_{i=1}^{M} \theta_{\xi_i, \eta_i} \Big) \zeta_n \Big\|_2 \leqslant \Big\| \phi(a) - \sum_{i=1}^{M} \theta_{\xi_i, \eta_i} \Big\| \|\zeta_n\|_2 \leqslant \varepsilon \sqrt{N}.
\end{aligned}
$$

Since $(x_n, y_n) \to (c,d)$ as $n \to \infty$, we have

$$
\Big| a(c) - \sum_{i=1}^{M} \xi_i(c,d) \overline{\eta_i(c,d)} \Big| \leqslant \varepsilon \sqrt{N}.
$$

On the other hand, consider $\zeta \in X$ satisfying $\zeta(c,d) = 1$, $0 \leqslant \zeta \leqslant 1$ and $\zeta(\gamma_j(d), d) = 0$ for j with $\gamma_j(d) \neq c$. Then

$$
\begin{aligned}
\Big| a(c) - \sum_{i=1}^{M} \xi_i(c,d) e(c,d) \overline{\eta_i(c,d)} \Big| &= \Big| a(c) - \sum_{i=1}^{M} \xi_i(c,d) \sum_{j=1}^{N} \overline{\eta_i(\gamma_j(d), d)} \zeta(\gamma_j(d), d) \Big| \\
&\leqslant \Big\| \Big(\phi(a) - \sum_{i=1}^{M} \theta_{\xi_i, \eta_i} \Big) \zeta \Big\|_2 \leqslant \varepsilon \sqrt{N}.
\end{aligned}
$$

Since $e(c,d) \geqslant 2$ and $a(c) = 1$, we have

$$\begin{aligned}\frac{1}{2} &\leqslant \Big| a(c) - \frac{1}{e(c,d)} a(c) \Big| \\ &\leqslant \Big| a(c) - \sum_{i=1}^{M} \xi_i(c,d)\overline{\eta_i(c,d)} \Big| + \Big| \sum_{i=1}^{M} \xi_i(c,d)\overline{\eta_i(c,d)} - \frac{1}{e(c,d)} a(c) \Big| \\ &\leqslant \varepsilon\sqrt{N} + \frac{1}{e(c,d)}\varepsilon\sqrt{N} \leqslant 2\varepsilon\sqrt{N} = \frac{2}{5}.\end{aligned}$$

This is a contradiction. Therefore $\phi(a) \notin K(X)$. ∎

COROLLARY 2.7. $^{\#}B(\gamma_1, \dots, \gamma_N) = \dim(A/I_X)$.

COROLLARY 2.8. *The closed set $B(\gamma_1, \dots, \gamma_N) = \emptyset$ if and only if $\phi(A)$ is contained in $K(X)$ if and only if X is a finitely generated projective right A module.*

3. SIMPLICITY AND PURE INFINTENESS

Let (K,d) be a compact metric space and $\gamma = (\gamma_1, \dots, \gamma_N)$ be a system of proper contractions on K. Assume that K is self-similar. Let $A = C(K)$ and $X = C(\mathcal{G})$. Define an endomorphism $\beta_i : A \to A$ by

$$(\beta_i(a))(y) = a(\gamma_i(y))$$

for $a \in A$, $y \in K$. We also define a unital completely positive map $E_\gamma : A \to A$ by

$$(E_\gamma(a))(y) := \frac{1}{N} \sum_{i=1}^{N} a(\gamma_i(y))$$

for $a \in A$, $y \in K$, that is, $E_\gamma = \frac{1}{N} \sum\limits_{i=1}^{N} \beta_i$. For the constant function $\xi_0 \in X$ with

$$\xi_0(x,y) := \frac{1}{\sqrt{N}}$$

we have

$$E_\gamma(a) = (\xi_0|\phi(a)\xi_0)_A \quad \text{and} \quad E_\gamma(I) = (\xi_0|\xi_0)_A = I.$$

We introduce an operator $D := S_{\xi_0} \in \mathcal{O}_\gamma(K)$.

LEMMA 3.1. *In the above situation, for $a \in A$, we have the following:*

$$D^* a D = E_\gamma(a) \text{ and in particular } D^*D = I.$$

Proof.

$$D^* a D = S_{\xi_0}^* a S_{\xi_0} = (\xi_0|\phi(a)\xi_0)_A = E_\gamma(a). \quad ∎$$

DEFINITION 3.2. Let (K,d) be a complete metric space and $\gamma = (\gamma_1, \dots, \gamma_N)$ be a system of proper contractions on K. Then $a \in A = C(K)$ is said to be γ-invariant if

$$a(\gamma_i(y)) = a(\gamma_j(y)) \quad \text{for any } y \in K \text{ and } i,j = 1, \dots, N.$$

Suppose that K is a self-similar set and $a \in A = C(K)$ is $(\gamma_w)_{w \in W_n}$-invariant, then a is $(\gamma_w)_{w \in W_{n-1}}$-invariant. In fact, for any $y \in K$ there exists $z \in K$ and i such that $y = \gamma_i(z)$, since K is self-similar. Then for any $w, v \in W_{n-1}$, we have

$$a(\gamma_w(y)) = a(\gamma_{wi}(z)) = a(\gamma_{vi}(z)) = a(\gamma_v(y)).$$

If a is $(\gamma_w)_{w \in W_n}$-invariant, then for any $k = 1, \dots, n$ we may write

$$\beta^k(a)(y) := a(\gamma_{w_1} \cdots \gamma_{w_k}(y)) \quad \text{for any } w \in W_k.$$

Since $\beta^k(a)(y)$ does not depend on the choice of $w \in W_k$, $\beta^k(a)(y)$ is well defined. We may write that $\beta(\beta^{k-1}(a))(y) = \beta^k(a)(y)$.

LEMMA 3.3. *In the same situation, if $a \in A$ is $(\gamma_w)_{w \in W_n}$-invariant, then for any $f_1, \dots, f_n \in X$, we have the following:*

$$aS_{f_1} \cdots S_{f_n} = S_{f_1} \cdots S_{f_n} \beta^n(a).$$

Proof. If $a \in A$ is $(\gamma_w)_{w \in W_n}$-invariant, then $\beta(a)$ is $(\gamma_w)_{w \in W_{n-1}}$-invariant. Therefore it is enough to show that $aS_f = S_f \beta(a)$ for $f \in X$. We have $aS_f = S_{\phi(a)f}$ and $S_f \beta(a) = S_{f\beta(a)}$. Since

$$\begin{aligned} f\beta(a)(\gamma_j(y), y) &= f(\gamma_j(y), y)(\beta(a))(y) = f(\gamma_j(y), y) a(\gamma_i(y)) \\ &= a(\gamma_j(y)) f(\gamma_j(y), y) = (\phi(a)f)(\gamma_j(y), y), \end{aligned}$$

we have $aS_f = S_f \beta(a)$. ∎

LEMMA 3.4. *Let (K,d) be a compact metric space and $\gamma = (\gamma_1, \dots, \gamma_N)$ be a system of proper contractions on K. Assume that K is self-similar. For any non-zero positive element $a \in A$ and for any $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $f \in X^{\otimes n}$ with $(f|f)_A = I$ such that*

$$\|a\| - \varepsilon \leqslant S_f^* a S_f \leqslant \|a\|.$$

Proof. Let x_0 be a point in K with $|a(x_0)| = \|a\|$. For any $\varepsilon > 0$ there exist an open neighbourhood U_0 of x_0 in K such that for any $x \in U_0$ we have $\|a\| - \varepsilon \leqslant a(x) \leqslant \|a\|$. Choose another open neighbourhood U_1 of x_0 in K and a compact subset $K_1 \subset K$ satisfying $U_1 \subset K_1 \subset U_0$. Then there exists $n \in \mathbb{N}$ and $v \in W_n$ such that $\gamma_v(K) \subset U_1$. We identify $X^{\otimes n}$ with $C(\mathcal{P}_n) \supset \rho^*(C(\mathcal{G}_n))$ as in Proposition 2.3. Define closed subsets F_1 and F_2 of $K \times K$ by

$$F_1 = \{(x,y) \in K \times K; x = \gamma_w(y), x \in K_1 \text{ for some } w \in W_n\},$$

$$F_2 = \{(x,y) \in K \times K; x = \gamma_w(y), x \in U_0^c \text{ for some } w \in W_n\}.$$

Since $F_1 \cap F_2 = \emptyset$, there exists $g \in C(\mathcal{G}_n)$ such that $0 \leqslant g(x,y) \leqslant 1$ and

$$g(x,y) = \begin{cases} 1 & (x,y) \in F_1, \\ 0 & (x,y) \in F_2. \end{cases}$$

Since $\gamma_v(K) \subset U_1$, for any $y \in K$ there exists $x_1 \in U_1$ such that $x_1 = \gamma_v(y) \in U_1 \subset K_1$, so that $(x_1, y) \in F_1$. Therefore

$$(g|g)_A(y) = \sum_{w \in W_n} |g(\gamma_w(y), y)|^2 \geqslant |g(x_1, y)|^2 \geqslant 1.$$

Let $b := (g|g)_A$. Then $b(y) = (g|g)_A(y) \geqslant 1$. Thus $b \in A$ is positive and invertible. We put $f := \rho^*(gb^{-1/2}) = \rho^*(g)b^{-1/2} \in X^{\otimes n}$. Then

$$(f|f)_A = (gb^{-1/2}|gb^{-1/2})_A = b^{-1/2}(g|g)_A b^{-1/2} = I.$$

For any $y \in K$ and any $w = (w_1, \dots, w_n) \in W_n$, let $x = \gamma_w(y)$. If $x \in U_0$, then $\|a\| - \varepsilon \leqslant a(x)$, and if $x \in U_0^c$, then

$$f(\gamma_{w_1,\dots,w_n}(y), \dots, \gamma_{w_n}(y), y) = g(x,y)b^{-1/2}(y) = 0,$$

because $(x,y) \in F_2$. Therefore

$$\begin{aligned} \|a\| - \varepsilon &= (\|a\| - \varepsilon)(f|f)_A(y) \\ &= (\|a\| - \varepsilon) \sum_{w \in W_n} |f(\gamma_{w_1,\dots,w_n}(y), \dots, \gamma_{w_n}(y), y)|^2 \\ &\leqslant \sum_{w \in W_n} a(\gamma_w(y))|f(\gamma_{w_1,\dots,w_n}(y), \dots, \gamma_{w_n}(y), y)|^2 \\ &= (f|af)_A(y) = S_f^* a S_f(y). \end{aligned}$$

We also have that

$$S_f^* a S_f = (f|af)_A \leqslant \|a\|(f|f)_A = \|a\|. \quad ∎$$

LEMMA 3.5. *Let (K,d) be a compact metric space and $\gamma = (\gamma_1, \dots, \gamma_N)$ be a system of proper contractions on K. Assume that K is self-similar. For any non-zero positive element $a \in A$ and for any $\varepsilon > 0$ with $0 < \varepsilon < \|a\|$, there exist $n \in \mathbb{N}$ and $u \in X^{\otimes n}$ such that*

$$\|u\|_2 \leqslant (\|a\| - \varepsilon)^{-1/2} \quad \textit{and} \quad S_u^* a S_u = I.$$

Proof. For any $a \in A$ and $\varepsilon > 0$ as above, we choose $f \in X^{\otimes n}$ as in Lemma 3.4. Put $c = S_f^* a S_f$. Since $0 < \|a\| - \varepsilon \leqslant c \leqslant \|a\|$, c is positive and invertible. Let $u := fc^{-1/2}$. Then

$$S_u^* a S_u = (u|au)_A = (fc^{-1/2}|afc^{-1/2})_A = c^{-1/2}(f|af)_A c^{-1/2} = I.$$

Since $\|a\| - \varepsilon \leqslant c$, we have $c^{-1/2} \leqslant (\|a\| - \varepsilon)^{-1/2}$. Hence

$$\|u\|_2 = \|fc^{-1/2}\|_2 \leqslant \|c^{-1/2}\|_2 \leqslant (\|a\| - \varepsilon)^{-1/2}. \quad ∎$$

We need the following easy fact: Let F be a closed subset of a topological space Z. Let $a : F \to \mathbb{C}$ be continuous. If $a(x) = 0$ for x in the boundary of F, then a can be extended to a continuous function on Z by putting $a(x) = 0$ for $x \notin F$.

LEMMA 3.6. *Let (K, d) be a compact metric space and $\gamma = (\gamma_1, \ldots, \gamma_N)$ be a system of proper contractions on K. Assume that K is self-similar and the system $\gamma = (\gamma_1, \ldots, \gamma_N)$ satisfies the open set condition in K. For any $n \in \mathbb{N}$, any $T \in L(X^{\otimes n})$ and any $\varepsilon > 0$, there exists a positive element $a \in A$ such that a is $\{\gamma_w : w \in W_n\}$-invariant,*

$$\|\phi(a)T\|^2 \geqslant \|T\|^2 - \varepsilon$$

and $\beta^p(a)\beta^q(a) = 0$ for $p, q = 1, \ldots, n$ with $p \neq q$.

Proof. For any $n \in \mathbb{N}$, any $T \in L(X^{\otimes n})$ and any $\varepsilon > 0$, there exists $f \in X^{\otimes n}$ such that $\|f\|_2 = 1$ and $\|T\|^2 \geqslant \|Tf\|_2^2 > \|T\|^2 - \varepsilon$. We still identify $X^{\otimes n}$ with $C(\mathcal{P}_n)$. Then there exists $y_0 \in K$ such that

$$\|Tf\|_2^2 = \sum_{w \in W_n} |(Tf)(\gamma_{w_1,\ldots,w_n}(y_0), \ldots, \gamma_{w_n}(y_0), y_0)|^2 > \|T\|^2 - \varepsilon.$$

Since $y \mapsto (Tf|Tf)_A(y)$ is continuous and

$$\|Tf\|_2^2 = \sup_{y \in K} \sum_{w \in W_n} |(Tf)(\gamma_{w_1,\ldots,w_n}(y), \ldots, \gamma_{w_n}(y), y)|^2,$$

there exists an open neighbourhood U_0 of y_0 such that for any $y \in U_0$

$$\sum_{w \in W_n} |(Tf)(\gamma_{w_1,\ldots,w_n}(y), \ldots, \gamma_{w_n}(y), y)|^2 > \|T\|^2 - \varepsilon.$$

Since $\gamma = (\gamma_1, \ldots, \gamma_N)$ satisfies the open set condition in K, there exists an open dense $V \subset K$ such that

$$\bigcup_{i=1}^{N} \gamma_i(V) \subset V \quad \text{and} \quad \gamma_i(V) \cap \gamma_j(V) = \emptyset \quad \text{for } i \neq j.$$

Then there exist $y_1 \in V \cap U_0$ and an open neighbourhood U_1 of y_1 with $U_1 \subset V \cap U_0$. Since the contractions are proper and K is self-similar, there exist $r \in \mathbb{N}$ and $(j_1, \ldots, j_r) \in W_r$ such that

$$\gamma_{j_1}\gamma_{j_2} \cdots \gamma_{j_r}(V) \subset U_1 \subset V \cap U_0.$$

Put $j_{r+1} = 2$ and $j_{r+2} = j_{r+3} = \cdots = j_{r+n} = 1$. Then

$$\emptyset \neq \gamma_{j_1}\gamma_{j_2} \cdots \gamma_{j_{r+n}}(V) \subset \gamma_{j_1}\gamma_{j_2} \cdots \gamma_{j_r}(V) \subset U_1 \subset V \cap U_0.$$

There exist $y_2 \in K$, an open neighbourhood U_2 of y_2 and a compact set L such that

$$y_2 \in U_2 \subset L \subset \gamma_{j_1}\gamma_{j_2} \cdots \gamma_{j_{r+n}}(V) \subset U_1 \subset V \cap U_0.$$

Choose a positive function $b \in A$ such that $0 \leqslant b \leqslant 1$, $b(y_2) = 1$ and $b|_{U_2^c} = 0$. Thus $\{x \in K; b(x) \neq 0\} \subset U_2$. For $w \in W_n$, we have

$$\gamma_w(y_2) \in \gamma_w(U_2) \subset \gamma_w(L) \subset \gamma_w(V).$$

Moreover for $w, v \in W_n$, by the open set condition,

$$\gamma_w(L) \cap \gamma_v(L) = \emptyset \quad \text{if } w \neq v.$$

Now we define a positive function a on K by

$$a(x) = \begin{cases} b(\gamma_w^{-1}(x)) & \text{if } x \in \gamma_w(L),\ w \in W_n; \\ 0, & \text{otherwise}. \end{cases}$$

Since $L' := \bigcup_{w \in W_n} \gamma_w(L)$ is compact, $U' := \bigcup_{w \in W_n} \gamma_w(U_2)$ is open and $\{x \in K; a(x) \neq 0\} \subset U' \subset L'$, a is continuous on L' and $a(x) = 0$ for x in the boundary of L'. Therefore a is continuous on K, i.e. $a \in A = C(K)$. By the construction, a is $(\gamma_w)_{w \in W_n}$-invariant.

For a natural number $p \leqslant n$ and $(i_1, \dots, i_p) \in W_p$, we have

$$\operatorname{supp}(\beta_{i_p}\beta_{i_{p-1}} \cdots \beta_{i_1}(a)) \subset \bigcup_{(i_{p+1},\dots,i_n) \in W_{n-p}} \gamma_{i_{p+1}} \cdots \gamma_{i_n}(\operatorname{supp} b).$$

In fact, if $a(\gamma_{i_1} \cdots \gamma_{i_p}(z)) \neq 0$, then there exists $(i_{p+1}, \dots, i_n) \in W_{n-p}$ and $y \in L$ satisfying $z = \gamma_{i_{p+1}} \cdots \gamma_{i_n}(y)$ by the definition of a. Moreover

$$a(\gamma_{i_1} \cdots \gamma_{i_p}(z)) = b(\gamma^{-1}_{(i_1,\dots,i_n)}(\gamma_{i_1} \cdots \gamma_{i_p})(z)) = b(\gamma^{-1}_{(i_{p+1},\dots,i_n)}(z)) \neq 0.$$

Hence $z \in \gamma_{i_{p+1}} \cdots \gamma_{i_n}(\operatorname{supp} b)$.

Since $\operatorname{supp} b \subset L \subset \gamma_{j_1}\gamma_{j_2} \cdots \gamma_{j_{r+n}}(V)$,

$$\operatorname{supp}(\beta_{i_p}\beta_{i_{p-1}} \cdots \beta_{i_1}(a)) \subset \bigcup_{(i_{p+1},\dots,i_n) \in W_{n-p}} \gamma_{i_{p+1}} \cdots \gamma_{i_n}\gamma_{j_1}\gamma_{j_2} \cdots \gamma_{j_{r+n}}(V).$$

For $1 \leqslant p \lneqq q \leqslant n$,

$$\operatorname{supp}(\beta^p(a)) \subset \bigcup_{(i_{p+1},\dots,i_n) \in W_{n-p}} \gamma_{i_{p+1}} \cdots \gamma_{i_n}\gamma_{j_1}\gamma_{j_2} \cdots \gamma_{j_{r+n}}(V)$$

and

$$\operatorname{supp}(\beta^q(a)) \subset \bigcup_{(i_{q+1},\dots,i_n) \in W_{n-q}} \gamma_{i_{q+1}} \cdots \gamma_{i_n}\gamma_{j_1}\gamma_{j_2} \cdots \gamma_{j_{r+n}}(V).$$

Since the $(n-p)+(r+1)$-th subscripts are different from $j_{r+1} = 2 \neq 1 = j_{r+1+(q-p)}$, we have $\operatorname{supp}(\beta^p(a)) \cap \operatorname{supp}(\beta^q(a)) = \emptyset$. Thus $\beta^p(a)\beta^q(a) = 0$.

Furthermore, we have

$$\begin{aligned}\|\phi(a)Tf\|_2^2 &= \sup_{y\in K}\sum_{w\in W_n}|a(\gamma_w(y))(Tf)(\gamma_{w_1,\dots,w_n}(y),\dots,\gamma_{w_n}(y),y)|^2\\ &= \sup_{y\in L}\sum_{w\in W_n}|b(y)(Tf)(\gamma_{w_1,\dots,w_n}(y),\dots,\gamma_{w_n}(y),y)|^2\\ &\geqslant \sum_{w\in W_n}|(Tf)(\gamma_{w_1,\dots,w_n}(y_2),\dots,\gamma_{w_n}(y_2),y_2)b(y_2)|^2\\ &= \sum_{w\in W_n}|(Tf)(\gamma_{w_1,\dots,w_n}(y_2),\dots,\gamma_{w_n}(y_2),y_2)|^2\\ &> \|T\|^2-\varepsilon\end{aligned}$$

because $y_2 \in L\cap U_2 \subset U_0$. Therefore we have $\|\phi(a)T\|^2 \geqslant \|T\|^2-\varepsilon$. ∎

Let $\mathcal{F}_n$ be the C^*-subalgebra of $\mathcal{F}_X$ generated by $K(X^{\otimes k})$, $k=0,1,\dots,n$ and let B_n be the C^*-subalgebra of $\mathcal{O}_X$ generated by

$$\bigcup_{k=1}^{n}\{S_{x_1}\cdots S_{x_k}S_{y_k}^*\cdots S_{y_1}^* : x_1,\dots,x_k,y_1,\dots,y_k\in X\}\cup A.$$

In the following Lemma, 3.7, we shall use the isomorphism $\varphi:\mathcal{F}_n\to B_n$ defined by the formula

$$\varphi(\theta_{x_1\otimes\cdots\otimes x_k,y_1\otimes\cdots\otimes y_k}) = S_{x_1}\cdots S_{x_k}S_{y_k}^*\cdots S_{y_1}^*.$$

See Pimsner [27] and Fowler-Muhly-Raeburn [10] for information about φ.

To simplify notation, we put $S_x = S_{x_1}\cdots S_{x_k}$ for $x = x_1\otimes\cdots\otimes x_k\in X^{\otimes k}$.

LEMMA 3.7. *In the above situation, let $b=c^*c$ for some $c\in\mathcal{O}_X^{\mathrm{alg}}$. We decompose $b=\sum_j b_j$ with $\alpha_t(b_j)=\mathrm{e}^{\mathrm{i}jt}b_j$. For any $\varepsilon>0$ there exists $P\in A$ with $0\leqslant P\leqslant I$ satisfying the following:*

(i) $Pb_jP=0$, $(j\neq 0)$.

(ii) $\|Pb_0P\|\geqslant\|b_0\|-\varepsilon$.

Proof. For $x\in X^{\otimes n}$, we define $\mathrm{length}(x)=n$ with the convention $\mathrm{length}(a)=0$ for $a\in A$. We write c as a finite sum $c=a+\sum_i S_{x_i}S_{y_i}^*$. Put $n=2\max\{\mathrm{length}(x_i),\mathrm{length}(y_i);i\}$. For $j>0$, each b_j is a finite sum of terms in the form such that

$$S_xS_y^*\quad x\in X^{\otimes(k+j)},\quad y\in X^{\otimes k}\quad 0\leqslant k+j\leqslant n.$$

In the case when $j<0$, b_j is a finite sum of terms in the form such that

$$S_xS_y^*\quad x\in X^{\otimes k},\quad y\in X^{\otimes(k+|j|)}\quad 0\leqslant k+|j|\leqslant n.$$

We shall identify b_0 with an element in $\mathcal{F}_{n/2}\subset\mathcal{F}_n\subset L(X^{\otimes n})$. Apply Lemma 3.6 with $T=(b_0)^{1/2}$. Then there exists a positive element $a\in A$ such that a is

$\{\gamma_w; w \in W_n\}$-invariant, $\|\phi(a)T\|^2 \geqslant \|T\|^2 - \varepsilon$, and $\beta^p(a)\beta^q(a) = 0$ for $p, q = 1, \dots, n$ with $p \neq q$. Define a positive operator $P = a \in A$. Then

$$\|Pb_0P\| = \|Pb_0^{1/2}\|^2 \geqslant \|b_0^{1/2}\|^2 - \varepsilon = \|b_0\| - \varepsilon.$$

For $j > 0$, we have

$$PS_xS_y^*P = aS_xS_y^*a = S_x\beta^{k+j}(a)\beta^k(a)S_y^* = 0.$$

For $j < 0$, we also have that $PS_xS_y^*P = 0$. Hence $Pb_jP = 0$ for $j \neq 0$. ∎

THEOREM 3.8. *Let (K, d) be a compact metric space and $\gamma = (\gamma_1, \dots, \gamma_N)$ be a system of proper contractions on K. Assume that K is self-similar and the system $\gamma = (\gamma_1, \dots, \gamma_N)$ satisfies the open set condition in K. Then the associated C^*-algebra $\mathcal{O}_\gamma(K)$ is simple and purely infinite.*

Proof. Let $w \in \mathcal{O}_X = \mathcal{O}_\gamma(K)$ be any non-zero positive element. We shall show that there exist $z_1, z_2 \in \mathcal{O}_\gamma(K)$ such that $z_1^* w z_2 = I$. We may assume that $\|w\| = 1$. Let $E : \mathcal{O}_\gamma(K) \to \mathcal{O}_\gamma(K)^\alpha$ be the canonical conditional expectation onto the fixed point algebra for the gauge action α. Since E is faithful, $E(w) \neq 0$. Choose ε such that

$$0 < \varepsilon < \frac{\|E(w)\|}{4} \quad \text{and} \quad \varepsilon\|E(w) - 3\varepsilon\|^{-1} \leqslant 1.$$

There exists an element $c \in \mathcal{O}_X^{\mathrm{alg}}$ such that $\|w - c^*c\| < \varepsilon$ and $\|c\| \leqslant 1$. Let $b = c^*c$. Then b is decomposed as a finite sum $b = \sum_j b_j$ with $\alpha_t(b_j) = \mathrm{e}^{\mathrm{i}jt} b_j$. Since $\|b\| \leqslant 1$, $\|b_0\| = \|E(b)\| \leqslant 1$. By Lemma 3.7, there exists $P \in A$ with $0 \leqslant P \leqslant I$ satisfying $Pb_jP = 0$ $(j \neq 0)$ and $\|Pb_0P\| \geqslant \|b_0\| - \varepsilon$. Then we have

$$\begin{aligned}\|Pb_0P\| &\geqslant \|b_0\| - \varepsilon = \|E(b)\| - \varepsilon \\ &\geqslant \|E(w)\| - \|E(w) - E(b)\| - \varepsilon \geqslant \|E(w)\| - 2\varepsilon.\end{aligned}$$

For $T := Pb_0P \in L(X^{\otimes m})$, there exists $f \in X^{\otimes m}$ with $\|f\| = 1$ such that

$$\|T^{1/2}f\|_2^2 = \|(f|Tf)_A\| \geqslant \|T\| - \varepsilon.$$

Hence we have $\|T^{1/2}f\|_2^2 \geqslant \|E(w)\| - 3\varepsilon$. Define $a = S_f^*TS_f = (f|Tf)_A \in A$. Then $\|a\| \geqslant \|E(w)\| - 3\varepsilon > \varepsilon$. By Lemma 3.5, there exists $n \in \mathbb{N}$ and $u \in X^{\otimes n}$ such that

$$\|u\|_2 \leqslant (\|a\| - \varepsilon)^{-1/2} \quad \text{and} \quad S_u^*aS_u = I.$$

Then $\|u\| \leqslant (\|E(w)\| - 3\varepsilon)^{-1/2}$. The rest of the proof is exactly the same as in Theorem 3.8 of [14]. We have

$$\|S_f^*PwPS_f - a\| \leqslant \|S_f\|^2\|P\|^2\|w - b\| < \varepsilon.$$

Therefore

$$\|S_u^*S_f^*PwPS_fS_u - I\| < \|u\|^2\varepsilon \leqslant \varepsilon\|E(w) - 3\varepsilon\|^{-1} \leqslant 1.$$

Hence $S_u^* S_f^* P w P S_f S_u$ is invertible. Thus there exists $v \in \mathcal{O}_X$ with $S_u^* S_f^* P w P S_f S_u v = I$. Put $z_1 = S_u^* S_f^* P$ and $z_2 = P S_f S_u v$. Then $z_1 w z_2 = I$. ∎

REMARK 3.9. J. Schweizer [30] showed that $\mathcal{O}_X$ is simple if the Hilbert bimodule X is minimal and non-periodic. Any X-invariant ideal J of A corresponds to a closed subset F of K with $\sum_i \gamma_i(F) \subset F$. Since such a closed set F is $\emptyset$ or K, X is minimal. Since A is commutative and $L(X_A)$ is non-commutative, X is non-periodic. Thus Schweizer's theorem also implies that $\mathcal{O}_\gamma(K)$ is simple. Our theorem gives simplicity and pure infiniteness with a direct proof.

PROPOSITION 3.10. *Let (K, d) be a compact separable metric space and $\gamma = (\gamma_1, \dots, \gamma_N)$ be a system of proper contractions on K. Assume that K is self-similar. Then the associated C^*-algebra $\mathcal{O}_\gamma(K)$ is separable and nuclear, and satisfies the Universal Coefficient Theorem.*

Proof. Since $\mathcal{J}_X$ and $\mathcal{T}_X$ are KK-equivalent to abelian C^*-algebras I_X and A, the quotient $\mathcal{O}_X \cong \mathcal{T}_X / \mathcal{J}_X$ satisfies the UCT. Also $\mathcal{O}_X$ is shown to be nuclear as in an argument of [8]. ∎

REMARK 3.11. In the above situation the isomorphisms class of $\mathcal{O}_\gamma(K)$ is completely determined by its K-theory together with the class of the unit, by the classification theorem by Kirchberg-Phillips [19], [26].

4. EXAMPLES

We collect some typical examples from a fractal geometry. We also give a general condition under which the associated C^*-algebra $\mathcal{O}_\gamma(K)$ is isomorphic to a Cuntz algebra $\mathcal{O}_N$.

We shall calculate the K-groups by the following six-term exact sequence due to Pimsner [27].

$$\begin{array}{ccccc} K_0(I_X) & \xrightarrow{\mathrm{id}-[X]} & K_0(A) & \xrightarrow{i_*} & K_0(\mathcal{O}_\gamma(K)) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(\mathcal{O}_\gamma(K)) & \xleftarrow[i_*]{} & K_1(A) & \xleftarrow[\mathrm{id}-[X]]{} & K_1(I_X) \end{array} .$$

EXAMPLE 4.1 (Cantor set). Let $\Omega = [0, 1]$ and γ_1 and γ_2 be the two contractions defined by

$$\gamma_1(y) = \frac{1}{3} y \quad \text{and} \quad \gamma_2(y) = \frac{1}{3} y + \frac{2}{3}.$$

Then the self-similar set $K = K(\gamma_1, \gamma_2)$ is the Cantor set and the associated C^*-algebra $\mathcal{O}_{(\gamma_1, \gamma_2)}(K)$ is isomorphic to a Cuntz algebra $\mathcal{O}_2$.

EXAMPLE 4.2 (Full Shift). The full N-shift space $\{1,2,\ldots,N\}^{\mathbb{N}}$ is the space of one-sided sequences $x=(x_n)_{n\in\mathbb{N}}$ of symbols $\{1,2,\ldots,N\}$. Define the system $\sigma=(\sigma_1,\ldots,\sigma_N)$ of N contractions on $\{1,2,\ldots,N\}^{\mathbb{N}}$ by

$$\sigma_j(x_1,x_2,\ldots,)=(j,x_1,x_2,\ldots,).$$

Then each σ_j is a proper contraction with Lipschitz constant $Lip(\sigma_j)=\frac{1}{2}$. The self-similar set $K(\sigma_1,\sigma_2,\ldots,\sigma_N)$ is the full product space $\{1,2,\ldots,N\}^{\mathbb{N}}$. The associated C^*-algebra $\mathcal{O}_\sigma(K)$ is isomorphic to a Cuntz algebra $\mathcal{O}_N$ as in Section 4 of [28].

DEFINITION 4.3. Recall that a system $\gamma=(\gamma_1,\ldots,\gamma_N)$ satisfies the *strong separation condition* in K if

$$K=\bigcup_{i=1}^{N}\gamma(K)\quad\text{and}\quad\gamma_i(K)\cap\gamma_j(K)=\varnothing\quad\text{for } i\neq j.$$

We say that a system $\gamma=(\gamma_1,\ldots,\gamma_N)$ satisfies the *graph separation condition* in K if

$$K=\bigcup_{i=1}^{N}\gamma(K)\quad\text{and}\quad\operatorname{cograph}\gamma_i\cap\operatorname{cograph}\gamma_j=\varnothing\quad\text{for } i\neq j,$$

where cograph $\gamma_i:=\{(x,y)\in K^2;x=\gamma_i(y)\}$. It is clear that:
(strong separation condition) $\Rightarrow$ (graph separation condition) and
(strong separation condition) $\Rightarrow$ (open set condition),
but the converses are not true in general.

If a system $\gamma=(\gamma_1,\ldots,\gamma_N)$ satisfies the strong separation condition in K, then the map $\pi:\{1,2,\ldots,N\}^{\mathbb{N}}\to K$ defined by $\{\pi(x)\}=\bigcap_{m\geqslant 1}K_{(x_1,\ldots,x_m)}$ is a homeomorphism. Since $\pi\circ\sigma_i=\gamma_i\circ\pi$ for $i=1,\ldots,N$, we can identify the system $\gamma=(\gamma_1,\ldots,\gamma_N)$ with the system of system $\{\sigma_j:j=1,2,\ldots,N\}$ in Example 4.2 (Full shift). Therefore it is trivial that the C^*-algebra $\mathcal{O}_\gamma(K)$ is isomorphic to a Cuntz algebra $\mathcal{O}_N$.

PROPOSITION 4.4. *Let (K,d) be a compact metric space and $\gamma=(\gamma_1,\ldots,\gamma_N)$ be a system of proper contractions on K. Assume that K is self-similar. If a system $\gamma=(\gamma_1,\ldots,\gamma_N)$ satisfies the graph separation condition, then the associated C^*-algebra $\mathcal{O}_\gamma(K)$ is isomorphic to a Cuntz algebra $\mathcal{O}_N$.*

Proof. For each $i=1,\ldots,N$, let ${}_{\beta_i}A$ be the bimodule associated with the endomorphism β_i of $A=C(K)$ defined in the introduction. Also, let $\mathcal{G}_i:=$ cograph γ_i. Then $C(\mathcal{G}_i)$ is a Hilbert bimodule over A by

$$(a\cdot f_i\cdot b)(\gamma_i(y),y)=a(\gamma_i(y))f(\gamma_i(y),y)b(y)$$

for $a,b\in A$ and $f_i\in C(\mathcal{G}_i)$. An A-valued inner product $(\cdot|\cdot)_A$ is defined by

$$(f_i|g_i)_A(y)=\overline{f(\gamma_i(y),y)}g(\gamma_i(y),y)$$

for $f_i, g_i \in C(\mathcal{G}_i)$ and $y \in K$. It is clear that there exists an A-A-bimodule isomorphism $\psi : {}_{\beta_i}A \to C(\mathcal{G}_i)$ preserving A-valued inner products subject to the property $\psi(f)(\gamma_i(y), y) = f(y)$ for $f \in {}_{\beta_i}A$ and $y \in K$. Since the system $\{\gamma_j : j = 1, 2, \ldots, N\}$ satisfies the graph separation condition, we have isomorphisms

$$C(\mathcal{G}) \cong \bigoplus_{i=1}^{N} C(\mathcal{G}_i) \cong \bigoplus_{i=1}^{N} {}_{\beta_i}A.$$

Since each γ_i is a proper contraction, the C^*-algebra $\mathcal{O}_\gamma(K)$ is isomorphic to a Cuntz algebra $\mathcal{O}_N$ by Section 4 of [28]. ∎

EXAMPLE 4.5 (Branches of the inverse of a tent map). A tent map $h : [0,1] \to [0,1]$ is defined by

$$h(x) = \begin{cases} 2x & 0 \leqslant x \leqslant \frac{1}{2}, \\ -2x + 2 & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

Let

$$\gamma_1(y) = \frac{1}{2}y \quad \text{and} \quad \gamma_2(y) = -\frac{1}{2}y + 1.$$

Then γ_1 and γ_2 are branches of h^{-1}. The self-similar set $K(\gamma_1, \gamma_2)$ is the interval $[0,1]$. The C^*-algebra $\mathcal{O}_{(\gamma_1,\gamma_2)}(K)$ is isomorphic to the C^*-algebra $\mathcal{O}_{z^2-2}$ associated to the polynomial $z^2 - 2$ [14]. Since the K-groups of $\mathcal{O}_\gamma$ and $\mathcal{O}_{z^2-2}$ are equal and since the position of the unit [1] in each K_0-group is the same, the algebras are isomorphic. Consequently, since $\mathcal{O}_{z^2-2}$ isomorphic to $\mathcal{O}_\infty$, so is $\mathcal{O}_\gamma$. The system (γ_1, γ_2) satisfies the open set condition but does not satisfies the graph separation condition.

We modify the example a bit. Let

$$\gamma_1'(y) = \frac{1}{2}y \quad \text{and} \quad \gamma_2'(y) = \frac{1}{2}y + \frac{1}{2}.$$

Then γ_1' and γ_2' are not branches of the inverse of a certain function, because $\gamma_1'(1) = \gamma_2'(0) = \frac{1}{2}$. The self-similar set $K(\gamma_1', \gamma_2') = [0,1]$. The system $(\gamma_1' \gamma_2')$ satisfies the graph separation condition but does not satisfy the strong separation condition. The C^*-algebra $\mathcal{O}_{(\gamma_1',\gamma_2')}(K)$ is isomorphic to the Cuntz algebra $\mathcal{O}_2$.

EXAMPLE 4.6 (Koch curve). Let $\omega = \frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{6} \in \mathbb{C}$. Consider the two contractions γ_1, γ_2 on the triangle domain $\triangle \subset \mathbb{C}$ with vertices $\{0, \omega, 1\}$ defined by $\gamma_1(z) = \omega\overline{z}$ and $\gamma_2(z) = (1 - \omega)(\overline{z} - 1) + 1$, for $z \in \mathbb{C}$. Then the self-similar set K is called the Koch curve. But these two contractions are not inverse branches of a map on K because $\gamma_1(1) = \gamma_2(0) = \omega$. We modify the construction of the contractions. Put $\widetilde{\gamma}_1 = \gamma_1$, $\widetilde{\gamma}_2 = \gamma_2 \circ \tau$, where τ is the reflection in the line $x = \frac{1}{2}$. Then $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are inverse branches of a map h on K. The C^*-algebra $\mathcal{O}_{(\gamma_1,\gamma_2)}(K)$ is isomorphic to the Cuntz algebra $\mathcal{O}_2$. Moreover, the C^*-algebra $\mathcal{O}_{(\gamma_w)_{w \in W_n}}(K)$ is isomorphic to the Cuntz algebra $\mathcal{O}_{2^n}$, while the C^*-algebra $\mathcal{O}_{(\widetilde{\gamma}_w)_{w \in W_n}}(K)$ is isomorphic to the purely infinite, simple C^*-algebra $\mathcal{O}_{T_{2^n}}([0,1])$, where T_n is the Tchebychev polynomial defined by the equation $\cos nz = T_n(\cos z)$; see Example 4.5 in [14]. Thus we have $K_0(\mathcal{O}_{(\widetilde{\gamma}_w)_{w\in W_n}}(K)) = \mathbb{Z}^{2^n-1}$ and $K_1(\mathcal{O}_{(\widetilde{\gamma}_w)_{w\in W_n}}(K)) = 0$.

EXAMPLE 4.7 (Sierpinski gasket). Recall that the usual Sierpinski gasket K is constructed with the three contractions $\gamma_1, \gamma_2, \gamma_3$ on the regular triangle T in $\mathbb{R}^2$ with three vertices $P = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $Q = (0,0)$ and $R = (1,0)$ such that $\gamma_1(x,y) = (\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{\sqrt{3}}{4})$, $\gamma_2(x,y) = (\frac{x}{2}, \frac{y}{2})$, $\gamma_3(x,y) = (\frac{x}{2} + \frac{1}{2}, \frac{y}{2})$. The self-similar set K is called a Sierpinski gasket. But these three contractions are not inverse branches of a map, because $\gamma_1(Q) = \gamma_2(P)$.

Ushiki [31] discovered a rational function whose Julia set is homeomorphic to the Sierpinski gasket. See also [15]. For example, let $R(z) = \frac{z^3 - \frac{16}{27}}{z}$. Then the Julia set J_R is homeomorphic to the Sierpinski gasket K and J_R contains three critical points. Therefore we need to modify the construction of contractions. Put $\widetilde{\gamma}_1 = \gamma_1$, $\widetilde{\gamma}_2 = \alpha_{-(2\pi/3)} \circ \gamma_2$, and $\widetilde{\gamma}_3 = \alpha_{2\pi/3} \circ \gamma_3$, where α_θ is a rotation by the angle θ. Then $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3$ are inverse branches of a map $h : K \to K$, which is conjugate to $R : J_R \to J_R$. Then C^*-algebra $\mathcal{O}_R \cong \mathcal{O}_{(\widetilde{\gamma}_1,\widetilde{\gamma}_2,\widetilde{\gamma}_3)}(K)$ is a purely infinite, simple C^*-algebra, and $K_0(\mathcal{O}_R)$ contains a torsion free element. But the C^*-algebra $\mathcal{O}_{(\gamma_1,\gamma_2,\gamma_3)}(K)$ is isomorphic to the Cuntz algebra $\mathcal{O}_3$ because the system $(\gamma_1, \gamma_2, \gamma_3)$ satisfies the graph separation condition. Therefore $\mathcal{O}_{(\gamma_1,\gamma_2,\gamma_3)}(K)$ and $\mathcal{O}_{(\widetilde{\gamma}_1,\widetilde{\gamma}_2,\widetilde{\gamma}_3)}(K)$ are not isomorphic. See [14].

EXAMPLE 4.8 (Sierpinski carpet). Recall that the usual Sierpinski carpet K is constructed using eight contractions $\gamma_1, \dots, \gamma_8$ on the regular square $S = [0,1] \times [0,1]$ in $\mathbb{R}^2$ with four vertices $P_1 = (0,1)$, $P_2 = (0,0)$, $P_3 = (1,0)$ and $P_4 = (1,1)$ such that $\gamma_1(x,y) = (\frac{x}{3}, \frac{y}{3})$, $\gamma_2(x,y) = (\frac{x}{3} + \frac{1}{3}, \frac{y}{3})$, $\gamma_3(x,y) = (\frac{x}{3} + \frac{2}{3}, \frac{y}{3})$, $\gamma_4(x,y) = (\frac{x}{3}, \frac{y}{3} + \frac{1}{3})$, $\gamma_5(x,y) = (\frac{x}{3} + \frac{2}{3}, \frac{y}{3} + \frac{1}{3})$, $\gamma_6(x,y) = (\frac{x}{3}, \frac{y}{3} + \frac{2}{3})$, $\gamma_7(x,y) = (\frac{x}{3} + \frac{1}{3}, \frac{y}{3} + \frac{2}{3})$, $\gamma_8(x,y) = (\frac{x}{3} + \frac{2}{3}, \frac{y}{3} + \frac{2}{3})$. Then the self-similar set K is called a Sierpinski carpet. But these eight contractions are not continuous branches of the inverse of any map $h : K \to K$, because $\gamma_1(P_1) = \gamma_4(P_2)$. We shall modify the construction of the contractions as follows: $\gamma_1'(x,y) = \gamma_1(x,y)$, $\gamma_2'(x,y) = (-\frac{x}{3} + \frac{2}{3}, \frac{y}{3})$, $\gamma_3'(x,y) = \gamma_3(x,y)$, $\gamma_4'(x,y) = (\frac{x}{3}, -\frac{y}{3} + \frac{2}{3})$, $\gamma_5'(x,y) = (\frac{x}{3} + \frac{2}{3}, -\frac{y}{3} + \frac{2}{3})$, $\gamma_6'(x,y) = \gamma_6(x,y)$, $\gamma_7'(x,y) = (-\frac{x}{3} + \frac{2}{3}, \frac{y}{3} + \frac{2}{3})$, $\gamma_8'(x,y) = \gamma_8(x,y)$. Then their self-similar set is the same Sierpinski carpet K as above and $\gamma_1', \dots, \gamma_8'$ are continuous branches of the inverse of a map $h : K \to K$. Since

$$B = B(\gamma_1', \dots, \gamma_8') = (([0, \frac{1}{3}] \cup [\frac{2}{3}, 1]) \times \{\frac{1}{3}, \frac{2}{3}\}) \cup (\{\frac{1}{3}, \frac{2}{3}\} \times ([0, \frac{1}{3}] \cup [\frac{2}{3}, 1])),$$

$K_0(C(B)) \cong \mathbb{Z}^4$ and $K_1(C(B)) \cong 0$. Since we have $K_0(C(K)) \cong \mathbb{Z}$ and $K_1(C(K)) \cong \mathbb{Z}^\infty$, $K_0(\mathcal{O}_{(\gamma_1',\dots,\gamma_8')}(K))$ contains a torsion free element. However, we observe that the C^*-algebra $\mathcal{O}_{(\gamma_1,\dots,\gamma_8)}(K)$ is isomorphic to the Cuntz algebra $\mathcal{O}_8$ because the system $(\gamma_1, \dots, \gamma_8)$ satisfies the graph separation condition. Therefore, the purely infinite, simple C^*-algebras $\mathcal{O}_{(\gamma_1,\dots,\gamma_8)}(K)$ and $\mathcal{O}_{(\gamma_1',\dots,\gamma_8')}(K)$ are not isomorphic.

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TSUYOSHI KAJIWARA, DEPARTMENT OF ENVIRONMENTAL AND MATHEMATICAL SCIENCES, OKAYAMA UNIVERSITY, TSUSHIMA, 700-8530, JAPAN
E-mail address: kajiwara@ems.okayama-u.ac.jp

YASUO WATATANI, DEPARTMENT OF MATHEMATICAL SCIENCES, KYUSHU UNIVERSITY, HAKOZAKI, FUKUOKA, 812-8581, JAPAN
E-mail address: watatani@math.kyushu-u.ac.jp

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