J. OPERATOR THEORY
56:2(2006), 423–449

COMPOSITION OPERATORS ON EMBEDDED DISKS

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Communicated by Kenneth R. Davidson

ABSTRACT. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ and let Ω be a domain in $\mathbb{C}^n$. Every holomorphic map $\varphi : \mathbb{D} \to \Omega$ induces a composition operator $C_\varphi : H(\Omega) \to H(\mathbb{D})$, where $H(\Omega)$ and $H(\mathbb{D})$ are the spaces of holomorphic functions in Ω and $\mathbb{D}$, respectively. We study the action of C_φ on the Hardy spaces $H^p(\Omega)$ and the weighted Bergman spaces $A^p_\alpha(\Omega)$ when Ω is the unit ball or the polydisc. More specifically, we determine the optimal range spaces, prove the boundedness of C_φ, and characterize the compactness of C_φ on these spaces.

KEYWORDS: *Composition operators, embedded disks, Bergman spaces, Hardy spaces, unit ball, polydisk.*

MSC (2000): Primary 47B33; Secondary 32A35, 32A36.

1. INTRODUCTION

Let $\mathbb{C}$ be the complex plane. For any positive integer n we let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the n-dimensional complex Euclidean space. If $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ are points in $\mathbb{C}^n$, we write

$$\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n, \quad \text{and} \quad |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

For any domain Ω in $\mathbb{C}^n$ we use $H(\Omega)$ to denote the space of holomorphic functions in Ω. Three domains will be used in the paper: the open unit disc in $\mathbb{C}$,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

the open unit ball in $\mathbb{C}^n$,

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\},$$

and the open unit polydisc in $\mathbb{C}^n$,

$$\mathbb{D}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_1| < 1, \dots, |z_n| < 1\}.$$

Although the unit disc is the one-dimensional version of $\mathbb{B}_n$ and $\mathbb{D}^n$, we use different notation for historic reasons. We use $\mathbb{T}$, $\mathbb{S}_n$, and $\mathbb{T}^n$ to denote the unit circle in $\mathbb{C}$, the unit sphere in $\mathbb{C}^n$, and unit torus in $\mathbb{C}^n$, respectively.

In the rest of the paper, if we do not say what Ω is, it means that it is either $\mathbb{B}_n$ or $\mathbb{D}^n$ in $\mathbb{C}^n$.

If $\varphi : \mathbb{D} \to \Omega$ is a holomorphic map, it obviously induces a composition operator

$$C_\varphi : H(\Omega) \to H(\mathbb{D}),$$

that is,

$$(C_\varphi f)(z) = (f \circ \varphi)(z) = f(\varphi(z)), \quad f \in H(\Omega), z \in \mathbb{D}.$$

We are going to study the action of C_φ on two types of subspaces of $H(\Omega)$: Hardy spaces $H^p(\Omega)$ and weighted Bergman spaces $A^p_\alpha(\Omega)$.

For $p > 0$ and $\alpha > -1$ the weighted Bergman space $A^p_\alpha(\mathbb{D})$ of the unit disc consists of all functions f in $H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f(z)|^p \, \mathrm{d}A_\alpha(z) < \infty \quad \text{where } \mathrm{d}A_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \, \mathrm{d}A(z),$$

and $\mathrm{d}A$ is area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. It is clear that $A^2_\alpha(\mathbb{D})$ is a Hilbert space with the following inner product:

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \, \overline{g(z)} \, \mathrm{d}A_\alpha(z).$$

See [6] and [17] for the theory of Bergman spaces in one complex variable.

Similarly, for $\alpha > -1$, we define a probability measure $\mathrm{d}v_\alpha$ on $\mathbb{B}_n$ and $\mathbb{D}^n$ as follows. On the unit ball $\mathbb{B}_n$, we set

$$\mathrm{d}v_\alpha(z) = c_\alpha (1-|z|^2)^\alpha \, \mathrm{d}v(z),$$

where $\mathrm{d}v$ is the normalized volume measure on $\mathbb{B}_n$ and c_α is a positive constant so that $v_\alpha(\mathbb{B}_n) = 1$. On the polydisc $\mathbb{D}^n$, we set

$$\mathrm{d}v_\alpha(z) = \mathrm{d}A_\alpha(z_1) \cdots \mathrm{d}A_\alpha(z_n) = (\alpha+1)^n \prod_{k=1}^n (1-|z_k|^2)^\alpha \mathrm{d}A(z_1) \cdots \mathrm{d}A(z_n).$$

Then the weighted Bergman space $A^p_\alpha(\Omega)$ is defined as

$$A^p_\alpha(\Omega) = H(\Omega) \cap L^p(\Omega, \mathrm{d}v_\alpha).$$

The special case $A^2_\alpha(\Omega)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\Omega} f(z) \overline{g(z)} \, \mathrm{d}v_\alpha(z).$$

See [18] for the theory of Bergman spaces in $\mathbb{B}_n$.

If $d\sigma$ denotes the normalized Lebesgue measure on $\mathbb{S}_n$ or $\mathbb{T}^n$, then for any $p > 0$, the Hardy space $H^p(\Omega)$ consists of functions f in $H(\Omega)$ such that

$$\sup_{0<r<1} \int_{\partial\Omega} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty,$$

where $\partial\Omega$ is the Shilov boundary of Ω, that is, $\partial\Omega = \mathbb{S}_n$ when $\Omega = \mathbb{B}_n$, and $\partial\Omega = \mathbb{T}^n$ when $\Omega = \mathbb{D}^n$. If $f \in H^p(\Omega)$, the radial limit

$$f(\zeta) = \lim_{r\to 1^-} f(r\zeta)$$

exists for almost every $\zeta \in \partial\Omega$, and

$$\sup_{0<r<1} \int_{\partial\Omega} |f(r\zeta)|^p \, d\sigma(\zeta) = \int_{\partial\Omega} |f(\zeta)|^p \, d\sigma(\zeta).$$

In particular, $H^2(\Omega)$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\partial\Omega} f(\zeta)\overline{g(\zeta)} \, d\sigma(\zeta).$$

See [12], [13], and [18] for more information on Hardy spaces of the unit ball and the polydisc.

We say that a sequence $\{f_k\}$ in H^p or A^p_α (of the polydisc or the unit ball) converges to 0 ultra-weakly if the sequence is bounded in norm and converges to 0 uniformly on compact subsets of $\mathbb{D}$ or $\mathbb{B}_n$. A bounded linear operator T from H^p or A^p_α into some L^p space is ultra-weakly compact if $\{Tf_k\}$ converges to 0 in norm whenever $\{f_k\}$ converges to 0 ultra-weakly.

When $p > 1$, it is easy to show that the ultra-weak topology on H^p or A^p_α is the same as the weak topology, which is also the same as the weak-star topology. Therefore, for $p > 1$, an operator from H^p or A^p_α into an L^p space is ultra-weakly compact if and only if it is compact in the usual sense. When $p = 1$, the ultra-weak topology on H^1 or A^1_α coincides with the weak-star topology, which is strictly weaker than the weak topology.

We can now state the main results of the paper.

THEOREM 1.1. *If $p > 0$ and φ is a holomorphic mapping from $\mathbb{D}$ into $\mathbb{B}_n$, then the composition operator C_φ maps $H^p(\mathbb{B}_n)$ boundedly into $A^p_{n-2}(\mathbb{D})$. Furthermore, the operator*

$$C_\varphi : H^p(\mathbb{B}_n) \to A^p_{n-2}(\mathbb{D})$$

is ultra-weakly compact if and only if

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

THEOREM 1.2. *If $p > 0$, $\alpha > -1$, and $\varphi : \mathbb{D} \to \mathbb{B}_n$ is holomorphic, then the composition operator C_φ maps $A^p_\alpha(\mathbb{B}_n)$ boundedly into $A^p_{n-1+\alpha}(\mathbb{D})$. Furthermore, the operator*

$$C_\varphi : A^p_\alpha(\mathbb{B}_n) \to A^p_{n-1+\alpha}(\mathbb{D})$$

is ultra-weakly compact if and only if

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

THEOREM 1.3. *If $p > 0$ and $\varphi = (\varphi_1, \dots, \varphi_n)$ is a holomorphic map from $\mathbb{D}$ into $\mathbb{D}^n$, then the composition operator C_φ maps $H^p(\mathbb{D}^n)$ boundedly into $A^2_{n-2}(\mathbb{D})$. If $p > 1$, then the operator*

$$C_\varphi : H^p(\mathbb{D}^n) \to A^p_{n-2}(\mathbb{D})$$

is compact if and only if

$$\lim_{|z|\to 1^-} \prod_{k=1}^{n} \frac{1-|z|^2}{1-|\varphi_k(z)|^2} = 0.$$

It is well known that the diagonal map

$$\Delta : H(\mathbb{D}^n) \to H(\mathbb{D})$$

defined by

$$(\Delta f)(z) = f(z, \dots, z), \quad f \in H(\mathbb{D}^n), z \in \mathbb{D},$$

maps $H^p(\mathbb{D}^n)$ boundedly onto $A^p_{n-2}(\mathbb{D})$; see Proposition 4.5 of [2]. Earlier papers on this problem include [8], [4], [15]. This result, together with the well-known theory of composition operators on Hardy spaces of the unit disk, shows that C_φ maps $H^p(\mathbb{D}^n)$ boundedly into $A^p_{n-2}(\mathbb{D})$. This also tells us that the range space $A^p_{n-2}(\mathbb{D})$ is the right choice for us here.

THEOREM 1.4. *If $p > 0$, $\alpha > -1$, and $\varphi = (\varphi_1, \dots, \varphi_n)$ is a holomorphic map from $\mathbb{D}$ into $\mathbb{D}^n$, then the composition operator C_φ maps $A^p_\alpha(\mathbb{D}^n)$ boundedly into $A^p_{n(\alpha+2)-2}(\mathbb{D})$. Furthermore, the operator*

$$C_\varphi : A^p_\alpha(\mathbb{D}^n) \to A^p_{n(\alpha+2)-2}(\mathbb{D})$$

is ultra-weakly compact if and only if

$$\lim_{|z|\to 1^-} \prod_{k=1}^{n} \frac{1-|z|^2}{1-|\varphi_k(z)|^2} = 0.$$

THEOREM 1.5. *For $p > 0$ and $\alpha > -1$ the diagonal map Δ maps the space $A^p_\alpha(\mathbb{D}^n)$ boundedly onto $A^p_{n(\alpha+2)-2}(\mathbb{D})$.*

We use a well-known technique involving Carleson type measures to reduce the proof of the theorems to the case $p = 2$. When $p = 2$, all spaces involved are Hilbert spaces, and we have reproducing kernels at our disposal.

2. CARLESON TYPE MEASURES

This section serves two purposes for us. First, the various characterizations for Carleson measures will enable us to reduce the proof of Theorems 1.1–1.4 to the case $p = 2$. Second, the geometric conditions for Carleson measures in the unit ball actually enables us to prove the boundedness and compactness for our composition operators.

Let $\beta(z,w)$ denote the Bergman metric on Ω. For any $z \in \Omega$ and $R > 0$ we use

$$D(z,r) = \{w \in \Omega : \beta(w,z) < R\}$$

for the Bergman metric ball at z with radius R. It is well known that for any fixed $R > 0$, we have:

$$v_\alpha(D(z,R)) \sim \prod_{k=1}^{n}(1-|z_k|^2)^{2+\alpha} \qquad \text{when } \Omega = \mathbb{D}^n, \tag{2.1}$$

$$v_\alpha(D(z,R)) \sim (1-|z|^2)^{n+1+\alpha} \qquad \text{when } \Omega = \mathbb{B}_n. \tag{2.2}$$

LEMMA 2.1. *Suppose $p > 0$, $\alpha > -1$, and $R > 0$. For any positive Borel measure μ on Ω the following conditions are equivalent:*

(i) *There exists a constant $C_1 > 0$ such that, for all $f \in A^p_\alpha(\Omega)$,*

$$\int_\Omega |f(z)|^p \, \mathrm{d}\mu(z) \leqslant C_1 \int_\Omega |f(z)|^p \, \mathrm{d}v_\alpha(z).$$

(ii) *There exists a constant $C_2 > 0$ such that, for all $z \in \Omega$,*

$$\mu(D(z,R)) \leqslant C_2 v_\alpha(D(z,R)).$$

Proof. The result actually holds for more general domains than the unit ball and the polydisc. For example, it is shown in [16] that the lemma holds for every bounded symmetric domain. ∎

If a measure μ on Ω satisfies the conditions in the above lemma, we say that μ is a *Carleson measure* for $A^p_\alpha(\Omega)$. The following is the little oh version of Lemma 2.1.

LEMMA 2.2. *Suppose $p > 0$, $\alpha > -1$, and $R > 0$. For any positive Borel measure μ on Ω the following conditions are equivalent*

(i) *The inclusion from $A^p_\alpha(\Omega)$ into $L^p(\Omega, \mathrm{d}\mu)$ is ultra-weakly compact.*

(ii) *The following limit exists and equals* 0*:*

$$\lim_{z\to\partial\Omega} \frac{\mu(D(z,R))}{v_\alpha(D(z,R))}.$$

If a measure μ on Ω satisfies the conditions in Lemma 2.2, we say that μ is a *vanishing Carleson measure* for $A^p_\alpha(\Omega)$.

In addition to the Bergman metric, we also need the following nonisotropic "metric" on $\mathbb{B}_n$:

$$d(z,w) = |1 - \langle z, w\rangle|, \quad z, w \in \mathbb{B}_n.$$

The function d itself is not a metric, but the restriction of $\sqrt{d}$ on $\mathbb{S}_n$ is. For $\zeta \in \mathbb{S}_n$ and $r > 0$ we write

$$Q_r(\zeta) = \{z \in \mathbb{B}_n : d(z,\zeta) < r\}.$$

For any fixed $\alpha > -1$, there exist positive constants c and C such that

$$cr^{n+1+\alpha} \leqslant v_\alpha(Q_r(\zeta)) \leqslant Cr^{n+1+\alpha} \tag{2.3}$$

for all $\zeta \in \mathbb{S}_n$ and all $r \in (0,1)$. See Corollary 5.24 of [18].

Carleson measures for weighted Bergman spaces of the unit ball (including the unit disc) can also be characterized in terms of the non-isotropic metric.

LEMMA 2.3. *Suppose $p > 0$, $\alpha > -1$, $r > 0$, and μ is a positive Borel measure on $\mathbb{B}_n$. Then μ is a Carleson measure for $A^p_\alpha(\mathbb{B}_n)$ if and only if*

$$\sup_{r,\zeta} \frac{\mu(Q_r(\zeta))}{r^{n+1+\alpha}} < \infty. \tag{2.4}$$

Similarly, μ is a vanishing Carleson measure for $A^p_\alpha(\mathbb{B}_n)$ if and only if

$$\lim_{r\to 0^+} \frac{\mu(Q_r(\zeta))}{r^{n+1+\alpha}} = 0 \quad \textit{uniformly for } \zeta \in \mathbb{S}_n. \tag{2.5}$$

Proof. It follows from (2.2), (2.3), and Lemma 5.23 of [18] that condition (2.4) here implies condition (ii) in Lemma 2.1.

If μ is a Carleson measure for $A^p_\alpha(\mathbb{B}_n)$, then there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}_n} \frac{(1-|a|^2)^{n+1+\alpha}\, d\mu(z)}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}} \leqslant C \tag{2.6}$$

for all $a \in \mathbb{B}_n$. In fact, this is what we get if we set $f(z) = \frac{(1-|a|^2)^{(n+1+\alpha)/p}}{(1-\langle z,a\rangle)^{2(n+1+\alpha)/p}}$ in condition (i) of Lemma 2.1, because $\int_{\mathbb{B}_n} \frac{(1-|a|^2)^{n+1+\alpha}\, dv_\alpha(z)}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}} = 1$ for every $a \in \mathbb{B}_n$.

For any $0 < r < 1$ and $\zeta \in \mathbb{S}_n$ we write $a = (1-r)\zeta$. By (2.6), we have

$$\int\limits_{Q_r(\zeta)} \frac{(1-|a|^2)^{n+1+\alpha}\,\mathrm{d}\mu(z)}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}} \leqslant C. \tag{2.7}$$

Since $1-|a|^2 = 1-(1-r)^2 = r(2-r) > r$, and for every $z \in Q_r(\zeta)$ we have $|1-\langle z,a\rangle| = |1-\langle z,\zeta\rangle + r\langle z,\zeta\rangle| \leqslant |1-\langle z,\zeta\rangle| + r|\langle z,\zeta\rangle| \leqslant r+r = 2r$, we deduce from (2.7) that

$$\frac{r^{n+1+\alpha}}{(2r)^{2(n+1+\alpha)}}\,\mu(Q_r(\zeta)) \leqslant C.$$

This proves (2.4) for $r < 1$. The case $r \geqslant 1$ is trivial.

A similar argument proves the characterization of vanishing Carleson measures for $A^p_\alpha(\mathbb{B}_n)$. ∎

The following result characterizes Carleson measures for Hardy spaces of the unit ball.

LEMMA 2.4. *Suppose $p > 0$ and μ is a positive Borel measure on $\mathbb{B}_n$. Then the following conditions are equivalent:*

(i) *There exists a constant $C_1 > 0$ such that, for all $f \in H^p(\mathbb{B}_n)$,*

$$\int\limits_{\mathbb{B}_n} |f(z)|^p\,\mathrm{d}\mu(z) \leqslant C_1 \int\limits_{\mathbb{S}_n} |f(\zeta)|^p\,\mathrm{d}\sigma(\zeta).$$

(ii) *There exists a constant $C_2 > 0$ such that, for all $\zeta \in \mathbb{S}_n$ and $r > 0$,*

$$\mu(Q_r(\zeta)) \leqslant C_2 r^n.$$

Proof. This follows from Hörmander's results in [7], which are valid for strongly pseudo-convex domains. See [10] or Section 5.2 of [18] for more details in the case of the unit ball. ∎

When a measure μ satisfies the conditions in Lemma 2.4, we say that μ is a Carleson measure for $H^p(\mathbb{B}_n)$. We will also need the little oh version of Lemma 2.4.

LEMMA 2.5. *Suppose $p > 0$ and μ is a positive Borel measure on $\mathbb{B}_n$. Then the following two conditions are equivalent:*

(i) *The identity map is ultra-weakly compact from the Hardy space $H^p(\mathbb{B}_n)$ into $L^p(\mathbb{B}_n, \mathrm{d}\mu)$.*

(ii) *The following limit holds uniformly for $\zeta \in \mathbb{S}_n$:*

$$\lim_{r\to 0} \frac{\mu(Q_r(\zeta))}{r^n} = 0.$$

Proof. See [10] or Section 5.3 of [18]. ∎

Measures satisfying the conditions in Lemma 2.5 are called *vanishing Carleson measures for* $H^p(\mathbb{B}_n)$.

The characterization of Carleson measures for Hardy spaces of the polydisc is slightly more involved. In particular, we have to restrict our attention to the case $p > 1$. The case $p = 1$ can be handled with some extra effort, but we are unable to go below $p = 1$.

We begin with "Carleson squares" on the unit disc. Given an open interval I on the unit circle, the Carleson square S_I is defined as follows:

$$S_I = \{z = r\zeta : 1 - |I| < r < 1, \zeta \in I\},$$

where $|I|$ is the normalized length of I (so that the unit circle has total length 1). It is obvious that the area of S_I is comparable to $|I|^2$. For

$$R = I_1 \times I_2 \times \cdots \times I_n$$

in $\mathbb{T}^n$, where each I_k is an open interval in the unit circle, let

$$S_R = S_{I_1} \times S_{I_2} \times \cdots \times S_{I_n}$$

and call it a Carleson region in $\mathbb{D}^n$. Recall that $d\sigma$ is the normalized Lebesgue measure on $\mathbb{T}^n$. So it is clear that

$$\sigma(R) = |I_1| \times |I_2| \times \cdots \times |I_n|.$$

The following result characterizes Carleson measures for Hardy spaces of the polydisc.

LEMMA 2.6. *Suppose $p > 1$ and μ is a positive Borel measure on $\mathbb{D}^n$. Then the following two conditions are equivalent:*

(i) *There exists a constant $C > 0$ such that, for all $f \in H^p(\mathbb{D}^n)$,*

$$\int_{\mathbb{D}^n} |f(z)|^p \, d\mu(z) \leqslant C \int_{\mathbb{T}^n} |f(\zeta)|^p \, d\sigma(\zeta).$$

(ii) *The limit*

$$\limsup_{\delta \to 0^+} \left\{ \frac{\mu(S(V))}{\sigma(V)} : V \subset \mathbb{T}^n, \sigma(V) < \delta \right\}$$

is finite, where V is open and

$$S(V) = \bigcup \{S_R : R = I_1 \times \cdots \times I_n \subset V\}.$$

Proof. See [1] for the case $n = 2$ and [9] for the general case. ∎

We have no intention of actually applying condition (ii) above. What we want is the fact that condition (ii) is independent of p, which implies that condition (i) holds for some $p > 1$ if and only if it holds for every $p > 1$. The same remark applies to the following little oh version of Lemma 2.6 as well.

LEMMA 2.7. *Suppose $p > 1$ and μ is a positive Borel measure on $\mathbb{D}^n$. Then the following two conditions are equivalent:*

(i) *The identity map is compact from the Hardy space $H^p(\mathbb{D}^n)$ into $L^p(\mathbb{D}^n, \mathrm{d}\mu)$.*

(ii) *The following limit equals 0:*

$$\limsup_{\delta\to 0^+}\Big\{\frac{\mu(S(V))}{\sigma(V)} : V \subset \mathbb{T}^n, \sigma(V) < \delta\Big\}.$$

Proof. See [1] and [9] again. ∎

We now make the connection between Carleson measures and composition operators.

Suppose $\alpha > -1$ and $\varphi : \mathbb{D} \to \Omega$ is holomorphic. We define a positive Borel measure $\mu_{\varphi,\alpha}$ on Ω as follows. If E is a Borel subset of Ω, we define

$$\mu_{\varphi,\alpha}(E) = A_\alpha(\varphi^{-1}(E)) = (\alpha+1)\int\limits_{\varphi^{-1}(E)} (1-|z|^2)^\alpha\,\mathrm{d}A(z).$$

It is then clear that we have the following change of variables formula.

LEMMA 2.8. *Suppose $p > 0$, $\alpha > -1$, and $\varphi : \mathbb{D} \to \Omega$ is holomorphic. Then*

$$\int\limits_{\mathbb{D}} |f(\varphi(z))|^p\,\mathrm{d}A_\alpha(z) = \int\limits_{\Omega} |f(z)|^p\,\mathrm{d}\mu_{\varphi,\alpha}(z), \tag{2.8}$$

where f is any holomorphic function in Ω.

COROLLARY 2.9. *Suppose $p > 1$, $\alpha > -1$, and $\varphi : \mathbb{D} \to \Omega$ is holomorphic. Then:*

(i) *The operator C_φ maps $H^p(\Omega)$ boundedly into $A^p_\alpha(\mathbb{D})$ if and only if the measure $\mu_{\varphi,\alpha}$ is Carleson for $H^p(\Omega)$.*

(ii) *The operator $C_\varphi : H^p(\Omega) \to A^p_\alpha(\mathbb{D})$ is ultra-weakly compact if and only if the measure $\mu_{\varphi,\alpha}$ is vanishing Carleson for $H^p(\Omega)$.*

COROLLARY 2.10. *Suppose $p > 0$, $\alpha > -1$, $\gamma > -1$, and φ is a holomorphic map from $\mathbb{D}$ into Ω. Then:*

(i) *The operator C_φ maps $A^p_\alpha(\Omega)$ boundedly into $A^p_\gamma(\mathbb{D})$ if and only if the measure $\mu_{\varphi,\gamma}$ is Carleson for $A^p_\alpha(\Omega)$.*

(ii) *The operator $C_\varphi : A^p_\alpha(\Omega) \to A^p_\gamma(\mathbb{D})$ is ultra-weakly compact if and only if the measure $\mu_{\varphi,\gamma}$ is vanishing Carleson for $A^p_\alpha(\Omega)$.*

Once again, it follows from results of this section that the boundedness or ultra-weak compactness of the operator

$$C_\varphi : H^p(\Omega) \to A^2_{n-2}(\mathbb{D})$$

is independent of the exponent p. Similarly, the boundedness or ultra-weak compactness of

$$C_\varphi : A^p_\alpha(\mathbb{B}_n) \to A^p_{n-1+\alpha}(\mathbb{D}) \quad \text{and} \quad C_\varphi : A^p_\alpha(\mathbb{D}^n) \to A^p_{n(\alpha+2)-2}(\mathbb{D})$$

is independent of p. This observation will be used numerous times later in the paper.

3. SOME ONE-DIMENSIONAL RESULTS

For any $r > 0$ we introduce the set

$$S_r = \{z \in \mathbb{D} : |1 - z| < r\}.$$

This is the one-dimensional version of $Q_r(\zeta)$ at the point 1 on the unit circle.

The following result is well known; see [3] in general and see Exercise 3.2.9 of [3] in particular. However, we offer a little more information here than the usual big oh statements, and we are going to need this extra information (about how the constant depends on φ) later when we prove the main theorems.

THEOREM 3.1. *Suppose $p > 0$, $\alpha > -1$, and $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic self-map of the unit disc. Then the operators*

$$C_\varphi : H^p(\mathbb{D}) \to H^p(\mathbb{D}), \quad C_\varphi : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D}),$$

are bounded. Furthermore, there exists a constant $C > 0$, independent of r and φ, such that, for all $r > 0$,

$$\mu_{\varphi,\alpha}(S_r) \leqslant C\Big(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\Big)^{2+\alpha} r^{2+\alpha}.$$

Proof. The boundedness of C_φ on $H^p(\mathbb{D})$ is a consequence of Littlewood's subordination principle. Also, if $\varphi(0) = 0$, then Littlewood's subordination principle along with integration in polar coordinates shows that C_φ has norm 1 on $A^2_\alpha(\mathbb{D})$. More generally, we consider the function

$$\psi(z) = \varphi_a \circ \varphi(z), \quad \text{where } a = \varphi(0) \quad \text{and} \quad \varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

is a Möbius map of the disk. Then ψ is an analytic self-map of the disk that fixes the origin. Therefore,

$$\int_{\mathbb{D}} |f(\psi(z))|^2 \, dA_\alpha(z) \leqslant \int_{\mathbb{D}} |f(z)|^2 \, dA_\alpha(z)$$

for all $f \in H(\mathbb{D})$. Replacing f by $f \circ \varphi_a$, we obtain

$$\int_{\mathbb{D}} |f(\varphi(z))|^2 \, dA_\alpha(z) \leqslant \int_{\mathbb{D}} |f \circ \varphi_a(z)|^2 \, dA_\alpha(z)$$

for all $f \in H(\mathbb{D})$. Changing variables two times and estimating in the denominator with the triangle inequality, we have

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^2 \,\mathrm{d}\mu_{\varphi,\alpha}(z) &= \int_{\mathbb{D}} |f(\varphi(z))|^2 \,\mathrm{d}A_\alpha(z) \leqslant \int_{\mathbb{D}} |f \circ \varphi_a(z)|^2 \,\mathrm{d}A_\alpha(z) \\ &= \int_{\mathbb{D}} |f(z)|^2 \frac{(1-|a|^2)^{2+\alpha}\,\mathrm{d}A_\alpha(z)}{|1-\bar{a}z|^{2(2+\alpha)}} \\ &\leqslant \frac{(1-|a|^2)^{2+\alpha}}{(1-|a|)^{2(2+\alpha)}} \int_{\mathbb{D}} |f(z)|^2 \,\mathrm{d}A_\alpha(z) \\ &= \Big(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\Big)^{2+\alpha} \int_{\mathbb{D}} |f(z)|^2 \,\mathrm{d}A_\alpha(z),\end{aligned}$$

where $f \in A^2_\alpha(\mathbb{D})$. The desired result then follows from Lemma 2.3. ∎

The following result is due to MacCluer and Shapiro; see [11] or Theorem 3.22 of [3]. We write down a streamlined proof here that will also be used later for other purposes.

THEOREM 3.2. *Suppose $p > 0$, $\alpha > -1$, and $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic self-map of the unit disc. Then the composition operator*

$$C_\varphi : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$$

is ultra-weakly compact if and only if

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0. \tag{3.1}$$

Proof. By Section 2, it suffices for us to prove the result when $p = 2$. In this case, we can consider the adjoint C^*_φ of C_φ, and it is well known (see Theorem 5.1) that

$$\|C^*_\varphi k_z\|^2 = \Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^{2+\alpha} \quad \text{where } k_z(w) = \frac{(1-|z|^2)^{(2+\alpha)/2}}{(1-w\bar{z})^{2+\alpha}}, \quad z, w \in \mathbb{D},$$

are the normalized reproducing kernels of $A^2_\alpha(\mathbb{D})$. Here and in the rest of this proof we use $\|\cdot\|$ to denote the norm in $A^2_\alpha(\mathbb{D})$. Since $\{k_z\}$ converges to 0 weakly in $A^2_\alpha(\mathbb{D})$ as $|z| \to 1^-$, we see that the compactness of C_φ on $A^2_\alpha(\mathbb{D})$ implies condition (3.1).

Next assume that condition (3.1) holds and $\{f_k\}$ is a sequence in $A^2_\alpha(\mathbb{D})$ that converges to 0 weakly. Then the sequence $\{f_k\}$ is bounded in norm and converges to 0 uniformly on compact subsets of $\mathbb{D}$. We proceed to show that

$$\lim_{k\to\infty} \|C_\varphi f_k\| = 0. \tag{3.2}$$

Given any positive number ε we choose a number $\delta \in (0,1)$ such that

$$1-|z|^2 < \varepsilon(1-|\varphi(z)|^2), \quad \delta < |z| < 1. \tag{3.3}$$

We can find a constant $C_1 > 0$, independent of φ and k, such that

$$\|C_\varphi f_k\|^2 \leqslant C_1\Big[|f_k(\varphi(0))|^2 + \int_{\mathbb{D}} |f_k'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^{\alpha+2}\mathrm{d}A(z)\Big].$$

It is clear that $f_k(\varphi(0)) \to 0$ as $k \to \infty$.

We write the integral $\int_{\mathbb{D}} |f_k'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^{\alpha+2}\,\mathrm{d}A(z)$ as the sum of

$$I_k = \int_{|z|<\delta} |f_k'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^{\alpha+2}\,\mathrm{d}A(z),$$

$$J_k = \int_{\delta<|z|<1} |f_k'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^{\alpha+2}\,\mathrm{d}A(z).$$

Since $\{f_k'\}$ converges to 0 uniformly on compact sets and $\varphi'(z)$ is bounded on $|z| \leqslant \delta$, we have $I_k \to 0$ as $k \to \infty$.

By (3.3), we have

$$J_k \leqslant 2\varepsilon^{\alpha+1} \int_{\delta<|z|<1} |f_k'(\varphi(z))|^2|\varphi'(z)|^2(1-|\varphi(z)|^2)^{\alpha+1}\log\frac{1}{|z|}\,\mathrm{d}A(z).$$

If $N_\varphi(z) = \sum\Big\{\log\frac{1}{|w|} : \varphi(w) = z\Big\}$ is the Nevanlinna counting function of φ, then a change of variables gives $J_k \leqslant 2\varepsilon^{\alpha+1}\int_{\mathbb{D}} |f_k'(z)|^2(1-|z|^2)^{\alpha+1}N_\varphi(z)\,\mathrm{d}A(z)$. The classical Littlewood's inequality states that $N_\varphi(z) \leqslant \log\Big|\frac{1-\overline{\varphi(0)}z}{\varphi(0)-z}\Big|$; see Theorem 2.29 of [3]. Since $\log\Big|\frac{1-\overline{\varphi(0)}z}{\varphi(0)-z}\Big|$ is comparable to $1-\Big|\frac{\varphi(0)-z}{1-\overline{\varphi(0)}z}\Big|^2 = \frac{(1-|\varphi(0)|^2)(1-|z|^2)}{|1-\overline{\varphi(0)}z|^2}$, we can find another constant $C_2 > 0$, independent of k and φ, such that

$$J_k \leqslant C_2\Big(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\Big)\varepsilon^{\alpha+1}\int_{\mathbb{D}} |f_k'(z)|^2(1-|z|^2)^{\alpha+2}\,\mathrm{d}A(z).$$

There exists a positive constant C_3, independent of k and φ, such that

$$\int_{\mathbb{D}} |f_k'(z)|^2(1-|z|^2)^{\alpha+2}\,\mathrm{d}A(z) \leqslant C_3\int_{\mathbb{D}} |f_k(z)|^2\,\mathrm{d}A_\alpha(z).$$

Therefore, there exists a positive constant C_4, independent of k and φ, such that

$$J_k \leqslant C_4\Big(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\Big)\varepsilon^{\alpha+1}$$

for all k. It follows that

$$\limsup_{k\to\infty} J_k \leqslant C_4\Big(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\Big)\varepsilon^{\alpha+1}.$$

Since ε is arbitrary, we must have $J_k \to 0$ as $k \to \infty$. This proves (3.2) and completes the proof of the theorem. ∎

4. BOUNDEDNESS

We begin with the case of the polydisc. So suppose $\varphi = (\varphi_1, \dots, \varphi_n)$ is a holomorphic map of the unit disc into the polydisc. It follows from the boundedness of each composition operator C_{φ_k} on the Hardy space $H^p(\mathbb{D})$ of the unit disc (see Theorem 3.1) that the operator

$$f(z_1, \dots, z_n) \mapsto f(\varphi_1(z_1), \dots, \varphi_n(z_n))$$

is bounded on $H^p(\mathbb{D}^n)$. On the other hand, it follows from the main results of [4] and [8] that the diagonal operator defined by

$$f(z_1, \dots, z_n) \mapsto f(z, \dots, z)$$

maps $H^p(\mathbb{D}^n)$ boundedly into $A^p_{n-2}(\mathbb{D})$, provided $n > 1$. Composing the action of these two operators, we obtain the following result.

THEOREM 4.1. *If $n > 1$, $p > 0$, and φ is a holomorphic map from $\mathbb{D}$ into $\mathbb{D}^n$. Then the composition operator C_φ maps $H^p(\mathbb{D}^n)$ boundedly into $A^p_{n-2}(\mathbb{D})$.*

To prove that C_φ maps $A^p_\alpha(\mathbb{D}^n)$ into $A^p_{n(\alpha+2)-2}(\mathbb{D})$, we first consider the case in which $p = 2$ and $\varphi(z) = (z, \dots, z)$.

LEMMA 4.2. *For any $\alpha > -1$ the diagonal map Δ maps the Bergman space $A^2_\alpha(\mathbb{D}^n)$ boundedly into the Bergman space $A^2_{n(\alpha+2)-2}(\mathbb{D})$.*

Proof. Suppose

$$f(z) = \sum_m a_m z^m$$

is the Taylor expansion of a function in $H(\mathbb{D}^n)$, where $m = (m_1, \dots, m_n)$ is a multi-index of nonnegative integers and

$$z^m = z_1^{m_1} \cdots z_n^{m_n}.$$

We have:

$$\int_{\mathbb{T}^n} |f(\zeta)|^2 \,\mathrm{d}\sigma(\zeta) = \sum_m |a_m|^2 = \sum_{k=0}^{\infty} \sum_{|m|=k} |a_m|^2 \quad \text{for } f \in H^2(\mathbb{D}^n), \text{ and}$$

$$\int_{\mathbb{D}^n} |f(z)|^2 \,\mathrm{d}v_\alpha(z) = \sum_m \frac{|a_m|^2}{(m_1+1)^{\alpha+1}\cdots(m_n+1)^{\alpha+1}}$$

$$= \sum_{k=0}^{\infty} \sum_{|m|=k} \frac{|a_m|^2}{(m_1+1)^{\alpha+1}\cdots(m_n+1)^{\alpha+1}} \quad \text{for } f \in A^2_\alpha(\mathbb{D}^n).$$

With the convention that $|m| = m_1 + \cdots + m_n$, we also have the following for the diagonal operator Δ:

$$\int_{\mathbb{D}} |\Delta f(z)|^2 \,\mathrm{d}A_{n(\alpha+2)-2}(z) = \sum_{k=0}^{\infty} \frac{\Big|\sum\limits_{|m|=k} a_m\Big|^2}{(k+1)^{n(\alpha+2)-1}},$$

$$\int_{\mathbb{D}} |\Delta f(z)|^2 \,\mathrm{d}A_{n-2}(z) = \sum_{k=0}^{\infty} \frac{\Big|\sum\limits_{|m|=k} a_m\Big|^2}{(k+1)^{n-1}}.$$

Since Δ maps $H^2(\mathbb{D}^n)$ boundedly into $A^2_{n-2}(\mathbb{D})$ (see [8]), there must exist a constant $C > 0$, independent of k and f, such that

$$\frac{\Big|\sum\limits_{|m|=k} a_m\Big|^2}{(k+1)^{n-1}} \leqslant C \sum_{|m|=k} |a_m|^2$$

for all k and f (this elementary identity can also be verified directly without appealing to the diagonal map). If $|m| = k$, it is obvious that

$$\frac{1}{(k+1)^{n(\alpha+1)}} \leqslant \frac{1}{(m_1+1)^{\alpha+1}\cdots(m_n+1)^{\alpha+1}}.$$

Therefore,

$$\sum_{k=0}^{\infty} \frac{\Big|\sum\limits_{|m|=k} a_m\Big|^2}{(k+1)^{n(\alpha+2)-1}} \leqslant C \sum_{k=0}^{\infty} \sum_{|m|=k} \frac{|a_m|^2}{(m_1+1)^{\alpha+1}\cdots(m_n+1)^{\alpha+1}},$$

or

$$\int_{\mathbb{D}} |\Delta f(z)|^2 \,\mathrm{d}A_{n(\alpha+2)-2}(z) \leqslant C \int_{\mathbb{D}^n} |f(z)|^2 \,\mathrm{d}v_\alpha(z),$$

completing the proof of the lemma. ∎

THEOREM 4.3. *For any $p > 0$ and $\alpha > -1$ the operator C_φ maps $A^p_\alpha(\mathbb{D}^n)$ boundedly into $A^p_{n(\alpha+2)-2}(\mathbb{D})$.*

Proof. By Section 2, we only need to prove the case $p = 2$. In this case, Lemma 4.2 tells us that the diagonal map Δ is a bounded operator from $A^2_\alpha(\mathbb{D}^n)$ into $A^2_{n(\alpha+1)-2}(\mathbb{D})$. Combining this with the fact that

$$f(z_1, \ldots, z_n) \mapsto f(\varphi_1(z_1), \ldots, \varphi_n(z_n))$$

is a bounded linear operator on $A^2_\alpha(\mathbb{D}^n)$ (which follows from the boundedness of composition operators on weighted Bergman spaces of the unit disc, see Theorem 3.1), we conclude that C_φ is bounded from $A^2_\alpha(\mathbb{D}^n)$ into $A^2_{n(\alpha+2)-2}(\mathbb{D})$. ∎

THEOREM 4.4. *For any $p > 0$ and $\alpha > -1$ the diagonal map Δ maps the Bergman space $A^p_\alpha(\mathbb{D}^n)$ boundedly onto the Bergman space $A^p_{n(\alpha+2)-2}(\mathbb{D})$.*

Proof. That Δ maps $A^p_\alpha(\mathbb{D}^n)$ boundedly into $A^p_{n(\alpha+2)-2}(\mathbb{D})$ follows from Theorem 4.3 by taking $\varphi(z) = (z, \ldots, z)$. We proceed to show that this map is actually onto.

First assume that $1 < p < \infty$ with $1/p + 1/q = 1$. Fix a function $f \in A^p_{n(\alpha+2)-2}(\mathbb{D})$ and define a function $F \in H(\mathbb{D}^n)$ by

$$F(z_1, \ldots, z_n) = \int_{\mathbb{D}} \frac{f(w)\, \mathrm{d}A_{n(\alpha+2)-2}(w)}{\prod_{k=1}^{n} (1 - z_k\overline{w})^{\alpha+2}}.$$

By Corollary 1.5 of [6], we have

$$\Delta F(z) = \int_{\mathbb{D}} \frac{f(w)\, \mathrm{d}A_{n(\alpha+2)-2}(w)}{(1 - z\overline{w})^{n(\alpha+2)}} = f(z).$$

To see $F \in A^p_\alpha(\mathbb{D}^n)$, we take an arbitrary function $G \in A^q_\alpha(\mathbb{D}^n)$ and use Fubini's theorem (by an approximation argument we may assume that G is bounded) to obtain

$$\begin{aligned}
\int_{\mathbb{D}^n} F\overline{G}\, \mathrm{d}v_\alpha &= \int_{\mathbb{D}^n} \overline{G}\, \mathrm{d}v_\alpha \int_{\mathbb{D}} \frac{f(w)\, \mathrm{d}A_{n(\alpha+2)-2}(w)}{\prod_{k=1}^{n} (1 - z_k\overline{w})^{\alpha+2}} \\
&= \int_{\mathbb{D}} f(w)\, \mathrm{d}A_{n(\alpha+2)-2}(w) \overline{\int_{\mathbb{D}^n} \frac{G(z_1, \ldots, z_n)\, \mathrm{d}v_\alpha(z)}{\prod_{k=1}^{n} (1 - w\overline{z}_k)^{\alpha+2}}} \\
&= \int_{\mathbb{D}} f(w)\overline{G(w, \ldots, w)}\, \mathrm{d}A_{n(\alpha+2)-2}(w).
\end{aligned}$$

The last equality above follows from iterated use of Corollary 1.5 of [6] again. Since the map $G(w_1, \ldots, w_n) \mapsto G(w, \ldots, w) = \Delta G(w)$ is bounded from $A^q_\alpha(\mathbb{D}^n)$

into $A^q_{n(\alpha+2)-2}(\mathbb{D})$, we can find a constant $C > 0$ such that

$$\Big|\int_{\mathbb{D}^n} F\overline{G}\,\mathrm{d}v_\alpha\Big| \leqslant C\Big[\int_{\mathbb{D}} |f|^p\,\mathrm{d}A_{n(\alpha+2)-2}\Big]^{1/p}\Big[\int_{\mathbb{D}^n} |G|^q\,\mathrm{d}v_\alpha\Big]^{1/q}.$$

It follows from the duality $(A^p_\alpha(\mathbb{D}^n))^* = A^q_\alpha(\mathbb{D}^n)$ that $F \in A^p_\alpha(\mathbb{D}^n)$, and so the diagonal map Δ maps $A^p_\alpha(\mathbb{D}^n)$ onto $A^p_{n(\alpha+2)-2}(\mathbb{D})$.

Next assume that $0 < p \leqslant 1$ and fix a function $f \in A^p_{n(\alpha+2)-2}(\mathbb{D})$. By the atomic decomposition for Bergman spaces (see [2] or [18]), we can write

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^{b-n(\alpha+2)/p}}{(1-z\overline{a}_k)^b},$$

where $\{a_k\}$ is a certain sequence in $\mathbb{D}$, $\{c_k\} \in l^p$, and b is a sufficiently large constant. Write $b = nt$ and define

$$F(z_1,\ldots,z_n) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^{nt-n(\alpha+2)/p}}{(1-z_1\overline{a}_k)^t\cdots(1-z_n\overline{a}_k)^t}.$$

Obviously, $\Delta F = f$. Also, since $0 < p \leqslant 1$, it follows from Hölder's inequality that

$$\begin{aligned}\int_{\mathbb{D}^n} |F|^p\,\mathrm{d}v_\alpha &\leqslant \sum_{k=1}^{\infty} |c_k|^p \int_{\mathbb{D}^n} \frac{(1-|a_k|^2)^{npt-n(\alpha+2)}}{\prod_{j=1}^{n} |1-z_j\overline{a}_k|^{pt}}\,\mathrm{d}v_\alpha(z)\\ &= \sum_{k=1}^{\infty} |c_k|^p (1-|a_k|^2)^{npt-n(\alpha+2)} \prod_{j=1}^{n} \int_{\mathbb{D}} \frac{\mathrm{d}A_\alpha(z_j)}{|1-z_j\overline{a}_k|^{pt}} \leqslant C\sum_{k=1}^{\infty} |c_k|^p.\end{aligned}$$

In the last inequality above we used Lemma 8.3 and the assumption that t is sufficiently large. This proves that $F \in A^p_\alpha(\mathbb{D}^n)$, so the diagonal map Δ sends $A^p_\alpha(\mathbb{D}^n)$ onto $A^p_{n(\alpha+2)-2}(\mathbb{D})$. ∎

Note that using the same ideas we can actually give an alternative proof that C_φ maps $H^p(\mathbb{D}^n)$ into $A^p_{n-2}(\mathbb{D})$. In particular, we can give an alternative proof that the diagonal map is bounded from $H^p(\mathbb{D}^n)$ into and onto $A^p_{n-2}(\mathbb{D})$.

Next we consider the case of the unit ball. Our trick here is to reduce the proof to the one-dimensional case.

THEOREM 4.5. *For any $p > 0$ and any holomorphic $\varphi : \mathbb{D} \to \mathbb{B}_n$ the composition operator C_φ maps $H^p(\mathbb{B}_n)$ boundedly into $A^p_{n-2}(\mathbb{D})$.*

Proof. According to Lemmas 2.4 and 2.8, it suffices for us to show that there exists a constant $C > 0$ such that, for all $r > 0$ and all $\zeta \in \mathbb{S}_n$,

$$\mu_{\varphi,n-2}(Q_r(\zeta)) \leqslant Cr^n.$$

Fix $\zeta \in \mathbb{S}_n$ and $r > 0$. Let us consider the preimage of the set $Q_r(\zeta)$ under φ:

$$\varphi^{-1}(Q_r(\zeta)) = \{z \in \mathbb{D} : |1 - \langle \varphi(z), \zeta\rangle| < r\} = \{z \in \mathbb{D} : |1 - \varphi_\zeta(z)| < r\} = \varphi_\zeta^{-1}(S_r),$$

where $\varphi_\zeta(z) = \langle \varphi(z), \zeta\rangle$ is an analytic self-map of the unit disk $\mathbb{D}$ and $S_r = \{z \in \mathbb{D} : |1 - z| < r\}$ is the one-dimensional version of $Q_r(\zeta)$ in $\mathbb{D}$ at the point 1. It follows that

$$\mu_{\varphi,n-2}(Q_r(\zeta)) = \mu_{\varphi_\zeta,n-2}(S_r) \quad \text{and} \quad \frac{1+|\varphi_\zeta(0)|}{1-|\varphi_\zeta(0)|} \leqslant \frac{1+|\varphi(0)|}{1-|\varphi(0)|}.$$

The desired result then follows from Theorem 3.1. ∎

THEOREM 4.6. *Suppose $p > 0$, $\alpha > -1$, and $\varphi : \mathbb{D} \to \mathbb{B}_n$ is holomorphic. Then the operator C_φ maps $A^p_\alpha(\mathbb{B}_n)$ boundedly into $A^p_{n-1+\alpha}(\mathbb{D})$.*

Proof. The proof is similar to that of Theorem 4.5. We omit the details. ∎

5. NECESSITY FOR COMPACTNESS

Our proof of necessity for compactness uses a standard method involving reproducing kernels.

THEOREM 5.1. *Suppose H_Ω is a Hilbert space of holomorphic functions in Ω with reproducing kernel $K^\Omega(z,w)$, $H_\mathbb{D}$ is a Hilbert space of holomorphic functions in $\mathbb{D}$ with reproducing kernel $K^\mathbb{D}(z,w)$, and $\varphi : \mathbb{D} \to \Omega$ is a holomorphic map with the property that the composition operator C_φ maps H_Ω boundedly into $H_\mathbb{D}$. Then the adjoint operator $C^*_\varphi : H_\mathbb{D} \to H_\Omega$ has the next property whenever $K^\mathbb{D}(z,z) \neq 0$, where $z \in \mathbb{D}$,*

$$\|C^*_\varphi k^\mathbb{D}_z\|^2 = \frac{K^\Omega(\varphi(z),\varphi(z))}{K^\mathbb{D}(z,z)}, \tag{5.1}$$

and the following is the normalized reproducing kernel of $H_\mathbb{D}$ at z:

$$k^\mathbb{D}_z(w) = \frac{K^\mathbb{D}(w,z)}{\sqrt{K^\mathbb{D}(z,z)}}.$$

Proof. For any $z \in \mathbb{D}$ we use $K^\mathbb{D}_z$ to denote the function

$$K^\mathbb{D}_z(w) = K^\mathbb{D}(w,z), \quad w \in \mathbb{D}.$$

Similarly, for any $z \in \Omega$ we write

$$K^\Omega_z(w) = K^\Omega(w,z), \quad w \in \Omega.$$

If $z \in \mathbb{D}$ and $w \in \Omega$, then

$$(C^*_\varphi K^\mathbb{D}_z)(w) = \langle C^*_\varphi K^\mathbb{D}_z, K^\Omega_w\rangle_{H_\Omega} = \langle K^\mathbb{D}_z, K^\Omega_w \circ \varphi\rangle_{H_\mathbb{D}} = \overline{K^\Omega_w(\varphi(z))} = K^\Omega_{\varphi(z)}(w).$$

It follows that $\|C_\varphi^* K_z^{\mathbb{D}}\|^2 = K^\Omega(\varphi(z),\varphi(z))$, and so

$$\|C_\varphi^* k_z^{\mathbb{D}}\|^2 = \frac{K^\Omega(\varphi(z),\varphi(z))}{K^{\mathbb{D}}(z,z)}. \quad \blacksquare$$

It is clear that the above result remains true if $\mathbb{D}$ is replaced by any other domain in one or several complex dimensions.

In each of the four corollaries below, the normalized reproducing kernels $\{k_z^{\mathbb{D}}\}$ all converge to 0 weakly as $|z| \to 1^-$. Therefore, the compactness of the composition operator C_φ implies that

$$\lim_{|z|\to 1^-} \frac{K^\Omega(\varphi(z),\varphi(z))}{K^{\mathbb{D}}(z,z)} = 0. \tag{5.2}$$

COROLLARY 5.2. *Suppose $n > 1$, $p > 0$, and $\varphi : \mathbb{D} \to \mathbb{B}_n$ is holomorphic. If the operator $C_\varphi : H^p(\mathbb{B}_n) \to A^p_{n-2}(\mathbb{D})$ is ultra-weakly compact, then*

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

Proof. If the operator $C_\varphi : H^p(\mathbb{B}_n) \to A^p_{n-2}(\mathbb{D})$ is ultra-weakly compact, then by Section 2, the operator $C_\varphi : H^2(\mathbb{B}_n) \to A^2_{n-2}(\mathbb{D})$ is compact. The reproducing kernels of $H^2(\mathbb{B}_n)$ and $A^2_{n-2}(\mathbb{D})$, respectively, are

$$K^\Omega(z,w) = \frac{1}{(1-\langle z,w\rangle)^n} \quad \text{and} \quad K^{\mathbb{D}}(z,z) = \frac{1}{(1-z\overline{w})^n}.$$

The desired result then follows from (5.2). $\blacksquare$

COROLLARY 5.3. *Suppose $p > 0$, $\alpha > -1$, and $\varphi : \mathbb{D} \to \mathbb{B}_n$ is holomorphic. If the operator $C_\varphi : A^p_\alpha(\mathbb{B}_n) \to A^p_{n-1+\alpha}(\mathbb{D})$ is ultra-weakly compact, then*

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

Proof. According Section 2, the ultra-weak compactness of the operator $C_\varphi : A^p_\alpha(\mathbb{B}_n) \to A^p_{n-1+\alpha}(\mathbb{D})$ implies the compactness of the operator $C_\varphi : A^2_\alpha(\mathbb{B}_n) \to A^2_{n-1+\alpha}(\mathbb{D})$. Since the reproducing kernels of $A^2_\alpha(\mathbb{B}_n)$ and $A^2_{n-1+\alpha}(\mathbb{D})$ are respectively,

$$K^\Omega(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}} \quad \text{and} \quad K^{\mathbb{D}}(z,w) = \frac{1}{(1-z\overline{w})^{n+1+\alpha}},$$

the desired result follows from (5.2). $\blacksquare$

COROLLARY 5.4. *Suppose $n > 1$, $p > 1$, and $\varphi = (\varphi_1, \dots, \varphi_n)$ is a holomorphic map from $\mathbb{D}$ into $\mathbb{D}^n$. If the operator $C_\varphi : H^p(\mathbb{D}^n) \to A^p_{n-2}(\mathbb{D})$ is compact, then*

$$\lim_{|z|\to 1^-} \prod_{k=1}^{n} \frac{1-|z|^2}{1-|\varphi_k(z)|^2} = 0.$$

Proof. By Section 2, the compactness of the operator $C_\varphi : H^p(\mathbb{D}^n) \to A^p_{n-2}(\mathbb{D})$ implies the compactness of the operator $C_\varphi : H^2(\mathbb{D}^n) \to A^2_{n-2}(\mathbb{D})$. The reproducing kernels of $H^2(\mathbb{D}^n)$ and $A^2_{n-2}(\mathbb{D})$ are respectively,

$$K^\Omega(z,w) = \frac{1}{\prod_{k=1}^{n}(1-z_k\overline{w}_k)} \quad \text{and} \quad K^{\mathbb{D}}(z,w) = \frac{1}{(1-z\overline{w})^n}.$$

The desired result follows from (5.2). ∎

COROLLARY 5.5. *Suppose $p > 0$, $\alpha > -1$, and $\varphi = (\varphi_1, \dots, \varphi_n)$ is a holomorphic map from $\mathbb{D}$ into $\mathbb{D}^n$. If the operator $C_\varphi : A^p_\alpha(\mathbb{D}^n) \to A^p_{n(\alpha+2)-2}(\mathbb{D})$ is ultra-weakly compact, then*

$$\lim_{|z|\to 1^-} \prod_{k=1}^{n} \frac{1-|z|^2}{1-|\varphi_k(z)|^2} = 0.$$

Proof. Again, the ultra-weak compactness of the operator $C_\varphi : A^p_\alpha(\mathbb{D}^n) \to A^p_{n(\alpha+2)-2}(\mathbb{D})$ implies the compactness of the operator $C_\varphi : A^2_\alpha(\mathbb{D}^n) \to A^2_{n(\alpha+2)-2}(\mathbb{D})$. The reproducing kernels of $A^2_\alpha(\mathbb{D}^n)$ and $A^2_{n(\alpha+2)-2}(\mathbb{D})$ are respectively,

$$K^\Omega(z,w) = \frac{1}{\prod_{k=1}^{n}(1-z_k\overline{w}_k)^{2+\alpha}} \quad \text{and} \quad K^{\mathbb{D}}(z,w) = \frac{1}{(1-z\overline{w})^{n(\alpha+2)}}.$$

The desired result follows from (5.2) again. ∎

6. TRACE FORMULAS

We obtain four trace formulas in this section and characterize when the operator C_φ, acting on $H^2(\Omega)$ or $A^2_\alpha(\Omega)$, is Hilbert-Schmidt.

LEMMA 6.1. *Suppose $\alpha > -1$ and T is a positive or trace-class operator on $A^2_\alpha(\mathbb{D})$. Then we have, with the inner product being in $A^2_\alpha(\mathbb{D})$,*

$$\operatorname{tr}(T) = \int_{\mathbb{D}} \langle T k^\alpha_z, k^\alpha_z \rangle \, \mathrm{d}\lambda(z), \tag{6.1}$$

where $k_z^\alpha(w)$ is the normalized reproducing kernel of $A^2_\alpha(\mathbb{D})$ at z, and respectively, $\mathrm{d}\lambda(z)$ *is the Möbius invariant measure on $\mathbb{D}$:*

$$k_z^\alpha(w)=\frac{(1-|z|^2)^{(2+\alpha)/2}}{(1-w\overline{z})^{2+\alpha}},\quad \mathrm{d}\lambda(z)=\frac{\mathrm{d}A(z)}{(1-|z|^2)^2}.$$

Proof. See Proposition 6.3.2 of [17] and Lemma 13 of [16]. ∎

Each of the following four theorems follows from (5.1) and (6.1). We omit the routine details.

THEOREM 6.2. *Suppose $n>1$ and $\varphi:\mathbb{D}\to\mathbb{B}_n$ is holomorphic. Then for the composition operator*

$$C_\varphi: H^2(\mathbb{B}_n)\to A^2_{n-2}(\mathbb{D})$$

we have

$$\operatorname{tr}(C_\varphi C_\varphi^*)=\int_{\mathbb{D}}\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^n\,\mathrm{d}\lambda(z).$$

Consequently, C_φ is Hilbert-Schmidt if and only if

$$\int_{\mathbb{D}}\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^n\,\mathrm{d}\lambda(z)<\infty.$$

THEOREM 6.3. *Suppose $\alpha>-1$ and $\varphi:\mathbb{D}\to\mathbb{B}_n$ is holomorphic. Then for the composition operator*

$$C_\varphi: A^2_\alpha(\mathbb{B}_n)\to A^2_{n-1+\alpha}(\mathbb{D})$$

we have

$$\operatorname{tr}(C_\varphi C_\varphi^*)=\int_{\mathbb{D}}\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^{n+1+\alpha}\,\mathrm{d}\lambda(z).$$

Consequently, C_φ is Hilbert-Schmidt if and only if

$$\int_{\mathbb{D}}\Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^{n+1+\alpha}\,\mathrm{d}\lambda(z)<\infty.$$

THEOREM 6.4. *Suppose $n>1$ and $\varphi=(\varphi_1,\dots,\varphi_n)$ is holomorphic from $\mathbb{D}$ into $\mathbb{D}^n$. Then for the composition operator*

$$C_\varphi: H^2(\mathbb{D}^n)\to A^2_{n-2}(\mathbb{D})$$

we have

$$\operatorname{tr}(C_\varphi C_\varphi^*)=\int_{\mathbb{D}}\Big(\prod_{k=1}^n\frac{1-|z|^2}{1-|\varphi_k(z)|^2}\Big)\,\mathrm{d}\lambda(z).$$

Consequently, C_φ is Hilbert-Schmidt if and only if

$$\int_{\mathbb{D}} \Big(\prod_{k=1}^{n} \frac{1-|z|^2}{1-|\varphi_k(z)|^2} \Big) \, \mathrm{d}\lambda(z) < \infty.$$

THEOREM 6.5. *Suppose $\alpha > -1$ and $\varphi = (\varphi_1, \dots, \varphi_n)$ is holomorphic from $\mathbb{D}$ into $\mathbb{D}^n$. Then for the composition operator*

$$C_\varphi : A^2_\alpha(\mathbb{D}^n) \to A^2_{n(\alpha+2)-2}(\mathbb{D})$$

we have

$$\operatorname{tr}(C_\varphi C_\varphi^*) = \int_{\mathbb{D}} \Big(\prod_{k=1}^{n} \frac{1-|z|^2}{1-|\varphi_k(z)|^2} \Big)^{\alpha+2} \mathrm{d}\lambda(z).$$

Therefore, the operator C_φ is Hilbert-Schmidt if and only if

$$\int_{\mathbb{D}} \Big(\prod_{k=1}^{n} \frac{1-|z|^2}{1-|\varphi_k(z)|^2} \Big)^{\alpha+2} \mathrm{d}\lambda(z) < \infty.$$

7. SUFFICIENCY FOR COMPACTNESS WHEN $\Omega = \mathbb{B}_n$

Recall from Section 2 that, for $\alpha > -1$ and $\varphi : \mathbb{D} \to \mathbb{B}_n$ holomorphic, the Borel measure $\mu_{\varphi,\alpha}$ on $\mathbb{B}_n$ is defined by

$$\mu_{\varphi,\alpha}(E) = A_\alpha(\varphi^{-1}(E)) = (\alpha+1) \int_{\varphi^{-1}(E)} (1-|z|^2)^\alpha \, \mathrm{d}A(z),$$

where E is any Borel set in $\mathbb{B}_n$. This definition includes the case $n = 1$ as well.

THEOREM 7.1. *Suppose $n > 1$, $p > 0$, and $\varphi : \mathbb{D} \to \mathbb{B}_n$ is holomorphic. If*

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0, \tag{7.1}$$

then the operator $C_\varphi : H^p(\mathbb{B}_n) \to A^p_{n-2}(\mathbb{D})$ is ultra-weakly compact.

Proof. According to Lemmas 2.3 and 2.8, it suffices for us to show that the next limit holds uniformly for $\zeta \in \mathbb{S}_n$:

$$\lim_{r\to 0^+} \frac{\mu_{\varphi,n-2}(Q_r(\zeta))}{r^n} = 0. \tag{7.2}$$

Using notation from the proof of Theorem 4.5, we have

$$\mu_{\varphi,n-2}(Q_r(\zeta)) = \mu_{\varphi_\zeta,n-2}(S_r), \quad r \in (0,1), \zeta \in \mathbb{S}_n.$$

Let $a = 1 - r$ and consider the normalized reproducing kernels k_a in $A^2_{n-2}(\mathbb{D})$, that is,

$$k_a(z) = \frac{(1-|a|^2)^{n/2}}{(1-z\overline{a})^n}, \quad z \in \mathbb{D}.$$

As $r \to 0^+$, we have $|a| \to 1^-$, and so $\{k_a\}$ converges to 0 weakly in $A^2_{n-2}(\mathbb{D})$. It is easy to find a positive constant C, independent of a and ζ, such that

$$\begin{aligned}\frac{\mu_{\varphi_\zeta,n-2}(S_r)}{r^n} &\leqslant C \int_{S_r} |k_a(z)|^2 \,\mathrm{d}\mu_{\varphi_\zeta,n-2}(z) \\ &\leqslant C \int_{\mathbb{D}} |k_a(z)|^2 \,\mathrm{d}\mu_{\varphi_\zeta,n-2}(z) + C \int_{\mathbb{D}} |k_a(\varphi_\zeta(z))|^2 \,\mathrm{d}A_{n-2}(z).\end{aligned}$$

Since

$$\frac{1-|z|^2}{1-|\varphi_\zeta(z)|^2} \leqslant \frac{1-|z|^2}{1-|\varphi(z)|^2},$$

carefully checking the proof of Theorem 3.2 shows that the limit (7.2) holds uniformly for $\zeta \in \mathbb{S}_n$. ∎

THEOREM 7.2. *Suppose $p > 0$, $\alpha > -1$, and $\varphi : \mathbb{D} \to \mathbb{B}_n$ is holomorphic. If*

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0,$$

then the next composition operator is ultra-weakly compact:

$$C_\varphi : A^p_\alpha(\mathbb{B}_n) \to A^p_{n-1+\alpha}(\mathbb{D}).$$

Proof. The proof is similar to that of Theorem 7.1. We omit the details. ∎

8. SUFFICIENCY FOR COMPACTNESS WHEN $\Omega = \mathbb{D}^n$

Our proof of the compactness of C_φ on $H^p(\mathbb{D}^n)$ and $A^p_\alpha(\mathbb{D}^n)$ depends on the following classical norm estimate for integral operators with positive kernel.

LEMMA 8.1. *If there exists a constant $C > 0$ and a positive function h on the unit disc $\mathbb{D}$ such that*

$$\int_{\mathbb{D}} K_r(z,w)h(w)\,\mathrm{d}A_\beta(w) \leqslant Ch(z)$$

for all $z \in \mathbb{D}$, then the integral operator T_r is bounded on $L^2(\mathbb{D}, \mathrm{d}A_\beta)$ and its norm satisfies $\|T_r\| \leqslant C$.

Proof. This is a special case of Schur's test. See 3.2.2 of [17]. ∎

We also need the following generalization of the classical Hölder's inequality.

LEMMA 8.2. *Let (X,μ) be a measure space. For each $1 \leqslant k \leqslant n$ let $p_k > 0$ and $f_k \in L^{p_k}(X, \mathrm{d}\mu)$. If*

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1,$$

then

$$\Big| \int_X \prod_{k=1}^n f_k(x)\, \mathrm{d}\mu(x) \Big| \leqslant \prod_{k=1}^n \Big[\int_X |f_k(x)|^{p_k}\, \mathrm{d}\mu(x) \Big]^{1/p_k}.$$

Proof. See [5] or Lemma 4.44 of [18]. ∎

Finally, we are going to need the following integral estimate, which has become indispensable for analysis on the unit ball.

LEMMA 8.3. *Suppose $t > -1$ and $\sigma > 0$. Then there exists a constant $C > 0$ such that, for all $z \in \mathbb{D}$, we have:*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t\, \mathrm{d}A(w)}{|1 - z\overline{w}|^{2+t+\sigma}} \leqslant \frac{C}{(1-|z|^2)^\sigma}.$$

Proof. See [13] or Lemma 4.22 of [18]. ∎

Suppose $\alpha \geqslant -1$ (yes, $\alpha = -1$ is permitted here) and set

$$\beta = n(\alpha+2) - 2.$$

We assume $n > 1$ in the rest of this section, so that we always have $\beta > -1$. Consider the integral operator

$$T : L^2(\mathbb{D}, \mathrm{d}A_\beta) \to L^2(\mathbb{D}, \mathrm{d}A_\beta)$$

defined by

$$Tf(z) = \int_{\mathbb{D}} \frac{f(w)\, \mathrm{d}A_\beta(w)}{\prod_{k=1}^n (1 - \varphi_k(z)\overline{\varphi_k(w)})^{\alpha+2}}.$$

For any $r \in (0,1)$ we let χ_r denote the characteristic function of the annulus $r \leqslant |z| < 1$ in the complex plane. We also consider the following integral operator on $L^2(\mathbb{D}, \mathrm{d}A_\beta)$:

$$T_r f(z) = \int_{\mathbb{D}} K_r(z,w) f(w)\, \mathrm{d}A_\beta(w), \quad \text{where } K_r(z,w) = \frac{\chi_r(z)\chi_r(w)}{\prod_{k=1}^n |1 - \varphi_k(z)\overline{\varphi_k(w)}|^{\alpha+2}}.$$

We are going to show that each T_r is bounded on $L^2(\mathbb{D}, \mathrm{d}A_\beta)$ and we are going to estimate the norm of

$$T_r : L^2(\mathbb{D}, \mathrm{d}A_\beta) \to L^2(\mathbb{D}, \mathrm{d}A_\beta)$$

in terms of the constant

$$M_r = \sup_{r\leqslant|z|<1}\prod_{k=1}^{n}\frac{1-|z|^2}{1-|\varphi_k(z)|^2}. \tag{8.1}$$

THEOREM 8.4. *There exist positive constants C and δ, independent of r, such that the norm of the operator*

$$T_r : L^2(\mathbb{D}, \mathrm{d}A_\beta) \to L^2(\mathbb{D}, \mathrm{d}A_\beta)$$

satisfies $\|T_r\| \leqslant CM_r^\delta$ for all $0 < r < 1$, where M_r is the constant defined in (8.1).

Proof. Let $h(z) = (1 - |z|^2)^{-\sigma}$, where σ is any positive number satisfying

$$t = \beta - \sigma = n(\alpha + 2) - 2 - \sigma > -1.$$

The existence of such a σ is guaranteed by the assumptions that $n > 1$ and $\alpha \geqslant -1$. Consider the integral

$$I_r(z) = \int_{\mathbb{D}} K_r(z, w)h(w)\,\mathrm{d}A_\beta(w), \quad z \in \mathbb{D}.$$

It is clear that $I_r(z) \leqslant c_\beta\chi_r(z)\int_{\mathbb{D}}\frac{(1-|w|^2)^t\,\mathrm{d}A(w)}{\prod_{k=1}^{n}|1-\varphi_k(z)\overline{\varphi_k(w)}|^{\alpha+2}}$. According to Lemma 8.2, we have $I_r(z) \leqslant c_\beta\chi_r(z)\prod_{k=1}^{n}\Big[\int_{\mathbb{D}}\frac{(1-|w|^2)^t\,\mathrm{d}A(w)}{|1-\varphi_k(z)\overline{\varphi_k(w)}|^{n(\alpha+2)}}\Big]^{1/n}$. By Theorem 3.1, there exists a constant $C_1 > 0$, independent of r and z, such that $\int_{\mathbb{D}}\frac{(1-|w|^2)^t\,\mathrm{d}A(w)}{|1-\varphi_k(z)\overline{\varphi_k(w)}|^{n(\alpha+2)}} \leqslant C_1\int_{\mathbb{D}}\frac{(1-|w|^2)^t\,\mathrm{d}A(w)}{|1-\varphi_k(z)\overline{w}|^{n(\alpha+2)}}$. By Lemma 8.3, there exists another constant $C_2 > 0$, independent of r and z, such that $\int_{\mathbb{D}}\frac{(1-|w|^2)^t\,\mathrm{d}A(w)}{|1-\varphi_k(z)\overline{w}|^{n(\alpha+2)}} \leqslant \frac{C_2}{(1-|\varphi_k(z)|^2)^\sigma}$ for all $z \in \mathbb{D}$ and $1 \leqslant k \leqslant n$. It follows that there exists a constant $C > 0$, independent of r and z, such that

$$I_r(z) \leqslant \frac{C\chi_r(z)}{\prod_{k=1}^{n}(1-|\varphi_k(z)|^2)^{\sigma/n}} = C\chi_r(z)\Big[\prod_{k=1}^{n}\frac{1-|z|^2}{1-|\varphi_k(z)|^2}\Big]^{\sigma/n}h(z) \leqslant CM_r^{\sigma/n}h(z).$$

We conclude from Lemma 8.1 that T_r is bounded on $L^2(\mathbb{D}, \mathrm{d}A_\beta)$ and its norm satisfies $\|T_r\| \leqslant CM_r^\delta$, where $\delta = \sigma/n$. ∎

We can now finish the proof of Theorems 1.3 and 1.4. Once again, we only need to consider the case $p = 2$.

THEOREM 8.5. *Suppose $n > 1$, $\alpha > -1$, and $\varphi = (\varphi_1, \ldots, \varphi_n) : \mathbb{D} \to \mathbb{D}^n$ satisfies the condition*

$$\lim_{|z|\to1^-}\prod_{k=1}^{n}\frac{1-|z|^2}{1-|\varphi_k(z)|^2} = 0. \tag{8.2}$$

Then the composition operator C_φ is compact from $H^2(\mathbb{D}^n)$ into $A^2_{n-2}(\mathbb{D})$; and it is also compact from $A^2_\alpha(\mathbb{D}^n)$ into $A^2_\beta(\mathbb{D})$, where $\beta = n(\alpha+2)-2$.

Proof. It is easy to represent the adjoint of the composition operator $C_\varphi : H^2(\mathbb{D}^n) \to A^2_{n-2}(\mathbb{D})$ as an integral operator, from which we easily obtain the following integral representation:

$$(C_\varphi C_\varphi^* f)(z) = \int_{\mathbb{D}} \frac{f(w)\,\mathrm{d}A_{n-2}(w)}{\prod_{k=1}^{n}(1-\varphi_k(z)\overline{\varphi_k(w)})}, \quad f \in A^2_{n-2}(\mathbb{D}), z \in \mathbb{D}.$$

Similarly, the composition operator $C_\varphi : A^2_\alpha(\mathbb{D}^n) \to A^2_{n(\alpha+2)-2}(\mathbb{D})$ has the following integral representation:

$$(C_\varphi C_\varphi^* f)(z) = \int_{\mathbb{D}} \frac{f(w)\,\mathrm{d}A_{n(\alpha+2)-2}(w)}{\prod_{k=1}^{n}(1-\varphi_k(z)\overline{\varphi_k(w)})^{\alpha+2}},$$

where $f \in A^2_{n(\alpha+2)-2}(\mathbb{D})$ and $z \in \mathbb{D}$. So it suffices for us to show that the integral operator T defined a little earlier is compact on the Bergman space $A^2_\beta(\mathbb{D})$. Here and below we want to allow α to be -1 in the definition of β, so that we can prove the compactness of

$$C_\varphi : H^2(\mathbb{D}^n) \to A^2_{n-2}(\mathbb{D}) \quad \text{and} \quad C_\varphi : A^2_\alpha(\mathbb{D}^n) \to A^2_{n(\alpha+2)-2}(\mathbb{D})$$

simultaneously. To this end, we suppose that $\{f_k\}$ is a sequence in $A^2_\beta(\mathbb{D})$ that converges to 0 weakly as $k \to \infty$. Then $\{Tf_k\}$ also converges to 0 weakly in $A^2_\beta(\mathbb{D})$. It is easy to see that a sequence in $A^2_\beta(\mathbb{D})$ converges to 0 weakly if and only if it is bounded in the norm topology and converges to 0 uniformly on compact subsets of $\mathbb{D}$.

Fix any $r \in (0,1)$ and write

$$\|Tf_k\|^2 = \int_{|z|<r} |Tf_k(z)|^2\,\mathrm{d}A_\beta(z) + \int_{\mathbb{D}} |\chi_r(z)Tf_k(z)|^2\,\mathrm{d}A_\beta(z), \tag{8.3}$$

where the norm is taken in the Bergman space $A^2_\beta(\mathbb{D})$ and χ_r is the characteristic function of the annulus $\{z \in \mathbb{D} : r < |z| < 1\}$. We have $\lim_{k\to\infty} \int_{|z|<r} |Tf_k(z)|^2\,\mathrm{d}A_\beta(z) = 0$, because $\{Tf_k\}$ converges to 0 uniformly on $|z| < r$.

On the other hand, we can write

$$\chi_r(z)Tf_k(z) = F_{r,k}(z) + G_{r,k}(z),$$

where

$$F_{r,k}(z) = \chi_r(z) \int_{|w|<r} \frac{f_k(w)\,\mathrm{d}A_\beta(w)}{\prod_{i=1}^{n}(1-\varphi_i(z)\overline{\varphi_i(w)})^{\alpha+2}}, \quad G_{r,k}(z) = \int_{\mathbb{D}} \frac{\chi_r(z)\chi_r(w)f_k(w)\,\mathrm{d}A_\beta(w)}{\prod_{i=1}^{n}(1-\varphi_i(z)\overline{\varphi_i(w)})^{\alpha+2}}.$$

It is clear that $\lim_{k\to\infty} F_{r,k}(z) = 0$ uniformly for $z \in \mathbb{D}$, so $\lim_{k\to\infty} \int_{\mathbb{D}} |F_{r,k}(z)|^2 \, dA_\beta(z) = 0$. It is also clear that $\int_{\mathbb{D}} |G_{r,k}(z)|^2 \, dA_\beta(z) \leqslant \int_{\mathbb{D}} |T_r(|f_k|)(z)|^2 \, dA_\beta(z)$. By Theorem 8.4 and the assumption that $\{f_k\}$ is bounded in $A^2_\beta(\mathbb{D})$, we can find positive constants C and δ, independent of r, such that $\int_{\mathbb{D}} |G_{r,k}(z)|^2 \, dA_\beta(z) \leqslant CM_r^\delta$ for all k and all r. Letting $k \to \infty$ in (8.3) now, we obtain $\limsup_{k\to\infty} \|Tf_k\|^2 \leqslant CM_r^\delta$. Since r is arbitrary, and since the condition in (8.2) implies that $M_r \to 0$ as $r \to 1^-$, we must have $\limsup_{k\to\infty} \|Tf_k\|^2 = 0$. This shows that $\lim_{k\to\infty} \|Tf_k\| = 0$, and so T is compact on $A^2_\beta(\mathbb{D})$. ∎

Acknowledgements. Research of Kehe Zhu is partially supported by the US National Science Foundation.

We thank the referee of the paper for pointing out that Theorem 1.5 has already appeared in the paper [14].

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Received February 10, 2005.