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INVOLUTIVE EQUIVALENCE BIMODULES AND INCLUSIONS OF C^*-ALGEBRAS WITH WATATANI INDEX 2

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ABSTRACT. Let A be a unital C^*-algebra. We shall introduce involutive A-A-equivalence bimodules and prove that any C^*-algebra containing A with Watatani index 2 is constructed by an involutive A-A-equivalence bimodule.

KEYWORDS: *Conditional expectations, equivalence bimodules, Goldman type theorem, unital C^*-algebras, Watatani index.*

MSC (2000): Primary 46L08; Secondary 46L40.

1. INTRODUCTION

V. Jones introduced an index theory for II_1 factors in [6]. One of his motivations is Goldman's theorem, which says that if M is a type II_1 factor and $N \subset M$ is a subfactor with the Jones index $[M : N] = 2$, then there is a crossed product decomposition $M = N \rtimes_\alpha \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the group $\mathbb{Z}/2\mathbb{Z}$ of order two. Since Jones index theory is extended to C^*-algebras by Y. Watatani, it is worth to investigate Goldman type theorems for inclusions of simple C^*-algebras. In the present paper, we shall study the inclusion $A \subset B$ of C^*-algebras with a conditional expectation $E : B \to A$ of Index $E = 2$. In Subsection 4.2, we shall show that a Goldman type theorem does not hold for inclusions of simple C^*-algebras in general by exhibiting examples of inclusions like a non-commutative sphere in an irrational rotation C^*-algebra A_θ and irrational rotation C^*-algebras $A_{2\theta} \subset A_\theta$ with different angles. Therefore there occurs the following natural question: What kind of structures are there in the inclusion of C^*-algebras with index 2? We shall answer the question in the present paper: Any inclusion of C^*-algebras with index two gives an involutive equivalence bimodule.

Let us explain the notion of involutive equivalence bimodules. Consider a typical situation, that is, the inclusion $A \subset B$ is given by the crossed product $B = A \rtimes_\alpha \mathbb{Z}_2$ by some action $\alpha : \mathbb{Z}_2 \to \mathrm{Aut}(A)$. Then the canonical conditional expectation $E : B \to A$ has Index $E = 2$. Moreover there exists the dual action $\widehat{\alpha} : \mathbb{Z}_2 \to \mathrm{Aut}(B)$ such that

$$(A \rtimes_\alpha \mathbb{Z}_2) \rtimes_{\widehat{\alpha}} \mathbb{Z}_2 \cong A \otimes M_2(\mathbb{C}),$$

where $M_2(\mathbb{C})$ is the 2×2-matrix algebra over $\mathbb{C}$. It is well known that the C^*-basic construction $C^*\langle B, e_A \rangle$ is exactly $(A \rtimes_\alpha \mathbb{Z}_2) \rtimes_{\widehat{\alpha}} \mathbb{Z}_2$. Then the Jones projection e_A corresponds to the projection $e_{11} = \mathrm{diag}(1,0)$ and $1 - e_A$ corresponds to $e_{22} = \mathrm{diag}(0,1)$, where $\mathrm{diag}(\lambda, \mu)$ is a 2×2-diagonal matrix with diagonal elements λ, μ. Let $X = e_{11}(A \otimes M_2(\mathbb{C}))e_{22}$. Then X is an A-A-equivalence bimodule in the natural way. There exists a natural involution on X such that

$$x^\sharp = \begin{pmatrix} 0 & z^* \\ 0 & 0 \end{pmatrix} \quad \text{for } x = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}.$$

We pick up these properties to define the notion of involutive equivalence bimodules. In Theorem 3.3.1, we shall show that even if B is not a crossed product of A, the inclusion of C^*-algebras with index 2 gives an involutive A-A-equivalence bimodule. Moreover the set of inclusions of C^*-algebras with index 2 has a one to one correspondence with the set of involutive A-A-equivalence bimodules up to isomorphisms.

In Proposition 4.1.2, we shall characterize the subclass such that B is the twisted crossed product of A by a partially inner C^*-dynamical system studied by Green, Olsen and Pedersen. The characterization is given by the von Neumann equivalence of e_A and $1 - e_A$ in $C^*\langle B, e_A \rangle$.

2. PRELIMINARIES

2.1. SOME RESULTS FOR INCLUSIONS WITH INDEX 2. Let B be a unital C^*-algebra and A a C^*-subalgebra of B with a common unit. Let E be a conditional expectation of B onto A with $1 < \mathrm{Index}E < \infty$. Then by Watatani [12] we have the C^*-basic construction $C^*\langle B, e_A \rangle$ where e_A is the Jones projection induced by E. Let $\widetilde{E}$ be the dual conditional expectation of $C^*\langle B, e_A \rangle$ onto B defined by

$$\widetilde{E}(ae_Ab) = \frac{1}{t}ab \quad \text{for any } a, b \in B,$$

where $t = \mathrm{Index}E$. Let F be a linear map of $(1 - e_A)C^*\langle B, e_A \rangle(1 - e_A)$ to $A(1 - e_A)$ defined by

$$F(a) = \frac{t}{t-1}(E \circ \widetilde{E})(a)(1 - e_A)$$

for any $a \in (1 - e_A)C^*\langle B, e_A \rangle(1 - e_A)$. By routine computations we can see that F is a conditional expectation of $(1 - e_A)C^*\langle B, e_A \rangle(1 - e_A)$ onto $A(1 - e_A)$.

LEMMA 2.1. *With the above notations, let* $\{(x_i, x_i^*)\}_{i=1}^n$ *be a quasi-basis for* E. *Then*

$$\{\sqrt{t-1}(1-e_A)x_j e_A x_i(1-e_A), \sqrt{t-1}(1-e_A)x_i^* e_A x_j^*(1-e_A)\}_{i,j=1}^n$$

is a quasi-basis for F. *Furthermore* $\mathrm{Index}F = (t-1)^2(1-e_A)$.

Proof. This is immediate by direct computations. ∎

COROLLARY 2.2. *We suppose that* $\mathrm{Index}E = 2$. *Then*

$$(1-e_A)C^*\langle B, e_A\rangle(1-e_A) = A(1-e_A) \cong A.$$

Proof. By Lemma 2.1 there is a conditional expectation F of $(1-e_A)C^*\langle B, e_A\rangle(1-e_A)$ onto $A(1-e_A)$ and

$$\mathrm{Index}F = (\mathrm{Index}E - 1)^2(1-e_A).$$

Since $\mathrm{Index}E = 2$, $\mathrm{Index}F = 1 - e_A$. Hence by Watatani [12],

$$(1-e_A)C^*\langle B, e_A\rangle(1-e_A) = A(1-e_A).$$

If $a(1-e_A) = 0$, for $a \in A$, then $a = 2\widetilde{E}(a(1-e_A)) = 0$. Therefore the map $a \mapsto a(1-e_A)$ is injective. And hence $A(1-e_A) \cong A$ as desired. ∎

LEMMA 2.3. *With the same assumptions as in Lemma* 2.1, *we suppose that* $\mathrm{Index}E = 2$. *Then for any* $b \in B$,

$$(1-e_A)b(1-e_A) = E(b)(1-e_A).$$

Proof. By Corollary 2.2 there exists $a \in A$ such that $(1-e_A)b(1-e_A) = a(1-e_A)$. Therefore $a = 2\widetilde{E}(a(1-e_A)) = 2\widetilde{E}((1-e_A)b(1-e_A)) = E(b)$. This completes the proof. ∎

PROPOSITION 2.4. *With the same assumptions as in Lemma* 2.1, *we suppose that* $\mathrm{Index}E = 2$. *Then there is a unitary element* $U \in C^*\langle B, e_A\rangle$ *satisfying the following conditions:*

(i) $U^2 = 1$;

(ii) $UbU^* = 2E(b) - b$ *for* $b \in B$.

Hence if we denote by β *the restriction of* $\mathrm{Ad}(U)$ *to* B, β *is an automorphism of* B *with* $\beta^2 = \mathrm{id}$ *and* $B^\beta = A$.

Proof. By Lemma 2.3, for any $b \in B$

$$(1-e_A)b(1-e_A) = E(b)(1-e_A) = E(b) - E(b)e_A.$$

On the other hand

$$(1-e_A)b(1-e_A) = b - e_A b - b e_A + E(b)e_A.$$

Therefore

$$E(b) = b - e_A b - b e_A + 2E(b)e_A.$$

Let U be a unitary element defined by $U = 2e_A - 1$. Then by the above equation for any $b \in B$

$$UbU^* = 2(b - e_A b - be_A + 2E(b)e_A) - b = 2E(b) - b. \quad \blacksquare$$

REMARK 2.5. By the above proposition, $E(b) = \frac{1}{2}(b + \beta(b))$.

LEMMA 2.6. *Let B be a unital C^*-algebra and A a C^*-subalgebra of B with a common unit. Let E be a conditional expectation of B onto A with* $\mathrm{Index}E = 2$. *Then we have*

$$C^*\langle B, e_A\rangle \cong B \rtimes_\beta \mathbb{Z}_2.$$

Proof. We may assume that $B \rtimes_\beta \mathbb{Z}_2$ acts on the Hilbert space $l^2(\mathbb{Z}_2, H)$ faithfully, where H is some Hilbert space on which B acts faithfully. Let W be a unitary element in $B \rtimes_\beta \mathbb{Z}_2$ with $\beta = \mathrm{Ad}(W)$, $W^2 = 1$. Let $e = \frac{1}{2}(W + 1)$. Then e is a projection in $B \rtimes_\beta \mathbb{Z}_2$ and $ebe = E(b)e$ for any $b \in B$. In fact,

$$ebe = \frac{1}{4}(WbW + bW + Wb + b).$$

On the other hand by Remark 2.5,

$$E(b)e = \frac{1}{2}(b + \beta(b))\frac{1}{2}(W + 1) = \frac{1}{4}(WbW + bW + Wb + b).$$

Hence $ebe = E(b)e$ for $b \in B$. Also $A \ni a \mapsto ae \in B \rtimes_\beta \mathbb{Z}_2$ is injective. In fact, if $ae = 0$, $aW + a = 0$. Let $\widehat{\beta}$ be the dual action of β. Then $0 = \widehat{\beta}(aW + a) = -aW + a$. Thus $2a = 0$, i.e., $a = 0$. Hence by Watatani ([12], Proposition 2.2.11) $C^*\langle B, e_A\rangle \cong B \rtimes_\beta \mathbb{Z}_2$. $\blacksquare$

REMARK 2.7. (i) By the proofs of Propositions 2.2.7 and 2.2.11 in [12] we see that $\kappa(b) = b$ for any $b \in B$, where κ is the isomorphism of $C^*\langle B, e_A\rangle$ onto $B \rtimes_\beta \mathbb{Z}_2$ in Lemma 2.6.

(ii) The above lemma is obtained in Kajiwara and Watatani ([7], Theorem 5.13).

By Lemma 2.6 and Remark 2.7, we regard $\widehat{\beta}$ as an automorphism of $C^*\langle B, e_A\rangle$ with $\widehat{\beta}(b) = b$ for any $b \in B$, $\widehat{\beta}^2 = \mathrm{id}$ and $\widehat{\beta}(e_A) = 1 - e_A$.

LEMMA 2.8. *With the same assumptions as in Lemma* 2.6,

$$C^*\langle B, e_A\rangle^{\widehat{\beta}} = B.$$

Proof. By Lemma 2.6 for any $x \in C^*\langle B, e_A\rangle$, we can write $x = b_1 + b_2U$, where $b_1, b_2 \in B$. We suppose that $\widehat{\beta}(x) = x$. Then $b_1 - b_2U = b_1 + b_2U$. Thus $b_2 = 0$. Hence $x = b_1 \in B$. Since it is clear that $B \subset C^*\langle B, e_A\rangle^{\widehat{\beta}}$, the lemma is proved. $\blacksquare$

2.2. INVOLUTIVE EQUIVALENCE BIMODULES. Let A be a unital C^*-algebra and $X(= {}_AX_A)$ an A-A-equivalence bimodule. X is *involutive* if there exists a conjugate linear map $x \mapsto x^\sharp$ on X, such that:

(1) $(x^\sharp)^\sharp = x, x \in X$;
(2) $(a \cdot x \cdot b)^\sharp = b^* \cdot x^\sharp \cdot a^*, x \in X, a, b \in A$;
(3) ${}_A\langle x, y^\sharp \rangle = \langle x^\sharp, y \rangle_A, x, y \in X$;

where ${}_A\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_A$ are the left and the right A-valued inner products on X, respectively. We call the above conjugate linear map an involution on X.

For an A-A-equivalence bimodule X, we define its dual bimodule. Let $\widetilde{X}$ be X itself when it is considered as a set. We write $\widetilde{x}$ when x is considered in $\widetilde{X}$. $\widetilde{X}$ is made into an equivalence A-A-bimodule as follows:

(1) $\widetilde{x} + \widetilde{y} = \widetilde{x + y}\ \lambda \widetilde{x} = \widetilde{\overline{\lambda} x}$ for any $x, y \in X$ and $\lambda \in \mathbb{C}$;
(2) $b \cdot \widetilde{x} \cdot a = \widetilde{a^* \cdot x \cdot b^*}$ for any $a, b \in A$ and $x \in X$;
(3) ${}_A\langle \widetilde{x}, \widetilde{y} \rangle = \langle x, y \rangle_A$, $\langle \widetilde{x}, \widetilde{y} \rangle_A = {}_A\langle x, y \rangle$ for any $x, y \in X$.

LEMMA 2.9. *Let V be a map of an involutive A-A-equivalence bimodule X onto its dual bimodule $\widetilde{X}$ defined by $V(x) = \widetilde{x^\sharp}$, where $\widetilde{x}$ means x as viewed as an element in $\widetilde{X}$. Then V is an A-A-equivalence bimodule isomorphism of X onto $\widetilde{X}$.*

Proof. This is immediate by routine computations. ∎

3. CORRESPONDENCE BETWEEN INVOLUTIVE EQUIVALENCE BIMODULES AND INCLUSIONS OF C^*-ALGEBRAS WITH INDEX 2

Let A be a unital C^*-algebra and we denote by (B, E) a pair of a unital C^*-algebra B including A as a C^*-subalgebra of B with a common unit and a conditional expectation E of B onto A with $\mathrm{Index}E = 2$. Let $\mathcal{L}$ be the set of all such pairs (B, E) as above. We define an equivalence relation $\sim$ in $\mathcal{L}$ as follows: for $(B, E), (B_1, E_1) \in \mathcal{L}$, $(B, E) \sim (B_1, E_1)$ if and only if there is an isomorphism π of B onto B_1 such that $\pi(a) = a$ for any $a \in A$ and $E_1 \circ \pi = E$. We denote by $[B, E]$ the equivalence class of (B, E).

Let $\mathcal{M}$ be the set of all involutive A-A-equivalence bimodules. We define an equivalence relation $\sim$ in $\mathcal{M}$ as follows: for $X, Y \in \mathcal{M}$, $X \sim Y$ if and only if there is an A-A-equivalence bimodule isomorphism ρ of X onto Y with $\rho(x^\sharp) = \rho(x)^\sharp$. We call ρ an involutive A-A-equivalence bimodule isomorphism of X onto Y. We denote by $[X]$ the equivalence class of X.

3.1. CONSTRUCTION OF A MAP FROM $\mathcal{L}/\sim$ TO $\mathcal{M}/\sim$. We shall use the same notations as in Section 2.

Let B be a unital C^*-algebra and A a C^*-subalgebra of B with a common unit. Let E be a conditional expectation of B onto A with $\mathrm{Index}E = 2$. Then, by Watatani [12] and Corollary 2.2, we have:

(1) $e_A C^*\langle B, e_A\rangle e_A = Ae_A \cong A$;
(2) $(1-e_A)C^*\langle B, e_A\rangle(1-e_A) = A(1-e_A) \cong A$.

Let ψ be an isomorphism of A onto Ae_A defined by $\psi(a) = ae_A$ for any $a \in A$ and ϕ an isomorphism of A onto $A(1-e_A)$ defined by $\phi = \widehat{\beta}\circ\psi$. Let $X_{(B,E)} = X_B = e_A C^*\langle B, e_A\rangle(1-e_A)$. We regard X_B as a Hilbert A-A-bimodule in the following way: for any $a, b \in A$ and $x \in X_B$, $a\cdot x\cdot b = \psi(a)x\phi(b) = axb$. For any $x, y \in X_B$, ${}_A\langle x, y\rangle = \psi^{-1}(xy^*)$, $\langle x, y\rangle_A = \phi^{-1}(x^*y)$.

LEMMA 3.1. *With the above notations, X_B is an A-A-equivalence bimodule.*

Proof. This is immediate by routine computations. ∎

Let $x \mapsto x^\sharp$ be a conjugate linear map on X_B defined by $x^\sharp = \widehat{\beta}(x^*)$ for any $x \in X_B$. Since $\widehat{\beta}^2 = \mathrm{id}$, $(x^\sharp)^\sharp = x$. Since $\widehat{\beta}(a) = a$ for any $a \in A$, $(a\cdot x\cdot b)^\sharp = \widehat{\beta}(b^*x^*a^*) = b^*\cdot x^\sharp\cdot a^*$ for $x \in X$, $a, b \in A$. Furthermore, for $x, y \in X_B$ ${}_A\langle x, y^\sharp\rangle = \langle x^\sharp, y\rangle_A$ by an easy calculation. Therefore X_B is an element in $\mathcal{M}$.

REMARK 3.2. $\widetilde{X}_B$ is isomorphic to $(1-e_A)C^*\langle B, e_A\rangle e_A$ as A-A-equivalence bimodules. Indeed, the map $(1-e_A)C^*\langle B, e_A\rangle e_A \ni (1-e_A)xe_A \mapsto \widetilde{e_A x^*(1-e_A)}$, $x \in C^*\langle B, e_A\rangle$ gives an A-A-equivalence bimodule isomorphism of $(1-e_A)C^*\langle B, e_A\rangle e_A$ onto $\widetilde{X}_B$, where $\widetilde{y}$ means y viewed as an element in $\widetilde{X}_B$ for any $y \in X_B$. Sometimes, we identify $\widetilde{X}_B$ with $(1-e_A)C^*\langle B, e_A\rangle e_A$.

Let $\mathcal{F}$ be a map from $\mathcal{L}/\sim$ to $\mathcal{M}/\sim$ defined by $\mathcal{F}([B, E]) = [X_B]$ for any $[B, E] \in \mathcal{L}/\sim$.

LEMMA 3.3. *With the above notations, $\mathcal{F}$ is well-defined.*

Proof. Let $(B, E), (B_1, E_1) \in \mathcal{L}$ with $(B, E) \sim (B_1, E_1)$. Let X_B and X_{B_1} be elements in $\mathcal{M}$ defined by (B, E) and (B_1, E_1), respectively. Since $(B, E) \sim (B_1, E_1)$, there is an isomorphism π of B onto B_1 such that $\pi(a) = a$ for any $a \in A$ and $E_1\circ\pi = E$. Let $\widetilde{\pi}$ be a homomorphism of the linear span of $\{be_Ac : b, c \in B\}$ to $C^*\langle B_1, e_{A,1}\rangle$ defined by $\widetilde{\pi}(be_Ac) = \pi(b)e_{A,1}\pi(c)$ for any $b, c \in B$. Then, for $b_i, c_i \in B$ $(i = 1, 2, \ldots, n)$ and $a \in B$, we have:

$$\begin{aligned}\Big\|\widetilde{\pi}\Big(\sum_{i=1}^n b_ie_Ac_i\Big)\pi(a)\Big\|^2 &= \Big\|\sum_{i=1}^n \pi(b_i)E_1(\pi(c_ia))\Big\|^2\\ &= \Big\|\sum_{i,j=1}^n E_1(\pi(a^*c_i^*))E_1(\pi(b_i^*b_j))E_1(\pi(c_ja))\Big\|\\ &= \Big\|\sum_{i,j=1}^n E(a^*c_i^*)E(b_i^*b_j)E(c_ja)\Big\|.\end{aligned}$$

On the other hand

$$\Big\|\sum_{i=1}^n b_ie_Ac_ia\Big\|^2 = \Big\|\sum_{i=1}^n b_iE(c_ia)\Big\|^2 = \Big\|\sum_{i,j=1}^n E(a^*c_i^*)E(b_i^*b_j)E(c_ja)\Big\|.$$

Hence

$$\Big\|\widetilde{\pi}\Big(\sum_{i=1}^n b_i e_A c_i\Big)\Big\| = \sup\Big\{\Big\|\widetilde{\pi}\Big(\sum_{i=1}^n b_i e_A c_i\Big)\pi(a)\Big\| : \|E_1(\pi(a)^*\pi(a))\| = 1, a\in B\Big\}$$

$$= \sup\Big\{\Big\|\sum_{i=1}^n b_i e_A c_i a\Big\| : \|E(a^*a)\| = 1, a\in B\Big\} = \Big\|\sum_{i=1}^n b_i e_A c_i\Big\|.$$

Thus $\widetilde{\pi}$ can be extended to an isomorphism of $C^*\langle B, e_A\rangle$ onto $C^*\langle B_1, e_{A,1}\rangle$. Hence $\widetilde{\pi}$ is an involutive A-A-equivalence bimodule isomorphism of X_B onto X_{B_1} since $\widetilde{\pi}(e_A) = e_{A,1}$. In fact, for $a\in A$ and $x\in C^*\langle B, e_A\rangle$

$$\widetilde{\pi}(a\cdot e_A x(1-e_A)) = e_{A,1}a\cdot \pi(x)(1-e_{A,1}) = a\cdot\widetilde{\pi}(e_A x(1-e_A)).$$

Similarly

$$\widetilde{\pi}(e_A x(1-e_A)\cdot a) = \widetilde{\pi}(e_A x(1-e_A))\cdot a.$$

Also, for $x, y\in C^*\langle B, e_A\rangle$, we have:

$$\begin{aligned} {}_A\langle\widetilde{\pi}(e_A x(1-e_A)), \widetilde{\pi}(e_A y(1-e_A))\rangle &= (\psi_1^{-1}\circ\widetilde{\pi}\Big)(e_A x(1-e_A)y^* e_A) \\ &= {}_A\langle e_A x(1-e_A), e_A y(1-e_A)\rangle, \\ \langle\widetilde{\pi}(e_A x(1-e_A)), \widetilde{\pi}(e_A y(1-e_A))\rangle_A &= \phi^{-1}((1-e_A)x^* e_A y(1-e_A)) \\ &= \langle e_A x(1-e_A), e_A y(1-e_A)\rangle_A, \end{aligned}$$

since $\psi_1^{-1} = \widetilde{\pi}\circ\psi$ and $\widetilde{\pi}\circ\widehat{\beta} = \widehat{\beta}_1\circ\widetilde{\pi}$. Furthermore, for any $x\in C^*\langle B, e_A\rangle$

$$\begin{aligned} \widetilde{\pi}((e_A x(1-e_A))^\sharp) &= \widetilde{\pi}(e_A\widehat{\beta}(x)^*(1-e_A)) \\ &= (e_{A,1}\widetilde{\pi}(x)(1-e_{A,1}))^\sharp = \widetilde{\pi}(e_A x(1-e_A))^\sharp. \end{aligned}$$

Therefore $X_B \sim X_{B_1}$ in $\mathcal{M}$. ∎

3.2. CONSTRUCTION OF A MAP FROM $\mathcal{M}/\sim$ TO $\mathcal{L}/\sim$. Let $X\in\mathcal{M}$. Following Brown, Green and Rieffel [2], we can define the linking algebra L for an A-A-equivalence bimodule X. Let

$$L_0 = \left\{\begin{bmatrix} a & x \\ \widetilde{y} & b\end{bmatrix} : a, b\in A, x, y\in X\right\},$$

where $\widetilde{y}$ means y viewed as an element in the dual bimodule $\widetilde{X}$ of X. In the same way as in Brown, Green and Rieffel [2] we can see that L_0 is a $*$-algebra. Also we regard L_0 as a $*$-subalgebra acting on the right Hilbert A-module $X\oplus A$. Hence we can define an operator norm in L_0 acting on $X\oplus A$. We define L as the above operator norm closure of L_0. But, since X is complete, in this case $L = L_0^- = L_0$. Let B_X be a subset of L defined by

$$B_X = \left\{\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a\end{bmatrix} : a\in A, x\in X\right\}.$$

By direct computations, we can see that B_X is a $*$-subalgebra of L and since X is complete, B_X is closed in L, that is, B_X is a C^*-subalgebra of L. We regard A

as a C^*-subalgebra $\left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in A \right\}$ of B_X. Let E_X be a linear map of B_X onto A defined by $E_X\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for any $\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \in B_X$. Then by easy computations E_X is a conditional expectation of B_X onto A.

LEMMA 3.4. *With the above notations,* $\mathrm{Index} E_X = 2$.

Proof. There are elements $z_1, \dots, z_n, y_1, \dots, y_n \in X$ such that $\sum_{i=1}^n \langle z_i, y_i \rangle_A = 1$ by Rieffel ([11], the proof of Proposition 2.1) since X is an A-A-equivalence bimodule. For $i = 1, 2, \dots, n$ let w_i be an element in X with $w_i = z_i^\sharp$. Then

$$\left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\} \cup \left\{ \left(\begin{bmatrix} 0 & w_i \\ \widetilde{w_i^\sharp} & 0 \end{bmatrix}, \begin{bmatrix} 0 & y_i \\ \widetilde{y_i^\sharp} & 0 \end{bmatrix} \right) : i = 1, 2, \dots, n \right\}$$

is a quasi-basis for E_X by direct computations. In fact, for $\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \in B_X$

$$E_X\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix},$$
$$E_X\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \begin{bmatrix} 0 & w_i \\ \widetilde{w_i^\sharp} & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & y_i \\ \widetilde{y_i^\sharp} & 0 \end{bmatrix} = \begin{bmatrix} 0 & {}_A\langle x, w_i^\sharp \rangle y_i \\ \langle x^\sharp, w_i \rangle_A \widetilde{y_i^\sharp} & 0 \end{bmatrix}.$$

Also,

$$\sum_{i=1}^n {}_A\langle x, w_i^\sharp \rangle y_i = \sum_{i=1}^n x \langle w_i^\sharp, y_i \rangle_A = x,$$
$$\sum_{i=1}^n \langle x^\sharp, w_i \rangle_A \widetilde{y_i^\sharp} = \sum_{i=1}^n {}_A\langle x, w_i^\sharp \rangle \widetilde{y_i^\sharp} = \sum_{i=1}^n V({}_A\langle x, w_i^\sharp \rangle y_i) = \widetilde{x^\sharp},$$

where V is an A-A-equivalence bimodule isomorphism defined in Lemma 2.9. Hence

$$E_X\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{i=1}^n E\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \begin{bmatrix} 0 & w_i \\ \widetilde{w_i^\sharp} & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & y_i \\ \widetilde{y_i^\sharp} & 0 \end{bmatrix} = \begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix}.$$

Similarly

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} E_X\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \right) + \sum_{i=1}^n \begin{bmatrix} 0 & w_i \\ \widetilde{w_i^\sharp} & 0 \end{bmatrix} E\left(\begin{bmatrix} 0 & y_i \\ \widetilde{y_i^\sharp} & 0 \end{bmatrix} \begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \right) = \begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix}.$$

Thus

$$\mathrm{Index} E_X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{i=1}^n \begin{bmatrix} 0 & w_i \\ \widetilde{w_i^\sharp} & 0 \end{bmatrix} \begin{bmatrix} 0 & y_i \\ \widetilde{y_i^\sharp} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \quad \blacksquare$$

REMARK 3.5. Let e be an element in $L(=L_0)$ defined by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then it is obvious that for any $b \in B_X, ebe = E_X(b)e$. Furthermore the map $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mapsto e\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ for $a \in A$ is injective. And hence L is the C^*-basic construction of $A \subset B$ by Watatani [12].

Let $\mathcal{G}$ be a map from $\mathcal{M}/\sim$ to $\mathcal{L}/\sim$ defined by $\mathcal{G}([X]) = [B_X, E_X]$ for any $[X] \in \mathcal{M}/\sim$.

LEMMA 3.6. *$\mathcal{G}$ is well-defined.*

Proof. Let $X, X_1 \in \mathcal{M}$ with $X \sim X_1$. Let (B_X, E_X) and (B_{X_1}, E_{X_1}) be elements in $\mathcal{L}$ induced by X and X_1, respectively. Since $X \sim X_1$, there is an involutive A-A-equivalence bimodule isomorphism ρ of X onto X_1. Let π be a map of B_X to B_{X_1} defined by for any $\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \in B_X$, $\pi\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix}\right) = \begin{bmatrix} a & \rho(x) \\ \widetilde{\rho(x)^\sharp} & a \end{bmatrix}$. Then it is clear that π is linear. For $\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \in B_X$,

$$\pi\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix}\right)^* = \begin{bmatrix} a & \rho(x) \\ \widetilde{\rho(x)}^\sharp & a \end{bmatrix}^* = \begin{bmatrix} a^* & \rho(x^\sharp) \\ \widetilde{\rho(x)} & a^* \end{bmatrix} = \pi\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix}^*\right).$$

Also for $\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix}$ and $\begin{bmatrix} b & y \\ \widetilde{y^\sharp} & b \end{bmatrix} \in B_X$,

$$\pi\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix}\begin{bmatrix} b & y \\ \widetilde{y^\sharp} & b \end{bmatrix}\right) = \begin{bmatrix} ab + {}_A\langle x, y^\sharp\rangle & \rho(ay+xb) \\ \widetilde{\rho(xb+ay)}^\sharp & \langle x^\sharp, y\rangle_A + ab \end{bmatrix},$$

and

$$\begin{aligned}\pi\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix}\right)\pi\left(\begin{bmatrix} b & y \\ \widetilde{y^\sharp} & b \end{bmatrix}\right) &= \begin{bmatrix} ab + {}_A\langle \rho(x), \rho(y^\sharp)\rangle & \rho(ay+xb) \\ \widetilde{\rho(xb+ay)}^\sharp & \langle \rho(x^\sharp), \rho(y)\rangle_A + ab \end{bmatrix} \\ &= \begin{bmatrix} ab + {}_A\langle x, y^\sharp\rangle & \rho(ay+xb) \\ \widetilde{\rho(xb+ay)}^\sharp & \langle x^\sharp, y\rangle_A + ab \end{bmatrix} \\ &= \pi\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix}\begin{bmatrix} b & y \\ \widetilde{y^\sharp} & b \end{bmatrix}\right).\end{aligned}$$

Hence π is a homomorphism of B_X to B_{X_1}. Furthermore, by the definition of π, π is a bijection and $\pi\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for any $a \in A$. And for $\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix} \in B_X$

$$(E_1 \circ \pi)\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix}\right) = E_1\left(\begin{bmatrix} a & \rho(x) \\ \widetilde{\rho(x)}^\sharp & a \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = E\left(\begin{bmatrix} a & x \\ \widetilde{x^\sharp} & a \end{bmatrix}\right). \quad \blacksquare$$

3.3. BIJECTION BETWEEN $\mathcal{L}/\sim$ AND $\mathcal{M}/\sim$. In this subsection, we shall show that $\mathcal{F}\circ\mathcal{G}=\mathrm{id}_{\mathcal{M}/\sim}$ and $\mathcal{G}\circ\mathcal{F}=\mathrm{id}_{\mathcal{L}/\sim}$.

LEMMA 3.7. *Let (B, E) be an element in $\mathcal{L}$ and $C^*\langle B, e_A\rangle$ the basic construction for (B, E). Then for each $x\in C^*\langle B, e_A\rangle$, there uniquely exists $b\in B$ such that $e_Ax = e_Ab$.*

Proof. Let $x=\sum_i b_ie_Ac_i$, where $b_i, c_i\in B$. Then $e_Ax=\sum_i e_Ab_ie_Ac_i=\sum_i e_AE(b_i)c_i$ $= e_A\sum_i E(b_i)c_i$. And hence $b=\sum_i E(b_i)c_i$. If $e_Ab=e_Ab'$, where $b, b'\in B$, then

$$b=\frac{1}{2}\widetilde{E}(e_Ab)=\frac{1}{2}\widetilde{E}(e_Ab')=b',$$

where $\widetilde{E}$ is the dual conditional expectation of $C^*\langle B, e_A\rangle$ onto B. Thus we obtain the conclusion. ∎

Let (B, E) be an element in $\mathcal{L}$. Let B_- be a linear subspace of B defined by

$$B_-=\{b\in B : E(b)=0\}=\{b\in B : \beta(b)=-b\},$$

where β is an automorphism of B defined in Proposition 2.4. By a routine computation we can see that B_- is an element in $\mathcal{M}$ with the involution $x^\sharp=x^*$ and the left and the right A-valued inner products defined by

$${}_A\langle x, y\rangle = E(xy^*),\quad \langle x, y\rangle_A = E(x^*y)\quad \text{for } x, y\in B_-.$$

LEMMA 3.8. *With the above notations, $B_-\sim X_B$ i.e., $[B_-]=[X_B]$ in $\mathcal{M}/\sim$.*

Proof. By Lemma 3.7, we can define a map φ from $C^*\langle B, e_A\rangle$ to B by $e_Ax = e_A\varphi(x)$. For $e_Ax(1-e_A)\in X_B$, we have

$$e_Ax(1-e_A)=e_A\varphi(x)-e_AE(\varphi(x))=e_A(\varphi(x)-E(\varphi(x))).$$

And hence $\varphi(e_Ax(1-e_A))=\varphi(x)-E(\varphi(x))\in B_-$. It is easy to see that $\varphi|_{X_B}$ is an A-A-bimodule isomorphism of X_B onto B_-. Furthermore for $e_Ax(1-e_A), e_Ay(1-e_A)\in X_B$,

$$\begin{aligned}{}_A\langle e_Ax(1-e_A), e_Ay(1-e_A)\rangle &= \psi^{-1}(E((\varphi(x)-E(\varphi(x)))(\varphi(y)-E(\varphi(y)))^*)e_A)\\ &= E((\varphi(x)-E(\varphi(x)))(\varphi(y)-E(\varphi(y)))^*)\\ &= {}_A\langle \varphi(x)-E(\varphi(x)), \varphi(y)-E(\varphi(y))\rangle.\end{aligned}$$

Similarly, $\langle e_Ax(1-e_A), e_Ay(1-e_A)\rangle_A = \langle\varphi(x)-E(\varphi(x)), \varphi(y)-E(\varphi(y))\rangle_A$. And

$$\begin{aligned}\varphi((e_Ax(1-e_A))^\sharp) &= \varphi(\widehat{\beta}(e_Ax(1-e_A))^*)=\varphi(\widehat{\beta}((1-e_A)\varphi(x)^*e_A))\\ &= \varphi(e_A\varphi(x)^*(1-e_A))=\varphi(x)^*-E(\varphi(x)^*)\\ &= (\varphi(x)-E(\varphi(x)))^*=\varphi(e_Ax(1-e_A))^*.\end{aligned}$$

Hence $X_B\sim B_-$ in $\mathcal{M}$. ∎

LEMMA 3.9. $\mathcal{G}\circ\mathcal{F}=\mathrm{id}_{\mathcal{L}/\sim}$.

Proof. For $(B, E) \in \mathcal{L}$, it is easy to see that $\mathcal{G}([B_-]) = [B, E]$. Since $[X_B] = [B_-]$ by the previous lemma, $\mathcal{G} \circ \mathcal{F}([B, E]) = \mathcal{G}([X_B]) = [B, E]$. Thus the lemma is proved. ∎

LEMMA 3.10. $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathcal{M}/\sim}$.

Proof. For $X \in \mathcal{M}$,

$$(B_X)_- = \{x \in B_X : E_X(x) = 0\} = \left\{ \begin{bmatrix} 0 & x \\ \widetilde{x^\sharp} & 0 \end{bmatrix} : x \in X \right\}.$$

So it is easy to see that $[(B_X)_-] = [X]$. And hence by Lemma 3.8

$$\mathcal{F} \circ \mathcal{G}([X]) = \mathcal{F}([B_X, E_X]) = [(B_X)_-] = [X]. \quad ∎$$

THEOREM 3.11. *There is a* 1-1 *correspondence between* $\mathcal{L}/\sim$ *and* $\mathcal{M}/\sim$.

Proof. This is immediate by Lemmas 3.9 and 3.10. ∎

4. APPLICATIONS

4.1. CONSTRUCTION OF INVOLUTIVE EQUIVALENCE BIMODULES BY $2\mathbb{Z}$-INNER C^*-DYNAMICAL SYSTEMS. Let A be a unital C^*-algebra and $(A, \mathbb{Z}, \alpha)$ a $2\mathbb{Z}$-inner C^*-dynamical system which means that $(A, \mathbb{Z}, \alpha)$ is a C^*-dynamical system and that there is a unitary element $z \in A$ with $\alpha(z) = z$ and $\alpha^2 = \mathrm{Ad}(z)$. In this case, we can form the restricted crossed product $A \rtimes_{\alpha/2\mathbb{Z}} \mathbb{Z}$ in the sense of P. Green [4]. Let X_α be the vector space A with the obvious left action of A on X_α and the obvious left A-valued inner product, but we define the right action of A on X_α by $x \cdot a = x\alpha(a)$ for any $x \in X_\alpha$ and $a \in A$, and the right A-valued inner product by $\langle x, y \rangle_A = \alpha^{-1}(x^* y)$ for any $x, y \in X_\alpha$.

LEMMA 4.1. *We can define an involution* $x \mapsto x^\sharp$ *on* X_α *by*

$$x^\sharp = \alpha(x^*)z,$$

where z *is a unitary element of* A *with* $\alpha(z) = z$ *and* $\alpha^2 = \mathrm{Ad}(z)$.

Proof. Since $\alpha(z) = z$ and $\alpha^2 = \mathrm{Ad}(z)$, by routine computations, we can see that the map $x \mapsto x^\sharp$ defined by $x^\sharp = \alpha(x^*)z$ is an involution on X_α. ∎

PROPOSITION 4.2. *With the above notations, we suppose that* A *is simple. Let* B_{X_α} *be a* C^**-algebra defined by* X_α *and* L *the linking algebra for* X_α *defined in Section* 2. *Then the following conditions are equivalent:*

(i) B_{X_α} *is simple;*
(ii) $A' \cap B_{X_\alpha} = \mathbb{C} \cdot 1$;
(iii) $B'_{X_\alpha} \cap L = \mathbb{C} \cdot 1$;
(iv) α *is an outer automorphism of* A.

Proof. (i) $\Rightarrow$ (ii): By Proposition 2.4, $B_{X_\alpha}^{\beta} = A$. Since A is simple, by Pedersen ([10], Proposition 8.10.12) β is outer. Hence by Pedersen ([10], Proposition 8.10.13) $A' \cap B_{X_\alpha} = \mathbb{C} \cdot 1$.

(ii) $\Leftrightarrow$ (iii): By Watatani ([12], Proposition 2.7.3) $A' \cap B_{X_\alpha}$ is anti-isomorphic to $B'_{X_\alpha} \cap C^*\langle B_{X_\alpha}, e_A \rangle$. This implies the conclusion.

(ii) $\Rightarrow$ (iv): We suppose that there is a unitary element $w \in A$ such that $\alpha = \mathrm{Ad}(w)$ Then for any $a \in A$

$$w \cdot a = w\alpha(a) = aw = a \cdot w.$$

So it is easy to see that

$$\begin{bmatrix} 0 & w \\ \widetilde{w}^{\sharp} & 0 \end{bmatrix} \in A' \cap B_{X_\alpha}.$$

This is a contradiction. Thus α is outer.

(iv) $\Rightarrow$ (i): We can identify L with the C^*-basic constraction of $A \subset B_{X_\alpha}$ by Remark 3.5. Let β be an automorphism of B_{X_α} defined in the same way as in Proposition 2.4 and let $\widehat{\beta}$ be its dual automorphism. Then $L^{\widehat{\beta}} = B_{X_\alpha}$ by Lemma 2.8. We suppose that $\widehat{\beta}$ is inner. Then there is a unitary element $w = \begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix} \in L$ such that $\widehat{\beta} = \mathrm{Ad}(w)$. Hence for any $c \in A$

$$\widehat{\beta}\left(\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & x \\ \widetilde{y} & b \end{bmatrix}^* .$$

Hence we obtain that

$$\begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} aca^* & ac \cdot y \\ \widetilde{ac^* \cdot y} & \langle c^* \cdot y, y \rangle_A \end{bmatrix}$$

for any $c \in A$. Put $c = 1$. Then $a = 0$ and $\langle y, y \rangle_A = 1$. Since w is a unitary element, by a routine computation we can see that $b = 0$ and ${}_A\langle y, y \rangle = 1$. This implies that y is a unitary element in A. Since $c = \langle c^* \cdot y, y \rangle_A = \alpha(y^* c y) = \alpha(y)^* \alpha(c) \alpha(y)$ for any $c \in A$, α is inner. This is a contradiction. Hence $\widehat{\beta}$ is outer. Since L and A are stably isomorphic by Brown, Green and Rieffel [2], L is simple. By Pedersen ([10], Theorem 8.10.12) $B_{X_\alpha} = L^{\widehat{\beta}}$ is simple. ∎

LEMMA 4.3. *Let $(A, \mathbb{Z}, \alpha)$ be a $2\mathbb{Z}$-inner dynamical system with $\alpha(z) = z$ and $\alpha^2 = \mathrm{Ad}(z)$, where z is a unitary element in A. Let B be the restricted crossed product $A \rtimes_{\alpha/2\mathbb{Z}} \mathbb{Z}$ associated with $(A, \mathbb{Z}, \alpha)$ and E the canonical conditional expectation of B onto A. Then $X_B \cong X_\alpha$ as involutive A-A-equivalence bimodules, where X_B is an involutive A-A-equivalence bimodule induced by (B, E).*

Proof. We may assume that A acts on a Hilbert space H. By Olesen and Pedersen ([9], Proposition 3.2) we also assume that B acts on the induced Hilbert space $\mathrm{Ind}_{2\mathbb{Z}}^{\mathbb{Z}}(H)$. Let

$$C = \left\{ \begin{bmatrix} a & x \\ \alpha(xz) & \alpha(a) \end{bmatrix} \in M_2(A) : a, x \in A \right\}.$$

Since A acts on H, we can put C as a C^*-algebra acting on $H \oplus H$. We claim that $B \cong C$. Indeed, let ρ be a map from $K(\mathbb{Z}, A, z)$ to C defined by for any $f \in K(\mathbb{Z}, A, z)$

$$\rho(f) = \begin{bmatrix} f(0) & f(1) \\ \alpha(f(1)z) & \alpha(f(0)) \end{bmatrix},$$

where $K(\mathbb{Z}, A, z)$ is a $*$-algebra of all functions $f : \mathbb{Z} \longrightarrow A$ satisfying that $f(n - 2m) = f(n)z^m$ for any $m, n \in \mathbb{Z}$ (see [9]). Then by routine computations ρ is a homomorphism of $K(\mathbb{Z}, A, z)$ to C. Let U be a map from $\mathrm{Ind}_{2\mathbb{Z}}^{\mathbb{Z}}(H)$ to $H \oplus H$ defined by $U\xi = \xi(0) \oplus \xi(1)$ for any $\xi \in K(\mathbb{Z}, A, z)$. Then by an easy computaion U is a unitary operator of $\mathrm{Ind}_{2\mathbb{Z}}^{\mathbb{Z}}(H)$ onto $H \oplus H$. Moreover, for any $f \in K(\mathbb{Z}, A, z)$, $\rho(f) = UfU^*$. Hence ρ is an isometry of $K(\mathbb{Z}, A, z)$ to C and we can extend ρ to an isomorphism of B onto C since $K(\mathbb{Z}, A, z)$ is dense in B. Thus $B \cong C$. Let F be a linear map of C onto A defined by $F\left(\begin{bmatrix} a & x \\ \alpha(xz) & \alpha(a) \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & \alpha(a) \end{bmatrix}$ for any $\begin{bmatrix} a & x \\ \alpha(xz) & \alpha(a) \end{bmatrix} \in C$, where we identify A with a C^*-algebra $\left\{ \begin{bmatrix} a & 0 \\ 0 & \alpha(a) \end{bmatrix} : a \in A \right\}$. Then by an easy computation $(B, E) \sim (C, F)$ in $\mathcal{L}$. Let $(B_{X_\alpha}, E_{X_\alpha})$ be an element in $\mathcal{L}$ induced by the involutive A-A-equivalence bimodule X_α. Let Φ be a map from C to B_{X_α} defined by

$$\Phi\left(\begin{bmatrix} a & x \\ \alpha(xz) & \alpha(a) \end{bmatrix}\right) = \begin{bmatrix} a & x \\ \widetilde{x}^{\natural} & a \end{bmatrix}$$

for any $\begin{bmatrix} a & x \\ \alpha(xz) & \alpha(a) \end{bmatrix} \in C$. Then by routine computations Φ is an isomorphism of C onto B_{X_α} with $F = E_{X_\alpha} \circ \Phi$. Thus $(B, E) \sim (B_{X_\alpha}, E_{X_\alpha})$. By Theorem 3.11, $X_B \sim X_\alpha$ in $\mathcal{M}$. ∎

Let B be a unital C^*-algebra and A a C^*-subalgebra of B with a common unit. Let E be a conditional expectation of B onto A with $\mathrm{Index}E = 2$. For any $n \in \mathbb{N}$ let M_n be the $n \times n$-matrix algebra over $\mathbb{C}$ and $M_n(A)$ the $n \times n$-matrix algebra over A. Let $\{(x_i, x_i^*)\}_{i=1}^n$ be a quasi-basis for E. We define $q = [q_{ij}] \in M_n(A)$ by $q_{ij} = E(x_i^* x_j)$. Then by Watatani [12], q is a projection and $C^*\langle B, e_A \rangle \simeq qM_n(A)q$. Let π be an isomorphism of $C^*\langle B, e_A \rangle$ onto $qM_n(A)q$ defined by

$$\pi(ae_A b) = [E(x_i^* a)E(bx_j)] \in M_n(A)$$

for any $a, b \in B$. Especially for any $b \in B$, $\pi(b) = [E(x_i^* bx_j)]$ since $\sum_{i=1}^n x_i e_A x_i^* = 1$.

PROPOSITION 4.4. *With the above notations, the following conditions are equivalent:*

(i) *e_A and $1-e_A$ are equivalent in $C^*\langle B, e_A\rangle$;*

(ii) *there exists a unitary element $u \in B$ such that $\{(1,1),(u,u^*)\}$ is a quasi-basis for E;*

(iii) *there exists a $2\mathbb{Z}$-inner C^*-dynamical system $(A,\mathbb{Z},\alpha)$ such that $X_\alpha \sim X_B$.*

Proof. (i) $\Rightarrow$ (ii): We suppose that there is a partial isometry $v \in C^*\langle B, e_A\rangle$ such that $v^*v = e_A$, $vv^* = 1-e_A$. Then $ve_Av^* = 1-e_A$. By Lemma 3.7, there exists an element u in B such that $ve_A = ue_A$ and hence $ue_Au^* = 1-e_A$. Let $\widetilde{E}$ be the dual conditional expectation for E. Then

$$uu^* = 2\widetilde{E}(ue_Au^*) = 2\widetilde{E}(1-e_A) = 1.$$

Therefore u is a co-isometry element in B. Since $e_Au^*ue_A = e_Av^*ve_A = e_A$, we have $E(u^*u) = 1$ and $E(1-u^*u) = 0$. And hence $u^*u = 1$ i.e., u is a unitary element in B. For any $x \in B$

$$xe_A = (e_A + ue_Au^*)xe_A = E(x)e_A + uE(u^*x)e_A = (E(x) + uE(u^*x))e_A.$$

Thus $x = E(x) + uE(u^*x)$ by Lemma 3.7. Similarly, $x = E(x) + E(xu)u^*$. This implies that $\{(1,1),(u,u^*)\}$ is a quasi-basis for E.

(ii) $\Rightarrow$ (i): We suppose that $\{(1,1),(u,u^*)\}$ is a quasi-basis for E and that u is a unitary element in B. Then

$$u = E(u) + uE(u^*u) = E(u) + u.$$

This implies that $E(u) = 0$. Hence

$$q = \begin{bmatrix} E(1\cdot 1) & E(u) \\ E(u^*) & E(u^*u) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore $C^*\langle B, e_A\rangle \simeq M_2(A)$. Furthermore

$$\pi(e_A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \pi(1-e_A) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

And hence $e_A \sim (1-e_A)$ in $C^*\langle B, e_A\rangle$.

(ii) $\Rightarrow$ (iii): We suppose that $\{(1,1),(u,u^*)\}$ is a quasi-basis for E and that u is a unitary element in B. Then in the same way as above $E(u) = 0$. For any $a \in A$

$$uau^* = E(uau^*) + E(uau^*u)u^* = E(uau^*) + E(u)au^* = E(uau^*).$$

Therefore $uAu^* = A$. Let α be an automorphism of A defined by $\alpha(a) = uau^*$ for any $a \in A$. Since $u^2 = E(u^2) + uE(u^*u^2) = E(u^2)$, u^2 is an element in A. Therefore $(A,\mathbb{Z},\alpha)$ is a $2\mathbb{Z}$-inner C^*-dynamical system. It is easy to see that

$$X_\alpha \sim Au = B_- = \{b \in B : E(b) = 0\}.$$

By Lemma 3.8, $X_\alpha \sim X_B$.

(iii) $\Rightarrow$ (ii) : We suppose that there exists a $2\mathbb{Z}$-inner C^*-dynamical system $(A,\mathbb{Z},\alpha)$ such that $X_\alpha \sim X_B$. By the previous lemma, we may suppose that $B = A \rtimes_{\alpha/2\mathbb{Z}} \mathbb{Z}$. Then there exists a unitary element $u \in B$ such that $\mathrm{Ad}(u) = \alpha$,

$u^2 \in A$ and $E(u) = 0$. By a routine computation we can see that $\{(1,1),(u,u^*)\}$ is a quasi-basis for E. ∎

COROLLARY 4.5. *Let θ be an irrational number in $(0,1)$ and A_θ the corresponding irrational rotation C^*-algebra. Let B be a unital C^*-algebra including A_θ as a C^*-subalgebra of B with a common unit. We suppose that there is a conditional expectation E of B onto A_θ with* $\mathrm{Index}E = 2$. *Then there is a $2\mathbb{Z}$-inner C^*-dynamical system $(A_\theta, \mathbb{Z}, \alpha)$ such that $(B, E) \sim (A_\theta \rtimes_{\alpha/2\mathbb{Z}} \mathbb{Z}, F)$, where F is the canonical conditional expectation of $A_\theta \rtimes_{\alpha/2\mathbb{Z}} \mathbb{Z}$ onto A_θ.*

Proof. Let e be the Jones projection induced by E. We can identify the basic construction $C^*\langle B, e\rangle$ with $qM_n(A_\theta)q$ in the same way as in the previous argument. Hence $C^*\langle B, e\rangle$ has the unique normalized trace τ and $\tau(e) = \tau(1-e) = \frac{1}{2}$. So it is easy to see that $e \sim 1-e$ in $C^*\langle B, e\rangle$ since A_θ has cancellation. Therefore we obtain the conclusion by the previous proposition. ∎

4.2. EXAMPLES. In this subsection, let A_θ be as in Corollary 4.5 and let u, v be two unitary generators satisfying the commutation relation:

$$uv = \mathrm{e}^{2\pi \mathrm{i}\theta} vu.$$

EXAMPLE 4.6. Let $A_{2\theta}$ be the C^*-subalgebra of A_θ generated by u^2 and v. Then we can denote $A_\theta = \{x + yu : x, y \in A_{2\theta}\}$. Let E be a map of A_θ onto $A_{2\theta}$ defined by $E(x + yu) = x$. It is easy to see that E is a conditional expectation of A_θ onto $A_{2\theta}$ with $\mathrm{Index}E = 2$ and a quasi-basis $\{(1,1),(u,u^*)\}$. Hence by Corollary 4.5, A_θ can be represented as the restricted crossed product $A_{2\theta} \rtimes_{\alpha/2\mathbb{Z}} \mathbb{Z}$, where α is an automorphism on $A_{2\theta}$ defined by $\alpha = \mathrm{Ad}(u)$.

Suppose that A_θ can be represented as a crossed product $A_{2\theta} \rtimes_\beta \mathbb{Z}_2$ for some $\mathbb{Z}_2$-action β on $A_{2\theta}$. Then there exists a self-adjoint unitary element w in A_θ satisfying that $\beta = \mathrm{Ad}(w)$ and $A_\theta = \{x + yw : x, y \in A_{2\theta}\}$. Let τ be the unique tracial state on A_θ. By the uniqueness of τ, we can see that $\tau(x + yw) = \tau(x)$. Let e be a projection in A_θ defined by $e = \frac{1}{2}(1 + w)$. Then $\tau(e) = \frac{1}{2}$. This contradicts that $\tau(A_\theta) = (\mathbb{Z} \cap \theta\mathbb{Z}) \cap (0,1)$. Therefore A_θ can not be represented as a crossed product $A_{2\theta} \rtimes_\beta \mathbb{Z}_2$ for any $\mathbb{Z}_2$-action β on $A_{2\theta}$.

EXAMPLE 4.7. Let σ be the involutive automorphism of A_θ determined by $\sigma(u) = u^*$ and $\sigma(v) = v^*$. Let C_θ denote the fixed point algebra $A_\theta^\sigma = \{x \in A_\theta : \sigma(x) = x\}$ and B_θ the crossed product $A_\theta \rtimes_\sigma \mathbb{Z}_2$. Then B_θ is the basic construction of $C_\theta \subset A_\theta$. By Kumjian [8], K_0-group of B_θ, $K_0(B_\theta)$ is isomorphic to $\mathbb{Z}^6$. By routine computations, we can see $[e] \neq [1-e]$ in $K_0(B_\theta)$, where e is the Jones projection for the inclusion $C_\theta \subset A_\theta$. Hence $e \not\sim 1-e$ in B_θ. Therefore the inclusion $C_\theta \subset A_\theta$ can not be represented as the restricted crossed product $C_\theta \subset C_\theta \rtimes_{\alpha/2\mathbb{Z}} \mathbb{Z}$ for any automorphism α on C_θ by Proposition 4.4.

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REFERENCES

[1] O. BRATTELI, G.A. ELLIOTT, D.E. EVANS, A. KISHIMOTO, Non-commutative spheres. I, *Internat. J. Math.* **2**(1991), 139–166.

[2] L.G. BROWN, P. GREEN, M.A. RIEFFEL, Stable isomorphism and strong Morita equivalence of C^*-algebra, *Pacific J. Math.* **71**(1977), 349–368.

[3] G.A. ELLIOTT, M. RØRDAM, The automorphism group of the irrational rotation algebra, *Comm. Math. Phys.* **155**(1993), 3–26.

[4] P. GREEN, The local structure of twisted covariance algebras, *Acta Math.* **140**(1978), 191–250.

[5] M. IZUMI, Inclusions of simple C^*-algebras, *J. Reine Angew. Math.* **547**(2002), 97–138.

[6] V. JONES, Index for subfactors, *Invent. Math.* **72**(1983), 1–25.

[7] T. KAJIWARA, Y. WATATANI, Jones index theory by Hilbert C^*-bimodules and K-theory, *Trans. Amer. Math. Soc.* **352**(2000), 3429–3472.

[8] A. KUMJIAN, On the K-theory of the symmetrized non-commutative torus, *C. R. Math. Rep. Acad. Sci. Canada* **12**(1990), 87–89.

[9] D. OLESEN, G.K. PEDERSEN, Partially inner C^*-dynamical systems, *J. Funct. Anal.* **66**(1986), 262–281.

[10] G.K. PEDERSEN, *C^*-Algebras and their Automorphism Groups*, Academic Press, London-New York 1979.

[11] M.A. RIEFFEL, C^*-algebra associated with irrational rotations, *Pacific J. Math.* **93**(1981), 415–429.

[12] Y. WATATANI, Index for C^*-subalgebras, *Mem. Amer. Math. Soc.* **424**(1990).

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