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THE BEHAVIOR OF THE RADICAL OF THE ALGEBRAS GENERATED BY A SEMIGROUP OF OPERATORS ON HILBERT SPACE

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ABSTRACT. Let $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ be a continuous semigroup of contractions on a Hilbert space. We define ***A(T)*** as the closure of the set $\{\widehat{f}(\boldsymbol{T}) : f \in L^1(\mathbb{R}_+)\}$ with respect to the operator-norm topology, where $\widehat{f}(\boldsymbol{T}) = \int\limits_0^\infty f(t)T(t)\mathrm{d}t$ is the Laplace transform of $f \in L^1(\mathbb{R}_+)$ with respect to the semigroup $\boldsymbol{T}$. Then, ***A(T)*** is a commutative Banach algebra. In this paper, we obtain some connections between the radical of ***A(T)*** and the set $\{R \in \boldsymbol{A(T)} : T(t)R \to 0$, strongly or in norm, as $t \to \infty\}$. Similar problems for the algebras generated by a discrete semigroup $\{T^n : n = 0, 1, 2, \dots\}$ is also discussed, where T is a contraction.

KEYWORDS: *Hilbert space, continuous (discrete) semigroup, Banach algebra, radical.*

MSC (2000): 47Dxx, 46J05.

1. INTRODUCTION

Let $\boldsymbol{A}$ be a complex commutative Banach algebra. Its structure space is $M_{\boldsymbol{A}} = \{\phi : \boldsymbol{A} \to \mathbb{C} :$ nonzero, continuous, linear, multiplicative$\}$ equipped with the w^*-topology. The Gelfand transform of $a \in \boldsymbol{A}$ is defined by $\widehat{a} : M_{\boldsymbol{A}} \to \mathbb{C}$, $\widehat{a}(\phi) = \phi(a)$. The *radical* of $\boldsymbol{A}$, denoted by $\mathrm{Rad}(\boldsymbol{A})$ is defined as

$$\{a \in \boldsymbol{A} : \widehat{a}(\phi) = 0,\ \phi \in M_{\boldsymbol{A}}\}.$$

$\mathrm{Rad}(\boldsymbol{A})$ is precisely the set of all quasinilpotent elements in $\boldsymbol{A}$. If $\mathrm{Rad}(\boldsymbol{A}) = \{0\}$, then $\boldsymbol{A}$ is said to be *semisimple*. By the definition, $\boldsymbol{A}$ is semisimple if and only if the Gelfand transform is injective.

Let X be a complex Banach space and $B(X)$, the algebra of all bounded linear operators on X. Let $\boldsymbol{A}$ be a closed commutative subalgebra of $B(X)$. It follows from the spectral radius formula that $\mathrm{Rad}(\boldsymbol{A}) = \{R \in \boldsymbol{A} : \sigma(R) = \{0\}\}$.

Hence, $\boldsymbol{A}$ is semisimple if and only if it does not contain a non-zero operator with zero spectrum.

If $T \in B(X)$, we let $A(T)$ denote the uniformly closed algebra generated by T and the identity operator I. Then, $A(T)$ is a commutative unital Banach algebra. The structure space of $A(T)$ can be identified with $\sigma_{A(T)}(T)$, where $\sigma_{A(T)}(T)$ is the spectrum of T with respect to the algebra $A(T)$. Note also that $\sigma_{A(T)}(T) \supset \sigma(T)$. By $W(T)$ we will denote the weak operator closure of $A(T)$. Clearly, $W(T) \subset \{T\}'$, the commutant of T.

Recall that a family $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ in $B(X)$ is called a C_0-*semigroup* (or *continuous semigroup*) if the following properties are satisfied:

(1) $T(0) = I$;
(2) $T(t+s) = T(t)T(s)$, for every $t, s \geqslant 0$;
(3) $\lim\limits_{t\to 0^+} \|T(t)x - x\| = 0$, for all $x \in X$.

The *generator* of the C_0-semigroup $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ is the linear operator A with domain $D(A)$ defined by

$$Ax = \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x), \quad x \in D(A).$$

The generator is always a closed, densely defined operator. The C_0-*groups* are defined analogously to C_0-semigroups, the only difference being that the role of the index family $t \geqslant 0$ is replaced by $t \in \mathbb{R}$. The generator of a C_0-group $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ is defined as the generator of the associated C_0-semigroup.

A C_0-semigroup $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ is said to be *bounded* if $\sup\limits_{t\geqslant 0} \|T(t)\| < \infty$. If $\boldsymbol{T}$ is a bounded C_0-semigroup on a Banach space X, then

$$\||x\|| = \sup_{t\geqslant 0} \|T(t)x\|$$

is an equivalent norm on X with respect to which $\boldsymbol{T}$ becomes a C_0-semigroup of contractions. Note also that if $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ is a C_0-semigroup of contractions on a Banach space X, then for every $x \in X$, the limit $\lim\limits_{t\to\infty} \|T(t)x\|$ exists and is equal to $\inf\limits_{t\geqslant 0} \|T(t)x\|$.

Now let $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ be a bounded C_0-semigroup with generator A. Then, the spectrum $\sigma(A)$ of A belongs to the closed left half-plane. For $Re\lambda > 0$, the resolvent is given by

$$R(\lambda, A) = \int_0^\infty \exp(-\lambda t)T(t)\mathrm{d}t$$

([12], p. 6). $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is called the *unitary spectrum* of the generator A.

Let $L^1(\mathbb{R}_+)$ be the space of all absolutely integrable measurable complex functions on the half-line $\mathbb{R}_+$. $L^1(\mathbb{R}_+)$ is a commutative Banach algebra when convolution is taken as the multiplication, where "convolution" is defined by the formula

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau.$$

$L^1(\mathbb{R}_+)$ can be considered (in the natural way) as a subalgebra of $L^1(\mathbb{R})$. The Fourier transform $\widehat{f}(z)$ of $f \in L^1(\mathbb{R}_+)$, where

$$\widehat{f}(z) = \int_0^\infty \exp(-\mathrm{i}tz)f(t)\mathrm{d}t,$$

is a function analytic in the open half-plane $\{z \in \mathbb{C} : \mathrm{Im} z < 0\}$ and is a bounded continuous function in the closed half-plane $\{z \in \mathbb{C} : \mathrm{Im} z \leqslant 0\}$. Every complex homomorphism ϕ on $L^1(\mathbb{R}_+)$ is of the form $\phi = \phi_z$, where $\phi_z(f) = \widehat{f}(z)$, $\mathrm{Im} z \leqslant 0$. In this sense, the maximal ideal space of $L^1(\mathbb{R}_+)$ can be identified with the closed left half-plane ([6], p. 115).

For a function $f \in L^1(\mathbb{R}_+)$, we put

$$\widehat{f}(\boldsymbol{T}) = \int_0^\infty f(t)T(t)\mathrm{d}t.$$

The map $f \to \widehat{f}(\boldsymbol{T})$ is a continuous algebra homomorphism of $L^1(\mathbb{R}_+)$ into $B(X)$. We define $\boldsymbol{A}(\boldsymbol{T})$ as the closure with respect to the operator-norm topology of the set $\{\widehat{f}(\boldsymbol{T}) : f \in L^1(\mathbb{R}_+)\}$. Then, ***A(T)*** is a commutative Banach algebra. The maximal ideal space of ***A(T)*** will be denoted by $M_{\boldsymbol{T}}$. If $R \in$ ***A(T)***, its Gelfand transform will be denoted as $\widehat{R}$. It can be easily verified that if the generator T of the C_0-semigroup $\boldsymbol{T}$ is bounded, then ***A(T)*** $= A(T)$.

We define ***W(T)*** as the closure with respect to the weak operator topology of ***A(T)***. It follows from the definition of the vector-valued integral that ***W(T)*** belongs to the closure with respect to the weak operator topology of all polynomials $c_1T(t_1) + \cdots + c_nT(t_n)$ in $\boldsymbol{T}$.

Let $e_n(s) = 2n\chi_{[0,1/n]}(s)$ $(n = 1, 2, \ldots)$, where $\chi_{[0,1/n]}(s)$ is the characteristic function of the interval $[0, 1/n]$. For $t \geqslant 0$, we define

$$e_n^t(s) = \begin{cases} e_n(s-t) & s \geqslant t; \\ 0 & 0 \leqslant s < t. \end{cases}$$

It is easy to see that

$$\int_0^\infty e_n^t(s)T(s)\mathrm{d}s \to T(t), \text{strongly}, \quad \text{as } n \to \infty.$$

This shows that $T(t) \in$ ***W(T)***, so that ***W(T)*** coincides with the weak operator closure of all polynomials in $\boldsymbol{T}$. Hence, ***W(T)*** is a commutative Banach algebra with the identity I.

In this paper, we study the semisimplicity problem for the Banach algebras defined above. In Section 2, we study the behavior of the radical of the algebras ***A(T)*** and ***W(T)***. In Section 3, similar problems for the algebras generated by a single bounded operator will be discussed.

2. THE BANACH ALGEBRA GENERATED BY A C_0-SEMIGROUP

We shall need the following preliminary lemmas.

LEMMA 2.1. *Let $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ be a bounded C_0-semigroup on a Banach space with generator A. If $z \in \sigma(A)$, then $\widehat{f}(\mathrm{i}z) \in \sigma(\widehat{f}(\boldsymbol{T}))$, for all $f \in L^1(\mathbb{R}_+)$.*

Proof. Let $f \in L^1(\mathbb{R}_+)$ be given. As is well known $\sigma(A) = \sigma_{\mathrm{a}}(A) \cup \sigma_{\mathrm{r}}(A)$, where $\sigma_{\mathrm{a}}(A)$ is the approximate point spectrum and $\sigma_{\mathrm{r}}(A)$, the residual spectrum of A. If $z \in \sigma_{\mathrm{a}}(A)$, then there is a sequence $(x_n)_{n\in\mathbb{N}}$ of norm one vectors in $D(A)$ such that $\|Ax_n - zx_n\| \to 0$. This implies ([12], Proposition 2.1.6),

$$\|T(t)x_n - \exp(zt)x_n\| \to 0, \quad \text{for all } t \geqslant 0.$$

Consequently, we have

$$\|\widehat{f}(\boldsymbol{T})x_n - \widehat{f}(\mathrm{i}z)x_n\| \leqslant \int_0^\infty \|T(t)x_n - \exp(zt)x_n\|\, |f(t)|\mathrm{d}t \to 0, \quad \text{as } n \to \infty.$$

This shows that $\widehat{f}(\mathrm{i}z) \in \sigma(\widehat{f}(\boldsymbol{T}))$.

If $z \in \sigma_{\mathrm{r}}(A)$, then $z \in \sigma_{\mathrm{p}}(A^*)$, the point spectrum of A^*. Consequently, $A^*x^* = zx^*$ for some nonzero $x^* \in D(A^*)$. It follows that $T(t)^*x^* = \mathrm{e}^{zt}x^*$ ([12], p. 31), for all $t \geqslant 0$. Hence, we have that $\widehat{f}(\boldsymbol{T})^*x^* = \widehat{f}(\mathrm{i}z)x^*$ and therefore, $\widehat{f}(\mathrm{i}z) \in \sigma(\widehat{f}(\boldsymbol{T}))$. ∎

LEMMA 2.2. *Let $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ be a bounded C_0-semigroup on a Banach space with generator A. Then, the map $z \to \phi_z$ homeomorphically identifies $\sigma(A)$ with a closed subset of $M_{\boldsymbol{T}}$, where $\phi_z : \boldsymbol{A(T)} \to \mathbb{C}$ is defined by*

$$\phi_z(\widehat{f}(\boldsymbol{T})) = \widehat{f}(\mathrm{i}z), \quad f \in L^1(\mathbb{R}_+).$$

Proof. By Lemma 2.1 we have $|\widehat{f}(\mathrm{i}z)| \leqslant \|\widehat{f}(\boldsymbol{T})\|$, for all $f \in L^1(\mathbb{R}_+)$ and $z \in \sigma(A)$. Since the set $\{\widehat{f}(\boldsymbol{T}) : f \in L^1(\mathbb{R}_+)\}$ is dense in ***A(T)***, it follows from the above inequality that the homomorphism $\phi_z : \widehat{f}(\boldsymbol{T}) \to \widehat{f}(\mathrm{i}z)$ can be extended to an element of $M_{\boldsymbol{T}}$. It is easy to see that the map $z \to \phi_z$, of $\sigma(A)$ into $M_{\boldsymbol{T}}$ is continuous and injective. Thus, it suffices to show that if $\{\phi_{z_n}\}_{n\in\mathbb{N}} \subset M_{\boldsymbol{T}}$ and $\phi_z \in M_{\boldsymbol{T}}$ are such that $\phi_{z_n} \to \phi_z$, in the usual topology of $M_{\boldsymbol{T}}$, then $z_n \to z$. But if $\phi_{z_n} \to \phi_z$, then for every $f \in L^1(\mathbb{R}_+)$ we have $\widehat{f}(\mathrm{i}z_n) \to \widehat{f}(\mathrm{i}z)$ and the conclusion follows from the fact that the maximal ideal space of $L^1(\mathbb{R}_+)$ is homeomorphically identified with $\{z \in \mathbb{C},\ \mathrm{Im} z \leqslant 0\}$. ∎

Let $\boldsymbol{T}$ be a bounded C_0-semigroup with generator A and let $R \in$ ***A(T)***. It follows from Lemma 2.2 that instead of $\widehat{R}(\phi_z)(= \phi_z(R))$, $z \in \sigma(A)$, we can (and will) write $\widehat{R}(z)$.

Let $\boldsymbol{U} = \{U(t)\}_{t \geqslant 0}$ be a C_0-semigroup of unitary operators on a Hilbert space. By the Stone Theorem

$$U(t) = \exp(-\mathrm{i}tQ) = \int_{\mathbb{R}} \exp(-\mathrm{i}t\lambda)\mathrm{d}E(\lambda),$$

where Q is a self-adjoint (possibly unbounded) operator and $E(\cdot)$, the spectral measure associated with Q which is supported on $\sigma(Q)$. It can be easily verified that

$$\widehat{f}(\boldsymbol{U}) = \int_{\mathbb{R}} \widehat{f}(\lambda)\mathrm{d}E(\lambda), \quad f \in L^1(\mathbb{R}_+).$$

From this and from Lemma 2.1 it follows that

$$\|\widehat{f}(\boldsymbol{U})\| = \sup_{\lambda \in \sigma(Q)} |\widehat{f}(\lambda)|. \tag{2.1}$$

This clearly implies that the algebra ***A(U)*** is semisimple.

The following example shows that there exists a bounded C_0-semigroup on a Hilbert space that generates a non-semisimple algebra.

EXAMPLE 2.3. Let V be the Volterra integration operator on the Hilbert space $L^2[0,1]$ and let $\boldsymbol{T} = \{\exp(-tV)\}_{t \geqslant 0}$. Notice that the exponential formula ([14], Theorem 1.8.3) yields $\|\exp(-tV)\| = 1$, for all $t \geqslant 0$. On the other hand, V is a nonzero quasinilpotent operator and $V \in A(V) =$ ***A(T)***. This shows that the algebra ***A(T)*** is not semisimple.

Let $\boldsymbol{T} = \{T(t)\}_{t \geqslant 0}$ be a C_0-semigroup of contractions on a Hilbert space H. As is known ([17], Chapter 1, Theorem 8.1), there exists a Hilbert space $K \supset H$ and a C_0-group of unitary operators $\boldsymbol{U} = \{U(t)\}_{t \in \mathbb{R}}$ on K such that

$$\langle T(t)x, y \rangle = \langle U(t)x, y \rangle,$$

for all $t \geqslant 0$ and $x, y \in H$. It follows that

$$\langle \widehat{f}(\boldsymbol{T})x, y \rangle = \langle \widehat{f}(\boldsymbol{U})x, y \rangle, \quad f \in L^1(\mathbb{R}_+).$$

From this and from the identity (2.1), we can write

$$\|\widehat{f}(\boldsymbol{T})\| \leqslant \|\widehat{f}(\boldsymbol{U})\| \leqslant \sup_{\lambda \in \mathbb{R}} |\widehat{f}(\lambda)|.$$

Thus, we have that

$$\|\widehat{f}(\boldsymbol{T})\| \leqslant \sup_{\lambda \in \mathbb{R}} |\widehat{f}(\lambda)|, \quad \text{for all } f \in L^1(\mathbb{R}_+).$$

This is the semigroup version of the von Neumann inequality.

PROPOSITION 2.4. *If $\boldsymbol{V} = \{V(t)\}_{t \geqslant 0}$ is a C_0-semigroup of isometries on a Hilbert space, then the algebra* ***A(V)*** *is semisimple.*

Proof. Let B be the generator of the semigroup $\boldsymbol{V}$. If $\mathrm{i}\mathbb{R} \not\subset \sigma(B)$, then $\boldsymbol{V}$ extends to a C_0-group of unitary operators with generator B ([12], Lemma 2.8) and therefore, the algebra ***A(V)*** is semisimple. Hence, we may assume that $\mathrm{i}\mathbb{R} \subset \sigma(B)$. Let $R \in \mathrm{Rad}$***A(V)***. Then, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(\mathbb{R}_+)$ such that $\|\widehat{f}_n(\boldsymbol{T}) - R\| \to 0$, as $n \to \infty$. It follows from Lemma 2.2 that $\phi_z(\widehat{f}_n(\boldsymbol{T})) = \widehat{f}_n(\mathrm{i}z) \to 0$, uniformly for $z \in \mathrm{i}\mathbb{R}$. By the semigroup version of the von Neumann inequality we have $\|\widehat{f}_n(\boldsymbol{T})\| \to 0$, so that $R = 0$. ∎

Recall that $T \in B(H)$ is called *cyclic* if it has a cyclic vector, that is, a vector $x \in H$ such that $\overline{\mathrm{span}}\{T^n x : n = 0, 1, 2, \ldots\} = H$. Let $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup on a Hilbert space H. $\boldsymbol{T}$ is called *cyclic* if it has a cyclic vector, that is, a vector $x \in H$ such that $\overline{\mathrm{span}}\{T(t)x : t \geqslant 0\} = H$. By $\{\boldsymbol{T}\}'$ we will denote the commutant of $\boldsymbol{T}$:

$$\{\boldsymbol{T}\}' = \{S \in B(H) : ST(t) = T(t)S, \text{ for all } t \geqslant 0\}.$$

Clearly, ***W(T)*** $\subset \{\boldsymbol{T}\}'$.

PROPOSITION 2.5. *If $\boldsymbol{V} = \{V(t)\}_{t\geqslant 0}$ is a cyclic C_0-semigroup of isometries on a Hilbert space, then $\{\boldsymbol{T}\}' =$ **W(V)** and the algebra **W(V)** is semisimple.*

For the proof, some further information is needed. Here and throughout the paper, we have written $D = \{z \in \mathbb{C} : |z| < 1\}$ and $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$. $A(D)$ will denote the disc-algebra. Now let $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup of contractions on a Hilbert space H with generator A. The operator T defined by $T = (A + I)/(A - I)$ is a contraction on H and is called the *cogenerator* of $\boldsymbol{T}$ ([17], Chapter 3, Section 8). For $t \geqslant 0$, let

$$e_t(z) = \exp\big(t\frac{z+1}{z-1}\big) = \sum_{k=1}^{\infty} c_k(t) z^k, \quad z \in D.$$

Then,

$$T(t) = e_t(T) : \ = \text{s-}\lim_{r\to 1^-} \sum_{k=1}^{\infty} c_k(t) r^k T^k$$

([17], Chapter 3, Theorem 8.1). It follows that if the semigroup $\boldsymbol{T}$ is cyclic, then its cogenerator T is also cyclic.

Now, we claim that $\{\boldsymbol{T}\}' = \{T\}'$. Recall ([17], Chapter 3, Theorem 8.1) that there exists a family $\{f_t(z)\}_{t\geqslant 0}$ in $A(D)$ such that

$$\text{s-}\lim_{t\to 0^+} f_t(T(t)) = T.$$

If $T(t)R = RT(t)$ for some $R \in B(H)$, then $f_t(T(t))R = Rf_t(T(t))$, for all $t \geqslant 0$. By letting $t \to 0^+$, we find that $TR = RT$. Conversely, if $TR = RT$, then $e_t(T)R = Re_t(T)$, so that $T(t)R = RT(t)$, for all $t \geqslant 0$.

Next, we claim that ***W(T)*** $= W(T)$. Recall that ***W(T)*** is the weak operator closure of all polynomials in $\boldsymbol{T}$. Since $T(t) = e_t(T)$, we have $T(t) \in W(T)$, for all

$t \geqslant 0$ and so $\boldsymbol{W(T)} \subset W(T)$. The reverse inclusion follows from the identity

$$T = \frac{A+I}{A-I} = \text{s-}\lim_{t\to 0^+} \frac{T(t) - I + tI}{T(t) - I - tI}.$$

Proof of Proposition 2.5. If $\boldsymbol{V} = \{V(t)\}_{t\geqslant 0}$ is a cyclic C_0-semigroup of isometries, then its cogenerator V is a cyclic isometry ([17], Chapter 3, Proposition 9.2). We know ([8]) that $\{V\}' = W(V)$ and the algebra $W(V)$ is semisimple. Since $\{\boldsymbol{V}\}' = \{V\}'$ and $\boldsymbol{W(V)} = W(V)$, we obtain that $\{\boldsymbol{V}\}' = \boldsymbol{W(V)}$ and the algebra $\boldsymbol{W(V)}$ is semisimple. ∎

One of the main results of this section is the following theorem.

THEOREM 2.6. *Let $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup of contractions on a Hilbert space H with generator A. If the Gelfand transform of $R \in \boldsymbol{A(T)}$ vanishes on the unitary spectrum of A, then for every $x \in H$,*

$$\lim_{t\to\infty} \|T(t)Rx\| = 0.$$

For the proof we need some preliminary results.

LEMMA 2.7. *Let H be a Hilbert space and let $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup of contractions on H with generator A. Let K be a Hilbert space and let $\boldsymbol{V} = \{V(t)\}_{t\geqslant 0}$ be a C_0-semigroup of isometries on K. Assume that the following conditions are satisfied:*

(i) *There exists a bounded linear operator $J : H \to K$ such that*

$$V(t)J = JT(t), \quad \text{for all } t \geqslant 0;$$

(ii) *$\|\widehat{f}(\boldsymbol{V})\| \leqslant \|\widehat{f}(\boldsymbol{T})\|$, for all $f \in L^1(\mathbb{R}_+)$.*

If the Gelfand transform of $R \in \boldsymbol{A(T)}$ vanishes on the unitary spectrum of A, then $JR = 0$.

Proof. Let B be the generator of $\boldsymbol{V}$. First, we claim that $\sigma(B) \cap \mathrm{i}\mathbb{R} \subset \sigma(A)$. Assume that $\mathrm{i}y \in \sigma(B)$, for some $y \in \mathbb{R}$. It follows from the condition (ii) that the mapping $\widehat{f}(\boldsymbol{T}) \to \widehat{f}(\boldsymbol{V})$ can be extended to a contractive homomorphism $h : \boldsymbol{A(T)} \to \boldsymbol{A(V)}$. We can see that $h^* M_{\boldsymbol{V}} \subset M_{\boldsymbol{T}}$. By Lemma 2.2 since $\phi_{\mathrm{i}y} \in M_{\boldsymbol{V}}$, we have $h^*\phi_{\mathrm{i}y} \in M_{\boldsymbol{T}}$ and

$$(h^*\phi_{\mathrm{i}y})(\widehat{f}(\boldsymbol{T})) = \widehat{f}(-y).$$

It follows that

$$|\widehat{f}(-y)| \leqslant \|\widehat{f}(\boldsymbol{T})\|, \quad \text{for all } f \in L^1(\mathbb{R}_+). \tag{2.2}$$

Let $\lambda = x + \mathrm{i}y$ be given, where $x > 0$. We put $f_\lambda(t) = \exp(-\lambda t)(t \geqslant 0)$. Then,

$$\widehat{f_\lambda}(\boldsymbol{T}) = \int_0^\infty \exp(-\lambda t) T(t) \mathrm{d}t = R(\lambda, A)$$

and $\widehat{f}_\lambda(-y) = (\lambda - \mathrm{i}y)^{-1} = 1/x$. In view of (2.2), we have

$$\frac{1}{x} \leqslant \|R(x+\mathrm{i}y, A)\|, \quad \text{for all } x > 0.$$

By letting $x \to 0^+$, we find that $\|R(\mathrm{i}y, A)\| = \infty$. This shows that $\mathrm{i}y \in \sigma(A)$.

Now let $R \in \boldsymbol{A(T)}$ be such that $\widehat{R}(z) = 0$ on $\sigma(A) \cap \mathrm{i}\mathbb{R}$. Assume first that $\mathrm{i}\mathbb{R} \subset \sigma(A)$. Since $R \in \boldsymbol{A(T)}$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(\mathbb{R}_+)$ such that $\|\widehat{f}_n(\boldsymbol{T}) - R\| \to 0$. Since $\widehat{R}(z) = 0$ on $\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\mathbb{R}$, it follows from Lemma 2.2 that $\phi_z(\widehat{f}_n(\boldsymbol{T})) = \widehat{f}_n(\mathrm{i}z) \to 0$, uniformly for $z \in \mathrm{i}\mathbb{R}$. By semigroup version of the von Neumann inequality, we have $\|\widehat{f}_n(\boldsymbol{T})\| \to 0$, so that $R = 0$. Hence, we may assume that $\mathrm{i}\mathbb{R} \not\subset \sigma(A)$.

Since $\sigma(B) \cap \mathrm{i}\mathbb{R} \subset \sigma(A)$, it follows that $\sigma(B) \cap \mathrm{i}\mathbb{R}$ is a proper subset of $\mathrm{i}\mathbb{R}$. By Lemma 2.8 of [12] $\boldsymbol{V}$ extends to a C_0-group of unitary operators $\boldsymbol{U} = \{U(t)\}_{t\in\mathbb{R}}$ with generator B. Also, since $\sigma(B) \subset \mathrm{i}\mathbb{R}$, we have $\sigma(B) \subset \sigma(A) \cap \mathrm{i}\mathbb{R}$. Further, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(\mathbb{R}_+)$ such that $\|\widehat{f}_n(\boldsymbol{T}) - R\| \to 0$. It follows that $\phi_z(\widehat{f}_n(\boldsymbol{T})) = \widehat{f}_n(\mathrm{i}z) \to 0$, uniformly for $z \in \sigma(A) \cap \mathrm{i}\mathbb{R}$. Consequently, $\widehat{f}_n(\mathrm{i}z) \to 0$, uniformly for $z \in \sigma(B)$. Since for every $f \in L^1(\mathbb{R}_+)$,

$$|\widehat{f}(\boldsymbol{U})\| = \sup_{z\in\sigma(B)} |\widehat{f}(\mathrm{i}z)|,$$

this implies $\|\widehat{f}_n(\boldsymbol{U})\| \to 0$. Now, using (i) we can write $\widehat{f}_n(\boldsymbol{U})J = J\widehat{f}_n(\boldsymbol{T})$. By letting $n \to \infty$, we obtain that $JR = 0$. ∎

A C_0-semigroup $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ on a Banach space X is *bounded away from zero* if $\inf_{t\geqslant 0} \|T(t)x\| > 0$, for all $x \in X\backslash\{0\}$ ([12], p. 180).

Let H be a Hilbert space. Recall that an operator $Y \in B(H)$ is said to be a *quasi-affinity* if Y has zero kernel and dense range. The operators $T, S \in B(H)$ are *quasi-similar* if there exist quasi-affinities $Y_1, Y_2 \in B(H)$ for which $TY_1 = Y_1S$ and $Y_2T = SY_2$.

LEMMA 2.8. *Let $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ be a bounded C_0-semigroup on a Hilbert space H. If $\boldsymbol{T}$ is bounded away from zero, then there exist a quasi-affinity Y and a C_0-semigroup of isometries $\boldsymbol{V} = \{V(t)\}_{t\geqslant 0}$ on H such that:*

(i) *$YT(t) = V(t)Y$, for all $t \geqslant 0$;*

(ii) *for every R in $\{\boldsymbol{T}\}'$, there exists a (unique) $\widetilde{R}$ in $\{\boldsymbol{V}\}'$ such that $\widetilde{R}Y = YR$ and $\|\widetilde{R}\| \leqslant \|R\|$.*

Proof. (i) Here, we follow basically the proof by Nagy-Foias ([17], Chapter 2, Proposition 5.3) given there for discrete semigroups. Let $C(\mathbb{R}_+)$ be the space of all bounded continuous functions on $\mathbb{R}_+$. It is well known that the semigroup $\mathbb{R}_+$ is amenable namely, there exists a functional $\Phi \in C(\mathbb{R}_+)^*$ such that:

(1) $\Phi(\mathbf{1}) = 1$, where $\mathbf{1}$ is the constant one function on $\mathbb{R}_+$;

(2) $\Phi(f) \geqslant 0$, for every $f \geqslant 0$;

(3) $\Phi(f^t) = \Phi(f)$, where $f^t(s) = f(s+t)$.

For given $x, y \in H$, let us consider the function $f_{x,y}$ on $\mathbb{R}_+$ defined by

$$f_{x,y}(s) = \langle T(s)x, T(s)y\rangle.$$

It can be seen that $f_{x,y} \in \mathbb{C}(\mathbb{R}_+)$. Note also that $\omega(x,y) = \Phi(f_{x,y})$ is a bounded sesquilinear form on H. Then, there exists $Z \in B(H)$ such that $\omega(x,y) = \langle Zx, y\rangle$. If $x = y \neq 0$, then we have

$$\langle Zx, x\rangle = \Phi(\|T(s)x\|^2) \geqslant \inf_s \|T(s)x\|^2 > 0.$$

Now, if we set $Y = Z^{1/2}$, clearly Y is a quasi-affinity and

$$\begin{aligned} \|Yx\|^2 &= \Phi(\|T(s)x\|^2) = \Phi(\|T(s+t)x\|^2) \\ &= \Phi(\|T(s)T(t)x\|^2) = \|YT(t)x\|^2, \quad x \in H. \end{aligned} \tag{2.3}$$

For given $t \geqslant 0$, we define an operator $V_0(t)$ on YH by $V_0(t)Yx = YT(t)x$, $x \in H$. Since $\|V_0(t)Yx\| = \|Yx\|$ and Y has dense range, $V_0(t)$ can be extended to an isometry $V(t)$ on H. Then, we have

$$YT(t) = V(t)Y, \quad \text{for all } t \geqslant 0.$$

It can be easily verified that $\boldsymbol{V} = \{V(t)\}_{t\geqslant 0}$ is a C_0-semigroup of isometries.

Next, we prove (ii). Let $R \in \{\boldsymbol{T}\}'$. Define an operator $\widetilde{R}_0$ on YH by $\widetilde{R}_0 Y = YR$. In view of (2.3), for any $x \in H$ we can write

$$\begin{aligned} \|\widetilde{R}_0 Yx\|^2 &= \|YRx\|^2 = \Phi(\|T(t)Rx\|^2) = \Phi(\|RT(t)x\|^2) \\ &\leqslant \|R\|^2 \Phi(\|T(t)x\|^2) = \|R\|^2\|Yx\|^2. \end{aligned}$$

Since Y has dense range, $\widetilde{R}_0$ can be extended to whole H. If denote this extension by $\widetilde{R}$, then we have $\widetilde{R}Y = YR$ and $\|\widetilde{R}\| \leqslant \|R\|$. It remains to show that $\widetilde{R} \in \{\boldsymbol{V}\}'$. Since Y has dense range, from the identities

$$\widetilde{R}V(t)Y = \widetilde{R}YT(t) = YRT(t) = YT(t)R = V(t)YR = V(t)\widetilde{R}Y \quad (t \geqslant 0),$$

we deduce that $\widetilde{R} \in \{\boldsymbol{V}\}'$. ∎

Now, we are in a position to prove Theorem 2.6.

Proof of Theorem 2.6. Assume that the Gelfand transform of $R \in$ ***A(T)*** vanishes on $\sigma(A) \cap \mathrm{i}\mathbb{R}$. Let $H_0 = \left\{x \in H : \lim_{t\to\infty} \|T(t)x\| = 0\right\}$. Then, H_0 is a closed subspace of H invariant under $\boldsymbol{T}$. We may assume that $H_0 \neq H$. Let $K = H/H_0$ and let $\pi : H \to K$ be the canonical surjection. Let $\overline{\boldsymbol{T}} = \{\overline{T}(t)\}_{t\geqslant 0}$ be the induced C_0-semigroup on K defined by $\overline{T}(t)\pi = \pi T(t)$. Then, $\overline{\boldsymbol{T}}$ is bounded away from zero and $\|\widehat{f}(\overline{\boldsymbol{T}})\| \leqslant \|\widehat{f}(\boldsymbol{T})\|$, for all $f \in L^1(\mathbb{R}_+)$. Now, apply Lemma 2.8(i) to this situation to obtain a quasi-affinity $Y : K \to K$ and a C_0-semigroup of isometries $\boldsymbol{V} = \{V(t)\}_{t\geqslant 0}$ on K such that $Y\overline{T}(t) = V(t)Y$. Hence we have $Y\pi T(t) = V(t)Y\pi$, for all $t \geqslant 0$. On the other hand, since $Y\widehat{f}(\overline{\boldsymbol{T}}) = \widehat{f}(\boldsymbol{V})Y$, it follows from Lemma 2.8(ii) that $\|\widehat{f}(\boldsymbol{V})\| \leqslant \|\widehat{f}(\overline{\boldsymbol{T}})\|$, so that $\|\widehat{f}(\boldsymbol{V})\| \leqslant \|\widehat{f}(\boldsymbol{T})\|$, for all $f \in L^1(\mathbb{R}_+)$. Finally, apply Lemma 2.7 to the situation $(H, \boldsymbol{T})$, $(K, \boldsymbol{V})$ and

$J = Y\pi$ to conclude that $Y\pi R = 0$. Since Y has zero kernel, we have that $\pi R = 0$, i.e., $RH \subset H_0$. ∎

As a corollary, we have the following special result.

COROLLARY 2.9. *Let $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup of contractions on a Hilbert space H with generator A. If $R \in A(\boldsymbol{T})$ is a compact operator, then the Gelfand transform of R vanishes on $\sigma(A) \cap \mathrm{i}\mathbb{R}$ if and only if*

$$\lim_{t\to\infty} \|T(t)R\| = 0.$$

Proof. Assume that $\|T(t)R\| \to 0$ $(t \to \infty)$, for some $R \in \boldsymbol{A(T)}$. Let $t \geqslant 0$, $\mathrm{i}y \in \sigma(A) (y \in \mathbb{R})$ and $f \in L^1(\mathbb{R}_+)$ be given. By Lemma 2.2, there exists a multiplicative functional $\phi_{\mathrm{i}y}$ on $\boldsymbol{A(T)}$ such that $\phi_{\mathrm{i}y}(\widehat{f}(\boldsymbol{T})) = \widehat{f}(-y)$. It follows that

$$\begin{aligned}\phi_{\mathrm{i}y}(T(t)\widehat{f}(\boldsymbol{T})) &= \phi_{\mathrm{i}y}(\widehat{f_t}(\boldsymbol{T})) = \widehat{f_t}(-y)\\ &= \exp(\mathrm{i}yt)\widehat{f}(-y) = \exp(\mathrm{i}yt)\phi_{\mathrm{i}y}(\widehat{f}(\boldsymbol{T})),\end{aligned}$$

where $f_t(s)$ is defined by $f_t(s) = f(s-t)$, if $s \geqslant t$ and $= 0$, if $0 \leqslant s < t$. Since the set $\{\widehat{f}(\boldsymbol{T}) : f \in L^1(\mathbb{R}_+)\}$ is dense in $\boldsymbol{A(T)}$, we have $\phi_{\mathrm{i}y}(T(t)R) = \exp(\mathrm{i}yt)\widehat{R}(\mathrm{i}y)$, for all $t \geqslant 0$. It follows that

$$|\widehat{R}(\mathrm{i}y)| = |\phi_{\mathrm{i}y}(T(t)R)| \leqslant \|T(t)R\| \to 0, \quad \text{as } t \to \infty.$$

Now, assume that $R \in \boldsymbol{A(T)}$ is a compact operator and $\widehat{R}(z)$ vanishes on $\sigma(A) \cap \mathrm{i}\mathbb{R}$. Fix $\varepsilon > 0$. Since the set $\{Rx : x \in H, \|x\| \leqslant 1\}$ is relatively compact, it has a finite ε-mesh, say $Rx_1, \ldots, Rx_n$, where $\|x_i\| \leqslant 1$ $(i = 1, \ldots, n)$. This clearly implies

$$\|T(t)R\| \leqslant \max\{\|T(t)Rx_i\| : i = 1, \ldots, n\} + \varepsilon, \qquad \text{for all } t \geqslant 0.$$

From this and from Theorem 2.6 it follows that $\|T(t)R\| \to 0$, as $t \to \infty$. ∎

For the bounded C_0-semigroups we have the following theorem.

THEOREM 2.10. *Let $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ be a bounded C_0-semigroup on a Hilbert space, which is bounded away from zero. If the Gelfand transform of $R \in \boldsymbol{A(T)}$ vanishes on the unitary spectrum of the generator of $\boldsymbol{T}$, then $R = 0$.*

Proof. This is an immediate consequence of Lemmas 2.7 and 2.8. ∎

THEOREM 2.11. *If $\boldsymbol{T} = \{T(t)\}_{t\geqslant 0}$ is a cyclic C_0-semigroup of contractions on a Hilbert space H, then for every quasinilpotent R in $\{\boldsymbol{T}\}'$ and $x \in H$,*

$$\lim_{t\to\infty} \|T(t)Rx\| = 0.$$

Proof. Let $H_0 = \left\{x \in H : \lim_{t\to\infty} \|T(t)x\| = 0\right\}$. Then, H_0 is a closed subspace of H invariant under $\{\mathbf{T}\}'$. We may assume that $H_0 \neq H$. Let $K = H/H_0$ and let $\pi : H \to K$ be the canonical surjection. Let $\overline{\boldsymbol{T}} = \{\overline{T}(t)\}_{t\geqslant 0}$ be the induced semigroup on K defined by $\overline{T}(t)\pi = \pi T(t)$. Then, $\overline{\boldsymbol{T}}$ is a bounded away from zero cyclic semigroup. Apply Lemma 2.8(i) to this situation to obtain a quasi-affinity $Y : K \to K$ and a C_0-semigroup of isometries $\boldsymbol{V} = \{V(t)\}_{t\geqslant 0}$ on K such that $Y\overline{T}(t) = V(t)Y$. It follows that $\boldsymbol{V}$ is a cyclic semigroup.

Note that any operator R in $\{\boldsymbol{T}\}'$ generates an operator $\overline{R}$ in $\{\overline{\boldsymbol{T}}\}'$ defined by $\overline{R}\pi = \pi R$. Since $\|\overline{R}\| \leqslant \|R\|$, it follows from the spectral radius formula that if $\sigma(R) = \{0\}$, then $\sigma(\overline{R}) = \{0\}$. On the other hand, by Lemma 2.8(ii), for every $\overline{R}$ in $\{\overline{\boldsymbol{T}}\}'$ there exists a unique $\tilde{\overline{R}}$ in $\{\boldsymbol{V}\}'$ such that $Y\overline{R} = \tilde{\overline{R}}Y$ and $\|\tilde{\overline{R}}\| \leqslant \|\overline{R}\|$. It follows that if $\sigma(\overline{R}) = \{0\}$, then $\sigma(\tilde{\overline{R}}) = \{0\}$.

Now let R be a quasinilpotent in $\{\boldsymbol{T}\}'$. Then, $\overline{R} \in \{\overline{\boldsymbol{T}}\}'$ and $\sigma(\overline{R}) = \{0\}$. Consequently, $\tilde{\overline{R}} \in \{\boldsymbol{V}\}'$ and $\sigma(\tilde{\overline{R}}) = \{0\}$. Since $\boldsymbol{V}$ is a cyclic semigroup, by Proposition 2.5 we have $\tilde{\overline{R}} = 0$. This implies that $Y\overline{R} = 0$. Since Y has zero kernel, we obtain $\overline{R} = 0$, so that $\pi R = 0$, i.e., $RH \subset H_0$. ∎

3. THE BANACH ALGEBRA GENERATED BY A SINGLE OPERATOR

In this section, we shall discuss some results concerning the semisimplicity problem for the algebras generated by a single bounded operator. For the proof of the main results of this section, we shall need some preliminary results with which we now proceed.

LEMMA 3.1. *Let X be a Banach space and let $T, R \in B(X)$, where $\sigma(R) = \{0\}$. If $(P_n(z))_{n\in\mathbb{N}}$ are polynomials such that $P_n(T) \to R$, in the operator-norm topology, then $P_n(z) \to 0$, uniformly for $z \in \sigma(T)$.*

Proof. Since for every $S \in A(T)$, $\sigma_{A(T)}(S) = \{\phi(S) : \phi \in M_{A(T)}\}$, it follows from the spectral radius formula that $\{\phi(R) : \phi \in M_{A(T)}\} = \sigma_{A(T)}(R) = \{0\}$. Also, since $\|P_n(T) - R\| \to 0$, it follows that $\phi(P_n(T)) \to \phi(R)$, uniformly with respect to $\phi \in M_{A(T)}$. Taking into account the relation

$$\phi(P_n(T)) = P_n(\phi(T)) \quad (\phi \in M_{A(T)}),$$

we have $P_n(z) \to 0$, uniformly for $z \in \sigma_{A(T)}(T)$. Since $\sigma_{A(T)}(T) \supset \sigma(T)$, we obtain that $P_n(z) \to 0$, uniformly for $z \in \sigma(T)$. ∎

LEMMA 3.2. *Let X be a Banach space and let $T \in B(X)$. Then, $A(T)$ is semisimple if and only if $A(T^*)$ is semisimple.*

Proof. Assume that $A(T^*)$ is semisimple. Let $R \in \mathrm{Rad}A(T)$. Then, there exists a sequence of polynomials $(P_n(z))_{n\in\mathbb{N}}$ such that $\|P_n(T) - R\| \to 0$. This implies $\|P_n(T^*) - R^*\| \to 0$. We see that $R^* \in A(T^*)$ and $\sigma(R^*) = \{0\}$. Since $A(T^*)$ is semisimple, we have $R^* = 0$, so that $R = 0$.

Now, assume that $A(T)$ is semisimple. Let $R \in \mathrm{Rad}A(T^*)$. Then, there exists a sequence of polynomials $(P_n(z))_{n\in\mathbb{N}}$ such that $\|P_n(T^*) - R\| \to 0$. This implies $\|P_n(T^{**}) - R^*\| \to 0$ and therefore, $\|P_n(T) - R^*|_{X}.\| \to 0$, where $R^*|_{X}.$ is the restriction of R^* to X. We see that $R^*|_{X^\cdot} \in A(T)$. On the other hand, since $\sigma(R^*) = \{0\}$, it follows from the spectral radius formula that $\sigma(R^*|_{X^\cdot}) = \{0\}$. Also, since $A(T)$ is semisimple, we have $R^*|_{X^\cdot} = 0$. Further, using the fact that X is dense in X^{**} in the $\sigma(X^{**}, X^*)$-topology, we obtain $R^* = 0$, and so $R = 0$. ∎

LEMMA 3.3. *Let S be a (possibly unbounded) normal operator on a Hilbert space H and let $Y, T \in B(H)$, where Y has zero kernel and*

$$TYx = YSx, \quad \text{for all } x \in D(S).$$

Then, $\sigma(S) \subset \sigma(T)$ and consequently, S is bounded.

Proof. Assume that there exists $\xi \in \sigma(S)$, but $\xi \notin \sigma(T)$. We put $\delta = \|(T-\xi)^{-1}\|^{-1}$. Choose $\varepsilon > 0$ such that $\varepsilon < \delta$. Let $\Delta_\varepsilon = \{z \in \mathbb{C} : |z - \xi| < \varepsilon\}$ and let $E(\cdot)$ be the spectral measure associated with S. Since $\sigma(S) \cap \Delta_\varepsilon \neq \emptyset$, we have $E(\Delta_\varepsilon) \neq 0$. Let $x \in E(\Delta_\varepsilon)H$ be such that $\|x\| = 1$. Then, $x \in D(S^n)$ for all $n = 1, 2, \ldots$. From the identities

$$(S-\xi)^n x = \int_{\Delta_\varepsilon} (z-\xi)^n \mathrm{d}E(z)x, \quad (n = 1, 2, \ldots),$$

we have

$$\|(S-\xi)^n x\| \leqslant \varepsilon^n.$$

On the other hand, we can write

$$(T-\xi)^n Yx = Y(S-\xi)^n x, \quad \text{for all } n = 1, 2, \ldots.$$

It follows that

$$\|(T-\xi)^n Yx\| \leqslant \varepsilon^n \|Y\|.$$

Consequently, we have

$$\|Yx\| \leqslant \|(T-\xi)^{-n}\| \, \|(T\text{-}\xi)^n Yx\| \leqslant \left(\frac{\varepsilon}{\delta}\right)^n \|Y\| \to 0, \quad \text{as } n \to \infty.$$

Hence, $Yx = 0$. Since Y has zero kernel, we obtain $x = 0$. This is a contradiction. ∎

Let $T_1, T_2 \in B(H)$. An operator $Y \in B(H)$ *intertwines* T_2 and T_1 if and only if $T_1 Y = Y T_2$.

One of the main results of this section is the following theorem.

THEOREM 3.4. *Let H be a Hilbert space and let $T, S \in B(H)$, where S is a normal operator. If there exists a quasi-affinity $Y \in B(H)$ intertwining S and T (or T and S), then the algebra $A(T)$ is semisimple.*

Proof. Let $R \in \operatorname{Rad} A(T)$. Then, there exists a sequence of polynomials $(P_n(z))_{n \in \mathbb{N}}$ such that $\|P_n(T) - R\| \to 0$. In view of Lemma 3.1, $P_n(z) \to 0$, uniformly for $z \in \sigma(T)$. By Lemma 3.3, $\sigma(S) \subset \sigma(T)$ and therefore, $P_n(z) \to 0$, uniformly for $z \in \sigma(S)$. Since S is a normal operator, $\|P_n(S)\| \to 0$. Further, from the identity $TY = YS$ we can write $P_n(T)Y = YP_n(S)$. By letting $n \to \infty$, we find that $RY = 0$. Since Y has dense range, we obtain $R = 0$.

If Y intertwines T and S, then Y^* intertwines S^* and T^*. Since S^* is a normal operator and Y^* is a quasi-affinity, it follows from what we showed above that $A(T^*)$ is semisimple. Hence, by Lemma 3.2 the algebra $A(T)$ is semisimple. ∎

Let H be a Hilbert space. $T \in B(H)$ is said to be essentially normal if $TT^* - T^*T$ is a compact operator. We know that normal operators generate a semisimple algebra. On the other hand, the Volterra operator is essentially normal but it generates an algebra that is not semisimple.

We say that (see [10]) the sequence $(T_n)_{n\in\mathbb{N}}$ in $B(H)$ *slowly converges to zero* if $T_n \to 0$ in the weak operator topology and $\inf_{n\to\infty} \|T_n x\| > 0$ for all $x \neq 0$.

COROLLARY 3.5. *Let T be an essentially normal operator. Assume that both $\{T\}'$ and $\{T^*\}'$ contain sequences which converge slowly to zero. Then, the algebra $A(T)$ is semisimple.*

Proof. As proved in [10], under the hypotheses of the corollary, T is quasi-similar to some normal operator. It remains to apply Theorem 3.4. ∎

It is a famous inequality of von Neumann that for every contraction T on a Hilbert space and every polynomial P, $\|P(T)\| \leqslant \sup_{z\in\Gamma} |P(z)|$. Von Neumann's inequality is equivalent to the existence of a contractive disc-algebra functional calculus. It follows that $A(T)$ (respectively $W(T)$) coincides with the closure of the set $\{f(T) : f \in A(D)\}$ in the uniform operator topology (respectively weak operator topology). Note also that for every $\xi \in \sigma(T)$, there exists a multiplicative functional ϕ_ξ on $A(T)$ such that $\phi_\xi(f(T)) = f(\xi)$, $f \in A(D)$.

THEOREM 3.6. *Let $T, S \in B(H)$, where S is a contraction and $\Gamma \subset \sigma(T)$. Assume that for S and T (respectively for T and S) there exists an intertwining operator Y with dense range (respectively with zero kernel). Then, the algebra $A(T)$ is semisimple.*

Proof. Let $R \in \mathrm{Rad}A(T)$. Then, there exists a sequence $(P_n(z))_{n\in\mathbb{N}}$ of polynomials such that $\|P_n(T) - R\| \to 0$. By Lemma 3.1, $P_n(z) \to 0$, uniformly for $z \in \sigma(T)$. Since $\Gamma \subset \sigma(T)$, $P_n(z) \to 0$, uniformly for $z \in \Gamma$. It follows from the von Neumann inequality that,

$$\|P_n(S)\| \leqslant \sup_{z\in\Gamma} |P_n(z)| \to 0, \quad \text{as } n \to \infty.$$

Further, from the identity $TY = YS$ we can write $P_n(T)Y = YP_n(S)$. By letting $n \to \infty$, we obtain $RY = 0$. Since Y has dense range, we have that $R = 0$. ∎

A theorem of Esterle-Strouse-Zouakia ([4], Theorem 3), states that if T is a contraction on a Hilbert space and $f \in A(D)$ vanishes on $\sigma(T) \cap \Gamma$, then it follows $\lim_{n\to\infty} \|T^n f(T)\| = 0$. We see that under these assumptions, the Lebesgue measure of $\sigma(T) \cap \Gamma$ is necessarily zero.

THEOREM 3.7. *Let T be a contraction on a Hilbert space such that $\sigma(T)\cap\Gamma$ has zero Lebesgue measure. Then, the Gelfand transform of $R\in A(T)$ vanishes on $\sigma(T)\cap\Gamma$ if and only if*

$$\lim_{n\to\infty}\|T^nR\|=0.$$

Proof. Assume that $\|T^nR\|\to 0$ $(n\to\infty)$, for some $R\in A(T)$. For an arbitrary $\xi\in\sigma(T)\cap\Gamma$ there exists a multiplicative functional ϕ_ξ on $A(T)$ such that $\phi_\xi(T)=\xi$ and so since ϕ_ξ has norm one,

$$|\widehat{R}(\xi)|=|\phi_\xi(T^nR)|\leqslant\|T^nR\|\to 0,\quad \text{as } n\to\infty.$$

Now let $R\in A(T)$ be such that $\widehat{R}(\xi)=0$ on $\sigma(T)\cap\Gamma$. Fix $\varepsilon>0$. Since $R\in A(T)$, there exists a function $f\in A(D)$ such that $\|R-f(T)\|<\varepsilon$. It follows that $\sup_{\xi\in\sigma(T)\cap\Gamma}|f(\xi)|<\varepsilon$. Since $\sigma(T)\cap\Gamma$ has zero Lebesgue measure, by the Rudin-Carleson Theorem ([1], Chapter 8, Theorem 7.4), there exists a function $g\in A(D)$ such that $f(\xi)=g(\xi)$, for all $\xi\in\sigma(T)\cap\Gamma$ and $\|g\|=\sup_{\xi\in\sigma(T)\cap\Gamma}|f(\xi)|<\varepsilon$. By the von Neumann inequality, $\|g(T)\|<\varepsilon$. We put $h=f-g$. Then, we can write

$$\|R-h(T)\|=\|R-f(T)+g(T)\|\leqslant\|R-f(T)\|+\|g(T)\|<2\varepsilon.$$

This implies

$$\|T^nR-T^nh(T)\|<2\varepsilon,\quad \text{for all } n=1,2,\dots,$$

so that

$$\|T^nR\|\leqslant\|T^nh(T)\|+2\varepsilon,\quad \text{for all } n=1,2,\dots.$$

Since $h(\xi)=0$ on $\sigma(T)\cap\Gamma$, by the Esterle-Strouse-Zouakia Theorem

$$\|T^nh(T)\|\to 0,\quad \text{as } n\to\infty.$$

Hence, we have that $\lim_{n\to\infty}\|T^nR\|\leqslant 2\varepsilon$. Since ε was arbitrary, the theorem is proved. ∎

Recall that for the contraction T on a Hilbert space the discrete version of Corollary 2.9 can be formulated as follows: If $R\in A(T)$ is a compact operator and if the Gelfand transform of R vanishes on $\sigma(T)\cap\Gamma$, then $\lim_{n\to\infty}\|T^nR\|\to 0$.

A partial converse of this fact is contained in the next theorem.

THEOREM 3.8. *Let T be a completely non-unitary contraction on a Hilbert space H such that $\sigma(T)\cap\Gamma$ has zero Lebesgue measure and let*

$$\dim(I-TT^*)H=\dim(I-T^*T)H=1.$$

If $\lim_{n\to\infty}\|T^nR\|\to 0$, then the Gelfand transform of $R\in A(T)$ vanishes on $\sigma(T)\cap\Gamma$ and R is a compact operator.

Proof. Assume that $\|T^n R\| \to 0$, as $n \to \infty$. As in the proof of Theorem 3.7 we can see that $\widehat{R}(\xi) = 0$, for all $\xi \in \sigma(T) \cap \Gamma$. It remains to show that R is a compact operator. Let us mention a theorem of Sz.-Nagy and Foias ([17], Chapter 2, Proposition 6.7) that if T is a completely non-unitary contraction with zero Lebesgue measure of $\sigma(T) \cap \Gamma$, then $T^n \to 0$ and $T^{*n} \to 0$ strongly. According to the well known model theorem of Sz.-Nagy and Foias, T is unitary equivalent to its model operator $M_\varphi = P_\varphi S|_{K_\varphi}$. acting on the model space $K_\varphi = H^2 \Theta \varphi H^2$, where φ is an inner function, $Sf = zf$ is the shift operator on the Hardy space H^2 and P_φ is the orthogonal projection of H^2 onto K_φ. It follows that for every $h \in A(D)$, the operator $h(T)$ is unitary equivalent to $h(M_\varphi) = P_\varphi h(S)|_{K_\varphi}$.

Fix $\varepsilon > 0$. Since $R \in A(T)$, there exists a function $f \in A(D)$ such that $\|R - f(T)\| < \varepsilon$. It follows that $\sup\limits_{\xi \in \sigma(T) \cap \Gamma} |f(\xi)| < \varepsilon$. Since $\sigma(T) \cap \Gamma$ has zero Lebesgue measure, by the Rudin-Carleson Theorem ([1], Chapter 8, Theorem 7.4) there exists a function $g \in A(D)$ such that $f(\xi) = g(\xi)$, for all $\xi \in \sigma(T) \cap \Gamma$ and $\|g\| = \sup\limits_{\xi \in \sigma(T) \cap \Gamma} |f(\xi)| < \varepsilon$. In view of the von Neumann inequality, we have $\|g(T)\| < \varepsilon$. Put $h = f - g$. Since $h(\xi) = 0$ on $\sigma(T) \cap \Gamma$, by the Esterle-Strouse-Zouakia Theorem, $\|T^n h(T)\| \to 0$, as $n \to \infty$. Hence, we have that $\|M_\varphi^n h(M_\varphi)\| \to 0$. Now, by the Hartman-Sarason Theorem ([13], p. 235), $h(M_\varphi)$ is a compact operator. Consequently, $h(T)$ is a compact operator. On the other hand,

$$\|R - h(T)\| = \|R - f(T) + g(T)\| \leqslant \|R - f(T)\| + \|g(T)\| < 2\varepsilon.$$

It follows that R is a compact operator. ∎

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