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CONNES-CHERN CHARACTERS OF HEXIC AND CUBIC MODULES

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Dedicated to George Elliott on his sixtieth birthday

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ABSTRACT. Let A_θ denote the rotation C^*-algebra generated by unitaries U, V satisfying $VU = e^{2\pi i\theta}UV$, where θ is a fixed real number. Let ρ denote the *hexic* transform of A_θ defined by $U \mapsto V \mapsto e^{-\pi i\theta}U^{-1}V$ (which has order six), let κ denote the *cubic* transform $\kappa = \rho^2$, and let $H_\theta := A_\theta \rtimes_\rho \mathbb{Z}_6$ and $C_\theta := A_\theta \rtimes_\kappa \mathbb{Z}_3$ denote the associated C^*-crossed products by corresponding cyclic groups. It is shown that for each θ there are canonical inclusions $\mathbb{Z}^{10} \hookrightarrow K_0(H_\theta)$ and $\mathbb{Z}^8 \hookrightarrow K_0(C_\theta)$ given explicitly by projections and "mysterious" modules (called *hexic* and *cubic* modules). We also find the unbounded traces on the canonical smooth dense $*$-subalgebras and so obtain Connes' cyclic cohomology groups of order zero $\mathrm{HC}^0(H_\theta) \cong \mathbb{C}^9$, $\mathrm{HC}^0(C_\theta) \cong \mathbb{C}^7$, when θ is irrational.

KEYWORDS: *Fourier transform, C^*-algebras, rotation algebras, automorphisms.*

MSC (2000): 46L80, 46L40.

1. INTRODUCTION

Let $\theta > 0$, $\lambda = e^{2\pi i\theta}$, and consider the rotation C^*-algebra A_θ generated by unitaries U, V satisfying $VU = \lambda UV$. The (noncommutative) *hexic* transform of A_θ is the canonical order six automorphism ρ defined by

$$\rho(U) = V, \quad \rho(V) = \lambda^{-\frac{1}{2}}U^{-1}V.$$

Its cube is the usual flip automorphism studied in [1], [2], [3]. Its square $\kappa := \rho^2$ is what we shall call the *cubic* transform:

$$\kappa(U) = \lambda^{-\frac{1}{2}}U^{-1}V, \quad \kappa(V) = U^{-1}.$$

The corresponding crossed product $H_\theta := A_\theta \rtimes_\rho \mathbb{Z}_6$ (the hexic C^*-algebra) is the universal C^*-algebra generated by unitaries U, V, W enjoying the commutation relations

$$VU = \lambda UV, \quad WUW^{-1} = V, \quad WVW^{-1} = \lambda^{-\frac{1}{2}} U^{-1} V, \quad W^6 = I. \tag{1.1}$$

One may view the crossed product $C_\theta = A_\theta \rtimes_\kappa \mathbb{Z}_3$ (the cubic C^*-algebra) as the C^*-subalgebra of H_θ generated by U, V, and $Z := W^2$. It can be viewed as the universal C^*-algebra generated by unitaries U, V, Z enjoying the commutation relations

$$VU = \lambda UV, \quad ZUZ^{-1} = \lambda^{-\frac{1}{2}} U^{-1} V, \quad ZVZ^{-1} = U^{-1}, \quad Z^3 = I. \tag{1.2}$$

We write H_θ^∞ and C_θ^∞ for their respective canonical smooth dense $*$-subalgebras. (For example, the elements of H_θ^∞ consist of sums of terms of the form aW^j where $a \in A_\theta^\infty$.)

The purpose of this paper is the construction of ten canonical classes in $K_0(H_\theta)$, eight canonical classes in $K_0(C_\theta)$, and show that they are independent over the integers so that there are injections $\mathbb{Z}^{10} \to K_0(H_\theta)$ and $\mathbb{Z}^8 \to K_0(C_\theta)$ for each $\theta > 0$. One of the classes in each case involves an exotic module, which we call the hexic and cubic modules $\mathcal{M}_6$ and $\mathcal{M}_3$ respectively, and the computations of their unbounded traces are lengthy and require detailed treatment with theta functions. Of course, we also obtain the unbounded traces on H_θ^∞ and C_θ^∞, thereby obtaining their Connes cyclic cohomology groups of order zero (see Theorem 1.3). The unbounded traces on the crossed products, denoted by T_{ij} in the hexic case and S_{ij} in the cubic case, arise from what we call "twisted" trace functionals on A_θ^∞ (as for example in [10], Section 2). These are determined in Section 3.

Throughout the paper we let $\omega := e(\frac{1}{6}) = \frac{1}{2}(1 + \mathrm{i}\sqrt{3})$ (a primitive 6th root of 1). We also adopt the convention $e(t) = \mathrm{e}^{2\pi \mathrm{i} t}$.

HEXIC CASE. Consider the projections

$$p_j = \frac{1}{6} \sum_{i=0}^{5} \omega^{ij} W^i, \quad q_j = \frac{1}{3} \sum_{i=0}^{2} \omega^{2ij} X^i, \quad r = \frac{1}{2}(I + UW^3)$$

where $X := \lambda^{\frac{1}{6}} UW^2$ is of order 3, and UW^3 has order 2. We prove that we have the character values in Table 1.

Consequently, one has

THEOREM 1.1. *The ten K_0 classes in Table* 1 *yield an inclusion $\mathbb{Z}^{10} \to K_0(H_\theta)$ for each $\theta > 0$.*

Table 1. Character table for the hexic case

K_0-class	τ	C_6	T_{10}	T_{20}	T_{21}	T_{30}	T_{31}
$[1]$	1	0	0	0	0	0	0
$[p_0]$	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$[p_1]$	$\frac{1}{6}$	0	$-\frac{1}{6}\omega^2$	$-\frac{1}{6}\omega$	$-\frac{1}{6}\omega$	$-\frac{1}{6}$	$-\frac{1}{6}$
$[p_2]$	$\frac{1}{6}$	0	$-\frac{1}{6}\omega$	$\frac{1}{6}\omega^2$	$\frac{1}{6}\omega^2$	$\frac{1}{6}$	$\frac{1}{6}$
$[p_3]$	$\frac{1}{6}$	0	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
$[p_4]$	$\frac{1}{6}$	0	$\frac{1}{6}\omega^2$	$-\frac{1}{6}\omega$	$-\frac{1}{6}\omega$	$\frac{1}{6}$	$\frac{1}{6}$
$[q_0]$	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$	0	0
$[q_1]$	$\frac{1}{3}$	0	0	0	$-\frac{1}{3}\omega$	0	0
$[r]$	$\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$
$[\mathcal{M}_6]$	$\frac{\theta}{6}$	-1	$\frac{1}{6}\omega$	$\frac{1}{6\sqrt{3}}\omega^{1/2}$	$\frac{1}{6}\sqrt{3}\omega^{1/2}$	$\frac{1}{12}$	$\frac{1}{3}$

CUBIC CASE. In this case, with Z denoting the canonical unitary of C_θ, we set $X = \lambda^{\frac{1}{6}}UZ$, $Y = \lambda^{\frac{2}{3}}U^2Z$ (unitaries of order three). Consider the polynomials (for $j = 0, 1$)

$$Q_j(x) = \tfrac{1}{3}(I + \omega^{2j}x + \omega^{4j}x^2).$$

THEOREM 1.2. *The eight K_0 classes in Table 2 yield an inclusion $\mathbb{Z}^8 \to K_0(C_\theta)$ for each $\theta > 0$.*

Table 2. Character table for the cubic case

K_0-class	τ	C_3	S_{10}	S_{11}	S_{12}
$[1]$	1	0	0	0	0
$[Q_0(Z)]$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0
$[Q_1(Z)]$	$\frac{1}{3}$	0	$-\frac{1}{3}\omega$	0	0
$[Q_0(Y)]$	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0
$[Q_1(Y)]$	$\frac{1}{3}$	0	0	$-\frac{1}{3}\omega$	0
$[Q_0(X)]$	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$
$[Q_1(X)]$	$\frac{1}{3}$	0	0	0	$-\frac{1}{3}\omega$
$[\mathcal{M}_3]$	$\frac{\theta}{3}$	-1	$\frac{1}{3\sqrt{3}}\omega^{1/2}$	$\frac{1}{3\sqrt{3}}\omega^{1/2}$	$\frac{1}{3\sqrt{3}}\omega^{1/2}$

Using Theorems 1.1 and 1.2, Polishchuk in [7] recently showed that one has isomorphisms $K_0(C_\theta) \cong \mathbb{Z}^8$ and $K_0(H_\theta) \cong \mathbb{Z}^{10}$ for all $\theta > 0$. In [5], the authors in turn use Polishchuk's result to show that the injection $\mathbb{Z}^{10} \to K_0(H_\theta)$ is an isomorphism, so that the ten canonical classes obtained herein do in fact form a basis for K_0 for each θ. We believe that similar computations show that the injection $\mathbb{Z}^8 \to K_0(C_\theta)$ is an isomorphism.

THEOREM 1.3. (See Corollaries 3.2 and 3.4) *For any irrational θ, one has the cyclic cohomology group of order zero*

$$HC^0(A_\theta \rtimes_\rho \mathbb{Z}_6) \cong HC^0(A_\theta^\rho) \cong \mathbb{C}^9, \quad HC^0(A_\theta \rtimes_\kappa \mathbb{Z}_3) \cong HC^0(A_\theta^\kappa) \cong \mathbb{C}^7.$$

Specific bases for these groups are given in Section 3.

2. BACKGROUND

We shall denote by $\mathbb{C}^+$ the set of complex numbers with positive real part. We will make frequent use of the identity

$$\int_{\mathbb{R}} e(Ax)\, \mathrm{e}^{-\pi a x^2}\, \mathrm{d}x \;=\; \frac{1}{\sqrt{a}} \mathrm{e}^{-\pi \frac{A^2}{a}} \tag{2.1}$$

where $a, A \in \mathbb{C}$ and $\mathrm{Re}(a) > 0$. The square root appearing here is the principal one (namely, if $z = r\mathrm{e}^{\mathrm{i}t}$ and $-\pi < t \leqslant \pi$, then $\sqrt{z} := \sqrt{r}\mathrm{e}^{\mathrm{i}\frac{t}{2}}$). For our purposes below it will be worthwhile noting that $\mathbb{C}^+$ is closed under addition, conjugation, and inversion; further, for $a, b \in \mathbb{C}^+$, one has $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and $\overline{\sqrt{a}} = \sqrt{\bar{a}}$.

As in Rieffel's construction in [9], one begins with a locally compact Abelian group M, forms the group $G = M \times \widehat{M}$ on which the canonical Heisenberg cocycle $\mathfrak{h}$ defined by $\mathfrak{h}((m, m'), (n, n')) = \langle m, n' \rangle$, where $\langle \cdot, \cdot \rangle$ is the canonical pairing on G. The Heisenberg (projective) representation $\pi : G \to \mathcal{L}(L^2(M))$ is given by $[\pi_{(m,s)} f](n) = \langle n, s \rangle f(n + m)$. It has the properties

$$\pi_x \pi_y = \mathfrak{h}(x, y) \pi_{x+y} = \mathfrak{h}(x, y)\overline{\mathfrak{h}(y, x)} \pi_y \pi_x, \quad \pi_x^* = \mathfrak{h}(x, x) \pi_{-x}$$

for $x, y \in G$. Given a lattice subgroup D of G, the C^*-algebra $C^*(D, \mathfrak{h}) = C^*(D)$ is generated by the unitaries π_x, for $x \in D$, and $C^*(D^\perp, \overline{\mathfrak{h}}) = C^*(D^\perp)$ is the opposite algebra of the C^*-algebra generated by the unitaries π_y^*, for $y \in D^\perp$. (Recall that the complement of D is $D^\perp = \{y \in G : \mathfrak{h}(x, y)\overline{\mathfrak{h}(y, x)} = 1, \ \forall x \in D\}$.) By Rieffel's Theorem 2.15 [9], the Schwartz space $\mathcal{S}(M)$ is an equivalence bimodule with the C^*-algebra $C^*(D)$ acting on the left and $C^*(D^\perp)$ acting on the right. We point out that the right action is $f\pi_y^* := \pi_y^*(f)$. Denoting the opposite multiplication by # (so *a*#*b* = *ba*), one has the module property $(f\pi_y^*)\pi_z^* = f(\pi_y^* \# \pi_z^*)$ for $f \in \mathcal{S}(M)$ and $y, z \in D^\perp$. The C^*-valued inner products are

$$\langle f, g \rangle_D = |G/D| \sum_{x \in D} \langle f, g \rangle_D (x) \pi_x, \quad \langle f, g \rangle_{D^\perp} = \sum_{y \in D^\perp} \langle f, g \rangle_{D^\perp} (y) \pi_y^*,$$

where $|G/D|$ is the Haar-Plancherel measure of a fundamental domain for D in G and, writing $x = (x', x'') \in D$ and $y = (y', y'') \in D^\perp$, one has

$$\langle f, g \rangle_D (x) = \langle f, \pi_x g \rangle_{L^2} = \int_M f(t)\overline{g(t + x')}\, \overline{\langle t, x'' \rangle}\, \mathrm{d}t$$

$$\langle f, g \rangle_{D^\perp} (y) = \langle \pi_y g, f \rangle_{L^2} = \int_M \overline{f(t)} g(t + y') \langle t, y'' \rangle\, \mathrm{d}t.$$

There are canonical normalized traces τ', τ on $C^*(D)$ and $C^*(D^\perp)$, respectively, satisfying

$$\tau'(\langle f, g\rangle_D) = |G/D|\,\tau(\langle g, f\rangle_{D^\perp}).$$

We shall mainly be interested in Rieffel's setup with $M = \mathbb{R}$, and for this purpose we shall also be interested in the following "square root" $\mathfrak{s}$ of the Heisenberg cocycle $\mathfrak{h}$ on $\mathbb{R}^2 \times \mathbb{R}^2$ given by $\mathfrak{s}((m, m'), (n, n')) = e(\frac{1}{2}mn')$, so that $\mathfrak{s}^2 = \mathfrak{h}$. In view of our interest in the hexic transform on the rotation C^*-algebra, we consider the order six map $H : G \to G$ given by $H(u, v) = (u + v, -u)$.

If α is an automorphism of a C^*-algebra A, then a linear functional ϕ (not necessarily norm continuous) defined on a dense α-invariant $*$-subalgebra A' of A is said to be α-trace if and only if

$$\phi(xy) = \phi(\alpha(y)x),$$

$\forall x, y \in A'$. Suppose α has finite order k. If ϕ is an α-invariant α^j-trace, then it induces a trace $\mathbf{T}$ on the smooth crossed product $A' \rtimes_\alpha \mathbb{Z}_k$ (dense in $A \rtimes_\alpha \mathbb{Z}_k$) given by

$$\mathbf{T}(a_0 + a_1 W + \cdots + a_{k-1}W^{k-1}) \;=\; \phi(a_{k-j}).$$

Recall the *adjoint* of a linear map ϕ is $\phi^*(x) := \overline{\phi(x^*)}$ (and ϕ is α-invariant if and only if its adjoint is α-invariant).

PROPOSITION 2.1. *Let X be a C-D-equivalence bimodule and let ρ_1, ρ_2 be automorphisms of C, D, respectively, such that there is a linear map $w : X \to X$ satisfying $\rho_1(\langle x, y\rangle_C) = \langle w(x), w(y)\rangle_C$ and $\rho_2(\langle x, y\rangle_D) = \langle w(x), w(y)\rangle_D$. Then there is a one-to-one correspondence between ρ_1-traces φ' on C and ρ_2-traces φ on D given by*

$$\varphi(\langle x, w^{-1}(y)\rangle_D) \;=\; \varphi'(\langle y, x\rangle_C). \tag{2.2}$$

Proof. It suffices to begin with a ρ_1-trace φ' on C and show the existence of a ρ_2-trace φ on D satisfying (2.2) — the proof of the converse being similar. (Since the inner products over C and D span C and D, respectively, φ will necessarily be unique.) As in the proof of Rieffel's Proposition 2.1 of [8], there is a positive integer n such that, with $E = M_n \otimes C$ and viewing X^n as an equivalence E-D-bimodule in the usual way, there is an element $z \in X^n$ such that $\langle z, z\rangle_D = 1$ (the identity of D). Extend w to X^n in the natural way by $w(x_1, \ldots, x_n) = (w(x_1), \ldots, w(x_n))$. It is easy to check that the hypothesis relating ρ_j to w yields the properties $w(cx) = \rho_1(c)w(x)$ and $w(xd) = w(x)\rho_2(d)$ for $x \in X$, $c \in C, d \in D$. Further, one has $(1 \otimes \rho_1)(\langle \xi, \eta\rangle_E) = \langle w(\xi), w(\eta)\rangle_E$ for $\xi, \eta \in X^n$.

Consider the linear map $\mu : D \to E$ given by $\mu(d) = \langle zd, w^{-1}(z)\rangle_E$. It is easy to check that the induced map $\psi : E \to \mathbb{C}$ defined by $\psi(m \otimes c) = \mathrm{Trace}(m)\varphi'(c)$, where $m \in M_n$, $c \in C$, is a $1 \otimes \rho_1$-trace on E. Now define the linear map $\varphi : D \to$

$\mathbb{C}$ by $\varphi(d) = \psi(\mu(d))$. We now check that (2.2) holds. For $\xi, \eta \in X^n$ one has

$$\begin{aligned}\varphi(\langle \xi, w^{-1}(\eta)\rangle_D) &= \psi(\mu(\langle \xi, w^{-1}(\eta)\rangle_D)) = \psi(\langle z\langle \xi, w^{-1}(\eta)\rangle_D, w^{-1}(z)\rangle_E) \\ &= \psi(\langle \langle z, \xi\rangle_E w^{-1}(\eta), w^{-1}(z)\rangle_E) = \psi(\langle z, \xi\rangle_E \langle w^{-1}(\eta), w^{-1}(z)\rangle_E) \\ &= \psi(\langle z, \xi\rangle_E (1\otimes\rho_1)^{-1}(\langle \eta, z\rangle_E))\end{aligned}$$

which by the $1\otimes\rho_1$-trace property of ψ is

$$\begin{aligned}&= \psi(\langle \eta, z\rangle_E \cdot \langle z, \xi\rangle_E) = \psi(\langle\langle \eta, z\rangle_E z, \xi\rangle_E) \\ &= \psi(\langle \eta\langle z, z\rangle_D, \xi\rangle_E) = \psi(\langle \eta, \xi\rangle_E).\end{aligned}$$

Specializing this equality to $\xi = (x, 0, \dots, 0)$, $\eta = (y, 0, \dots, 0)$ it gives rise to (2.2). It remains to check that φ is a ρ_2-trace. To do this, it is enough to check it using inner products $\langle\cdot, \cdot\rangle_D$ since they span D. Rewriting (2.2) as $\varphi(\langle x, y\rangle_D) = \varphi'(\langle w(y), x\rangle_C)$, one gets

$$\begin{aligned}&\varphi(\langle x_1, y_1\rangle_D \cdot \langle x_2, y_2\rangle_D) \\ &\quad= \varphi(\langle x_1, y_1\langle x_2, y_2\rangle_D\rangle_D) = \varphi(\langle x_1, \langle y_1, x_2\rangle_C y_2\rangle_D) = \varphi'(\langle w[\langle y_1, x_2\rangle_C y_2], x_1\rangle_C) \\ &\quad= \varphi'(\langle \rho_1(\langle y_1, x_2\rangle_C) w(y_2), x_1\rangle_C) = \varphi'(\rho_1(\langle y_1, x_2\rangle_C)\langle w(y_2), x_1\rangle_C)\end{aligned}$$

which by the ρ_1-trace property of φ' is

$$\begin{aligned}&= \varphi'(\langle w(y_2), x_1\rangle_C \cdot \langle y_1, x_2\rangle_C) = \varphi'(\langle\langle w(y_2), x_1\rangle_C\, y_1, x_2\rangle_C) \\ &= \varphi'(\langle w(y_2)\langle x_1, y_1\rangle_D, x_2\rangle_C);\end{aligned}$$

using (2.2) this becomes

$$\begin{aligned}&= \varphi(\langle x_2, w^{-1}[w(y_2)\langle x_1, y_1\rangle_D]\rangle_D) = \varphi(\langle x_2, y_2\rho_2^{-1}(\langle x_1, y_1\rangle_D)\rangle_D) \\ &= \varphi(\langle x_2, y_2\rangle_D \cdot \rho_2^{-1}(\langle x_1, y_1\rangle_D)).\end{aligned}$$

This shows that for each $a, b \in D$ one has $\varphi(ab) = \varphi(b\rho_2^{-1}(a))$. Replacing a by $\rho_2(a)$ one obtains the ρ_2-trace property of φ. ∎

3. UNBOUNDED TRACES AND THE ZEROTH CYCLIC COHOMOLOGY GROUPS

In this section we calculate the unbounded traces and obtain the zeroth cyclic cohomology groups of the crossed products H_θ and C_θ (which are the same as their first Hochschild homology groups). But first let us point out that there is a conceptual basis behind our result for obtaining the traces on the (smooth) crossed products in both the hexic and cubic cases. This stems from looking at the case $\theta = 0$ so that one could in fact apply the result of Brylinski and Nistor [4] to obtain these traces in terms of conjugacy classes of the underlying group ($\mathbb{Z}_6$ and $\mathbb{Z}_3$ in our case) and their fixed points under their action on the 2-torus — as their result applies to crossed products of smooth commutative C^*-algebras by a finite group action. What we do here for the case $\theta \neq 0$ is a kind of noncommutative analogue of that (though it cannot be directly derived from [4]).

THE ρ-TRACE. For simplicity write $\Lambda(n) = \lambda^n$. First, observe that

$$\rho(U^pV^q) = \rho(U)^p\rho(V)^q = \Lambda(-q^2/2 - pq)U^{-q}V^{p+q}.$$

Now, let ϕ be a ρ-trace on $A := A_\theta^\infty$. Then $\phi(U^mV^nU^pV^q) = \phi(\rho(U^pV^q)U^mV^n)$ and applying the formula for $\rho(U^pV^q)$ this becomes $\Lambda(np) \cdot \phi(U^{m+p}V^{n+q}) = \Lambda(-q^2/2 - pq + m(p+q)) \cdot \phi(U^{m-q}V^{n+p+q})$ and so we have $\phi(U^{m+p}V^{n+q}) = \Lambda(-np - q^2/2 - pq + m(p+q)) \cdot \phi(U^{m-q}V^{n+p+q})$. Replacing m by $m-p$ and n by $n-q$ gives

$$\phi(U^mV^n) = \Lambda(-(n-q)p - q^2/2 - pq + (m-p)(p+q)) \cdot \phi(U^{m-p-q}V^{n+p}).$$

Now, for any $m, n \in \mathbb{Z}$, take $p = -n$ and $q = m - p = m + n$. Then

$$\phi(U^mV^n) = \Lambda(m^2/2 + n^2/2)$$

where we have set $\phi(1) = 1$. Thus we have one basic ρ-trace functional, which we normalize, $\psi_{10}(1) = 1$, given by

$$\psi_{10}(U^mV^n) = \lambda^{\frac{1}{2}(m^2+n^2)}.$$

Observe that

$$\begin{aligned}\psi_{10}(\rho(U^mV^n)) &= \Lambda(-n^2/2 - mn)\psi_{10}(U^{-n}V^{m+n}) \\ &= \Lambda(-n^2/2 - mn + n^2/2 + (m+n)^2/2) \\ &= \Lambda(m^2/2 + n^2/2) = \psi_{10}(U^mV^n)\end{aligned}$$

so that ψ_{10} is a ρ-invariant ρ-trace, as expected.

THE ρ^2-TRACES. Now consider the cubic automorphism ρ^2. First, observe that

$$\rho^2(U^pV^q) = \rho^2(U)^p\rho^2(V)^q = \Lambda(-p^2/2 - pq)U^{-p-q}V^p.$$

Letting ϕ be a ρ^2-trace on A, we have $\phi(U^mV^nU^pV^q) = \phi(\rho^2(U^pV^q)U^mV^n)$ and applying the formula for $\rho^2(U^pV^q)$ this becomes $\Lambda(np)\phi(U^{m+p}V^{n+q}) = \Lambda(-p^2/2 - pq) \cdot \phi(U^{-p-q}V^pU^mV^n)$ or $\phi(U^{m+p}V^{n+q}) = \Lambda(-np - p^2/2 - pq + mp) \cdot \phi(U^{m-p-q}V^{p+n})$. Upon substituting $m-p$ for m and $n-q$ for n this gives

$$\phi(U^mV^n) = \Lambda(-p(n-q) - p^2/2 - pq + p(m-p))\phi(U^{m-2p-q}V^{p+n-q}).$$

Now, for any $m, n \in \mathbb{Z}$, let $q = p + n$, so that $\phi(U^mV^n) = \Lambda(-3p^2/2 + p(m-n)) \cdot \phi(U^{m-n-3p})$. Let $k = 0, 1, 2$, where $m - n \equiv k \bmod 3$, and write $m - n = k + 3p$ for some $p \in \mathbb{Z}$. Then $\phi(U^mV^n) = \Lambda([(m-n)^2 - k^2]/6) \cdot \phi(U^k)$ and so we have three basic, independent ρ^2-trace functionals, normalized as follows

$$\phi_{2k}(U^j) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 0, 1, 2$. So, the value of $\phi_{2k}(U^mV^n)$ depends on the value of $m - n - k$ mod 3. For convenience of notation, we introduce the *divisor delta function* defined by

$$\delta_n^m = \begin{cases} 1 & \text{if } n \mid m, \\ 0 & \text{if } n \nmid m, \end{cases}$$

for $m, n \in \mathbb{Z}$. Thus under this notation, $\phi_{2\ell}(U^k) = \delta_3^{k-\ell}$, and observing that for $k \equiv m - n$ mod 3, $\Lambda([(m-n)^2 - k^2]/6)\delta_3^{k-\ell} = \Lambda([(m-n)^2 - \ell^2]/6)\delta_3^{m-n-\ell}$, we obtain the formula $\phi_{2\ell}(U^mV^n) = \lambda^{\frac{1}{6}((m-n)^2-\ell^2)}\delta_3^{m-n-\ell}$ where $\ell = 0, 1, 2$. Now, observe that $\phi_{20}(\rho(U^mV^n)) = \Lambda(-n^2/2 - mn)\phi_{20}(U^{-n}V^{m+n}) = \Lambda((m-n)^2/6)\delta_3^{m+2n}$ and observing that $\delta_3^{m+2n} = \delta_3^{m-n}$, it follows that $\phi_{20}(\rho(U^mV^n)) = \phi_{20}(U^mV^n)$. Thus, ϕ_{20} is a ρ-invariant ρ^2-trace. We let $\psi_{20}=\phi_{20}$, so that $\psi_{20}(U^mV^n) = \lambda^{\frac{1}{6}(m-n)^2}\delta_3^{m-n}$. It is readily seen that neither ϕ_{21} nor ϕ_{22} are ρ-invariant. However, observe that

$$\begin{aligned}
(\phi_{21} + \lambda^{\frac{1}{2}}\phi_{22})(\rho(U^mV^n)) &= \lambda^{-\frac{1}{2}n^2 - mn + \frac{1}{6}(m+2n)^2 - \frac{1}{6}}(\delta_3^{m+2n+1} + \delta_3^{m+2n+2}) \\
&= \lambda^{\frac{1}{6}(m-n)^2 - \frac{1}{6}}(\delta_3^{m-n-1} + \delta_3^{m-n-2}) \\
&= \lambda^{\frac{1}{6}(m-n)^2 - \frac{1}{6}}\delta_3^{m-n-1} + \lambda^{\frac{1}{2}}\lambda^{\frac{1}{6}(m-n)^2 - \frac{2}{3}}\delta_3^{m-n-2} \\
&= (\phi_{21} + \lambda^{\frac{1}{2}}\phi_{22})(U^mV^n)
\end{aligned}$$

and so $\phi_{21} + \lambda^{\frac{1}{2}}\phi_{22}$ is a ρ-invariant ρ^2-trace. Now, let $\psi_{21} = \lambda^{\frac{1}{6}}(\phi_{21} + \lambda^{\frac{1}{2}}\phi_{22}) + \psi_{20}$. Then

$$\psi_{21}(U^mV^n) = \lambda^{\frac{1}{6}(m-n)^2}\delta_3^{m-n-1} + \lambda^{\frac{1}{6}(m-n)^2}\delta_3^{m-n-2} + \lambda^{\frac{1}{6}(m-n)^2}\delta_3^{m-n} = \lambda^{\frac{1}{6}(m-n)^2}.$$

Since ψ_{20} and $\phi_{21} + \lambda^{\frac{1}{2}}\phi_{22}$ are ρ-invariant, ψ_{21} is also ρ-invariant. Thus, we have two independent, ρ-invariant ρ^2-trace functionals, ψ_{20} and ψ_{21}. Further, any ρ-invariant ρ^2-trace is a linear combination of ψ_{20} and ψ_{21}, as can easily be seen.

THE ρ^3-TRACES. Since ρ^3 is the flip automorphism, the ρ^3-traces are

$$\phi_{ij}(U^mV^n) = \lambda^{-\frac{1}{2}mn}\delta_2^{m-i}\delta_2^{n-j}$$

where $i, j = 0, 1$. Observe that $\phi_{ij}(\rho(U^mV^n)) = \lambda^{-\frac{1}{2}mn}\delta_2^{n+i}\delta_2^{m+n-j}$ from which it follows that $\phi_{ij} = \phi_{ij} \circ \rho$ if and only if $\delta_2^{m-i}\delta_2^{n-j} = \delta_2^{n+i}\delta_2^{m+n-j}$. Now, ϕ_{00} is non-zero if and only if m and n are both even. If this is the case, then $\phi_{00} \circ \rho$ is non-zero as $m + n$ is even. If at least one of m or n is odd, then both are clearly zero. Thus, ϕ_{00} is ρ-invariant. We let $\psi_{30} = \phi_{00}$, so that

$$\psi_{30}(U^mV^n) = \lambda^{-\frac{1}{2}mn}\delta_2^m\delta_2^n.$$

We observe also that ϕ_{01} is non-zero if and only if m is even and n is odd; that ϕ_{10} is non-zero if and only if m is odd and n is even; and that ϕ_{11} is non-zero if and only if m and n are both odd. It is then readily verified that ϕ_{ij} is not ρ-invariant if at least one of i, j is non-zero. Next, observe that $\phi_{10} \circ \rho = \lambda^{-\frac{1}{2}mn}\delta_2^{n+1}\delta_2^{m+n}$ which is non-zero if and only if m and n are both odd. Thus, $\phi_{10} \circ \rho = \phi_{11}$. Similarly, $\phi_{11} \circ \rho$ is non-zero if and only if m is even and n is odd, so $\phi_{11} \circ \rho = \phi_{01}$, and $\phi_{01} \circ \rho$ is non-zero if and only if m is odd and n is even, so $\phi_{01} \circ \rho = \phi_{10}$. It follows that $(\phi_{01} + \phi_{10} + \phi_{11}) \circ \rho = \phi_{10} + \phi_{11} + \phi_{01}$ and so $\phi_{10} + \phi_{01} + \phi_{11}$ is ρ-invariant. Let $\phi := \phi_{01} + \phi_{10} + \phi_{11}$, so that

$$\phi(U^m V^n) = \lambda^{-\frac{1}{2}mn}(\delta_2^m\delta_2^{n-1} + \delta_2^{m-1}\delta_2^n + \delta_2^{m-1}\delta_2^{n-1})$$

from which it follows that $\phi(U^m V^n) = 0$ if and only if m and n are both even, and $\phi(U^m V^n) = \lambda^{-\frac{1}{2}mn}$ otherwise. Hence, we can re-write the equation for ϕ as

$$\phi(U^m V^n) = \lambda^{-\frac{1}{2}mn}(1 - \delta_2^m\delta_2^n).$$

Next, we observe that if $\xi := a\phi_{01} + b\phi_{10} + c\phi_{11}$ is any ρ-invariant linear combination of the ϕ_{ij}, i, j not both zero, then it is a scalar multiple of ϕ. Indeed, ρ-invariance gives

$$\xi(U^m V^n) = \xi(\rho(U^m V^n)) = \lambda^{-\frac{1}{2}mn}(a\delta_2^n\delta_2^{m+n-1} + b\delta_2^{n+1}\delta_2^{m+n} + c\delta_2^{n+1}\delta_2^{m+n-1}).$$

From this equality and the substitutions $m = 1, n = 0$ we get $a = b$, and from the substitutions $m = 0, n = 1$ we get $a = c$. Thus $a = b = c$ and so $\xi = a(\phi_{01} + \phi_{10} + \phi_{11}) = a\phi$. Hence, ϕ is the unique basic ρ-invariant ρ^3-trace other than $\psi_{30} = \phi_{00}$. Now, let $\psi_{31} = \psi_{30} + \phi$, so that

$$\psi_{31}(U^m V^n) = \lambda^{-\frac{1}{2}mn}.$$

Since ψ_{30} and ϕ are ρ-invariant, ψ_{31} is also ρ-invariant. Thus, we have two independent, ρ-invariant ρ^3-trace functionals, ψ_{30} and ψ_{31}.

THE ρ^4-TRACES AND ρ^5-TRACE. The ρ^4-traces and ρ^5-trace do not provide any new K-theoretical data not already given by the ρ-trace and ρ^2-traces. This is because given a ρ-invariant ρ^n-trace ϕ, $1 \leqslant n \leqslant 5$, the map $\phi^*(x) = \overline{\phi(x^*)}$ defines a ρ^m-trace, where $m = 6 - n$. This follows from

$$\phi^*(xy) = \overline{\phi(y^*x^*)} = \overline{\phi(\rho^n(x^*)y^*)} = \overline{\phi(x^*\rho^m(y^*))} = \overline{\phi((\rho^m(y)x)^*)} = \phi^*(\rho^m(y)x).$$

Hence, there is a one-to-one correspondence between the ρ^n-traces and the ρ^m-traces. By applying this to the ρ^2-traces, one obtains the ρ^4-traces

$$\psi_{40}(U^m V^n) = \lambda^{-mn-\frac{1}{6}(n-m)^2}\delta_3^{n-m}, \quad \psi_{41}(U^m V^n) = \lambda^{-mn-\frac{1}{6}(n-m)^2}.$$

Applying the results to the ρ-trace gives the ρ^5-trace $\psi_{50}(U^m V^n) = \lambda^{-\frac{1}{2}(m+n)^2}$. It is readily checked that $\psi_{40} = \psi_{20}^*$, $\psi_{41} = \psi_{21}^*$, and $\psi_{50} = \psi_{10}^*$. It is also clear that $\psi_{30} = \psi_{30}^*$ and $\psi_{31} = \psi_{31}^*$.

THE CYCLIC COHOMOLOGY OF ORDER ZERO FOR THE HEXIC CASE. We summarize the results obtained in the following theorem.

THEOREM 3.1. *Along with the canonical bounded trace τ, one has the following 9-dimensional basis of the vector space of all unbounded traces on the fixed point subalgebra A^ρ. More specifically, $\{\psi_{10}\}$ is a basis of ρ-invariant ρ-traces on A, $\{\psi_{20}, \psi_{21}\}$ a basis of ρ-invariant ρ^2-traces, $\{\psi_{30}, \psi_{31}\}$ a basis of ρ-invariant ρ^3-traces, $\{\psi_{40}, \psi_{41}\}$ a basis of ρ-invariant ρ^4-traces, and $\{\psi_{50}\}$ a basis of ρ-invariant ρ^5-traces, and are given by*

$$\psi_{10}(U^m V^n) = \lambda^{\frac{1}{2}(m^2+n^2)}, \qquad \psi_{31}(U^m V^n) = \lambda^{-\frac{1}{2}mn},$$
$$\psi_{20}(U^m V^n) = \lambda^{\frac{1}{6}(m-n)^2} \delta_3^{m-n}, \qquad \psi_{40}(U^m V^n) = \lambda^{-mn-\frac{1}{6}(n-m)^2} \delta_3^{n-m},$$
$$\psi_{21}(U^m V^n) = \lambda^{\frac{1}{6}(m-n)^2}, \qquad \psi_{41}(U^m V^n) = \lambda^{-mn-\frac{1}{6}(n-m)^2},$$
$$\psi_{30}(U^m V^n) = \lambda^{-\frac{1}{2}mn} \delta_2^m \delta_2^n, \qquad \psi_{50}(U^m V^n) = \lambda^{-\frac{1}{2}(m+n)^2}.$$

Now, the unbounded traces T_{ij} on $H_\theta^\infty = A \rtimes_\rho \mathbb{Z}_6$ are given by

$$T_{ij}(a_0 + a_1 W + a_2 W^2 + a_3 W^3 + a_4 W^4 + a_5 W^5) = \psi_{ij}(a_{6-i}),$$

for $i = 1,2,3,4,5$ and j ranges from 0 to n_i, where $n_1 = n_5 = 1$ and $n_2 = n_3 = n_4 = 2$. Therefore, using Theorem 3.1, one obtains all the traces on H_θ^∞ giving its cyclic cohomology group of order zero.

COROLLARY 3.2. *Let $A = A_\theta^\infty$ where θ is irrational. Then one has the cyclic cohomology group of order zero*

$$HC^0(A \rtimes_\rho \mathbb{Z}_6) \cong HC^0(A^\rho) \cong \mathbb{C}^9.$$

The group on the left is generated by τ and T_{ij}, while the middle group is generated by τ and ψ_{ij} (restricted to A^ρ), where τ is the canonical bounded trace in each case.

UNBOUNDED κ-INVARIANT κ-TRACES ON A.

THE κ-TRACES. Now, consider the cubic automorphism $\kappa = \rho^2$ of A. Clearly, the κ-traces (which are κ-invariant) are just the ρ^2-traces $\phi_{2\ell}$ computed earlier, so we have the three κ-traces

$$\phi_{2\ell}(U^m V^n) = \lambda^{\frac{1}{6}((m-n)^2-\ell^2)} \delta_3^{m-n-\ell}$$

where $\ell = 0,1,2$. We let $\varphi_{1\ell} = \phi_{2\ell}$ for $\ell = 0,1,2$. Thus, we have three independent κ-trace functionals, φ_{10}, φ_{11} and φ_{12}.

THE κ^2-TRACES. The κ^2-traces may be obtained from the κ-traces in the same manner from which the ρ^4-traces and ρ^5-trace were obtained from the ρ-trace and ρ^2-traces. There is a one-to-one correspondence between the κ^2-traces and the κ-traces (and hence the κ^2-traces do not yield any new K-theoretical data not already provided by the κ-traces). From the equation $\varphi_{2i}(x) = \overline{\varphi_{1i}(x^*)}$ we obtain the κ^2-traces

$$\varphi_{2\ell}(U^m V^n) = \lambda^{-mn-\frac{1}{6}((m-n)^2-\ell^2)} \delta_3^{m-n-\ell}.$$

THE CYCLIC COHOMOLOGY OF ORDER ZERO FOR THE CUBIC CASE. The results obtained give a theorem analogous to Theorem 3.1 for the κ-traces.

THEOREM 3.3. *One has the following 6-dimensional basis of the vector space of all unbounded traces on the fixed point subalgebra A^κ. More specifically, $\{\varphi_{10}, \varphi_{11}, \varphi_{12}\}$ is a basis of κ-traces, and $\{\varphi_{20}, \varphi_{21}, \varphi_{22}\}$ a basis of κ^2-traces, and are given by*

$$\varphi_{10}(U^mV^n) = \lambda^{\frac{1}{6}((m-n)^2)}\delta_3^{m-n}, \qquad \varphi_{20}(U^mV^n) = \lambda^{-mn-\frac{1}{6}(n-m)^2}\delta_3^{n-m},$$
$$\varphi_{11}(U^mV^n) = \lambda^{\frac{1}{6}((m-n)^2-1)}\delta_3^{m-n-1}, \qquad \varphi_{21}(U^mV^n) = \lambda^{-mn-\frac{1}{6}((n-m)^2-1)}\delta_3^{n-m-1},$$
$$\varphi_{12}(U^mV^n) = \lambda^{\frac{1}{6}((m-n)^2-4)}\delta_3^{m-n-2}, \qquad \varphi_{22}(U^mV^n) = \lambda^{-mn-\frac{1}{6}((n-m)^2-4)}\delta_3^{n-m-2}.$$

The unbounded traces S_{ij} on the *cubic algebra* $C_\theta^\infty = A \rtimes_\kappa \mathbb{Z}_3$ are given by

$$S_{ij}(a_0 + a_1Z + a_2Z^2) = \varphi_{ij}(a_{3-i})$$

for $i = 1, 2,\ j = 0, 1, 2$. Therefore, using Theorem 3.3, one obtains all the traces on C_θ^∞ giving its cyclic cohomology group of order zero.

COROLLARY 3.4. *Let $A = A_\theta^\infty$ where θ is irrational. One has the cyclic cohomology group of order zero*

$$HC^0(A \rtimes_\kappa \mathbb{Z}_3) \cong HC^0(A^\kappa) \cong \mathbb{C}^7.$$

The group on the left is generated by τ and S_{ij}, while the middle group is generated by τ and φ_{ij} (restricted to A^κ), where τ is the canonical bounded trace in each case.

4. SECOND ORDER CONNES-CHERN CHARACTER

In this section we let $k = 3, 6$ so as to handle both the cubic and hexic cases together, and obtain the values of C_3 and C_6 in Tables 1 and 2. For simplicity, we let $L_3 = C_\theta$ and $L_6 = H_\theta$. One has the unital $*$-embedding

$$\Psi : L_k \to M_k(A_\theta)$$

given by

$$\Psi\Big(\sum_{j=0}^{k-1} a_jW^j\Big) = [\sigma^{k-i}(a_{i-j})]_{i,j=0}^{k-1} = \begin{bmatrix} a_0 & a_{k-1} & \cdots & a_1 \\ \sigma^{k-1}(a_1) & \sigma^{k-1}(a_0) & \cdots & \sigma^{k-1}(a_2) \\ \sigma^{k-2}(a_2) & \sigma^{k-2}(a_1) & \cdots & \sigma^{k-2}(a_3) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(a_{k-1}) & \sigma(a_{k-2}) & \cdots & \sigma(a_0) \end{bmatrix}$$

where $i - j$ is reduced mod k and where $a_j \in A_\theta$ and where σ here stands for ρ or κ. This already shows that the range of the canonical (normalized) trace on $K_0(L_k)$ is contained in $\frac{1}{k}(\mathbb{Z} + \mathbb{Z}\theta)$. The hexic/cubic module $\mathcal{M}_k$ can be shown to have trace $\frac{\theta}{k}$ (in exactly the same way as in Proposition 3.3 of [10]). Further, since one has a projection in L_k of trace $\frac{1}{k}$, one obtains the equality

$$\tau_*(K_0(L_k)) = \tfrac{1}{k}(\mathbb{Z} + \mathbb{Z}\theta)$$

where τ is the canonical trace of L_k ($k = 3, 6$). The embedding Ψ induces the map

$$\Psi_* : K_0(L_k) \to K_0(A_\theta).$$

Thus, if e is a projection in a matrix algebra over L_k, then one takes the class of $\Psi(e)$ in $K_0(A_\theta)$. So, for example, the identity 1 of L_k maps to the $k \times k$ identity matrix in $M_k(A_\theta)$, so that $\Psi_*[1] = k[1]'$ in $K_0(A_\theta)$. (For the sake of clarity, we shall write $[e]'$ for classes in $K_0(A_\theta)$ and unprimed brackets $[e]$ for classes in $K_0(L_k)$.)

Recall Connes' canonical cyclic 2-cocycle χ (see III.2.β of [6]) on the smooth rotation algebra

$$\chi(x^0, x^1, x^2) = \frac{1}{2\pi \mathrm{i}} \tau(x^0[\delta_1(x^1)\delta_2(x^2) - \delta_2(x^1)\delta_1(x^2)])$$

where δ_i are the canonical derivations of A_θ under the canonical action of $\mathbb{T}^2$ (2-torus). More specifically, $\delta_1 = \delta_U$, $\delta_2 = \delta_V$. (The cocycle χ is known to give one of two basis elements for the cyclic cohomology group $HC^2(A_\theta^\infty) \cong \mathbb{C}^2$; see III.2.$\beta$ of [6].) This cocycle implements the canonical map $c_1 : K_0(A_\theta) \to \mathbb{Z}$. Using the cup product, c_1 is given as follows: if E is a projection in $M_n(A_\theta)$ then $c_1(E) = (\chi \# \mathrm{Tr}_n)(E, E, E)$ where Tr_n is the usual (non-normalized) trace on $M_n(\mathbb{C})$ and $\chi \# \mathrm{Tr}_n$ is the unique cyclic 2-cocycle on $M_n(A_\theta)$ given by

$$(\chi \# \mathrm{Tr}_n)(x^0 \otimes a^0, x^1 \otimes a^1, x^2 \otimes a^2) = \chi(x^0, x^1, x^2) \cdot \mathrm{Tr}_n(a^0 a^1 a^2) \tag{4.1}$$

where $a^j \in M_n(\mathbb{C})$ and $x^j \in A_\theta$. If e is a projection in A_θ, then

$$c_1[e] = \chi(e, e, e) = \frac{1}{2\pi \mathrm{i}} \tau(e[\delta_1(e), \delta_2(e)])$$

where τ is the canonical trace of A_θ. The map c_1 has the property that if $[e]' = m[1]' + n[e_\theta]'$ (in $K_0(A_\theta) = \mathbb{Z}^2$) for some integers m, n, then $c_1[e] = -n$ (see [6], p. 601). The invariant that is of interest for us is the composition

$$C_k := c_1 \circ \Psi_* : K_0(L_k) \to \mathbb{Z}.$$

For θ in $(0, 1)$, the map C_k has the property that if $[e] \in K_0(L_k)$ is such that $\Psi_*[e] = m[1]' + n[e_\theta]'$ in $K_0(A_\theta)$, so that its trace in L_k is $\frac{1}{k}(m + n\theta)$, then $C_k[e] = -n$. This follows immediately from the above since $C_k[e] = c_1(\Psi_*[e]) = c_1(m[1]' + n[e_\theta]') = -n$.

It is not hard to check that the values of C_k on the first nine projections in Table 1 (for the hexic case) are zero, as well as for the first seven projections in Table 2 for the cubic case. For the cubic/hexic module $\mathcal{M}_k$ (whose trace is $\frac{\theta}{k}$) one therefore has $C_k[\mathcal{M}_k] = -1$. This clearly follows for θ irrational by the above property. For the rational case it can be shown to follow from the construction of the module $\mathcal{M}_k$ (since, considered as an A_θ-module, it is a Heisenberg module whose c_1-character value is -1, as can be seen from Connes' computation [6], Theorem 7 and following).

For the hexic case since $\Psi(p_j)$ is a scalar matrix projection one immediately has $C_6(p_j) = 0$. To see that $C_6(q_j) = 0$ note that the entries of the matrix $\Psi(q_j(X))$ are scalars times one of the unitaries $U^{\pm 1}, V, U^{-1}V$. It is easy to check that $\chi(U^{a_0}V^{b_0}, U^{a_1}V^{b_1}, U^{a_2}V^{b_2}) \neq 0$ only if $a_0 + a_1 + a_2 = 0$ and $b_0 + b_1 + b_2 = 0$, and as the latter are not satisfied for any three pairs (a_j, b_j) in the set $\{(\pm 1, 0), (0, 1), (-1, 1)\}$, one has $C_6(q_j) = 0$. Similarly, one also has $C_6(r_0) = 0$. In exactly the same way one checks that C_3 is zero for the first seven projections Q_j in the cubic case. This yields all the C_3 and C_6 values in Tables 1 and 2.

All the traces and Connes Chern character invariants can be put together to form the Connes Chern character for each case. We write

$$\mathbf{T}_6 = (\tau,\ C_6;\ T_{10};\ T_{20}, T_{21};\ T_{30}, T_{31}), \quad \mathbf{T}_3 = (\tau,\ C_3;\ S_{10}, S_{11}, S_{12})$$

where τ is the canonical (bounded) trace, C_k the canonical second order Connes Chern character, and T_{ij}, S_{ij} are the unbounded traces. The map $\mathbf{T}_k$ defines a group homomorphism $K_0(L_k) \to \mathbb{R} \times \mathbb{Z} \times \mathbb{D}$ where $\mathbb{D}$ is some lattice subgroup of $\mathbb{R}^n$, where $n = 5$ when $k = 6$, and $n = 3$ when $k = 3$.

5. THE HEXIC AND CUBIC MODULES

Let $\beta = \frac{1}{\sqrt{\theta}}$. Let $M = \mathbb{R}$ and $G = M \times \widehat{M}$. Consider the lattice D in G given by

$$D: \ \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix},$$

It is clearly invariant under the map $H(u, v) = (u + v, -u)$ discussed earlier. A fundamental domain for D is $[0, \beta) \times [0, \beta)$ so that the covolume of D is $|G/D| = \beta^2$. The C^*-algebra $C^*(D)$ is generated by the canonical unitaries $U_1 = \pi_{\varepsilon_1}$, $U_2 = \pi_{\varepsilon_2}$ whose commutation relation is

$$U_1 U_2 U_1^* U_2^* = \mathfrak{h}(\varepsilon_1, \varepsilon_2)\overline{\mathfrak{h}(\varepsilon_2, \varepsilon_1)} = e(\beta^2) =: \lambda_1$$

so that $U_1 U_2 = \lambda_1 U_2 U_1$. We consider the associated hexic transform given by $\rho_1(U_2) = U_1$, $\rho_1(U_1) = \lambda_1^{-\frac{1}{2}} U_2^{-1} U_1$. We can define ρ_1 in a more invariant way by verifying that it satisfies $\rho_1(\pi_x) = \mu(x)\pi_{Hx}$ where $\mu(x) := \mathfrak{s}(Hx, Hx)\overline{\mathfrak{s}(x, x)}$. (This is easily verified for $x = \varepsilon_1, \varepsilon_2$.)

The complementary lattice $D^\perp$ is easily checked to be generated by the basis elements

$$D^\perp: \ \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

where $\alpha = \frac{1}{\beta} = \sqrt{\theta}$.

In general one has $\pi_{x+y} = \overline{\mathfrak{h}(x,y)}\pi_x\pi_y$ and $\pi_{\delta_j}\pi_{\delta_k} = \lambda_{jk}\overline{\lambda}_{kj}\,\pi_{\delta_k}\pi_{\delta_j}$ where $\lambda_{jk} = \mathfrak{h}(\delta_j,\delta_k)$. In our case, $\lambda_{11} = \lambda_{22} = \lambda_{21} = 1$, $\lambda_{12} = e(\alpha^2) = e(\theta) = \lambda$. Let $V_j = \pi^*_{\delta_j} = \lambda_{jj}\pi_{-\delta_j} = \pi_{-\delta_j}$ and $\pi_{n\delta_j} = \lambda_{jj}^{-\frac{1}{2}n(n-1)}\pi^n_{\delta_j} = \pi^n_{\delta_j}$. One thus has $V_1V_2 = \pi^*_{\delta_1}\pi^*_{\delta_2} = \lambda_{12}\overline{\lambda}_{21}\,\pi^*_{\delta_2}\pi^*_{\delta_1} = \lambda\, V_2V_1$. The C^*-algebra $C^*(D^\perp)$ is the opposite algebra of the one generated by the unitaries V_1, V_2. Letting # denote the opposite multiplication ($a\#b = ba$), we can identify the rotation algebra A_θ with $C^*(D^\perp)$ by the identification $U \leftrightarrow V_1$, $V \leftrightarrow V_2$. (Thus, for example, $UV = V_1\#V_2$.) (Recall that $C^*(D^\perp)$ is given the opposite multiplication in order that the module property $(fa)b = f(a\#b)$ be satisfied, where $a, b \in C^*(D^\perp)$, $f \in \mathcal{S}(\mathbb{R})$.) We recall that the right action of $C^*(D^\perp)$ on $\mathcal{S}(\mathbb{R})$ is given by $f\pi^*_x = \pi^*_x(f)$, where the Heisenberg representation of G on $\mathcal{S}(\mathbb{R})$ is given by $\pi_x(f)(t) := e(tx'')f(t + x')$ for $x = (x'; x'') \in G$. In view of Rieffel's Theorem 2.15 in [9], when completed, the Schwartz space $\mathcal{S}(\mathbb{R})$ becomes an equivalence $A_{\frac{1}{\theta}}$-A_θ bimodule. Thus the right action of A_θ on $\mathcal{S}(\mathbb{R})$ is given by

$$(fU)(t) = (fV_1)(t) = f(t-\alpha), \quad (fV)(t) = (fV_2)(t) = e(-\alpha t)f(t).$$

From the relations $\mathfrak{h}(u+v,w) = \mathfrak{h}(u,w)\mathfrak{h}(v,w)$, $\mathfrak{h}(u,v+w) = \mathfrak{h}(u,v)\mathfrak{h}(u,w)$, one gets the following

$$\begin{aligned}&\mathfrak{h}(\delta_j, n\delta_k) = \mathfrak{h}(\delta_j,\delta_k)^n,\\ &\mathfrak{h}(n_1\delta_1 + n_2\delta_2,\ n_1\delta_1 + n_2\delta_2) = \lambda^{n_1n_2},\\ &\pi_{n\delta_j} = \pi^n_{\delta_j}.\end{aligned}$$

Thus $\pi_{n_1\delta_1+n_2\delta_2} = \lambda^{-n_1n_2}\pi^{n_1}_{\delta_1}\pi^{n_2}_{\delta_2}$ and $\pi^*_{n_1\delta_1+n_2\delta_2} = \lambda^{n_1n_2}\,V_2^{n_2}V_1^{n_1} = V_1^{n_1}V_2^{n_2}$.

The $D^\perp$ inner product therefore becomes

$$\langle f, g\rangle_{D^\perp} = \sum_{m,n}\langle f, g\rangle_{D^\perp}(m\delta_1 + n\delta_2)\, V_1^mV_2^n$$

where

$$\langle f, g\rangle_{D^\perp}(m\delta_1 + n\delta_2) = \langle f, g\rangle_{D^\perp}(\alpha m;\ \alpha n) = \int_{\mathbb{R}} \overline{f(x)}g(x+\alpha m)e(\alpha n x)\,\mathrm{d}x.$$

The inner product over D is given by

$$\langle f, g\rangle_D = |G/D|\sum_{m,n}\langle f, g\rangle_D(m\varepsilon_1 + n\varepsilon_2)\, U_2^nU_1^m$$

(since $\pi_{m\varepsilon_1+n\varepsilon_2} = U_2^nU_1^m$), where

$$\langle f, g\rangle_D(m\varepsilon_1 + n\varepsilon_2) = \langle f, g\rangle_D(m\beta;\ n\beta) = \int_{\mathbb{R}} f(x)\overline{g(x+m\beta)}e(-xn\beta)\,\mathrm{d}x.$$

The hexic automorphism ρ is given by $\rho(U) = V$ and $\rho(V) = \lambda^{-\frac{1}{2}}U^{-1}V$. The crossed product $H_\theta = A_\theta \rtimes_\rho \mathbb{Z}_6$ is the universal C^*-algebra generated by unitaries

U, V, W satisfying

$$VU = \lambda UV, \quad WUW^* = V, \quad WVW^* = \lambda^{-\frac{1}{2}} U^{-1} V, \quad W^6 = I.$$

The two middle relations can be re-written as

$$VW = WU, \quad UWV = \lambda^{-\frac{1}{2}} VW. \tag{5.1}$$

As was shown in [12], by defining the action of W on $\mathcal{S}(\mathbb{R})$ to be given by the hexic transform

$$(fW)(t) = \mathrm{i}^{\frac{1}{6}} \int_{-\infty}^{\infty} f(x) e(tx - \tfrac{1}{2}x^2) \mathrm{d}x$$

(where we have taken $\mu = \frac{1}{2}$ in [12]), one can extend the Heisenberg A_θ^∞-module $\mathcal{S}(\mathbb{R})$ into a right H_θ^∞-module. Indeed, letting W act as some transform $(fW)(t) = \int f(x) K(x,t) \mathrm{d}x$, the first relation gives (keeping in mind the opposite multiplication of $C^*(D^\perp)$)

$$[f(VW)](t) = [f(V_2 \# W)](t) = [(fV_2)W](t) = \int_{-\infty}^{\infty} e(-\alpha x) f(x) K(x,t) \mathrm{d}x$$

and

$$[f(WU)](t) = [f(W \# V_1)](t) = [(fW)V_1](t) = \int_{-\infty}^{\infty} f(x) K(x, t-\alpha) \mathrm{d}x$$

thus one requires

$$K(x, t-\alpha) = e(-\alpha x) K(x,t), \quad K(x+\alpha, t) = e(-\tfrac{\theta}{2}) e(\alpha t) K(x, t-\alpha)$$

where the latter equality arises similarly from the second relation in (5.1). It is easy to check that the kernel function $K(x,t) = \mathrm{i}^{\frac{1}{6}} e(tx - \frac{1}{2}x^2)$ satisfies these two relations (as $\alpha = \sqrt{\theta}$). As was done in [11] for the Fourier module, one has a natural A_θ^∞-valued inner product on $\mathcal{S}(\mathbb{R})$ to a H_θ^∞-valued inner product by

$$\langle f, g \rangle_{H_\theta^\infty} = \sum_{j=0}^{5} \langle f, gW^{-j} \rangle_{A_\theta^\infty} W^j$$

turning it into an appropriate equivalence bimodule (in the sense of Rieffel), finitely generated projective, H_θ^∞-module which we shall denote by $\mathcal{M}_6$. It therefore gives a class in $K_0(H_\theta^\infty)$. Almost exactly as in [11], and using an argument of Rieffel one can show that for the unbounded traces T_{ij} on H_θ^∞, one has $T_{ij}[\mathcal{M}_6] = \frac{1}{6}\widetilde{\psi}_{ij}(1)$ where $\widetilde{\psi}_{ij}$ is the twisted trace dual to ψ_{ij} (as given by Proposition 2.1).

Similarly, for the cubic case we have the C_θ^∞-module $\mathcal{M}_3$ and its unbounded traces are $S_{ij}[\mathcal{M}_3] = \frac{1}{3}\widetilde{\varphi}_{ij}(1)$ where $\widetilde{\varphi}_{ij}$ is the twisted trace dual to φ_{ij}.

To end this section, let us establish the following equality

$$\rho^{-1}(\pi_y) = \mu(y) \pi_{Hy}$$

for $y \in D^{\perp}$. Writing $y = m\delta_1 + n\delta_2$ so that $Hy = (m+n)\delta_1 - m\delta_2$, and noting from above that $\rho(\pi_{\delta_1}) = \pi_{\delta_2}$ and $\rho(\pi_{\delta_2}) = \lambda^{\frac{1}{2}}\pi_{\delta_1}^{*}\pi_{\delta_2}$, one has $\mu(y) = e(-\frac{1}{2}m^2\alpha^2 - mn\alpha^2)$ and

$$\begin{aligned}\mu(y)\rho(\pi_{Hy}) &= e(-\tfrac{1}{2}m^2\alpha^2 - mn\alpha^2)\rho(\pi_{(m+n)\delta_1 - m\delta_2})\\ &= e(-\tfrac{1}{2}m^2\alpha^2 - mn\alpha^2)e(m^2\alpha^2 + mn\alpha^2)\rho(\pi_{\delta_1}^{m+n}\pi_{\delta_2}^{-m})\\ &= e(\tfrac{1}{2})m^2\alpha^2\pi_{\delta_2}^{m+n}[\lambda^{\frac{1}{2}}\pi_{\delta_1}^{*}\pi_{\delta_2}]^{-m} = \pi_y\end{aligned}$$

the last equality being easy to verify.

LEMMA 5.1. *One has the equalities*

$$S\pi_w = \mu(w)\pi_{Hw}S, \quad \rho^{-1}(\langle f, g\rangle_{D^{\perp}}) = \langle Sf, Sg\rangle_{D^{\perp}}, \quad \rho_1(\langle f, g\rangle_{D}) = \langle Sf, Sg\rangle_{D}.$$

Proof. Let us establish the first equality. Writing $w = (u, v) \in \mathbb{R}^2$, one has

$$\begin{aligned}\mu(w)(\pi_{Hw}Sf)(t) &= \mu(w)\pi_{(u+v,-u)}(Sf)(t) = \mu(w)e(-ut)(Sf)(t+u+v)\\ &= \mathrm{i}^{-\frac{1}{6}}e(-\tfrac{1}{2})u^2 - uv)e(-ut)\int f(x)e((t+u+v)x - \tfrac{1}{2}x^2)\mathrm{d}x\end{aligned}$$

and on the other hand

$$\begin{aligned}(S\pi_w f)(t) &= \mathrm{i}^{-\frac{1}{6}}\int(\pi_w f)(x)e(tx - \tfrac{1}{2}x^2)\mathrm{d}x = \mathrm{i}^{-\frac{1}{6}}\int e(vx)f(x+u)e(tx - \tfrac{1}{2}x^2)\mathrm{d}x\\ &= \mathrm{i}^{-\frac{1}{6}}\int e(vx - vu)f(x)e(t(x-u) - \tfrac{1}{2}(x-u)^2)\mathrm{d}x\end{aligned}$$

from which it is easy to see that the two expressions are equal. Now we show the third equality in the statement of the proposition. To do this we first need to show

$$\langle Sf, Sg\rangle_{D}(Hw) \;=\; \mu(w)\langle f, g\rangle_{D}(w).$$

We have (using the fact that S is a unitary operator on $L^2(\mathbb{R})$)

$$\begin{aligned}\langle Sf, Sg\rangle_{D}(Hw) &= \langle Sf, \pi_{Hw}Sg\rangle_{L^2} = \langle Sf, \overline{\mu(w)}S\pi_w g\rangle_{L^2} = \mu(w)\langle Sf, S\pi_w g\rangle_{L^2}\\ &= \mu(w)\langle f, \pi_w g\rangle_{L^2} = \mu(w)\langle f, g\rangle_{D}(w).\end{aligned}$$

Hence

$$\begin{aligned}\rho_1(\langle f, g\rangle_{D}) &= |G/D|\sum_{x\in D}\langle f, g\rangle_{D}(x)\,\rho_1(\pi_x) = |G/D|\sum_{x\in D}\mu(x)\langle f, g\rangle_{D}(x)\,\pi_{Hx}\\ &= |G/D|\sum_{x\in D}\langle Sf, Sg\rangle_{D}(Hx)\,\pi_{Hx} \;=\; \langle Sf, Sg\rangle_{D}.\end{aligned}$$

Similarly, $\langle Sf, Sg\rangle_{D^{\perp}}(Hw) = \overline{\mu(w)}\langle f, g\rangle_{D^{\perp}}(w)$ and

$$\begin{aligned}\rho^{-1}(\langle f, g\rangle_{D^{\perp}}) &= \sum_{y\in D^{\perp}}\langle f, g\rangle_{D^{\perp}}(y)\,\rho^{-1}(\pi_y^{*}) = \sum_{y\in D^{\perp}}\langle f, g\rangle_{D^{\perp}}(y)\,\overline{\mu(y)}\,\pi_{Hy}^{*}\\ &= \sum_{y\in D^{\perp}}\langle Sf, Sg\rangle_{D^{\perp}}(Hy)\,\pi_{Hy}^{*} \;=\; \langle Sf, Sg\rangle_{D^{\perp}}\end{aligned}$$

which gives the second equality in the statement of the proposition. ∎

6. CALCULATION OF $T_{10}([\mathcal{M}_6])$

Rewriting the relations in Lemma 5.1 as $\rho(\langle f, g\rangle_{D^\perp}) = \langle fW^{-1}, gW^{-1}\rangle_{D^\perp}$, $\rho_1^{-1}(\langle f, g\rangle_D) = \langle fW^{-1}, gW^{-1}\rangle_D$ (with $w(f) = fW^{-1}$), Lemma 5.1 says that for the basic ρ-trace ψ_{10} on A_θ^∞, there is a ρ_1^{-1}-trace φ on $A_{1/\theta}^\infty$ such that $\psi_{10}(\langle f, gW\rangle_{D^\perp}) = \varphi(\langle g, f\rangle_D)$. The value of the unbounded trace T_{10} on the hexic module $\mathcal{M}_6$ is $\frac{1}{6}\varphi(1)$, which we now seek to calculate. Since φ is a ρ_1^{-1}-trace, its adjoint $\varphi^*(x)$ is a ρ_1-trace, and thus there is a complex constant c such that $\varphi^* = c\psi'_{10}$, where ψ'_{10} is the basic ρ_1-trace on $A_{1/\theta}^\infty$. We then have

$$\psi_{10}(\langle f, gW\rangle_{D^\perp}) \;=\; \bar{c}\;\overline{\psi'_{10}(\langle f, g\rangle_D)}$$

and $\varphi(1) = \bar{c}$. Thus, we want to calculate

$$T_{10}([\mathcal{M}_6]) \;=\; \frac{1}{6}\varphi(1) \;=\; \frac{\psi_{10}(\langle f, gW\rangle_{D^\perp})}{6\overline{\psi'_{10}(\langle f, g\rangle_D)}}$$

for suitable f, g. For these, we pick $f(x) = \mathrm{e}^{-\pi a x^2}$, $g(x) = \mathrm{e}^{-\pi b x^2}$ where $a, b \in \mathbb{C}^+$ (the right-half plane).

A simple calculation (using (2.1)) shows that $\langle f, g\rangle_D \;=\; \beta^2 \sum_{m,n} b_{m,n}\, U_2^n U_1^m$ where $b_{m,n} = \langle f, g\rangle_D(m\varepsilon_1 + n\varepsilon_2) = \frac{1}{\sqrt{a+\bar{b}}}\mathrm{e}^{-\pi\bar{b}\beta^2 m^2}\mathrm{e}^{-\pi\beta^2\frac{(\mathrm{i}\bar{b}m-n)^2}{a+\bar{b}}}$. Further, we check that $(gW)(t) = \frac{\mathrm{i}^{\frac{1}{6}}}{\sqrt{b+\mathrm{i}}}\mathrm{e}^{-\pi\frac{t^2}{b+\mathrm{i}}}$ hence

$$a_{m,n} = \langle f, gW\rangle_{D^\perp}(m\delta_1 + n\delta_2) = \frac{\mathrm{i}^{\frac{1}{6}}}{\sqrt{\gamma(b+\mathrm{i})}}\mathrm{e}^{-\pi\frac{\alpha^2 m^2}{b+\mathrm{i}}}\;\mathrm{e}^{-\pi\frac{\alpha^2}{\gamma}\left(n+\frac{\mathrm{i}m}{b+\mathrm{i}}\right)^2}$$

where $\gamma = \frac{\bar{a}(b+\mathrm{i})+1}{b+\mathrm{i}}$. (Here we used the fact that the principal square root is a multiplicative function on the right-half plane — γ and $b+\mathrm{i}$ being there.)

The basic ρ_1-trace ψ'_{10} on the rotation algebra $C^*(D)$ is given by $\psi'_{10}(U_2^n U_1^m) = \lambda_1^{\frac{1}{2}(m^2+n^2)}$, where $\lambda_1 = e(\beta^2)$. Thus

$$\psi'_{10}(\langle f, g\rangle_D) = \beta^2 \sum_{m,n} b_{m,n}\,\mathrm{e}^{\pi\mathrm{i}\beta^2(m^2+n^2)} = \frac{\beta^2}{\sqrt{a+\bar{b}}}\sum_{m,n}\mathrm{e}^{-\pi\nu m^2}\mathrm{e}^{-\pi\mu n^2}\mathrm{e}^{2\pi d m n}$$

where $\nu = \beta^2\left(\frac{a\bar{b}}{a+\bar{b}} - \mathrm{i}\right)$, $\mu = \beta^2\left(\frac{1}{a+\bar{b}} - \mathrm{i}\right)$, $d = \frac{\mathrm{i}\beta^2\bar{b}}{a+\bar{b}}$.

On the other hand, since the basic ρ-trace on A_θ is given by $\psi_{10}(U^m V^n) = \lambda^{\frac{1}{2}(m^2+n^2)}$, (and noting the opposite multiplication of $C^*(D^\perp)$) one has $\psi_{10}(V_1^m V_2^n)$

$= \lambda^{mn}\lambda^{\frac{1}{2}(m^2+n^2)}$. Thus, from $\langle f, gW\rangle_{D^\perp} = \sum_{m,n} a_{m,n}\, V_1^m V_2^n$ one gets

$$\psi_{10}(\langle f, gW\rangle_{D^\perp}) = \sum_{m,n} a_{m,n}\, \lambda^{mn}\lambda^{\frac{1}{2}(m^2+n^2)} = \frac{\mathrm{i}^{\frac{1}{6}}}{\sqrt{\gamma(b+\mathrm{i})}} \sum_{m,n} \mathrm{e}^{-\pi\tau m^2}\, \mathrm{e}^{-\pi\zeta n^2} \mathrm{e}^{2\pi\delta mn}$$

where $\tau = \alpha^2\Big(\frac{1}{b+\mathrm{i}} - \frac{1}{\gamma(b+\mathrm{i})^2} - \mathrm{i}\Big), \zeta = \alpha^2\Big(\frac{b-\mathrm{i}\overline{a}(b+\mathrm{i})}{\overline{a}(b+\mathrm{i})+1}\Big), \delta = \frac{\mathrm{i}\alpha^2\overline{a}(b+\mathrm{i})}{\overline{a}(b+\mathrm{i})+1}$. Let Σ denote the sum appearing in $\psi_{10}(\langle f, gW\rangle_{D^\perp})$ and Σ' the sum in $\psi'_{10}(\langle f, g\rangle_D)$. We shall relate these sums as follows. First, a bit of algebra shows that $\tau = -\mathrm{i}\alpha^2\Big(\frac{\overline{a}(b+2\mathrm{i})+1}{\overline{a}(b+\mathrm{i})+1}\Big)$, $\zeta\tau - \delta^2 = \frac{-\mathrm{i}\alpha^4(\overline{a}+b)}{\overline{a}(b+\mathrm{i})+1}$ hence

$$\frac{\tau}{\zeta\tau-\delta^2} = \beta^2\Big(\frac{\overline{a}(b+2\mathrm{i})+1}{\overline{a}+b}\Big) =: \overline{\mu}_0, \quad \frac{\zeta}{\zeta\tau-\delta^2} = \overline{\nu}, \quad \frac{\delta}{\zeta\tau-\delta^2} = \frac{-\beta^2\overline{a}(b+\mathrm{i})}{\overline{a}+b} =: \overline{d}_0.$$

Therefore, upon applying the 2-dimensional inversion formula, Lemma 10.1, to Σ, one obtains $\Sigma = \frac{1}{\sqrt{\zeta\tau-\delta^2}}\overline{\Sigma}_0$ where

$$\Sigma_0 := \sum_{m,n} \mathrm{e}^{-\pi\nu m^2}\, \mathrm{e}^{-\pi\mu_0 n^2}\, \mathrm{e}^{2\pi d_0 mn}.$$

To this sum apply the substitution $m \to m - n$ so that it becomes

$$\Sigma_0 = \sum_{m,n} \mathrm{e}^{-\pi\nu m^2}\, \mathrm{e}^{-\pi\mu' n^2}\, \mathrm{e}^{2\pi d' mn}$$

where $\mu' = \nu + \mu_0 + 2d_0$ and $d' = \nu + d_0$. It is easy to check that $\mu' = \mu$ and $d' = -d$. This shows that $\Sigma_0 = \Sigma'$. In addition, one can choose suitable a, b so that $\Sigma' \neq 0$ — for example, when $0 < \theta < 1$, by taking $b = 1$ and $a = \beta^2 - 1 > 0$ one sees that $d = \mathrm{i}$ so Σ' reduces to a product of two theta functions of the form $\vartheta_3(0, \mathrm{i}\nu)$ which is not zero. Hence

$$T_{10}([\mathcal{M}_6]) = \frac{1}{6}\varphi(1) = \frac{\mathrm{i}^{\frac{1}{6}}\overline{\Sigma}_0}{6\sqrt{\gamma(b+\mathrm{i})(\zeta\tau-\delta^2)}} \cdot \frac{\sqrt{\overline{a}+b}}{\beta^2\,\overline{\Sigma}_0} = \frac{1}{6}\mathrm{i}^{\frac{2}{3}} = \tfrac{1}{12}(1+\mathrm{i}\sqrt{3}) = \frac{1}{6}\omega$$

which gives the corresponding entry in Table 1 (in the Introduction).

7. CALCULATION OF $T_{2k}([\mathcal{M}_6])$

From Lemma 5.1 it follows that for the basic ρ^2-trace $\psi_{2,1-j}$ on A_θ^∞, there is a ρ_1^{-2}-trace φ on $A_{1/\theta}^\infty$ such that $\psi_{2,1-j}(\langle f, gW^2\rangle_{D^\perp}) = \varphi(\langle g, f\rangle_D)$. The value of the unbounded trace $T_{2,1-j}$ on the hexic module $\mathcal{M}_6$ is $\frac{1}{6}\varphi(1)$, which we now seek to calculate. Since φ is a ρ_1^{-2}-trace, its adjoint $\varphi^*(x)$ is a ρ_1^2-trace, and thus there are complex constants c_j, d_j such that $\varphi^* = c_j\psi'_{20} + d_j\psi'_{21}$, where ψ'_{20} and ψ'_{21} are the basic ρ_1^2-traces on $A_{1/\theta}^\infty$. We then have

$$\psi_{2,1-j}(\langle f, gW^2\rangle_{D^\perp}) = \overline{c}_j\overline{\psi'_{20}(\langle f, g\rangle_D)} + \overline{d}_j\overline{\psi'_{21}(\langle f, g\rangle_D)}$$

and $\varphi(1) = \overline{c}_j\overline{\psi'_{20}(1)} + \overline{d}_j\overline{\psi'_{21}(1)} = \overline{c}_j + \overline{d}_j$.

Let $f, g \in \mathcal{S}(\mathbb{R})$ such that $f(x) = \mathrm{e}^{-\pi a x^2}$ and $g(x) = \mathrm{e}^{-\pi b x^2}$, where $a, b \in \mathbb{C}^+$ (the right half-plane). We observe that

$$gW^2(t) = \mathrm{i}^{-\frac{1}{6}} e(t^2/2) \int_{-\infty}^{\infty} g(x)e(tx)\, \mathrm{d}x = \frac{\mathrm{i}^{-\frac{1}{6}}}{\sqrt{b}} \mathrm{e}^{-\pi(\frac{1}{b}-\mathrm{i})t^2}.$$

As before we have $\langle f, gW^2\rangle_{D^\perp} = \sum_{m,n} c_{m,n} V_1^m V_2^n$ where $c_{m,n} = \langle f, gW^2\rangle_{D^\perp}(m\delta_1 + n\delta_2) = \frac{\mathrm{i}^{-\frac{1}{6}}}{\sqrt{b\tau}} \mathrm{e}^{-\pi(\frac{1}{b}-\mathrm{i})\alpha^2 m^2} \mathrm{e}^{-\frac{\pi}{\tau}\alpha^2(n+m(1+\frac{\mathrm{i}}{b}))^2}$ with $\tau = \frac{\overline{a}b-\mathrm{i}b+1}{b}$. Using the formula $\psi_{2,1-j}(V_2^m V_1^n) = \lambda^{\frac{1}{6}(m-n)^2}\delta_3^{j(m-n)}$ we observe that

$$\psi_{2,1-j}(\langle f, gW^2\rangle_{D^\perp}) = \sum_{m,n} c_{m,m+n}\lambda^{m(m+n)}\lambda^{\frac{1}{6}n^2}\delta_3^{jn} = \sum_{m,n} c_{m,m+3^j n}\lambda^{m(m+3^j n)}\lambda^{\frac{1}{6}3^{2j}n^2}.$$

Substitution shows that $c_{m,m+3^j n} = \frac{\mathrm{i}^{-\frac{1}{6}}}{\sqrt{\overline{a}b-\mathrm{i}b+1}} \mathrm{e}^{-\frac{\pi}{\tau}\alpha^2 A_0 m^2} \mathrm{e}^{-\frac{\pi}{\tau}\alpha^2 B_0 n^2} \mathrm{e}^{-\frac{2\pi}{\tau}\alpha^2 C_0 mn}$ where $A_0 = (\overline{a}+2\mathrm{i})(\frac{1}{b}-\mathrm{i})+1$, $B_0 = 3^{2j}$, and $C_0 = 3^j(2+\frac{\mathrm{i}}{b})$. Combining all terms gives

$$\psi_{2,1-j}(\langle f, gW^2\rangle_{D^\perp}) = \frac{\mathrm{i}^{-\frac{1}{6}}}{\sqrt{\overline{a}b-\mathrm{i}b+1}} \sum_{m,n} \mathrm{e}^{-\pi\gamma m^2}\mathrm{e}^{-\pi\delta n^2}\mathrm{e}^{2\pi\zeta mn}$$

where $\gamma = \frac{\alpha^2(b+\overline{a}-3\mathrm{i}\overline{a}b)}{\overline{a}b-\mathrm{i}b+1}$, $\delta = \frac{3^{2j-1}\alpha^2(2b-\mathrm{i}\overline{a}b-\mathrm{i})}{\overline{a}b-\mathrm{i}b+1}$, $\zeta = \frac{3^j\alpha^2(\mathrm{i}\overline{a}b-b)}{\overline{a}b-\mathrm{i}b+1}$. Applying Lemma 10.1 allows us to transform the previous sum into the equivalent sum

$$\psi_{2,1-j}(\langle f, gW^2\rangle_{D^\perp}) = \Delta\sum_{m,n} \mathrm{e}^{-\pi\frac{\gamma}{\gamma\delta-\zeta^2}m^2}\mathrm{e}^{-\pi\frac{\delta}{\gamma\delta-\zeta^2}n^2}\mathrm{e}^{2\pi\frac{\zeta}{\gamma\delta-\zeta^2}mn}$$

where $\Delta = \frac{\mathrm{i}^{-\frac{1}{6}}}{\sqrt{\overline{a}b-\mathrm{i}b+1}\sqrt{\gamma\delta-\zeta^2}}$. Computation shows that $\gamma\delta = \frac{-3^{2j-1}\alpha^4}{(\overline{a}b-\mathrm{i}b+1)^2}(7\mathrm{i}\overline{a}b^2 + 3\overline{a}^2b^2 + \overline{a}b + \mathrm{i}\overline{a}^2b + \mathrm{i}\overline{a} - 2b^2 + \mathrm{i}b)$ and $\zeta^2 = \frac{-3^{2j}\alpha^4}{(\overline{a}b-\mathrm{i}b+1)^2}(\overline{a}^2b^2 + 2\mathrm{i}\overline{a}b^2 - b^2)$ from which it follows that $\gamma\delta - \zeta^2 = \frac{-3^{2j-1}\mathrm{i}\alpha^4(\overline{a}+b)}{(\overline{a}b-\mathrm{i}b+1)}$. This gives

$$\Delta = \frac{\mathrm{i}^{-\frac{1}{6}}}{\sqrt{(\overline{a}b-\mathrm{i}b+1)(\gamma\delta-\zeta^2)}} = \frac{3^{\frac{1}{2}-j}\beta^2\mathrm{i}^{\frac{1}{3}}}{\sqrt{\overline{a}+b}}$$

and the equalities

$$\begin{aligned}
\frac{\gamma}{\gamma\delta-\zeta^2} &= \frac{3^{1-2j}\mathrm{i}\beta^2(b+\overline{a}-3\mathrm{i}\overline{a}b)}{\overline{a}+b},\\
\frac{\delta}{\gamma\delta-\zeta^2} &= \frac{\beta^2(\overline{a}b+2\mathrm{i}b+1)}{\overline{a}+b},\\
\frac{\zeta}{\gamma\delta-\zeta^2} &= \frac{-3^{1-j}\beta^2(\overline{a}b+\mathrm{i}b)}{\overline{a}+b}.
\end{aligned}$$

Letting $\gamma' = \frac{i\beta^2(b+\bar{a}-3i\bar{a}b)}{\bar{a}+b}, \delta' = \frac{\beta^2(\bar{a}b+2ib+1)}{\bar{a}+b}, \zeta' = \frac{-\beta^2(\bar{a}b+ib)}{\bar{a}+b}$ so that $\frac{\gamma}{\gamma\delta-\zeta^2} = 3^{1-2j}\gamma', \frac{\delta}{\gamma\delta-\zeta^2} = \delta', \frac{\zeta}{\gamma\delta-\zeta^2} = 3^{1-j}\zeta'$, we may write explicitly (for $j = 0, 1$)

$$\psi_{20}(\langle f, gW^2\rangle_{D^\perp}) = \frac{\beta^2 i^{\frac{1}{3}}}{\sqrt{3}\sqrt{\bar{a}+b}} \sum_{m,n} e^{-\frac{\pi}{3}\gamma' m^2} e^{-\pi\delta' n^2} e^{2\pi\zeta' mn},$$

$$\psi_{21}(\langle f, gW^2\rangle_{D^\perp}) = \frac{\sqrt{3}\beta^2 i^{\frac{1}{3}}}{\sqrt{\bar{a}+b}} \sum_{m,n} e^{-3\pi\gamma' m^2} e^{-\pi\delta' n^2} e^{6\pi\zeta' mn}.$$

We will return to these shortly.

Again we have $\langle f, g\rangle_D = \beta^2 \sum\limits_{m,n} b_{m,n} U_2^n U_1^m$ where $b_{m,n} = \langle f, g\rangle_D(m\varepsilon_1 + n\varepsilon_2) = \frac{1}{\sqrt{\bar{a}+b}} e^{-\pi\bar{b}\beta^2 m^2} e^{-\pi\beta^2 \frac{(i\bar{b}m-n)^2}{a+\bar{b}}}$.

Using the formula $\psi'_{2,1-s}(U_2^n U_1^m) = \lambda_1^{\frac{1}{6}(n-m)^2} \delta_3^{s(n-m)}$ we observe that for $s = 0, 1$,

$$\psi'_{2,1-s}(\langle f, g\rangle_D) = \beta^2 \sum_{m,n} b_{m,m+3^s n} \lambda_1^{\frac{1}{6}3^{2s}n^2} = \frac{\beta^2}{\sqrt{a+\bar{b}}} \sum_{m,n} e^{-\pi\bar{\gamma}_1 m^2} e^{-\pi\bar{\delta}_1 n^2} e^{2\pi\bar{\zeta}_1 mn}$$

where $\gamma_1 = \frac{\beta^2(\bar{a}b+2ib+1)}{\bar{a}+b}, \delta_1 = \frac{3^{2s-1}\beta^2(3+i\bar{a}+ib)}{\bar{a}+b}, \zeta_1 = \frac{-3^s\beta^2(ib+1)}{\bar{a}+b}$. So, we have

$$\overline{\psi'_{2,1-s}(\langle f, g\rangle_D)} = \frac{\beta^2}{\sqrt{\bar{a}+b}} \sum_{m,n} e^{-\pi\gamma_1 m^2} e^{-\pi\delta_1 n^2} e^{2\pi\zeta_1 mn}.$$

Letting $\delta_2 = \frac{\beta^2(1+\frac{1}{3}i(\bar{a}+b))}{\bar{a}+b}, \zeta_2 = \frac{-\beta^2(ib+1)}{\bar{a}+b}$ we may write (for $s = 0, 1$)

$$\overline{\psi'_{20}(\langle f, g\rangle_D)} = \frac{\beta^2}{\sqrt{\bar{a}+b}} \sum_{m,n} e^{-\pi\gamma_1 m^2} e^{-9\pi\delta_2 n^2} e^{6\pi\zeta_2 mn},$$

$$\overline{\psi'_{21}(\langle f, g\rangle_D)} = \frac{\beta^2}{\sqrt{\bar{a}+b}} \sum_{m,n} e^{-\pi\gamma_1 m^2} e^{-\pi\delta_2 n^2} e^{2\pi\zeta_2 mn}.$$

CALCULATION OF $T_{20}([\mathcal{M}_6])$. We first concern ourselves with the equation

$$\psi_{20}(\langle f, gW^2\rangle_{D^\perp}) = \bar{c}_1 \overline{\psi'_{20}(\langle f, g\rangle_D)} + \bar{d}_1 \overline{\psi'_{21}(\langle f, g\rangle_D)}.$$

Our goal is to find constants $\bar{c}_1, \bar{d}_1$ such that this equation becomes an identity for any $a, b \in \mathbb{C}^+$. First, we observe that $\gamma_1 = \delta'$, and so after simplifying we may write the equation as

$$\frac{i^{\frac{1}{3}}}{\sqrt{3}} \sum_{m,n} e^{-\frac{\pi}{3}\gamma' m^2} e^{-\pi\gamma_1 n^2} e^{2\pi\zeta' mn}$$
$$= \bar{c}_1 \sum_{m,n} e^{-\pi\gamma_1 m^2} e^{-9\pi\delta_2 n^2} e^{6\pi\zeta_2 mn} + \bar{d}_1 \sum_{m,n} e^{-\pi\gamma_1 m^2} e^{-\pi\delta_2 n^2} e^{2\pi\zeta_2 mn}.$$

Consider the left-hand side of this equation. After implementing the substitutions $m \mapsto n$, $n \mapsto m$ it becomes $\frac{\mathrm{i}^{\frac{1}{3}}}{\sqrt{3}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-\frac{\pi}{3}\gamma' n^2} \mathrm{e}^{2\pi\zeta' mn}$. Next, after the substitution $m \mapsto m - n$ it becomes $\frac{\mathrm{i}^{\frac{1}{3}}}{\sqrt{3}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 (m-n)^2} \mathrm{e}^{-\frac{\pi}{3}\gamma' n^2} \mathrm{e}^{2\pi\zeta'(m-n)n}$ which simplifies to

$$\frac{\mathrm{i}^{\frac{1}{3}}}{\sqrt{3}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-\pi B n^2} \mathrm{e}^{2\pi C mn}$$

where $B = \gamma_1 + \frac{\gamma'}{3} + 2\zeta' = \frac{\beta^2}{3(\bar{a}+b)}(\mathrm{i}\bar{a} + \mathrm{i}b + 3) = \delta_2$ and $C = \zeta' + \gamma_1 = \frac{\beta^2}{(\bar{a}+b)}(\mathrm{i}b + 1) = -\zeta_2$. Now, the left-hand side becomes $\frac{\mathrm{i}^{\frac{1}{3}}}{\sqrt{3}} \sum \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-\pi\delta_2 n^2} \mathrm{e}^{-2\pi\zeta_2 mn}$. Upon making the substitution $n \mapsto -n$ this becomes $\frac{\mathrm{i}^{\frac{1}{3}}}{\sqrt{3}} \sum \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-\pi\delta_2 n^2} \mathrm{e}^{2\pi\zeta_2 mn}$, and so by taking $\bar{d}_1 = \frac{\mathrm{i}^{\frac{1}{3}}}{\sqrt{3}}$ and $\bar{c}_1 = 0$, the equation becomes an identity for all $a, b \in \mathbb{C}^+$. It follows that

$$T_{20}([\mathcal{M}_6]) = \frac{1}{6}\varphi(1) = \frac{1}{6}\bar{d}_1 = \frac{\mathrm{i}^{\frac{1}{3}}}{6\sqrt{3}} = \frac{\omega^{\frac{1}{2}}}{6\sqrt{3}}.$$

CALCULATION OF $T_{21}([\mathcal{M}_6])$. Now, we consider the equation

$$\psi_{21}(\langle f, gW^2\rangle_{D^\perp}) = \bar{c}_0 \overline{\psi'_{20}(\langle f, g\rangle_D)} + \bar{d}_0 \overline{\psi'_{21}(\langle f, g\rangle_D)}.$$

Our goal is to find constants $\bar{c}_0, \bar{d}_0$ such that this equation becomes an identity for any $a, b \in \mathbb{C}^+$. After simplifying we may write the equation as

$$\begin{aligned}&\sqrt{3}\mathrm{i}^{\frac{1}{3}} \sum_{m,n} \mathrm{e}^{-3\pi\gamma' m^2} \mathrm{e}^{-\pi\gamma_1 n^2} \mathrm{e}^{6\pi\zeta' mn} \\ &\quad = \bar{c}_0 \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-9\pi\delta_2 n^2} \mathrm{e}^{6\pi\zeta_2 mn} + \bar{d}_0 \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-\pi\delta_2 n^2} \mathrm{e}^{2\pi\zeta_2 mn}.\end{aligned}$$

Consider the left-hand side of this equation. After implementing the substitutions $m \mapsto n, n \mapsto m$, it becomes $\sqrt{3}\mathrm{i}^{\frac{1}{3}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-3\pi\gamma' n^2} \mathrm{e}^{6\pi\zeta' mn}$. Next, after the substitution $m \mapsto m - 3n$ it becomes $\sqrt{3}\mathrm{i}^{\frac{1}{3}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 (m-3n)^2} \mathrm{e}^{-3\pi\gamma' n^2} \mathrm{e}^{6\pi\zeta'(m-3n)n}$ which simplifies to

$$\sqrt{3}\mathrm{i}^{\frac{1}{3}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-\pi Y n^2} \mathrm{e}^{2\pi Z mn}$$

where $Y = 9\delta' + 3\gamma' + 18\zeta' = \frac{3\beta^2}{(\bar{a}+b)}(\mathrm{i}b + \mathrm{i}\bar{a} + 3) = 9\delta_2$ and $Z = 3\zeta' + 3\delta' = \frac{\beta^2}{(\bar{a}+b)}(3\mathrm{i}b + 3) = -3\zeta_2$.

Now, the left-hand side becomes $\sqrt{3}\mathrm{i}^{\frac{1}{3}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-9\pi\delta_2 n^2} \mathrm{e}^{-6\pi\zeta_2 mn}$. Upon making the substitution $n \mapsto -n$ this becomes $\sqrt{3}\mathrm{i}^{\frac{1}{3}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-9\pi\delta_2 n^2} \mathrm{e}^{6\pi\zeta_2 mn}$, and so by taking $\overline{d}_0 = 0$ and $\overline{c}_0 = \sqrt{3}\mathrm{i}^{\frac{1}{3}}$, the equation becomes an identity for all $a, b \in \mathbb{C}^+$. It follows that

$$T_{21}([\mathcal{M}_6]) = \frac{1}{6}\varphi(1) = \frac{1}{6}\overline{c}_0 = \frac{\sqrt{3}}{6}\mathrm{i}^{\frac{1}{3}} = \frac{\sqrt{3}}{6}\omega^{\frac{1}{2}}.$$

8. CALCULATION OF $T_{3k}([\mathcal{M}_6])$

Again, from Lemma 5.1 it follows that for the basic ρ^3-trace $\psi_{3,1-j}$ on A_θ^∞, there is a ρ_1^{-3}-trace φ on $A_{1/\theta}^\infty$ such that $\psi_{3,1-j}(\langle f, gW^3\rangle_{D^\perp}) = \varphi(\langle g, f\rangle_D)$.

The value of the unbounded trace $T_{3,1-j}$ on the hexic module $\mathcal{M}_6$ is $\frac{1}{6}\varphi(1)$, which we now seek to calculate. Since φ is a ρ_1^{-3}-trace, its adjoint $\varphi^*(x)$ is a ρ_1^3-trace, and thus there are complex constants c_j, d_j such that $\varphi^* = c_j\psi'_{30} + d_j\psi'_{31}$, where ψ'_{30} and ψ'_{31} are the basic ρ_1^3-traces on $A_{1/\theta}^\infty$. We then have

$$\psi_{3,1-j}(\langle f, gW^3\rangle_{D^\perp}) = \overline{c}_j\overline{\psi'_{30}(\langle f, g\rangle_D)} + \overline{d}_j\overline{\psi'_{31}(\langle f, g\rangle_D)} \tag{8.1}$$

and $\varphi(1) = \overline{c}_j\overline{\psi'_{30}(1)} + \overline{d}_j\overline{\psi'_{31}(1)} = \overline{c}_j + \overline{d}_j$.

As before, let $f, g \in \mathcal{S}(\mathbb{R})$ such that $f(x) = \mathrm{e}^{-\pi a x^2}$ and $g(x) = \mathrm{e}^{-\pi b x^2}$, where $a, b \in \mathbb{C}^+$. Since W^3 is the flip operator, we observe that $gW^3(t) = g(-t) = g(t)$, since g is an even function. We have

$$\langle f, gW^3\rangle_{D^\perp} = \sum_{m,n} c_{m,n} V_1^m V_2^n$$

where $c_{m,n} = \langle f, gW^3\rangle_{D^\perp}(m\delta_1 + n\delta_2) = \frac{1}{\sqrt{\overline{a}+b}}\mathrm{e}^{-\frac{\pi\alpha^2\overline{a}b}{\overline{a}+b}m^2}\mathrm{e}^{-\frac{\pi\alpha^2}{\overline{a}+b}n^2}\mathrm{e}^{-\frac{2\pi\alpha^2\mathrm{i}b}{\overline{a}+b}mn}$. From the formula $\psi_{30}(V_2^n V_1^m) = \lambda^{-\frac{1}{2}mn}\delta_2^m\delta_2^n$ we compute $\psi_{30}(\langle f, gW^3\rangle_{D^\perp}) = \sum_{m,n} c_{2m,2n}\lambda^{2mn}$ where $c_{2m,2n} = \frac{1}{\sqrt{\overline{a}+b}}\mathrm{e}^{-\pi\frac{4\alpha^2}{\overline{a}+b}\overline{a}bm^2}\mathrm{e}^{-\pi\frac{4\alpha^2}{\overline{a}+b}n^2}\mathrm{e}^{-\pi\frac{8\alpha^2\mathrm{i}b}{\overline{a}+b}mn}$. So,

$$\psi_{30}(\langle f, gW^3\rangle_{D^\perp}) = \frac{1}{\sqrt{\overline{a}+b}}\sum_{m,n}\mathrm{e}^{-\pi\gamma m^2}\mathrm{e}^{-\pi\delta n^2}\mathrm{e}^{2\pi\zeta mn}$$

where $\gamma = \frac{4\alpha^2\overline{a}b}{\overline{a}+b}$, $\delta = \frac{4\alpha^2}{\overline{a}+b}$, $\zeta = \frac{2\mathrm{i}\alpha^2(\overline{a}-b)}{\overline{a}+b}$. As before, applying Lemma 10.1 allows us to transform the previous sum into the equivalent sum

$$\psi_{30}(\langle f, gW^3\rangle_{D^\perp}) = \Delta\sum_{m,n}\mathrm{e}^{\pi\frac{\gamma}{\gamma\delta-\zeta^2}m^2}\mathrm{e}^{-\pi\frac{\delta}{\gamma\delta-\zeta^2}n^2}\mathrm{e}^{2\pi\frac{\zeta}{\gamma\delta-\zeta^2}mn}$$

where $\Delta = \frac{1}{\sqrt{\overline{a}+b}\sqrt{\gamma\delta-\zeta^2}}$. Computation shows that $\gamma\delta = \frac{16\alpha^4\overline{a}b}{(\overline{a}+b)^2}$, $\zeta^2 = \frac{-4\alpha^4}{(\overline{a}+b)^2}(\overline{a}^2 - 2\overline{a}b + b^2)$ from which it follows that $\gamma\delta - \zeta^2 = \frac{4\alpha^4}{(\overline{a}+b)^2}(\overline{a}^2 + 2\overline{a}b + b^2) = \frac{4\alpha^4}{(\overline{a}+b)^2}(\overline{a} +$

$b)^2 = 4\alpha^4$. This gives

$$\Delta = \frac{1}{\sqrt{(\bar{a}+b)(\gamma\delta-\zeta^2)}} = \frac{\beta^2}{2\sqrt{\bar{a}+b}}$$

and the equalities

$$\frac{\gamma}{\gamma\delta-\zeta^2} = \frac{\beta^2\bar{a}b}{\bar{a}+b}, \quad \frac{\delta}{\gamma\delta-\zeta^2} = \frac{\beta^2}{\bar{a}+b}, \quad \frac{\zeta}{\gamma\delta-\zeta^2} = \frac{\mathrm{i}\beta^2(\bar{a}-b)}{2(\bar{a}+b)}.$$

An analogous calculation may be done for ψ_{31}, where $\psi_{31}(V_2^n V_1^m) = \lambda^{-\frac{1}{2}mn}$. In this case we have

$$\psi_{31}(\langle f, gW^3\rangle_{D^\perp}) = \sum_{m,n} c_{m,n}\psi_{31}(V_1^m V_2^n) = \frac{1}{\sqrt{\bar{a}+b}}\sum_{m,n} \mathrm{e}^{-\pi\gamma_0 m^2}\mathrm{e}^{-\pi\delta_0 n^2}\mathrm{e}^{2\pi\zeta_0 mn}$$

where $\gamma_0 = \frac{\alpha^2\bar{a}b}{\bar{a}+b}, \delta_0 = \frac{\alpha^2}{\bar{a}+b}, \zeta_0 = \frac{\mathrm{i}\alpha^2(\bar{a}-b)}{2(\bar{a}+b)}$. Applying Lemma 10.1 now gives

$$\psi_{31}(\langle f, gW^3\rangle_{D^\perp}) = \Delta_0 \sum_{m,n} \mathrm{e}^{-\pi\frac{\gamma_0}{\gamma_0\delta_0-\zeta_0^2}m^2}\mathrm{e}^{-\pi\frac{\delta_0}{\gamma_0\delta_0-\zeta_0^2}n^2}\mathrm{e}^{2\pi\frac{\zeta_0}{\gamma_0\delta_0-\zeta_0^2}mn}$$

where $\Delta_0 = \frac{1}{\sqrt{\bar{a}+b}\sqrt{\gamma_0\delta_0-\zeta_0^2}}$. Computation shows that

$$\gamma_0\delta_0 = \frac{\alpha^4\bar{a}b}{(\bar{a}+b)^2}, \quad \zeta_0^2 = \frac{-\alpha^4}{4(\bar{a}+b)^2}(\bar{a}^2 - 2\bar{a}b + b^2)$$

from which it follows that $\gamma_0\delta_0 - \zeta_0^2 = \frac{\alpha^4}{4(\bar{a}+b)^2}(\bar{a}^2 + 2\bar{a}b + b^2) = \frac{\alpha^4}{4(\bar{a}+b)^2}(\bar{a}+b)^2 = \frac{\alpha^4}{4}$. This gives

$$\Delta_0 = \frac{1}{\sqrt{(\bar{a}+b)(\gamma_0\delta_0-\zeta_0^2)}} = \frac{2\beta^2}{\sqrt{\bar{a}+b}}$$

and the equalities

$$\frac{\gamma_0}{\gamma_0\delta_0-\zeta_0^2} = \frac{4\beta^2\bar{a}b}{\bar{a}+b}, \quad \frac{\delta_0}{\gamma_0\delta_0-\zeta_0^2} = \frac{4\beta^2}{\bar{a}+b}, \quad \frac{\zeta_0}{\gamma_0\delta_0-\zeta_0^2} = \frac{2\mathrm{i}\beta^2(\bar{a}-b)}{\bar{a}+b}.$$

We now consider the right hand side of equation (8.1). With $\langle f, g\rangle_D$ and $b_{m,n}$ defined as in Section 6, we have

$$\psi_{30}'(\langle f, g\rangle_D) = \beta^2\sum_{m,n} b_{m,n}\psi_{30}'(U_2^n U_1^m) = \beta^2\sum_{m,n} b_{2m,2n}\lambda_1^{-2mn}$$

where $b_{2m,2n} = \frac{1}{\sqrt{a+\bar{b}}}\mathrm{e}^{-\frac{4\pi a\bar{b}\beta^2}{a+\bar{b}}m^2}\mathrm{e}^{-\frac{4\pi\beta^2}{a+\bar{b}}n^2}\mathrm{e}^{\frac{8\mathrm{i}\pi\beta^2\bar{b}}{a+\bar{b}}mn}$. So

$$\psi_{30}'(\langle f, g\rangle_D) = \frac{\beta^2}{\sqrt{a+\bar{b}}}\sum_{m,n} \mathrm{e}^{-\pi\bar{\gamma}_1 m^2}\mathrm{e}^{-\pi\bar{\delta}_1 n^2}\mathrm{e}^{2\pi\bar{\zeta}_1 mn}$$

where $\gamma_1 = \frac{4\beta^2\bar{a}b}{\bar{a}+b}, \delta_1 = \frac{4\beta^2}{\bar{a}+b}, \zeta_1 = \frac{2i\beta^2(\bar{a}-b)}{\bar{a}+b}$. Similarly,

$$\psi'_{31}(\langle f,g\rangle_D) = \beta^2 \sum_{m,n} b_{m,n}\psi'_{31}(U_2^n U_1^m) = \frac{\beta^2}{\sqrt{a+\bar{b}}} \sum_{m,n} \mathrm{e}^{-\pi\overline{\gamma}_2 m^2}\mathrm{e}^{-\pi\overline{\delta}_2 n^2}\mathrm{e}^{2\pi\overline{\zeta}_2 mn}$$

where $\gamma_2 = \frac{\beta^2\bar{a}b}{\bar{a}+b}, \delta_2 = \frac{\beta^2}{\bar{a}+b}, \zeta_2 = \frac{i\beta^2(\bar{a}-b)}{2(\bar{a}+b)}$. So,

$$\begin{aligned}\overline{\psi'_{30}(\langle f,g\rangle_D)} &= \frac{\beta^2}{\sqrt{\bar{a}+b}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2}\mathrm{e}^{-\pi\delta_1 n^2}\mathrm{e}^{2\pi\zeta_1 mn}\\ \overline{\psi'_{31}(\langle f,g\rangle_D)} &= \frac{\beta^2}{\sqrt{\bar{a}+b}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_2 m^2}\mathrm{e}^{-\pi\delta_2 n^2}\mathrm{e}^{2\pi\zeta_2 mn}.\end{aligned}$$

CALCULATIONS OF $T_{30}([\mathcal{M}_6])$ AND $T_{31}([\mathcal{M}_6])$. It now follows readily that

$$\frac{\gamma}{\gamma\delta-\zeta^2} = \gamma_2, \quad \frac{\delta}{\gamma\delta-\zeta^2} = \delta_2, \quad \frac{\zeta}{\gamma\delta-\zeta^2} = \zeta_2$$

and

$$\frac{\gamma_0}{\gamma_0\delta_0-\zeta_0^2} = \gamma_1 = 4\gamma_2, \quad \frac{\delta_0}{\gamma_0\delta_0-\zeta_0^2} = \delta_1 = 4\delta_2, \quad \frac{\zeta_0}{\gamma_0\delta_0-\zeta_0^2} = \zeta_1 = 4\zeta_2.$$

This implies that $\bar{c}_1 = 0$, $\bar{d}_1 = \frac{1}{2}$, $\bar{c}_0 = 2$, and $\bar{d}_0 = 0$. We hence obtain the identities

$$\psi_{30}(\langle f,gW^3\rangle_{D^\perp}) = \frac{1}{2}\overline{\psi'_{31}(\langle f,g\rangle_D)}, \quad \psi_{31}(\langle f,gW^3\rangle_{D^\perp}) = 2\overline{\psi'_{30}(\langle f,g\rangle_D)},$$

from which it follows that

$$T_{30}([\mathcal{M}_6]) = \frac{1}{6}\varphi(1) = \frac{1}{6}\bar{d}_1 = \frac{1}{12}, \quad T_{31}([\mathcal{M}_6]) = \frac{1}{6}\varphi(1) = \frac{1}{6}\bar{c}_0 = \frac{1}{3}.$$

9. CALCULATION OF $S_{1k}([\mathcal{M}_3])$

From Lemma 5.1 it follows that for the basic κ-trace φ_{1j} on A_θ^∞, there is a κ_1^{-1}-trace ψ on $A_{1/\theta}^\infty$ such that $\varphi_{1j}(\langle f,gZ\rangle_{D^\perp}) = \psi(\langle g,f\rangle_D)$. The value of the unbounded trace S_{1j} on the cubic module $\mathcal{M}_3$ is $\frac{1}{3}\psi(1)$, which we now seek to calculate. Since ψ is a κ_1^{-1}-trace, its adjoint ψ^* is a κ_1-trace, and thus there are complex constants c_j, d_j, e_j such that $\psi^* = c_j\varphi'_{10} + d_j\varphi'_{11} + e_j\varphi'_{12}$, where the φ'_{1j} are the basic κ_1-traces on $A_{1/\theta}^\infty$. We then have

$$\varphi_{1j}(\langle f,gZ\rangle_{D^\perp}) = \bar{c}_j\overline{\varphi'_{10}(\langle f,g\rangle_D)} + \bar{d}_j\overline{\varphi'_{11}(\langle f,g\rangle_D)} + \bar{e}_j\overline{\varphi'_{12}(\langle f,g\rangle_D)}$$

and $\psi(1) = \bar{c}_j\overline{\varphi'_{10}(1)} + \bar{d}_j\overline{\varphi'_{11}(1)} + \bar{e}_j\overline{\varphi'_{12}(1)} = \bar{c}_j$.

The φ_{1j} traces are defined by

$$\varphi_{1j}(U^m V^n) = \lambda^{\frac{1}{6}((m-n)^2-j^2)}\delta_3^{m-n-j}.$$

We take φ_{10} and φ_{11} to assume their usual definitions, but define φ_{12} by

$$\varphi_{12}(U^m V^n) = \lambda^{\frac{1}{6}((m-n)^2-1)}\delta_3^{m-n+1},$$

which corresponds to the choice $j = -1$ and is non-zero at exactly those values where the usual φ_{12} is non-zero.

As before, let $f(x) = \mathrm{e}^{-\pi a x^2}$ and $g(x) = \mathrm{e}^{-\pi b x^2}$, where $a, b \in \mathbb{C}^+$. Using a previous calculation, we have

$$gZ(t) = gW^2(t) = \frac{\mathrm{i}^{-\frac{1}{6}}}{\sqrt{b}}\mathrm{e}^{-\pi(\frac{1}{b}-\mathrm{i})t^2}.$$

We then have $\langle f, gZ\rangle_{D^\perp} = \sum_{m,n} c_{m,n} V_1^m V_2^n$ where again using a previous calculation $c_{m,n} = \langle f, gZ\rangle_{D^\perp}(m\delta_1 + n\delta_2) = \frac{\mathrm{i}^{-\frac{1}{6}}}{\sqrt{b\tau}}\mathrm{e}^{-\pi(\frac{1}{b}-\mathrm{i})\alpha^2 m^2}\mathrm{e}^{-\frac{\pi}{\tau}\alpha^2(n+m(1+\frac{\mathrm{i}}{b}))^2}$ with $\tau = \frac{\bar{a}b-\mathrm{i}b+1}{b}$. We observe that $\psi_{20}(U^m V^n) = \varphi_{10}(U^m V^n)$, from which we immediately obtain the equalities

$$\psi_{20}(\langle f, gW^2\rangle_{D^\perp}) = \varphi_{10}(\langle f, gZ\rangle_{D^\perp}) \quad \text{and} \quad \psi'_{20}(\langle f, g\rangle_D) = \varphi'_{10}(\langle f, g\rangle_D).$$

Earlier calculations then yield (after appropriate inversions and substitutions)

$$\varphi_{10}(\langle f, gZ\rangle_{D^\perp}) = \frac{-\omega^{\frac{1}{2}}\beta^2}{\sqrt{3}\sqrt{\bar{a}+b}}\sum_{m,n}\mathrm{e}^{-\pi\gamma_1 m^2}\mathrm{e}^{-\frac{\pi}{9}\delta_1 n^2}\mathrm{e}^{\frac{2\pi}{3}\zeta_1 mn}$$

and

$$\overline{\varphi'_{10}(\langle f, g\rangle_D)} = \frac{\beta^2}{\sqrt{\bar{a}+b}}\sum_{m,n}\mathrm{e}^{-\pi\gamma_1 m^2}\mathrm{e}^{-\pi\delta_1 n^2}\mathrm{e}^{2\pi\zeta_1 mn}$$

where $\gamma_1 = \frac{\beta^2(\bar{a}b+2\mathrm{i}b+1)}{\bar{a}+b}, \delta_1 = \frac{3\beta^2(\mathrm{i}\bar{a}+\mathrm{i}b+3)}{\bar{a}+b}, \zeta_1 = \frac{-3\beta^2(\mathrm{i}b+1)}{\bar{a}+b}$. Next, we have

$$\varphi_{11}(\langle f, gZ\rangle_{D^\perp}) = \sum_{m,n} c_{m,n}\lambda^{mn}\varphi_{11}(V_1^m \# V_2^n) = \sum_{m,n} c_{m,m+3n-1}\lambda^{m(m+3n-1)}\lambda^{\frac{1}{6}(9n^2-6n)}$$

where

$$\begin{aligned} c_{m,m+3n-1} &= \frac{\mathrm{i}^{-\frac{1}{6}}}{\sqrt{b\tau}}\mathrm{e}^{-\pi(\frac{1}{b}-\mathrm{i})\alpha^2 m^2}\mathrm{e}^{-\frac{\pi}{\tau}\alpha^2(3n+m(2+\frac{\mathrm{i}}{b})-1)^2} \\ &= \frac{\mathrm{i}^{-\frac{1}{6}}\mathrm{e}^{-\frac{\pi}{\tau}\alpha^2}}{\sqrt{b\tau}}\mathrm{e}^{-\pi A_0 m^2}\mathrm{e}^{-\pi B_0 n^2}\mathrm{e}^{\frac{2\pi}{3}E_0 m}\mathrm{e}^{-\frac{2\pi}{3}B_0 n}\mathrm{e}^{-\pi E_0 mn} \end{aligned}$$

where $A_0 = \frac{\alpha^2(-\mathrm{i}\bar{a}b+\bar{a}+3b+2\mathrm{i})}{b\tau}, B_0 = \frac{9\alpha^2}{\tau}, E_0 = \frac{-3\alpha^2(2b+\mathrm{i})}{b\tau}$. Combining all terms gives

$$\varphi_{11}(\langle f, gZ\rangle_{D^\perp}) = \frac{\mathrm{i}^{-\frac{1}{6}}\mathrm{e}^{-\frac{\pi}{\tau}\alpha^2}}{\sqrt{b\tau}}\sum_{m,n}\mathrm{e}^{-\gamma m^2}\mathrm{e}^{-\pi\delta n^2}\mathrm{e}^{-\frac{2\pi}{3}\zeta m}\mathrm{e}^{\frac{2\pi}{3}\delta n}\mathrm{e}^{2\pi\zeta mn}$$

where $\gamma = \frac{\alpha^2(\bar{a}+b-3\mathrm{i}\bar{a}b)}{b\tau}, \delta = \frac{3\alpha^2(-\mathrm{i}\bar{a}b+2b-\mathrm{i})}{b\tau}, \zeta = \frac{3\alpha^2(\mathrm{i}\bar{a}b-b)}{b\tau}$. Considering now φ'_{11}, we have as before $\langle f,g\rangle_D = \beta^2 \sum_{m,n} b_{m,n} U_2^n U_1^m$ with $b_{m,n} = \langle f,g\rangle_D(m\varepsilon_1 + n\varepsilon_2) = \frac{1}{\sqrt{a+\bar{b}}} \mathrm{e}^{-\pi\bar{b}\beta^2 m^2} \mathrm{e}^{-\pi\beta^2\frac{(\mathrm{i}\bar{b}m-n)^2}{a+\bar{b}}}$. It follows that $\varphi'_{11}(\langle f,g\rangle_D) = \beta^2 \sum_{m,n} b_{m,n}\varphi'_{11}(U_2^n U_1^m) = \beta^2 \sum_{m,n} b_{m,m+3n+1}\lambda_1^{\frac{1}{6}(9n^2+6n)}$ where $b_{m,m+3n+1} = \frac{1}{\sqrt{a+\bar{b}}} \mathrm{e}^{-\pi\beta^2\bar{b}m^2}\ \mathrm{e}^{-\pi\beta^2\frac{(\mathrm{i}\bar{b}m-m-3n-1)^2}{a+\bar{b}}}$. We hence obtain

$$\varphi'_{11}(\langle f,g\rangle_D) = \frac{\beta^2 \mathrm{e}^{-\pi\frac{\beta^2}{a+\bar{b}}}}{\sqrt{a+\bar{b}}} \sum_{m,n} \mathrm{e}^{-\pi\bar{\gamma}_1 m^2} \mathrm{e}^{-\pi\bar{\delta}_1 n^2} \mathrm{e}^{\frac{2\pi}{3}\bar{\zeta}_1 m} \mathrm{e}^{-\frac{2\pi}{3}\bar{\delta}_1 n} \mathrm{e}^{2\pi\bar{\zeta}_1 mn},$$

with γ_1, δ_1, and ζ_1 as previously. This gives

$$\overline{\varphi'_{11}(\langle f,g\rangle_D)} = \frac{\beta^2 \mathrm{e}^{-\pi\frac{\beta^2}{\bar{a}+b}}}{\sqrt{\bar{a}+b}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-\pi\delta_1 n^2} \mathrm{e}^{\frac{2\pi}{3}\zeta_1 m} \mathrm{e}^{-\frac{2\pi}{3}\delta_1 n} \mathrm{e}^{2\pi\zeta_1 mn}.$$

Now, applying Lemma 10.1 to $\varphi_{11}(\langle f, gZ\rangle_{D^\perp})$ gives the equivalent form

$$\Delta \sum_{m,n} \mathrm{e}^{-\pi\frac{\gamma}{\gamma\delta-\zeta^2}m^2} \mathrm{e}^{-\pi\frac{\delta}{\gamma\delta-\zeta^2}n^2} \mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}m} \mathrm{e}^{2\pi\frac{\zeta}{\gamma\delta-\zeta^2}mn}$$

where $\Delta = \frac{\mathrm{i}^{-\frac{1}{6}}\mathrm{e}^{-\frac{\pi}{\tau}\alpha^2}\mathrm{e}^{\frac{\pi}{9}\delta}}{\sqrt{b\tau(\gamma\delta-\zeta^2)}}$. A routine calculation shows that

$$\gamma\delta - \zeta^2 = \frac{3\alpha^4}{b^2\tau^2}(\bar{a}+b-3\mathrm{i}\bar{a}b)(-\mathrm{i}\bar{a}b+2\mathrm{i}b-\mathrm{i}) - \frac{9\alpha^4}{b^2\tau^2}(\mathrm{i}\bar{a}b-b)^2 = \frac{-3\mathrm{i}\alpha^4(\bar{a}+b)}{\bar{a}b-\mathrm{i}b+1}.$$

This gives

$$\Delta = \frac{\omega^{\frac{1}{2}}\beta^2\mathrm{e}^{\frac{\pi}{9}\delta-\frac{\pi}{\tau}\alpha^2}}{\sqrt{3}\sqrt{\bar{a}+b}} = \frac{\omega^{\frac{1}{2}}\beta^2\lambda^{-\frac{1}{2}}}{\sqrt{3}\sqrt{\bar{a}+b}}$$

and the equalities

$$\begin{aligned}
\frac{\gamma}{\gamma\delta-\zeta^2} &= \frac{\beta^2(3\bar{a}b+\mathrm{i}\bar{a}+\mathrm{i}b)}{3(\bar{a}+b)},\\
\frac{\delta}{\gamma\delta-\zeta^2} &= \frac{\beta^2(\bar{a}b+2\mathrm{i}b+1)}{\bar{a}+b} = \gamma_1,\\
\frac{\zeta}{\gamma\delta-\zeta^2} &= \frac{\beta^2\mathrm{i}(\mathrm{i}\bar{a}b-b)}{\bar{a}+b}.
\end{aligned}$$

Interchanging m and n and making the substitution $m \mapsto m-n$ produces the series

$$\sum_{m,n} \mathrm{e}^{-\pi\gamma_1(m-n)^2} \mathrm{e}^{-\pi\frac{\gamma}{\gamma\delta-\zeta^2}n^2} \mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}n} \mathrm{e}^{2\pi\frac{\zeta}{\gamma\delta-\zeta^2}(m-n)n} = \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-\pi R n^2} \mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}n} \mathrm{e}^{2\pi S mn}$$

where $R = \gamma_1 + \frac{\gamma}{\gamma\delta-\zeta^2} + 2\frac{\zeta}{\gamma\delta-\zeta^2} = \frac{\beta^2}{3(\bar{a}+b)}(\mathrm{i}\bar{a} + \mathrm{i}b + 3) = \frac{1}{9}\delta_1$ and $S = \gamma_1 + \frac{\zeta}{\gamma\delta-\zeta^2} = \frac{\beta^2}{\bar{a}+b}(\bar{a}b + 2\mathrm{i}b + 1 - \bar{a}b - \mathrm{i}b) = \frac{\beta^2}{\bar{a}+b}(\mathrm{i}b+1) = -\frac{1}{3}\zeta_1$. Finally, the substitution $m \mapsto -m$ gives

$$\varphi_{11}(\langle f, gZ\rangle_{D^\perp}) = \frac{\omega^{\frac{1}{2}}\beta^2\lambda^{-\frac{1}{2}}}{\sqrt{3}\sqrt{\bar{a}+b}}\sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2}\mathrm{e}^{-\frac{\pi}{9}\delta_1 n^2}\mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}n}\mathrm{e}^{\frac{2\pi}{3}\zeta_1 mn}.$$

We now partition this series into three parts, depending on the value of k where $k \equiv n \bmod 3$. Explicitly,

$$\sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2}\mathrm{e}^{-\frac{\pi}{9}\delta_1 n^2}\mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}n}\mathrm{e}^{\frac{2\pi}{3}\zeta_1 mn} = L_0 + L_1 + L_{-1}$$

where $n \equiv k \bmod 3$ on L_k. The substitutions $n \mapsto 3n + k$ give

$$\begin{aligned}
L_0 &= \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2}\mathrm{e}^{-\pi\delta_1 n^2}\mathrm{e}^{-2\pi\mathrm{i}n}\mathrm{e}^{2\pi\zeta_1 mn} = \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2}\mathrm{e}^{-\pi\delta_1 n^2}\mathrm{e}^{2\pi\zeta_1 mn},\\
L_1 &= \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2}\mathrm{e}^{-\frac{\pi}{9}\delta_1(3n+1)^2}\mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}(3n+1)}\mathrm{e}^{\frac{2\pi}{3}\zeta_1 m(3n+1)}\\
&= -\omega\mathrm{e}^{-\frac{\pi}{9}\delta_1}\sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2}\mathrm{e}^{-\pi\delta_1 n^2}\mathrm{e}^{\frac{2\pi}{3}\zeta_1 m}\mathrm{e}^{-\frac{2\pi}{3}\delta_1 n}\mathrm{e}^{2\pi\zeta_1 mn},\\
L_{-1} &= \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2}\mathrm{e}^{-\frac{\pi}{9}\delta_1(3n-1)^2}\mathrm{e}^{-\frac{2\pi}{3}\mathrm{i}(3n-1)}\mathrm{e}^{\frac{2\pi}{3}\zeta_1 m(3n-1)}\\
&= \omega^2\mathrm{e}^{-\frac{\pi}{9}\delta_1}\sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2}\mathrm{e}^{-\pi\delta_1 n^2}\mathrm{e}^{-\frac{2\pi}{3}\zeta_1 m}\mathrm{e}^{\frac{2\pi}{3}\delta_1 n}\mathrm{e}^{2\pi\zeta_1 mn}.
\end{aligned}$$

Making the substitution $m, n \mapsto -m, -n$ in L_{-1} shows that $L_{-1} = -\omega L_1$. Hence, $\varphi_{11}(\langle f, gZ\rangle_{D^\perp}) = \frac{\omega^{\frac{1}{2}}\beta^2\lambda^{-\frac{1}{2}}}{\sqrt{3}\sqrt{\bar{a}+b}}(L_0 + (1-\omega)L_1)$. Considering finally φ_{12},

$$\varphi_{12}(\langle f, gZ\rangle_{D^\perp}) = \sum_{m,n} c_{m,n}\lambda^{mn}\varphi_{12}(V_1^m \# V_2^n) = \sum_{m,n} c_{m,m+3n+1}\lambda^{m(m+3n+1)}\lambda^{\frac{1}{6}(9n^2+6n)}$$

where

$$\begin{aligned}
c_{m,m+3n+1} &= \frac{\mathrm{i}^{-\frac{1}{6}}}{\sqrt{b\tau}}\mathrm{e}^{-\pi(\frac{1}{b}-\mathrm{i})\alpha^2 m^2}\mathrm{e}^{-\frac{\pi}{\tau}\alpha^2(3n+m(2+\frac{\mathrm{i}}{b})+1)^2}\\
&= \frac{\mathrm{i}^{-\frac{1}{6}}\mathrm{e}^{-\frac{\pi}{\tau}\alpha^2}}{\sqrt{b\tau}}\mathrm{e}^{-\pi A_0 m^2}\mathrm{e}^{-\pi B_0 n^2}\mathrm{e}^{-\frac{2\pi}{3}E_0 m}\mathrm{e}^{\frac{2\pi}{3}B_0 n}\mathrm{e}^{-\pi E_0 mn}
\end{aligned}$$

where $A_0 = \frac{\alpha^2(-\mathrm{i}\bar{a}b+\bar{a}+3b+2\mathrm{i})}{b\tau}$, $B_0 = \frac{9\alpha^2}{\tau}$, $E_0 = \frac{-3\alpha^2(2b+\mathrm{i})}{b\tau}$. Combining all terms gives

$$\varphi_{12}(\langle f, gZ\rangle_{D^\perp}) = \frac{\mathrm{i}^{-\frac{1}{6}}\mathrm{e}^{-\frac{\pi}{\tau}\alpha^2}}{\sqrt{b\tau}}\sum_{m,n} \mathrm{e}^{-\pi\gamma m^2}\mathrm{e}^{-\pi\delta n^2}\mathrm{e}^{2\frac{2\pi}{3}\zeta m}\mathrm{e}^{-\frac{2\pi}{3}\delta n}\mathrm{e}^{2\pi\zeta mn}$$

with γ, δ, and ζ as before. The substitution $m, n \mapsto -m, -n$ then gives the equality $\varphi_{12}(\langle f, gZ\rangle_{D^\perp}) = \varphi_{11}(\langle f, gZ\rangle_{D^\perp})$. Considering now φ'_{12}, we have as before $\langle f, g\rangle_D = \beta^2 \sum_{m,n} b_{m,n} U_2^n U_1^m$ with $b_{m,n} = \langle f, g\rangle_D(m\varepsilon_1 + n\varepsilon_2) = \frac{1}{\sqrt{a+\bar{b}}} \mathrm{e}^{-\pi\bar{b}\beta^2 m^2}$ $\mathrm{e}^{-\pi\beta^2 \frac{(\mathrm{i}\bar{b}m-n)^2}{a+\bar{b}}}$. It follows that $\varphi'_{12}(\langle f, g\rangle_D) = \beta^2 \sum_{m,n} b_{m,n} \varphi'_{12}(U_2^n U_1^m) = \beta^2 \sum_{m,n} b_{m,m+3n-1}$ $\lambda_1^{\frac{1}{6}(9n^2-6n)}$ where $b_{m,m+3n-1} = \frac{1}{\sqrt{a+\bar{b}}} \mathrm{e}^{-\pi\beta^2\bar{b}m^2} \mathrm{e}^{-\pi\beta^2 \frac{(\mathrm{i}\bar{b}m-m-3n+1)^2}{a+\bar{b}}}$. We hence obtain

$$\varphi'_{12}(\langle f, g\rangle_D) = \frac{\beta^2 \mathrm{e}^{-\pi\frac{\beta^2}{a+\bar{b}}}}{\sqrt{a+\bar{b}}} \sum_{m,n} \mathrm{e}^{-\pi\overline{\gamma}_1 m^2} \mathrm{e}^{-\pi\bar{\delta}_1 n^2} \mathrm{e}^{-\frac{2\pi}{3}\bar{\zeta}_1 m} \mathrm{e}^{\frac{2\pi}{3}\bar{\delta}_1 n} \mathrm{e}^{2\pi\bar{\zeta}_1 mn},$$

with γ_1, δ_1, and ζ_1 as previously. This gives

$$\overline{\varphi'_{12}(\langle f, g\rangle_D)} = \frac{\beta^2 \mathrm{e}^{-\pi\frac{\beta^2}{\bar{a}+b}}}{\sqrt{\bar{a}+b}} \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-\pi\delta_1 n^2} \mathrm{e}^{-\frac{2\pi}{3}\zeta_1 m} \mathrm{e}^{\frac{2\pi}{3}\delta_1 n} \mathrm{e}^{2\pi\zeta_1 mn}.$$

Again, the substitution $m, n \mapsto -m, -n$ gives $\overline{\varphi'_{12}(\langle f, g\rangle_D)} = \overline{\varphi'_{11}(\langle f, g\rangle_D)}$. Our problem is then reduced to determining constants $\bar{c}_k, \bar{r}_k$, where $k = 0, 1$ and $\bar{r}_k = \bar{d}_k + \bar{e}_k$, such that the equations

$$\varphi_{10}(\langle f, gZ\rangle_{D^\perp}) = \bar{c}_0 \overline{\varphi'_{10}(\langle f, g\rangle_D)} + \bar{r}_0 \overline{\varphi'_{11}(\langle f, g\rangle_D)}$$

and

$$\varphi_{11}(\langle f, gZ\rangle_{D^\perp}) = \varphi_{12}(\langle f, gZ\rangle_{D^\perp}) = \bar{c}_1 \overline{\varphi'_{10}(\langle f, g\rangle_D)} + \bar{r}_1 \overline{\varphi'_{11}(\langle f, g\rangle_D)}$$

become identities. For convenience, let

$$\Sigma_0 = \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-\pi\delta_1 n^2} \mathrm{e}^{2\pi\zeta_1 mn}, \quad \Sigma_1 = \sum_{m,n} \mathrm{e}^{-\pi\gamma_1 m^2} \mathrm{e}^{-\pi\delta_1 n^2} \mathrm{e}^{\frac{2\pi}{3}\zeta_1 m} \mathrm{e}^{-\frac{2\pi}{3}\delta_1 n} \mathrm{e}^{2\pi\zeta_1 mn}.$$

In this notation we may write

$$\varphi_{10}(\langle f, gZ\rangle_{D^\perp}) = \frac{\omega^{\frac{1}{2}}\beta^2}{\sqrt{3}\sqrt{\bar{a}+b}}(\Sigma_0 + 2\mathrm{e}^{-\frac{\pi}{9}\delta_1}\Sigma_1),$$
$$\varphi_{11}(\langle f, gZ\rangle_{D^\perp}) = \frac{\omega^{\frac{1}{2}}\beta^2\lambda^{-\frac{1}{2}}}{\sqrt{3}\sqrt{\bar{a}+b}}(\Sigma_0 + \mathrm{e}^{-\frac{\pi}{9}\delta_1}(\omega^2 - \omega)\Sigma_1),$$

and

$$\overline{\varphi'_{10}(\langle f, g\rangle_D)} = \frac{\beta^2}{\sqrt{\bar{a}+b}}\Sigma_0, \quad \overline{\varphi'_{11}(\langle f, g\rangle_D)} = \frac{\beta^2 \mathrm{e}^{-\frac{\pi\beta^2}{\bar{a}+b}}}{\sqrt{\bar{a}+b}}\Sigma_1.$$

CALCULATION OF $S_{10}([\mathcal{M}_3])$. We consider the first of our two equations, which may be written using this new notation, after some simplification, as

$$\frac{\omega^{\frac{1}{2}}}{\sqrt{3}}(\Sigma_0 + 2\mathrm{e}^{-\frac{\pi}{9}\delta_1}\Sigma_1) = \bar{c}_0\Sigma_0 + \bar{r}_0\mathrm{e}^{-\frac{\pi\beta^2}{\bar{a}+b}}\Sigma_1$$

which implies that $\bar{c}_0 = \frac{\omega^{\frac{1}{2}}}{\sqrt{3}}$ and $\bar{r}_0 = \frac{2\omega^{\frac{1}{2}}\lambda_1^{-\frac{1}{6}}}{\sqrt{3}}$. Hence $S_{10}([\mathcal{M}_3]) = \frac{1}{3}\psi(1) = \frac{1}{3}\bar{c}_0 = \frac{\omega^{\frac{1}{2}}}{3\sqrt{3}}$.

CALCULATION OF $S_{11}([\mathcal{M}_3]) = S_{12}([\mathcal{M}_3])$. We now consider the second of our two equations, which may be written as

$$\frac{\omega^{\frac{1}{2}}\beta^2\lambda^{-\frac{1}{2}}}{\sqrt{3}\sqrt{\bar{a}+b}}(\Sigma_0 + \mathrm{e}^{-\frac{\pi}{9}\delta_1}(\omega^2-\omega)\Sigma_1) = \bar{c}_1\Sigma_0 + \overline{r_1}\mathrm{e}^{-\frac{\pi\beta^2}{\bar{a}+b}}\Sigma_1$$

which implies that $\bar{c}_1 = \frac{\omega^{\frac{1}{2}}\lambda^{-\frac{1}{6}}}{\sqrt{3}}$ and $\overline{r_1} = \frac{\omega^{\frac{1}{2}}(\omega^2-\omega)\lambda^{-\frac{1}{6}}\lambda_1^{-\frac{1}{6}}}{\sqrt{3}}$. Since S_{11} and S_{12} are normalized by $\lambda^{\frac{1}{6}}$, it follows that $S_{11}([\mathcal{M}_3]) = \frac{1}{3}\lambda^{\frac{1}{6}}\psi(1) = \frac{1}{3}\lambda^{\frac{1}{6}}\overline{c_1} = \frac{\omega^{\frac{1}{2}}}{3\sqrt{3}}$ and, from the equalities between φ_{10} and φ_{11} discussed earlier, that $S_{12}([\mathcal{M}_3]) = S_{11}([\mathcal{M}_3]) = \frac{\omega^{\frac{1}{2}}}{3\sqrt{3}}$.

10. APPENDIX

For $z, t \in \mathbb{C}$, where $\mathrm{Im}(t) > 0$, we define the theta function ϑ_3 by

$$\vartheta_3(z,t) = \sum_n \mathrm{e}^{\pi \mathrm{i} t n^2}\mathrm{e}^{\mathrm{i}2nz}$$

where the summation is over the integers. The Jacobi transformation formula for ϑ_3 is

$$\vartheta_3(z,t) = (-\mathrm{i}t)^{-\frac{1}{2}}\mathrm{e}^{\frac{z^2}{\pi \mathrm{i} t}}\vartheta_3(\frac{z}{t}, -\frac{1}{t}).$$

We prove the following formula for two-dimensional theta functions (where we have a sum double-indexed over the integers in independent variables m and n).

LEMMA 10.1. *Let $\gamma, \delta, \mu, \nu, \zeta \in \mathbb{C}$ with positive real parts, and let $\Delta = \gamma\delta - \zeta^2$. Then*

$$\sum_{m,n}\mathrm{e}^{-\pi\gamma m^2}\mathrm{e}^{-\pi\delta n^2}\mathrm{e}^{-\pi\mu m}\mathrm{e}^{-\pi\nu n}\mathrm{e}^{2\pi\zeta mn}$$
$$= \Phi \sum_{m,n}\mathrm{e}^{-\pi\frac{\gamma}{\Delta}m^2}\mathrm{e}^{-\pi\frac{\delta}{\Delta}n^2}\mathrm{e}^{\pi \mathrm{i}\frac{\gamma\nu+\zeta\mu}{\Delta}m}\mathrm{e}^{-\pi \mathrm{i}\frac{\delta\mu+\zeta\nu}{\Delta}n}\mathrm{e}^{2\pi\frac{\zeta}{\Delta}mn}$$

provided the series converge, where $\Phi = \frac{1}{\sqrt{\Delta}} e^{\frac{\pi}{4\Delta}(\mu(\delta\mu+\zeta\nu)+\nu(\gamma\nu+\zeta\mu))}$.

Proof. We begin by separating the sum as

$$\sum_n e^{-\pi\delta n^2} e^{-\pi\nu n} \sum_m e^{-\pi\gamma m^2} e^{(2\pi\zeta n-\pi\mu)m}$$

and replacing the second sum by a theta function to get

$$\sum_n e^{-\pi\delta n^2} e^{-\pi\nu n} \vartheta_3\left(\frac{2\pi\zeta n-\pi\mu}{2\mathrm{i}}, \mathrm{i}\gamma\right).$$

We now apply the Jacobi inversion formula to obtain

$$\sum_n e^{-\pi\delta n^2} e^{-\pi\nu n} \gamma^{-\frac{1}{2}} e^{\frac{(2\pi\zeta n-\pi\mu)^2}{4\pi\gamma}} \vartheta_3\left(\frac{\pi\mu-2\pi\zeta n}{2\gamma}, \frac{\mathrm{i}}{\gamma}\right)$$

which simplifies to

$$\gamma^{-\frac{1}{2}} e^{\frac{4\mu^2}{4\gamma}} \sum_n e^{-\pi(\delta-\frac{\zeta^2}{\gamma})n^2} e^{-\pi(\nu+\frac{\zeta\mu}{\gamma})n} \sum_m e^{-\frac{\pi}{\gamma}m^2} e^{\frac{\mathrm{i}\pi(2\zeta n-\mu)}{\gamma}m}.$$

We now wish to perform inversion on n. Separating the sums as

$$\gamma^{-\frac{1}{2}} e^{\frac{-\pi\mu^2}{4\gamma}} \sum_m e^{-\frac{\pi}{\gamma}m^2} e^{-\frac{\mathrm{i}\pi\mu}{\gamma}m} \sum_n e^{-\pi(\delta-\frac{\zeta^2}{\gamma})n^2} e^{-\pi(\nu+\frac{\zeta\mu}{\gamma}-\frac{2\mathrm{i}\zeta m}{\gamma})n}$$

allows us to write $\gamma^{-\frac{1}{2}} e^{\frac{\pi\mu^2}{4\gamma}} \sum_m e^{-\frac{\pi}{\gamma}m^2} e^{-\frac{\mathrm{i}\pi\mu}{\gamma}m} \vartheta_3\left(\frac{\mathrm{i}\pi\nu\gamma+\mathrm{i}\pi\zeta\mu+2\pi\zeta m}{2\gamma}, \frac{\mathrm{i}\Delta}{\gamma}\right)$. Applying the Jacobi inversion to this yields

$$\Delta^{-\frac{1}{2}} e^{\frac{\pi\mu^2}{4\gamma}} \sum_m e^{-\frac{\pi}{\gamma}m^2} e^{-\frac{\mathrm{i}\pi\mu}{\gamma}m} e^{-\frac{\pi(\mathrm{i}\gamma\nu+\mathrm{i}\zeta\mu+2\zeta m)^2}{4\gamma\Delta}} \vartheta_3\left(\frac{\pi(\gamma\nu+\zeta\mu-2\mathrm{i}\zeta m)}{2\Delta}, \frac{\mathrm{i}\gamma}{\Delta}\right).$$

Expanding the theta function into a sum gives

$$\Delta^{-\frac{1}{2}} e^{\frac{\pi\mu^2}{4\gamma}} \sum_m e^{-\frac{\pi}{\gamma}m^2} e^{-\frac{\mathrm{i}\pi\mu}{\gamma}m} e^{-\frac{\pi\mathrm{i}(\gamma\nu+\zeta\mu-2\mathrm{i}\zeta m)^2}{4\gamma\Delta}} \sum_n e^{-\frac{\pi\gamma}{\Delta}n^2} e^{\frac{\pi\mathrm{i}(\gamma\nu+\zeta\mu-2i\zeta m)}{\Delta}n}.$$

Collecting terms and returning to the double summation gives

$$\frac{1}{\sqrt{\Delta}} e^{\frac{\pi\mu^2}{4\gamma}} e^{\frac{\pi(\gamma\nu+\zeta\mu)^2}{4\gamma\Delta}} \sum_{m,n} e^{-\pi\frac{\delta}{\Delta}m^2} e^{-\pi\frac{\mathrm{i}(\delta\mu+\zeta\nu)}{\Delta}m} e^{-\pi\frac{\gamma}{\Delta}n^2} e^{\pi\frac{\mathrm{i}(\gamma\nu+\zeta\mu)}{\Delta}n} e^{2\pi\frac{\zeta}{\Delta}mn},$$

and upon expanding the constant terms and making the substitutions $m \mapsto n$, $n \mapsto m$, we obtain, with Φ as defined earlier,

$$\Phi \sum_{m,n} e^{-\pi\frac{\gamma}{\Delta}m^2} e^{-\pi\frac{\delta}{\Delta}n^2} e^{\pi\mathrm{i}\frac{(\gamma\nu+\zeta\mu)}{\Delta}m} e^{-\pi\mathrm{i}\frac{(\delta\mu+\zeta\nu)}{\Delta}n} e^{2\pi\frac{\zeta}{\Delta}mn}. \quad \blacksquare$$

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